EXPECTATION OF PRODUCT OF QUADRATIC FORMS

By AMOD KUMAR

Indian Statistical Institute

SUMMARY. If x is $1 \times m$ vector and $x \sim n(0, 1)$, $A_1, A_2, \ldots, A_n$ are a $m \times m$ symmetric matrices, then an expression for expectation of $((xA_1x')(xA_2x') \ldots (xA_nx'))$ is given in terms of traces of products of matrices A_i.

1. Notation and introduction

E stands for expectation of the stochastic variable. Trace of a matrix is denoted by tr. When $r \geqslant 3$, $S(j_1, j_2, \ldots, j_r)$ denotes $\Sigma A_{j_{i_1}} A_{j_{i_2}} \ldots A_{j_{i_r}}$, the summation being over the $(r-1)!/2$ distinct permutations $(i_1, i_2, \ldots, i_r)$ of $(1, 2, \ldots, r)$ where the permutations $(i_1, i_2, \ldots, i_r)$, $(i_2, i_3, \ldots, i_r, i_1)$, $(i_r, i_{r-1}, \ldots, i_2, i_1)$ are considered to be the same. When $r = 2$, $S(j_1, j_2) = A_{j_1} A_{j_2}$ and $S(j_1) = A_{j_1}$. $p(n)$ is the number of solutions in integers of

$$n = 1y_1 + 2y_2 + \ldots + ny_n, \quad y_i \geqslant 0, \quad i = 1, 2, \ldots, n. \qquad \ldots (1.1)$$

For a particular solution $y_1, y_2, \ldots, y_n$ of (1.1) and a permutation $(j_1, j_2, \ldots, j_n)$ of $(1, 2, \ldots, n)$, we denote the product

$$\prod_{k_1=1}^{y_1} \{\operatorname{tr} S(j_{k_1})\} \prod_{k_2=1}^{y_2} \{\operatorname{tr} S(j_{y_1+2k_2-1}, j_{y_1+2k_2})\} \ldots$$

$$\prod_{k_r=1}^{y_r} \{\operatorname{tr} S(j_{y_1+2y_2+\ldots+(r-1)y_{r-1}+rk_r-r+1}\, j_{y_1+2y_2+\ldots+(r-1)y_{r-1}+rk_r-r+2} \cdots j_{y_1+2y_2+\ldots+(r-1)y_{r-1}+rk_r})$$

$$\cdot \quad \cdot \quad \cdot$$

$$\prod_{k_n=1}^{y_n} \{\operatorname{tr} S(j_{y_1+2y_2+\ldots+(n-1)y_{n-1}+nk_n-n+1}\, j_{y_1+2y_2+\ldots+(n-1)y_{n-1}+nk_n-n+2} \cdots j_{y_1+2y_2+\ldots+(n-1)y_{n-1}+nk_n})$$

by $P(j_1, j_2, \ldots, j_n;\, y_1, y_2, \ldots, y_n)$. Observe that if $y_r = 0$, the factor

$$\prod_{k_r=1}^{y_r} \{\operatorname{tr} S(j_{y_1+2y_2+\ldots+(r-1)y_{r-1}+rk_r-r+1}\, j_{y_1+2y_2+\ldots+(r-1)y_{r-1}+rk_r-r+2} \cdots j_{y_1+2y_2+\ldots+(r-1)y_{r-1}+rk_r})$$

will be absent in $P(j_1, j_2, \ldots, j_n;\, y_1, y_2 \ldots y_n)$.

Nagar (1959) has obtained the expectation of the product of two quadratic forms of normally distributed stochastic variables in terms of traces of matrices. Neudecker (1968) used the method of Kronecker product and was successful in obtaining the expectation of the product of two and three quadratic forms of normally distributed stochastic variables in terms of traces of matrices. The author of the present paper feels that his technique is much simpler than that of Neudecker (1968). Recently Dr. P. N. Mishra has written a paper entitled "Recurrence formula for mathematical expectation of products of matrices of structural disturbances" (*Sankhyā*, Series B, 34, 379-384, 1972). We can obtain expectation of products of three quadratic forms from Dr. Mishra's paper. But when $n = 4$, it is extremely difficult to obtain the expectation using Mishra's result. For the case of general n the expressions become too involved and derivation of expectation by using Dr. Mishra's result becomes unmanagable. But from the present results, it is easy to obtain the expectation for any n without any recurrence relation.

2. The expectation of a product of two quadratic forms of normally distributed independent stochastic variables

$E\{(xA_1x')(xA_2x')\}$ is the coefficient of $t_1 t_2$ in

$$E(\exp(t_1 xA_1x' + t_2xA_2x')) \quad \dots \quad (2.1)$$

(2.1) is same as $E[\exp\{x(t_1A_1+t_2A_2)x'\}]$.

