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# EXTENSIONS OF A DUALITY THEOREM CONCERNING g-INVERSES OF MATRICES*

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*SUMMARY.* Rao and Mitra (1971a,b) and Sibuya (1970) have shown that the conjugate transpose of a minimium norm g-inverse of a matrix is a least squares g-inverse of its conjugate transpose under the dual norm. In this paper this duality relation is examined for the minimum seminorm and semileast squares inverses.

## 1. Introduction

We use $\mathcal{E}^n$ to denote the vector space of complex $n$-tuples. For integers $m$ and $n$ let the seminorm of $x \epsilon \mathcal{E}^n$ and $y \epsilon \mathcal{E}^m$ be defined by

$$\|x\|_n = (x^*Nx)^{\frac{1}{2}}, \quad \|y\|_m = (y^*My)^{\frac{1}{2}} \qquad \dots \quad (1.1)$$

where $M$ and $N$ are nonnegative definite matrices. As in Rao and Mitra (1971a, b) and Mitra and Rao (1974), we define the following.

(a) $G$ is a g-inverse of $A$ if $x = Gy$ is a solution of the consistent equation $Ax = y$, $\forall y \epsilon \mathcal{M}(A)$, the column space of $A$. We represent such an inverse by $A^-$, the entire class by $\{A^-\}$, and the subclass satisfying $A^-AA^- = A^-$ by $\{A_r^-\}$.

(b) $G$ is a minimum $N$-seminorm g-inverse of $A$ if $\forall y \epsilon \mathcal{M}(A)$, $x = Gy$ is a solution of the equation $Ax = y$, and if $u$ is any other solution then $\|Gy\|_n \leqslant \|u\|_n$. We represent $G$ by $A^-_{m(N)}$ and the class by $\{A^-_{m(N)}\}$.

(c) $G$ is a $M$-semileast squares inverse of $A$ if $\forall y \epsilon \mathcal{E}^m$, $x = Gy$ provides a minimum of $\|y - Ax\|_m$ for all $x \epsilon \mathcal{E}^n$. We represent $G$ by $A_{l(M)}$ and the class by $\{A_{l(M)}\}$. Note that there is a subclass of $\{A_{l(M)}\} \subset \{A^-\}$ which we denote by $\{A^-_{l(M)}\}$.

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Rao and Mitra (1971a, b) and Sibuya (1970) established the following duality relationship between minimum norm and least squares g-inverses and indicated the key role it plays in the Gauss-Markoff Theory of linear estimation. If $M$ and $\Lambda$ are positive definite matrices such that $M\Lambda = I$, then

$$\{(A^*)^-_{m(M)}\} = \{[A^-_{l(\Lambda)}]^*\}. \qquad \dots \quad (1.2)$$

In this paper we examine the nature of the duality relationship when $M$ and/or $\Lambda$ are possibly positive semidefinite.

## 2. Some lemmas

We reproduce here (without proof) from Rao and Mitra (1971a,b) two basic results we need in our present study.

Lemma 1: *A matrix $G$ is $A^-_{m(N)}$ if and only if*

$$AGA = A \text{ and } (GA)^*N = NGA. \qquad \dots \quad (2.1)$$

*If $G_0$ is a particular solution of (2.1), a general solution is given by*

$$G = G_0 + W(I - AG_0) + (I - G_0A)V \qquad \dots \quad (2.2)$$

*where $W$ is arbitrary and $V$ is an arbitrary solution of the equation*

$$N(I - G_0A)V = 0. \qquad \dots \quad (2.3)$$

*The matrix $(N + A^*A)^-A^*[A(N + A^*A)^-A^*]^-$ is one choice for $G_0$.*

Lemma 2: *A matrix $G$ is $A_{l(M)}$ if and only if*

$$MA = G^*A^*MA \qquad \dots \quad (2.4)$$

*or equivalently*

$$MAGA = MA \text{ and } (AG)^*M = MAG. \qquad \dots \quad (2.5)$$

*If $G_0$ is a particular solution of (2.4), a general solution is given by*

$$G = G_0 + [I - (A^*MA)^-A^*MA]U \qquad \dots \quad (2.6)$$

*where $U$ is arbitrary. The matrix $(A^*MA)^-A^*M$ is one choice for $G_0$.*

Lemma 3: *$A^-_{l(M)}$ exists. If $G_0$ is one choice of $A^-_{l(M)}$, a general solution to $A^-_{l(M)}$ is given by*

$$G = G_0 + [I - (A^*MA)^-A^*MA]U \qquad \dots \quad (2.7)$$

*where $U$ is an arbitrary solution of*

$$[A - A(A^*MA)^-A^*MA]UA = 0. \qquad \dots \quad (2.8)$$

*The matrix $A^- + (A^*MA)^-A^*M(I - AA^-)$ is one choice for $G_0$.*

*Proof*: The proof of Lemma 3 is easy and is therefore omitted.

