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ON OPTIMUM INVARIANT TEST OF INDEPENDENCE OF TWO SETS OF VARIATES WITH ADDITIONAL INFORMATION ON COVARIANCE MATRIX

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Abstract

Test of independence of two sets of variates has been considered under the assumpotion that a part of the covariance matrix is known. This has been interpreted as that of testing the problem with incomplete data. LRT for the problem has been obtained by Olkin and Sylvan (1977). We have derived an optimum invariant test which is LMPI and locally minimax but the test is not LRT. However, under special situation LRT has been shown to be UMPI.

Key words

Incomplete data, independence of two sets of variates, optimum invariant test, locally minimax.

1. Introduction

Let $\underset{\sim}{X}_\alpha(p \times 1), \alpha = 1, \ldots, N$ be N independent observations from $N_p(\mu, \sum)$. Let us partition $\underset{\sim}{X}_\alpha = (\underset{\sim}{X}'_{1\alpha}, \underset{\sim}{X}'_{2\alpha})$, where $\underset{\sim}{X}_{i\alpha}$ is a $p_i \times 1$ vector, $i = 1, 2, p_1 + p_2 = p$. Similarly partition

$$\underset{\sim}{\mu} = \begin{pmatrix} \underset{\sim}{\mu}_1 \\ \underset{\sim}{\mu}_2 \end{pmatrix}, \quad \sum = \begin{pmatrix} \sum_{11} & \sum_{12} \\ \sum_{21} & \sum_{22} \end{pmatrix} \tag{1.1}$$

Let us assume that the elements of $\sum_{22}$ are known and hence, without any loss of generality, we assume that $\sum_{22} = I_{p_2}$. Under this set up, the problem is to test

$$H_O[\sum_{21} = 0] \quad \text{against} \quad H[\sum_{21} \neq 0] \tag{1.2}$$

The data of this kind have been considered by Olkin and Sylvan (1977), where they have studied the problems of estimation and testing concerning correlations and $\sum$.

Now this type of model may be interpreted in terms of the model with missing (or extra) observations as follows:

Consider N observations on $\underset{\sim}{X}_1$ and $N + M$ observations on $\underset{\sim}{X}_2$ and all the observations are independent. This means there are M extra observations on $\underset{\sim}{X}_2$. This can be regarded as a special case of monotone sample defined generally by Bhargava (1962). Now for large M, $\sum_{22}$ may be assumed to be a known matrix and we have the above model. Eaton and Kariya(1974) considered the case when M is finite.

It has been shown by Olkin and Sylvan that the likelihood ratio test (LRT) of the problem (1.2) is the same as that obtained when $\sum$ is unknown and arbitrary. Thus extra information on $\underset{\sim}{X}_2$ components i.e., $\sum_{22}$ known has no effect on the LRT of this problem. In this article we have derived an optimum invariant test for (1.2) which is locally most powerful invariant (LMPI) and locally minimax level α test but this is not LRT. Further for $p_2 = 1$, the LRT is uniformly most powerful invariant (UMPI) level α test for this problem.

2. Reduction of the data

To construct an optimum invariant test for the problem (1.2), we reduce the given data of Section 1 by sufficiency and translation, under which testing problem remains invariant It is known that a sufficient statistic for $(\underset{\sim}{\mu}, \sum)$ is $(\bar{X}, S)$,

where $\bar{\underset{\sim}{X}} = \frac{1}{N} \sum_{\alpha=1}^{N} \underset{\sim}{X}_\alpha$ and $S = \sum_{\alpha=1}^{N} (\underset{\sim}{X}_\alpha - \bar{\underset{\sim}{X}})(\underset{\sim}{X}_\alpha - \bar{\underset{\sim}{X}})$.

Since the problem is invariant under $\bar{\underset{\sim}{X}} \to \bar{\underset{\sim}{X}} + \underset{\sim}{a}$ and $S \to S$, where $\underset{\sim}{a}$ is a $p \times 1$ vector, the reduced sample space is S and the corresponding parameter space is $\sum > 0, \sum_{22} = I_{p_2}$. Hence, without any loss of generality, we consider the data $(S_{11.2}, S_{21}, S_{22})$, when $S_{11.2} = S_{11} - S_{12} S_{22}^{-1} S_{21}$, which is $1-1$ to S and where

$$\begin{aligned} S_{11.2} &\sim W_{p_1}\left(n_1, p_1, \textstyle\sum_{11.2}\right), \quad n_1 = N - p_2 - 1 \\ S_{21}|S_{22} &\sim N\left(S_{22} \textstyle\sum_{21}, S_{22} \otimes \sum_{11.2}\right) \\ S_{22} &\sim W_{p_2}(n, P_2, I_{p_2}), n = N - 1 \end{aligned} \tag{2.1}$$

and $S_{11.2}$ and (S_{21}, S_{22}) are independently distributed.