Now $$E[\exp\{x(t_1A_1+t_2A_2)x'\}] = |I-2(t_1A_1+t_2A_2)|^{-\frac{1}{2}}$$

$$= \exp\left\{-\frac{1}{2}\log|I-2(t_1A_1+t_2A_2)|\right\}.$$

We can easily show that for some positive values of h_1, h_2 such that $|t_1| < h_1, |t_2| < h_2$, we have

$$\log(|I-2(t_1A_1+t_2A_2)|) = -\text{tr}\{2(t_1A_1+t_2A_2)\} - \text{tr}\{2(t_1A_1+t_2A_2)\}^2/2 - \dots$$
$$-\text{tr}\{2(t_1A_1+t_2A_2)\}^r/r - \dots \text{ ad inf}$$

$$\therefore \quad (2.1) = \exp\frac{1}{2}[\text{tr}(t_1A_1+t_2A_2))/1 + \text{tr}\{2(t_1A_1+t_2A_2)\}^2/2 + \dots + \text{tr}\{2(t_1A_1+t_2A_2)\}^r/r + \dots \text{ad inf}].$$

Now collecting coefficients t_1t_2 in (2.1) we get

$$E\{(xA_1x')(xA_2x')\} = 2\,\text{tr}(A_1A_2) + \text{tr}\,A_1\,\text{tr}\,A_2.$$

3. The expectation of a product of three quadratic forms of normally distributed independent stochastic variables

$$E\{(xA_1x')(xA_2x')(xA_3x')\}$$

is the coeficient of $t_1t_2t_3$ in $\quad E(\exp(t_1xA_1x'+t_2xA_2x'+t_3xA_3x'))$.

We proceed as in Section 2 and obtain that

$$E\{(xA_1x')(xA_2x')(xA_3x')\} = 8\,\text{tr}(A_1A_2A_3) + 2\,\text{tr}\,A_1\,\text{tr}\,A_2A_3 + 2\,\text{tr}\,A_2\,\text{tr}\,A_3A_1$$
$$+2\,\text{tr}\,A_3\,\text{tr}\,A_1A_2 + \text{tr}\,A_1\,\text{tr}\,A_2\,\text{tr}\,A_3.$$

4. The expectation of a product of $n(>3)$ quadratic forms of normally distributed independent stochastic variables

$$E\{(xA_1x')(xA_2x')\dots(xA_nx')\}$$

is the coefficient of $t_1t_2\dots t_n$ in $\quad E(\exp(t_1xA_1x'+t_2xA_2x'+\dots+t_nxA_nx')) \quad \dots \quad (4.1)$

We proceed as in Section 2 and obtain that for some positive values of $h_1, h_2, \dots, h_n$ such $|t_1| < h_1$, $|t_2| < h_2, \dots, |t_n| < h_n$: we have

$$(4.1) = \exp\frac{1}{2}[\text{tr}\{2(t_1A_1+t_2A_2+\dots+t_nA_n)\}/1 + \text{tr}(2(t_1A_1+t_2A_2+\dots+t_nA_n))^2/2 + \dots$$
$$+\text{tr}\{2(t_1A_1+t_2A_2+\dots+t_nA_n)\}^r/r + \dots \text{ ad inf}] \quad \dots \quad (4.2)$$

Expanding (4.2) in exponential series and collecting the terms involving $t_1 t_2 \dots t_n$, we see that we have to consider the terms

$$\{1/(y_1+y_2+\dots+y_n)!\}\{(y_1+y_2+\dots+y_n)!/y_1!y_2! \dots y_n!\}\left(\frac{1}{2}\right)^{y_1+y_2+\dots+y_n}$$

$$\times[\mathrm{tr}\,\{2(t_1A_1+t_2A_2+\dots+t_nA_n)\}/1]^{y_1}\times[\mathrm{tr}\{2(t_1A_1+t_2A_2+\dots+t_nA_n)\}^2/2]^{y_2}$$

$$\dots\times[\mathrm{tr}\{2(t_1A_1+t_2A_2+\dots+t_nA_n)\}^r/r]^{y_r}\times\dots\times[\mathrm{tr}\{2(t_1A_1+t_2A_2+\dots+t_nA_n)\}^n/n]^{y_n} \quad \dots \quad (4.3)$$

for all solutions $(y_1, y_2, \dots, y_n)$ of (1.1).

Now to find the coefficient of $t_1 t_2 \dots t_n$ in (4.3), we see that for any partition

$$\{j_1\}, \dots, \{j_{y_1}\}, \{j_{y_1+1}\, j_{y_1+2}\}, \dots, \{j_{y_1+2y_2-1}\, j_{y_1+2y_2}\}, \dots,$$

$$\{j_{y_1+2y_2+\dots+(r-1)y_{r-1}+1}\, j_{y_1+2y_2+\dots+(r-1)y_{r-1}+2} \dots j_{y_1+2y_2\dots+(r-1)y_{r-1}+r}\}, \dots,$$

$$\{j_{y_1+2y_2+\dots+ry_r-r+1}\, j_{y_1+2y_2+\dots+ry_r-r+2} \dots j_{y_1+2y_2+\dots ry_r}\}, \dots,$$

$$\{j_{y_1+2y_2+\dots+ny_n-n+1}\, j_{y_1+2y_2+\dots+ny_n-n+2} \dots j_{y_1+2y_2+\dots+ny_n}\} \quad \dots \quad (4.4)$$

arising from the permutation $(j_1, j_2, \dots, j_n)$ of $(1, 2, \dots, n)$ we have a contribution.

