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Covariance identities for exponential and related distributions

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Abstract

Bobkov and Houdre (1997) proved that if ξ, η and ζ are independent standard exponential random variables, then for any two absolutely continuous functions f and g such that $E|f(\xi)|^2 < \infty$ and $E|g(\xi)|^2 < \infty$, the equality $\operatorname{Cov}(f(\xi), g(\xi)) = Ef'(\xi+\eta)g'(\xi+\zeta)$ holds. We prove that the identity holds if and only if ξ, η and ζ or $-\xi, -\eta$ and $-\zeta$ are standard exponential random variables.

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1. Main result

Suppose ξ, η and ζ are independent standard exponential random variables. Then for any two absolutely continuous functions f, g such that $E|f(\xi)|^2 < \infty$ and $E|g(\xi)|^2 < \infty$ are finite, the identity

$$\operatorname{Cov}[f(\xi), g(\xi)] = E[f'(\xi+\eta)g'(\xi+\zeta)] \tag{1.1}$$

holds. This result is due to Bobkov and Houdre (1997). We now prove that the identity (1.1) characterizes the standard exponential up to a sign, that is, either ξ or $-\xi$ have a standard exponential distribution.

Theorem. *Suppose that ξ, η, ζ are independent and identically distributed random variables such that the identity* (1.1) *holds for all absolutely continuous functions f and g such that $E|f(\xi)|^2$ and $E|g(\xi)|^2$ are finite, then either ξ and hence η and ζ are standard exponential random variables or $-\xi$ and hence $-\eta$, and $-\zeta$ are standard exponential random variables.*

Proof. Suppose that the relation (1.1) holds. Let $f(x) = \exp(itx)$ and $g(y) = \exp(isy)$ for some real t and s. Let $\phi_\xi(t)$ denote the characteristic function of ξ. Then Eq. (1.1) reduces to

$$\phi_\xi(t+s) - \phi_\xi(t)\phi_\xi(s) = -ts\phi_\xi(t)\phi_\xi(s)\phi_\xi(t+s) \tag{1.2}$$

for all $-\infty < t,\ s < \infty$. We claim that $\phi_\xi(t) \neq 0,\ -\infty < t < \infty$. In other words $\phi_\xi(.)$ is nonvanishing on the real line. On the contrary suppose that $\phi_\xi(t) = 0$ for some $t = t_0$. Then it follows from (1.2) that

$$\phi_\xi(t_0 + s) = 0, \quad -\infty < s < \infty.$$

Hence $\phi_\xi(t) = 0,\ -\infty < t < \infty$ which is impossible since $\phi_\xi(0) = 1$. Let $\psi(t) = [\phi_\xi(t)]^{-1}$. Then $\psi(t)$ is well defined since $\phi_\xi(t)$ is nonvanishing. Note that $\psi(t)$ is continuous, $\psi(0) = 1$ and $\overline{\psi(t)} = \psi(-t)$. Eq. (1.2) can be written in the form

$$\psi(t)\psi(s) - \psi(t+s) = -ts, \quad -\infty < t,\ s < \infty.$$

Define $\psi(t) = A(t) + \mathrm{i}B(t)$. Then $A(t) = A(-t)$ and $B(t) = -B(-t)$ since $\overline{\psi(t)} = \psi(-t)$. Furthermore, $A(t)$ and $B(t)$ are both continuous with $A(0) = 1$ and $B(0) = 0$. The above equation implies that

$$(A(t) + \mathrm{i}B(t))(A(s) + \mathrm{i}B(s)) - (A(t+s) + \mathrm{i}B(t+s)) = -ts, \quad -\infty < t,\ s < \infty. \tag{1.3}$$

Equating the real and imaginary parts of this equation, it follows that

$$A(t)A(s) - B(t)B(s) - A(t+s) = -ts, \quad -\infty < t,\ s < \infty \tag{1.4}$$

and

$$A(t)B(s) + B(t)A(s) - B(t+s) = 0, \quad -\infty < t,\ s < \infty. \tag{1.5}$$

Replacing s by $-s$ in (1.4), we have

$$A(t)A(-s) - B(t)B(-s) - A(t-s) = ts, \quad -\infty < t,\ s < \infty \tag{1.6}$$

or equivalently

$$A(t)A(s) + B(t)B(s) - A(t-s) = ts, \quad -\infty < t,\ s < \infty \tag{1.7}$$

since $A(s) = A(-s)$ and $B(s) = -B(-s)$. Adding (1.4) and (1.7) lead to the equation

$$2A(t)A(s) - A(t+s) - A(t-s) = 0, \quad -\infty < t,\ s < \infty \tag{1.8}$$

or equivalently

$$A(t+s) + A(t-s) = 2A(t)A(s), \quad -\infty < t,\ s < \infty, \tag{1.9}$$

where $A(t)$ is continuous, $A(0) = 1$, and $A(t) = A(-t)$. Applying the theorem on p. 120 of Aczel (1966), it follows that the function $A(t)$ has to be of the form $A(t) = 0$ for all t or $A(t) = \cosh bt$ or $A(t) = \cos bt$ for some real constant b. The solution $A(t) = 0$ for all t is not possible since $A(0) = 1$.