Reduction by invariance

The problem (1.2) remains invariant under the group G of transformations, where

$$G = \left\{ g = \begin{pmatrix} g_1 & O \\ O & g_2 \end{pmatrix} \right\} \tag{2.2}$$

where $g_1 \varepsilon Gl(p_1)$, $g_2 \varepsilon O(p_2)$. The group action on sample space is

$$S_{11.2} \rightarrow g_1 S_{11.2} g_1^1, \; S_{21} \rightarrow g_2 S_{21} g_1', \; S_{22} \rightarrow g_2 S_{22} g_2' \tag{2.3}$$

and that on parameter space

$$(\textstyle\sum_{11.2}, \sum_{21}, I_{p_2}) \text{ is } \sum_{11.2} \rightarrow$$

$$\bar{g}_1 \textstyle\sum_{11.2} \bar{g}_1', \sum_{21} \rightarrow \bar{g}_2 \sum_{21} \bar{g}_1', \sum_{22} \rightarrow \bar{g}_2 \sum_{22} \bar{g}_2' = I_{p_2} \tag{2.4}$$

Proposition 1: A maximal invariant in the parameter space is $\delta_1 \geq \ldots \geq \delta_t$, $t = \min(p_1, p_2)$, where $\delta_1, \ldots, \delta_t$ are the ordered characteristic roots of the matrix $\sum_{21} \sum_{11.2}^{-1} \sum_{12}$. Let $\beta(p_2 \times p_1)$ be a diagonal matrix such that the diagonal element $\beta_{ii} = \sqrt{\delta_i}, i = 1, \ldots, t$. Then $tr\, \beta \beta' = \sum_{i=1}^{t} \delta_i = \delta(say)$.

The proof of the proposition is straightforward and hence omitted. Under the proposition 1, the hypothesis (1.2) can be written as

$$HO[\delta = 0] Vs\, H[\delta > 0] \tag{2.5}$$

Since the power function of an invariant test depends only on th ... ivariants in the parameter space, without any loss of generality, we may assume the data are such that from (2.1),

$$\begin{aligned} S_{11.2} &\sim W(n_1, p_1, I_{p_1}) \\ S_{21} | S_{22} &\sim N(S_{22}\beta, S_{22}(\otimes) I_{p_1}) \\ S_{22} &\sim W(n, p_2, I_{p_2}) \end{aligned} \tag{2.6}$$

where β is as defined in proposition 1.

In order to construct optimum invariant test for the problem (2.5) we consider the well-known Wijsman's representation theorem (1967, Theorem 4, eq. 3, Page 394) of the probability ratio of the maximal invariant in the sample space.

To apply the theorem we assume

$$S_{21} = X, \quad S_{11.2} = YY', \quad S_{22} = uu'.$$

Then from (2.6) we have,

$$\begin{aligned} X|u &\sim N(uu'\beta, \; uu' \otimes I_{p_1}) \\ Y &\sim N(O, I_{p_1} \otimes I_{n_1}) \\ u &\sim N(O, I_{p_2} \otimes I_n) \end{aligned} \tag{2.7}$$

3. Optimum invariant test for (2.5)

It has been shown by Olkin and Sylvan (1977) that the LRT for the problem rejects H_O for small values of the statistic

$$|I - S_{11}^{-1} S_{12} S_{22}^{-1} s_{21}| \tag{3.1}$$

For $p_2 = 1$, this becomes $1 - R^2$ where $R^2 = S_{21} S_{11}^{-1} S_{12}/S_{22}$, the square of the multiple correlation of X_2 on X_1 Hence the test which rejects for large values of R^2 can be shown to be UMPI level α test (as shown in theorem 2 below). In general, however, for $p_2 > 1$, (3.1) does not provide on UMPI test for the problem. To construct an optimum invariant test for this problem, we have from (2.7) the joint density of (X, Y, u) w.r.t. Lebesgue measure,

$$\begin{aligned} p(x, y, u) = C|uu'|^{-p_1/2} \exp[-\frac{1}{2} tr\{X'(uu')^{-1}X + YY'\} \\ + tr X\beta' - \frac{1}{2} tr \beta' uu' \beta - \frac{1}{2} tr\, uu'] \end{aligned} \tag{3.2}$$

