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Indicator method for a recurrence relation for order statistics

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Abstract: An indicator method is used to derive a recurrence relation satisfied by the distribution of order statistics from n random variables having an arbitrary joint distribution.

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1. Introduction

Let $X_1, X_2, \dots, X_n$ be n random variables having an arbitrary joint distribution, with $X_{1:n} \leqslant X_{2:n} \leqslant \cdots \leqslant X_{n:n}$ as the order statistics obtained by arranging these n variables. Let $F_{r:n}(x_1)$ denote the distribution function of the order statistic $X_{r:n}$. Further, let $F_{r:n-1}^{[i]}(x_1)$ denote the distribution function of the order statistic $X_{r:n-1}^{[i]}$, where $X_{r:n-1}^{[i]}$ denotes the rth order statistic in $n-1$ variables obtained by dropping X_i from the original n variables.

For $-\infty < x_1 < \infty$, let us define the events A_i as

$$A_i = \{X_i \leqslant x_1\},$$

for $i = 1, 2, \dots, n$. For any event A, let χ_A be the indicator variable taking on the value 1 when A occurs and 0 otherwise.

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By considering the special case when X_i's are independent and non-identically distributed, Balakrishnan (1988) and Bapat and Beg (1988) derived some recurrence relations satisfied by distributions of order statistics. Recently, Sathe and Dixit (1990) proved these relations for the general case when the X_i's are arbitrarily distributed by using set theoretic arguments. Their results were used by Balakrishnan et al. (1992) to establish some general relations and identities satisfied by order statistics arising from n arbitrarily distributed variables. In this note, we use indicator functions of sets to prove these recurrence relations for order statistics. This approach, in addition to being simpler, lends itself to easy generalizations for higher orders.

2. Relations for single order statistic

Here, we prove the 'triangle rule' (Arnold and Meeden, 1975; Arnold, 1977) for order statistics from arbitrary variables in the following theorem.

Theorem 1. *For* $1 \leqslant r \leqslant n-1$ *and* $x_1 \in \mathbb{R}$,

$$rF_{r+1:n}(x_1) + (n-r)F_{r:n}(x_1) = \sum_{i=1}^{n} F_{r:n-1}^{[i]}(x_1). \tag{2.1}$$

Proof. Let us consider

$$\phi = E\prod_{j=1}^{n}\left(t_1\chi_{A_j} + t_2(1-\chi_{A_j})\right) \tag{2.2}$$

$$= \sum_{r=0}^{n} t_1^r t_2^{n-r} \Pi_r, \tag{2.3}$$

where Π_r is the probability of exactly r of the A_j's happening. It may be noted that for $r = 0, 1, 2, \ldots, n$,

$$\Pi_r = F_{r:n}(x_1) - F_{r+1:n}(x_1), \tag{2.4}$$

or, equivalently,

$$F_{r:n}(x_1) = \sum_{j=r}^{n} \Pi_j, \tag{2.5}$$

with the convention that $F_{0:n}(x_1) \equiv 1$ and $F_{n+1:n}(x_1) \equiv 0$. From (2.3), we obtain

$$\frac{\partial\phi}{\partial t_1} + \frac{\partial\phi}{\partial t_2} = \sum_{r=0}^{n}\left\{rt_1^{r-1}t_2^{n-r} + (n-r)t_1^r t_2^{n-r-1}\right\}\Pi_r$$

$$= \sum_{r=0}^{n-1} t_1^r t_2^{n-r-1}\left\{(r+1)\Pi_{r+1} + (n-r)\Pi_r\right\}. \tag{2.6}$$

Alternatively, from (2.2) we directly find

$$\frac{\partial\phi}{\partial t_1} + \frac{\partial\phi}{\partial t_2} = \sum_{i=1}^{n} E\prod_{\substack{j=1\\ j\neq i}}^{n}\left(t_1\chi_{A_j} + t_2(1-\chi_{A_j})\right),$$

which, upon using (2.3), yields

$$\frac{\partial\phi}{\partial t_1} + \frac{\partial\phi}{\partial t_2} = \sum_{i=1}^{n}\sum_{r=0}^{n-1} t_1^r t_2^{n-r-1}\Pi_r^{[i]}, \tag{2.7}$$

where

$$\Pi_r^{[i]} = F_{r:n-1}^{[i]}(x_1) - F_{r+1:n-1}^{[i]}(x_1).$$

Upon comparing the coefficients of $t_1^R t_2^{n-R-1}$ in (2.6) and (2.7), we get

$$(R+1)\Pi_{R+1} + (n-R)\Pi_R = \sum_{i=1}^{n} \Pi_R^{[i]},$$

which may be rewritten as

$$\left\{(R+1)\Pi_{R+1} - R\Pi_R\right\} + n\Pi_R = \sum_{i=1}^{n} \Pi_R^{[i]}. \tag{2.8}$$

The triangle rule in (2.1) follows from (2.8) by adding over R from r to n. □

3. Conclusion

The indicator function method used in this paper for deriving a recurrence relation for order statistics from arbitrary variables can be suitably adopted to extend many other known recurrence relations and identities for order statistics in the i.i.d. case to the arbitrary case. Work in this direction is currently being done.

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References

Arnold, B.C. (1977), Recurrence relations between expectations of functions of order statistics, *Scand. Actuar. J.*, 169–174.

Arnold, B.C. and G. Meeden (1975), Characterization of distributions by sets of moments of order statistics, *Ann. Statist.* 3, 754–758.

Balakrishnan, N. (1988), Recurrence relations for order statistics from n independent and non-identically distributed random variables, *Ann. Inst. Statist. Math.* **40**, 273–277.

Balakrishnan, N., S.M. Bendre and H.J. Malik (1992), General relations and identities for order statistics from non-independent non-identical variables, to appear in: *Ann. Inst. Statist. Math.*

Bapat, R.B. and M.I. Beg (1988), Order statistics for nonidentically distributed variables and permanents, *Sankhyā* **51**, 79–93.

Sathe, Y.S. and U.J. Dixit (1990), On a recurrence relation for order statistics, *Statist. Probab. Lett.* **9**, 1–4.