Lemma 4 : $$\{A_{l(M)}\} = \{A^{-}_{\bar{l}(M)}\} \qquad \ldots \quad (2.9)$$

*if and only if*

$$Rank\ (A^{*}MA) = Rank\ (A). \qquad \ldots \quad (2.10)$$

*If* (2.10) *holds, AG is unique for each* $G\epsilon\{A_{l(M)}\}$. *The unique expression for AG is* $A(A^{*}MA)^{-}A^{*}M$.

*Proof*: Let $G_0$ be a particular choice for $A^{-}_{\bar{l}(M)}$ naturally of $A_{l(M)}$. A comparison of the expressions (2.6) and (2.7) for the respective general solutions shows that (2.9) is true if and only if arbitrary matrices $U$ satisfy (2.8) for which it is necessary and sufficient that

$$A - A(A^{*}MA)^{-}A^{*}MA = 0 \Longleftrightarrow (2.10).$$

The uniqueness of $AG$ under (2.10) is easily established.

## 3. Main theorems

Theorem 1 : *Let* $A$ *be a* $m\times n$ *matrix and* $M,\Lambda$ *be positive semidefinite matrices of order* $m\times m$ *each. Then,*

(a) $$\{[A^{-}_{\bar{l}(M)}]^{*}\} \subset \{(A^{*})^{-}_{m(\Lambda)}\} \qquad \ldots \quad (3.1)$$

*if and only if one of the following conditions* (i) *or* (ii) *is true*

$$\text{(i)}\quad Rank\ (A^{*}MA) < Rank\ A \qquad \ldots \quad (3.2)$$

$$\mathcal{M}(\Lambda) \subset \mathcal{M}(A) \qquad \ldots \quad (3.3)$$

$$\text{(ii)}\quad Rank\ (A^{*}MA) = Rank\ A \qquad \ldots \quad (3.4)$$

$$A^{*}M\Lambda Q = 0 \qquad \ldots \quad (3.5)$$

*where* $Q$ *is a matrix such that* $\mathcal{M}(Q) = \mathcal{N}(A^{*})$, *the nullspace of* $A^{*}$.

(b) *For a given* $M$, *if* (3.4) *is true a general nonnegative definite solution* $\Lambda$ *of* (3.5) *is*

$$\Lambda = \Lambda_0 + (I-H^{*})\Lambda_1(I-H) \qquad \ldots \quad (3.6)$$

*where* $\Lambda_0$ *and* $\Lambda_1$ *are arbitrary nonnegative definite matrices of order* $m\times m$ *each with*

$$\mathcal{M}(\Lambda_0) \subset \mathcal{M}(A) \qquad \ldots \quad (3.7)$$

*and*

$$H = MA(A^{*}MA)^{-}A^{*}. \qquad \ldots \quad (3.8)$$

(c) *For a given* $\Lambda$, *if* (3.3) *holds,* (3.1) *is true for arbitrary nonnegative definite matrices* $M$. *If* (3.3) *is untrue a general nonnegative definite solution* $M$ *of* (3.4) *and* (3.5) *is*

$$M = E^{-}[\Lambda QQ^{*}\Lambda U_1\Lambda QQ^{*}\Lambda + AA^{*}U_2AA^{*}](E^{-})^{*} + (I-E^{-}E)U_3(I-E^{-}E)^{*} \quad (3.9)$$

*where* $E = \Lambda QQ^*\Lambda + AA^*$, $E^-$ *is an arbitrary g-inverse of* $E$, $U_1$ *and* $U_3$ *are arbitrary nonnegative definite matrices and* $U_2$ *is arbitrary positive definite.*

*Proof of* (a) : Consider the general solution to $A^-_{l(M)}$ given in Lemma 3. If (3.1) holds, $AG\Lambda$ is hermitian for every $G$ determined by (2.7) and (2.8). This implies that

$$[A - A(A^*MA)^- A^*MA]U\Lambda$$

is hermitian for every $U$ satisfying (2.8).

$$\begin{aligned}
&[A - A(A^*MA)^- A^*MA]U\Lambda U^*[A - A(A^*MA)^- A^*MA]^* \\
&\quad = [A - A(A^*MA)^- A^*MA]U[A - A(A^*MA)^- A^*MA]U\Lambda \\
&\quad = 0 \text{ for every } U \text{ satisfying (2.8)} \\
&\quad \Longrightarrow [A - A(A^*MA)^- A^*MA]U\Lambda = 0 \qquad \ldots . \quad (3.10)
\end{aligned}$$

for every $U$ satisfying (2.8). For which either (3.3) or (3.4) is necessary. Also when (3.4) is true, by Lemma 4, $AG$ is unique and equal to $H^*$ for every $G \epsilon \{A_{l(M)}\} = \{A^-_{l(M)}\}$.