In order to apply Wijsman's theorem, let ϑ be the left invariant Haar measure under G defined in (2.2.), $|J|$ the Jacobian of the transformation, where $|J| = |g_1 g_1'|^{-n/2}$ and R_δ the probability ratio of the maximal invariant in the sample space. Now $X'(uu')^{-1}X + YY'$ being non-singular, there exists a unique $g_0 \varepsilon G_T^+(p_1)$, a group of lower triangle matrix with positive diagonals, such the $g_O(X'(uu')^{-1}X + YY')g_O' = I_{p_1}$. Then substituting $(g_1 g_O, g_2)$ for (g_1, g_2) without changing the value of R_δ, we have from (3.2), after simplification.

$$R_\delta = D_1^{-1} \int_{G_t(p_1)} |g_1 g_1'|^{n/2} \exp[-\frac{1}{2} tr\, g_1 g_1'] \wedge (g_1) \vartheta (dg_1) \tag{3.3}$$

Where $D_1 = \int_{G_t(p_1)} |g_1 g_1'|^{n/2} \exp[-\frac{1}{2} tr g_1 g_1'] \vartheta(dg_1)$

$\wedge(g_1) = \int_{O(P_2)} \exp[-\frac{1}{2}\, tr\, \beta\, \beta'\, g_2\, uu'\, g_2' + tr\, X\, g_0'\, g_1'\, \beta'\, g_2]\, \vartheta\, (dg_2).$

Since explicit evaluation of (3.3) for $p_2 > 1$ is difficult for general alternatives, we consider local alternatives of (2.5). To evaluate R_δ explicitly under local alternatives we require the following results due to James (1960, 1961):

Lemma 1: Let $H \epsilon O(p)$ be orthogonal matrix in an orthogonal group $O(p)$ and $\vartheta(dH)$ is the invariant Haar measure on $O(p)$. Then

$$(i) \quad \int_{O(p)} tr\,(AH)^{2j+1}\, \vartheta(dH) = 0, j = 0, 1, \ldots$$

$$(ii) \quad \int_{O(p)} tr\, B_1\, H\, B_2\, H^1\, \vartheta(dH) = \frac{1}{p}\, tr\, B_1\, tr\, B_2 \tag{3.4}$$

$$(iii) \quad \int_{O(p)} (trB_1 H)^2 \vartheta(dH) = \frac{1}{p} trB_1 B_1^1$$

Where A, B_1, B_2 are matrices conformable for multiplication.

Expanding the integrand in $\wedge(g_1)$ of (3.3) and applying the results of lemma 1, we obtain

$$\wedge(g_1) = 1 - \frac{1}{2p_2}\, tr\, \beta\, \beta'\, tr\, uu' + \frac{1}{2p_2} trg_0 X' X g_0' g_1' \beta' \beta g_1 + O(tr\, \beta\, \beta') \tag{3.5}$$

Hence from (3.3) and (3.5), we have,

$$R_\delta = 1 - \frac{1}{2p_2}\, tr\, \beta'\, \beta\, tr\, uu' + \frac{1}{2p_2} D_1^{-1} \int_{G_t(p_1)} |g_1 g_1'|^{n/2} \exp[-\frac{1}{2}\, tr\, g_1\, g_1']\, tr\, (g_0\, X'\, X\, g_0'\, g_1\, \beta'\, \beta\, g_1)\, \nu\, (dg_1) + O(tr\, \beta'\, \beta) \tag{3.6}$$

Now let $g_1 = h_1 k_1$, where $h_1 \epsilon G_T^+(p_1)$, $k_1 \epsilon O(p_1)$. Indtroducing this in the integral of (3.6) and on repeated application of (ii) and (iii) of lemma 1, and remembering that $h_1 h_1' \sim W_{p_1}(n, I_{p_1})$, we have from (3.6) after simplification,

$$R_\delta = 1 - \frac{1}{2p_2} tr\,\beta'\beta\, tr\, uu' + \frac{n}{2p_1p_2} tr\beta'\beta tr g_0 X'Xg_0' + o(tr\beta'\beta) \tag{3.7}$$