Replacing s by $-s$ in (1.5), it follows that

$$A(t)B(-s) + B(t)A(-s) - B(t-s) = 0, \quad -\infty < t,\ s < \infty. \tag{1.10}$$

Adding (1.5) and (1.10) and using the fact that $A(t) = A(-t)$ and $B(s) + B(-s) = 0$, we have

$$2B(t)A(s) - B(t+s) - B(t-s) = 0, \quad -\infty < t,\ s < \infty \tag{1.11}$$

or equivalently

$$B(t+s) + B(t-s) = 2B(t)A(s), \quad -\infty < t,\ s < \infty, \tag{1.12}$$

where $A(t)$ and $B(t)$ are continuous with $A(0) = 1$ and $B(t) = -B(-t)$. Applying Theorem 1 on p. 170 of Aczel (1966), the most general continuous solutions of (1.12) are of the form

$$B(t) = 0 \quad \text{for all } t \text{ and } A(t) \text{ arbitrary}$$

or

$$B(t) = c\cos bt + C\sin bt \quad \text{and} \quad A(t) = \cos bt$$

or

$$B(t) = c\cosh bt + C\sinh bt \quad \text{and} \quad A(t) = \cosh bt$$

or

$$B(t) = c + Ct \quad \text{and} \quad A(t) = 1 \quad \text{for all } t,$$

where b, c and C are arbitrary real constants. In view of the earlier remarks, the last three cases are the only possible solutions of (1.9) and (1.12).

If the second case holds, then it follows that $A(t) = \cos bt$ where b is not zero and $B(t) = c\cos bt + C\sin bt$. Since $B(t) = -B(-t)$, it follows that $B(t) = 2c\cos bt$, $-\infty < t < \infty$. Since $B(0) = 0$, we have $c = 0$. Hence $B(t) = 0$ for all t.

If the third case holds, then it follows that $A(t) = \cosh bt$ where b is not zero and $B(t) = c\cosh bt + C\sinh bt$. Since $B(t) = -B(-t)$, it follows that $B(t) = 2c\cosh bt$, $-\infty < t < \infty$. Since $B(0) = 0$, we have $c = 0$. Hence $B(t) = 0$ for all t.

If the last case holds, then $A(t) = 1$ for all t and $B(t) = c + Ct$. Since $B(0) = 0$, it follows that $c = 0$ and hence $B(t) = Ct$, $-\infty < t < \infty$.

Hence a complex-valued function $\psi(t) = A(t) + iB(t)$ with $\psi(0) = 1$ and $\psi(t) = \overline{\psi(-t)}$ is a solution of the functional equation (1.3) if and only if $\psi(t) = \cos bt$ or $\psi(t) = \cosh bt$ for some constant b different from zero or $\psi(t) = 1 + iCt$ for some real constant C.

Since $\psi(t)$ is the reciprocal of a characteristic function, it follows that $|\psi(t)| \geqslant 1$ for all t. Clearly this implies that $\psi(t)$ cannot be equal to $\cos bt$ for some constant b not equal to zero. On the other hand suppose that $\psi(t) = \cosh bt$ where b is not equal to zero. Then it follows that

$$\frac{(e^{-bt} - e^{bt})(e^{bs} - e^{-bs})}{4} = -ts, \quad -\infty < t < \infty$$

from Eq. (1.3). Let $s = -t$. Then it follows that

$$(e^{-bt} - e^{bt})^2 = 4t^2, \quad -\infty < t < \infty$$

where b is not equal to zero. This is impossible. Hence

$$\psi(t) = 1 + iCt, \quad -\infty < t < \infty$$

for some real constant C. Let $s = t$ in Eq. (1.4). Then we have

$$A^2(t) - B^2(t) - A(2t) = -t^2, \quad -\infty < t < \infty. \tag{1.13}$$

Since $A(t) = 1$ for all t and $B(t) = Ct$, it follws that $-C^2t^2 = -t^2$ or $C^2 = 1$. Hence

$$\psi(t) = 1 + it$$

for all t or

$$\psi(t) = 1 - it$$

for all t.