Hence (3.1) $\Longleftrightarrow$

$$H^*\Lambda = \Lambda H \Longleftrightarrow (3.5). \qquad \ldots \quad (3.11)$$

This shows both the necessity and sufficiency of (3.5) under (3.4). Sufficiency of (3.3) is easily established.

*Proof of* (b) : To obtain a general nonnegative definite solution $\Lambda$ of (3.5) or equivalently of (3.11), observe that a nonnegative definite matrix $\Lambda$ can always be expressed as $\Lambda = CC^*$ for some $C$. Also, since

$$\mathscr{M}(H^*) \oplus \mathscr{M}(I - H^*) = \mathscr{E}^m,$$

$$C = H^*U_1 + (I - H^*)U_2$$

for some $U_1$ and $U_2$. Hence

$$\Lambda = H^*U_1U_1^*H + H^*U_1U_2^*(I-H) + (I-H)^*U_2U_1^*H + (I-H)^*U_2U_2^*(I-H).$$

However,

$$H^*\Lambda(I-H) = H^*U_1U_2^*(I-H) = 0 \Longrightarrow$$

$$\Lambda = H^*U_1U_1^*H + (I-H^*)U_2U_2^*(I-H)$$

which is of the required form (3.6). Conversely, if $\Lambda$ is expressible as in (3.6)

$$H^*\Lambda = H^*\Lambda_0 = A(A^*MA)^- A^*M\Lambda_0 = \Lambda_0$$

in view of (3.7). Hence $\Lambda$ satisfies (3.11).

*Proof of* (c) : The first part of (c) is easy. To show that (3.9) is the general solution to a nonnegative definite $M$ satisfyng (3.4) and (3.5), check first by direct multiplication that since $\mathcal{M}(\Lambda Q)$ and $\mathcal{M}(A)$ are virtually disjoint, for a matrix $M$ determined by (3.9), $A^*M\Lambda Q = 0$. Also $A^*MA = A^*U_2A$ is of same rank as $A$. Conversely if $M_0$ satisfies (3.4) and (3.5), $M = M_0$ is a nonnegative definite solution of

$$EME^* = F \qquad \ldots \quad (3.12)$$

where $E = \Lambda QQ^*\Lambda + AA^*$ and $F = \Lambda QQ^*\Lambda M_0\Lambda QQ^*\Lambda + AA^*M_0AA^* = \Lambda QQ^*\Lambda M_0\Lambda QQ^*\Lambda + AA^*(M_0+QQ^*)AA^*$. The expression (3.9) therefore follows from Lemma 2.1 of Khatri and Mitra (1975) where we identify $M_0$ with $U_1$ and $M_0+QQ^*$ with $U_2$. That $M_0+QQ^*$ is positive definite is seen as follows. We note first that $\mathcal{M}(M_0A)$ and $\mathcal{M}(Q)$ are virtually disjoint. Also since $M_0$ satisfies (3.4)

$$\mathcal{M}(M_0A : Q) = \mathcal{M}(M_0A) \oplus \mathcal{M}(Q) = \mathcal{E}^m.$$

Further, $\mathcal{M}(M_0A : Q) \subset \mathcal{M}(M_0 : Q) = \mathcal{M}(M_0+QQ^*) \subset \mathcal{E}^m$. Hence $M_0+QQ^*$ which is clearly nonnegative definite is also of full rank. This concludes the proof of Theorem 1.

*Note* 1 : An alternative expression for a general solution to (3.4) and (3.5) was given by Rao (1971, 1973) as follows :

$$M = (\Lambda + AUA^*)^- + K \qquad \ldots \quad (3.13)$$

where $U$ and $K$ are arbitrary Hermitian matrices subject to the conditions that $M$ is nonnegative definite,

$$\mathcal{M}(\Lambda + AUA^*) = \mathcal{M}(\Lambda : A),\ A^*KA = 0 \text{ and } \Lambda K\Lambda = 0.$$

Theorem 2 : *Let $A$ be a $m \times n$ matrix, $Q$ be defined as in Theorem* 1 *and $M, \Lambda$ be positive semidefinite matrices of order $m \times m$ each. Then*

(a) $$\{(A^*)^-_{m(\Lambda)}\} \subset \{[A^-_{l(M)}]^*\} \qquad \ldots \quad (3.14)$$