It is easy to show that the remainder $o(tr\beta'\beta)$ is uniform in (X, Y, u).
Since $tr\, g_0 X'Xg_0' = tr\, S_{21}S_{11}^{-1}$ and $tr\, uu' = tr\, S_{22}$, applying Neyman - Pearson lemma we have the following:

Theorem 1: Let $\varphi \epsilon \partial_\alpha$ be the level α test function in a class of all invariant level α test functions ∂_α such that

$$\varphi = 1, \text{if } \frac{n}{p_1} tr\, S_{21}S_{11}^{-1}S_{12} - tr\, S_{22} > K$$
$$0, \text{otherwise} \tag{3.8}$$

where K is chosen to make φ level α. Then φ is unique locally most prowerful invariant (LMPI) test for H_0.

Theorem 2: When $p_2 = 1$, the test which rejects H_0 for large values of $U = S_{21}S_{11.2}^{-1}S_{12}$ is UMP invariant level α test in a class of level α invariant tests in ∂_α.

Proof: For $p_2 = 1$, g_2 in (2.2) is a scalar and in this case the group under which the problem remains invariant is

$$G = \left\{ g = \begin{pmatrix} g_1 & O \\ O & \pm 1 \end{pmatrix} \right\} \tag{3.9}$$

Under this situation, from (3.3) the explicit form of R_δ can be easily shown to be

$$R_\delta = \exp[-\frac{1}{2}\delta S_{22}] \sum_{j=0}^{\infty} \frac{\delta^j}{2^j j!} \frac{(\frac{n}{2})_j}{(\frac{p_1}{2})_j} (S_{21}S_{11}^{-1}S_{12})^j$$
$$= \exp[-\frac{1}{2}\delta S_{22}] \sum_{j=0}^{\infty} \frac{(\delta S_{22})^j}{2^j j!} \frac{(\frac{n}{2})_j}{(\frac{p_1}{2})_j} (\frac{U}{1+U})^j$$

where $U = \frac{S_{21}S_{11.2}^{-1}S_{12}}{S_{22}} = \frac{R^2}{1-R^2}$, where R^2 is the square of the multiple correlation of X_2 on $\underset{\sim}{X}_1$.
From R_δ above, the joint *p.d.f.* of (S_{22}, U) can be easily obtained and hence the marginal *p.d.f* of U is obtained as follows.

$$f_\delta(U) = \sum_{j=0}^{\infty} \frac{\delta^j}{(1+\delta)^{\frac{n}{2}+j}} \frac{\Gamma\frac{n}{2}+j\Gamma\frac{n}{2}+j}{\Gamma\frac{n}{2}\Gamma\frac{p_1+2}{2}\Gamma\frac{n-p_1}{2}} \frac{U^{\frac{p_1}{2}+j}}{(1+U)^{\frac{n}{2}+j}} \tag{3.10}$$

It is easy to show that $f_\delta(U)/f_0(U)$ has a monotone likelihood ratio in U and δ . Hence the test which rejects H_O for large values of U is unconditionally $UMPI$ level α test in ∂_α which is LRT as stated in (3.1)

3.1 Local minimaxity of the test (3.8)

To demonstrate that the test (3.8) is locally minimax in the sense of Giri and Kiefer (1964), the first step is to reduce the original problem, using Hunt - Stein theorm. It is easy to show that the group

$$G = \left\{ g = \begin{pmatrix} g_1 & O \\ O & g_2 \end{pmatrix} \right\} \tag{3.11}$$

where $g_1 \varepsilon G_T(p_1)$ is non-singular lower triangular matrix and $g_2 \varepsilon O(p_2)$ is an orthogonal matrix, which leaves the origninal problem invariant, will satisfy the conditions of Hunt - Stein theorem.
To obtain the probability ratio of the maximal invariant R_δ under G_o, we observe that a left-invariant measure on G_T is

$$\vartheta(dg_1) = \pi_{i=1}^{p_1} g_{ii}^{-(p_1-i+1)} \pi_{i \geq j} dg_{ij}$$

and the Jacobian of the transformations is $|J| = \pi_{i=1}^{p_1} g_{ii}^{-n}$.
Then from (3.3), R_δ under G_o may be written

$$R_\delta = D_1^{-1} \int_{G_T(p_1)} |g_1 g_1'|^{\frac{n}{2}} \exp[-\frac{1}{2} tr\, g_1 g_1'] \int_{O(p_2)} \exp[-\frac{1}{2} tr\, \beta \beta' g_2 uu' g_2' + tr\, X g_0' g_1' \beta' g_2] \nu(dg_1) \nu(dg_2) \tag{3.12}$$