For this particular partition (4.4) the coefficient of $t_1 t_2 \dots t_n$ from (4.3) is

$$\{1/(y_1+y_2+\dots+y_n)!\}\left\{(y_1+y_2+\dots+y_n)!/y_1!y_2! \dots y_n!\left(\frac{1}{2}\right)\right\}^{y_1+y_2+\dots+y_n}(2^{y_1}/1)$$

$$\times(2^2\times 2/2)^{y_2}(2^3\times 2\times 3/3)^{y_3}(2^4\times 2\times 4/4)^{y_4}\dots(2^r\times 2\times r/r)^{y_r}\dots$$

$$(2^n\times 2\times n/n)^{y_n}\times y_1!\, y_2! \dots y_n!\times P(j_1, j_2, \dots, j_n;\, y_1, y_2, \dots, y_n)$$

$$= 2^{n-y_1-y_2}P(j_1, j_2, \dots, j_n;\; y_1, y_2, \dots, y_n).$$

Now taking into account all the $n!/\{(1!)^{y_1}y!(2!)^{y_2}y_2! \dots (n!)^{y_n}y_n!\}$ modes of partition for a particular solution $y_1, y_2, \dots, y_n$ of (1.1) we get the coefficient of $t_1 t_2 \dots t_n$ from (4.3) as

$$= 2^{n-y_1-y_2}\Sigma P(j_1, j_2, \dots, j_n;\; y_1, y_2, \dots, y_n)$$

where the summation is over all $n!/(1!)^{y_1}y_1!(2!)^{y_2}y_2! \dots (n!)^{y_n}y_n!$ permutations $(j_1, j_2, \dots, j_n)$ of $(1, 2, \dots, n)$ giving rise to distinct partition (4.4).

Hence the coefficient of $t_1 t_2 \dots t_n$ in (4.2) is

$$= \Sigma 2^{n-y_1-y_2}P(j_1, j_2, \dots, j_n;\; y_1, y_2, \dots, y_n)$$

where the summation is over all solutions $y_1, y_2, \dots, y_n$ of (1.1) and (for fixed $y_1, y_2, \dots, y_n$) all permutations $(j_1, \dots, j_n)$ of $(1, 2, \dots, n)$ giving rise to distinct partition (4.4) which is our expression for $E\{(xA_1x')(xA_2x') \dots (xA_nx')\}$.

5. ILLUSTRATION

We will obtain $E\{(xA_1x')(xA_2x')(xA_3x')(xA_4x')\}$. Here $p(n) = 5$. A recurrence formula for the values of $p(n)$ for $n = 1, 2, \ldots, 100$ are given in Hall, Marshall Jr. (1967, pp. 32).

$$\therefore \quad E\{(xA_1x')(xA_2x')(xA_3x')(xA_4x')\}$$

$$= 16 \operatorname{tr}\{(A_1A_2A_3A_4)+(A_1A_2A_4A_3)+(A_1A_4A_2A_3)\}+8 \operatorname{tr} A_1 \operatorname{tr} A_2A_3A_4+8 \operatorname{tr} A_2 \operatorname{tr} A_3A_4A_1$$

$$+8 \operatorname{tr} A_3 \operatorname{tr} A_4A_2A_1+8 \operatorname{tr} A_4 \operatorname{tr} A_1A_2A_3+4 \operatorname{tr} A_1A_2 \operatorname{tr} A_3A_4+4 \operatorname{tr} A_1A_3 \operatorname{tr} A_2A_4+$$

$$+4 \operatorname{tr} A_1A_4 \operatorname{tr} A_2A_3+2 \operatorname{tr} A_1A_2 \operatorname{tr} A_3 \operatorname{tr} A_4+2 \operatorname{tr} A_1A_3 \operatorname{tr} A_4 \operatorname{tr} A_2+2 \operatorname{tr} A_1A_4 \operatorname{tr} A_2 \operatorname{tr} A_3$$

$$+2 \operatorname{tr} A_2A_3 \operatorname{tr} A_1 \operatorname{tr} A_4+2 \operatorname{tr} A_2A_4 \operatorname{tr} A_1 \operatorname{tr} A_3+2 \operatorname{tr} A_3A_4 \operatorname{tr} A_1 \operatorname{tr} A_2+\operatorname{tr} A_1 \operatorname{tr} A_2 \operatorname{tr} A_3 \operatorname{tr} A_4.$$

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