*if and only if*

$$\Lambda + AA^* \textit{ is positive definite} \qquad \ldots \quad (3.15)$$

*or equivalently*

$$\textit{Rank } (Q^*\Lambda Q) = \textit{Rank } Q \qquad \ldots \quad (3.16)$$

*and*

$$A^*M\Lambda Q = 0. \qquad \ldots \quad (3.5)$$

(b) *For a given* $\Lambda$, *if* (3.15) *is true, a general nonnegative definite solution $M$ of* (3.5) *is given by*

$$M = H^*\Lambda_0 H + (I-H)^*\Lambda_1(I-H)$$

*where $\Lambda_0$ and $\Lambda_1$ are arbitrary nonnegative definite matrices and*

$$H^* = (\Lambda + AA^*)^{-1}A[A^*(\Lambda + AA^*)^{-1}A]^- A^* \qquad \ldots \quad (3.17)$$

(c) *For a given $M$, a general nonnegative definite solution $\Lambda$ of* (3.5) *and* (3.16) *is given by*

$$\Lambda = B^{-}[MAA^{*}MU_1MAA^{*}M+QQ^{*}U_2QQ^{*}](B^{-})^{*} + (I-B^{-}B)U_3(I-B^{-}B)^{*} \quad \ldots \quad (3.18)$$

*where $B = MAA^{*}M+QQ^{*}$, $B^{-}$ is an arbitrary g-inverse of $B$, $U_1$ and $U_3$ are arbitrary nonnegative definite matrices and $U_2$ is arbitrary positive definite.*

*Proof of* (a) : For arbitrary choice of $(\Lambda+AA^{*})^{-}$

$$(\Lambda+AA^{*})^{-}A[A^{*}(\Lambda+AA^{*})^{-}A]^{-} \in \{(A^{*})^{-}_{m(\Lambda)}\}$$

Hence if (3.14) is true

$$A^{*}MA[A^{*}(\Lambda+AA^{*})^{-}A]^{-}A^{*}(\Lambda+AA^{*})^{-} = A^{*}M. \quad \ldots \quad (3.19)$$

The left hand side of (3.19) is therefore invariant under choice of $(\Lambda+AA^{*})^{-}$ which can hold iff (3.15) is true (see in Rao, Mitra and Bhimasankaram (1972)).

Also (3.19) $\Longrightarrow A^{*}MA[A^{*}(\Lambda+AA^{*})^{-}A]^{-}A^{*} = A^{*}M(\Lambda+AA^{*})$

$\Longrightarrow$ (3.5).

These show the necessity of (3.5) and (3.15).

For the sufficiency part assume now that (3.5) and (3.15) hold and let $G$ satisfy the conditions

$$AGA = A, \; AG\Lambda = \Lambda G^{*}A^{*} \quad \ldots \quad (3.20)$$

that is, let $G^{*} \in \{(A^{*})^{-}_{m(\Lambda)}\}$. (3.20) and (3.5) $\Longrightarrow$

$$A^{*}MAG\Lambda = A^{*}MAG^{*}A^{*} = A^{*}M\Lambda$$

$\Longrightarrow A^{*}MAG(\Lambda+AA^{*}) = A^{*}M(\Lambda+AA^{*})$, which on account of (3.15)

$\Longrightarrow A^{*}MAG = A^{*}M \Longrightarrow G \in \{A^{-}_{l(M)}\}$.

Hence (3.5) and (3.15) $\Longrightarrow$ (3.14).

*Proofs of* (b) *and* (c) : Proofs of (b) and (c) are similar to that of the corresponding results in Theorem 1 and are therefore omitted. The following two corollaries are easily established.

Corollary 1 : *Let $A$ be a $m\times n$ matrix and $M,\Lambda$ be nonnegative definite matrices satisfying* (3.4), (3.5) *and* (3.16). *Then*

$$\{(A^{*})^{-}_{m(\Lambda)}\} = \{[A^{-}_{l(M)}]^{*}\}. \quad \ldots \quad (1.2)$$

Conversely if Rank $A < m$, then (1.2) $\Longrightarrow$ (3.4), (3.5) and (3.16).

Corollary 2 : *Let $A$ be a $m\times n$ matrix, $M$. $\Lambda$ be positive semidefinite matrices of order $m\times m$ each and $Q$ be defined as in Theorem* 1. *Then*

$$\{(A^{*})^{-}_{m(\Lambda)}\} \subset \{[A^{-}_{l(M)}]^{*}\} \Longrightarrow \{[Q^{-}_{l(\Lambda)}]^{*}\} \subset \{(Q^{*})^{-}_{m(M)}\}. \quad \ldots \quad (3.14)$$

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