Let $v' = X g_o', \theta = \beta' g_2$ and we first integrate over $G_T(p_1)$ for fixed $g_2 \varepsilon O(p_2)$. Then using $\nu(dg_1)$ and $|J|$ as obtained above, we have from (3.12),

$$R_\delta = \int_{O(p_2)} D_1^{-1}\{ \int_{G_T(p_1)} \pi_{i=1}^{p_1} g_{ii}^{(n-p_1+i-1)} \exp[-\frac{1}{2}\sum_{i\geq j=1}^{p_1} g_{ij}^2$$

$$+ \sum_{i\geq j=1}^{p_1} (\sum_k \theta_{ik}\nu_{jk}) g_{ji}] \pi_{i\geq j}^{p_1} dg_{ij}\} \exp[-\frac{1}{2} tr\,\beta\beta' g_2 uu' g_2'] \nu(dg_2)$$

$$= \int_{O(p_2)} [\exp\left\{\frac{1}{2}\sum_{i>j}(\sum_{k=1}^{p_1}\theta_{ik}v_{jk})^2\right\} \pi_{j=1}^{p_1} F_1(\frac{n-p_2+i-1}{2}), \frac{1}{2}, \frac{1}{2}(\sum_{k=1}^{p_i}\theta_{jk}v_{jk})^2]$$

$$\exp[-\frac{1}{2} tr\,\beta\beta' g_2 uu' g_2'] \nu(dg_2).$$

For local minimaxity, we write

$$R_\delta = 1 + \frac{1}{2}\int_{O(p_2)} [\sum_{i>j}(\sum_k \theta_{ik} v_{jk})^2 + \sum_{j=1}^{p_1}(n-p_1+j-1)(\sum_k \theta_{jk} v_{jk})^2$$

$$- tr\,\beta\beta' g_2 uu' g_2' + R] \nu(dg_2)$$

$$= 1 + \frac{\delta}{2}[\int_{O(p_2)} \sum_{i>j}(\sum_k \eta_{ik} v_{jk})^2 + \sum_{j=1}^{p_1}(n-p_1+j-1)(\sum_k \eta_{jk} v_{jk})^2$$

$$- \frac{1}{\delta} tr\,\beta\beta' g_2 uu' g_2' + R]\nu(dg_2) \qquad (3.13)$$

where $\eta_{ij} = \theta_{ij}/\delta$.

Now choosing

$$\underset{\sim}{\eta}_i \underset{\sim}{\eta}_i' = \varepsilon(n-p_1+i-1)^{-1}(n-p_1+i)^{-1} p_1^{-1}(n-p_1) n_1$$

where $\underset{\sim}{\eta}_i = (\eta_{i1}, \ldots \eta_{i+1})$, and transforming $\eta \to gH$, where H is uniformly distributed over $O(p_1)$, we have (following Schewartz (1967) on averaging over $O(p_1)$, the quantity

$$\delta E_g[\sum_{i>j}(\sum_k g_{ik} v_{jk})^2 + \sum_j (n-p_1+i-1)(\sum_k g_{jk} v_{jk})^2]$$

$$= \delta\varepsilon p_1^{-1}\; n\; tr\; v'\; v \qquad (3.14)$$

Where $\delta\varepsilon = \delta\; tr\; \eta\eta' = tr\; \theta\theta' - tr\; \beta\beta'$

Thus on taking expectation over g, (3.14) becomes independent on g_2. Hence substituting (3.14) in (3.13) and integrating over $g_2 \varepsilon O(p_2)$ and using (ii) of lemma 1, we have

$$R_\delta = 1 + \frac{n}{2p_1} tr\,\beta\,\beta'\, tr\, v' v - \frac{1}{2} tr\,\beta\,\beta'\, tr\, uu' + o(tr\,\beta\,\beta')$$
$$= 1 + \frac{\delta}{2}[\frac{n}{p_1} tr\, S_{21}\, S_{11}^{-1}\, S_{12} - tr\, S_{22}] + o(\delta) \qquad (3.15)$$

Hence from Giri and Kiefer (1964) we have the following:

Theorem 3: For testing $H_0[\delta = 0]$ against $H[g > 0]$, the test (3.8), which is LMPI, is locally minimax in the sense of Giri and Kiefer (1964).

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