This proves that either

$$\phi_\xi(t) = \{(1 + it)^{-1}\} \tag{1.14}$$

or

$$\phi_\xi(t) = \{(1 - it)^{-1}\}. \tag{1.15}$$

Hence either ξ or $-\xi$ is a standard exponential random variable. This completes the proof of the theorem. □

Remark. (1) It is evident from the proof of the theorem that it is sufficient if identity (1.1) holds for functions of the type $f(x)=e^{itx}, g(x)=e^{isx}, -\infty < t,\ s<\infty$ for the validity of the theorem.

(2) Suppose ξ has an exponential distribution with parameter λ, that is, the density function of ξ is given by

$$p_\xi(x)=\lambda e^{-\lambda x},\quad 0<x<\infty,$$
$$p_\xi(x)=0\quad \text{otherwise}$$

for some fixed $\lambda>0$. It is easy to check that for any two absolutely continuous functions f and g such that $E|f(\xi)|^2<\infty$ and $E|g(\xi)|^2<\infty$,

$$\lambda^2\,\mathrm{Cov}[f(\xi),g(\xi)]=E[f'(\xi+\eta)g'(\xi+\eta)], \tag{1.16}$$

whenever ξ, η and ζ are independent exponential random variables with parameter $\lambda>0$. It is easy to show that the above relation holds for all such f and g if and only if ξ, η and ζ are independent standard exponentials with parameter $|\lambda|$ or $-\xi$, $-\eta$ and $-\zeta$ are independent standard exponentials with parameter $|\lambda|$. In general for any absolutely continuous functions f,g and h with ξ, η, ζ standard exponential random variables such that $E|f(h(\xi))|^2<\infty$ and $E|g(h(\xi))|^2<\infty$,

$$\mathrm{Cov}(f(h(\xi)),g(h(\xi))=E[f'(h(\xi+\eta))h'(\xi+\eta)g'(h(\xi+\zeta))h'(\xi+\zeta)]. \tag{1.17}$$

Conversely, if this identity holds for ξ, η and ζ i.i.d. for all absolutely continuous functions f,g and a fixed absolutely continuous function h with $h'(x)$ not equal to zero almost everywhere, then $h(\xi)$, $h(\eta)$ and $h(\zeta)$ are i.i.d. where ξ, η and ζ are i.i.d. standard exponentials or $-\xi$, $-\eta$ and $-\zeta$ are i.i.d. standard exponentials. This can be seen by an application of the theorem for the functions $f(h(.))$ and $g(h(.))$.

2. Extensions

We assume that all the expectations of random variables discussed in this section exist and E_ξ,Cov_ξ, etc. denote the expectation and the covariance etc. with respect to the distribution of ξ.

Suppose that ξ_1 and ξ_2 are independent random variables with ξ_1 as a standard exponential random variable. Let $f(x,y)$ and $g(x,y)$ be real-valued functions such that $f_x=\partial f/\partial x$ and $g_x=\partial g/\partial x$ exist almost everywhere. Then

$$\begin{aligned}\mathrm{Cov}[f(\xi_1,\xi_2),g(\xi_1,\xi_2)]&=E_{\xi_2}[\mathrm{Cov}_{\xi_1}(f(\xi_1,\xi_2),g(\xi_1,\xi_2))]\\&\quad+\mathrm{Cov}_{\xi_2}(E_{\xi_1}f(\xi_1,\xi_2),E_{\xi_1}g(\xi_1,\xi_2))\\&=E_{\xi_2}[E_{\xi_1,\eta_1,\zeta_1}[f_x(\xi_1+\eta_1,\xi_2)g_x(\xi_1+\zeta_1,\xi_2)]]\\&\quad+\mathrm{Cov}_{\xi_2}(E_{\xi_1}f(\xi_1,\xi_2),E_{\xi_1}g(\xi_1,\xi_2))\quad(\text{for } \xi_1,\eta_1, \text{ and } \zeta_1 \text{ are i.i.d. as } \xi_1)\\&=E[f_x(\xi_1+\eta_1,\xi_2)g_x(\xi_1+\zeta_1,\xi_2)]\\&\quad+\mathrm{Cov}_{\xi_2}(E_{\xi_1}f(\xi_1,\xi_2),E_{\xi_1}g(\xi_1,\xi_2)).\end{aligned} \tag{2.1}$$

In general if ξ_1 and $(\xi_2,\ldots,\xi_k)$ are independent and ξ_1 is a standard exponential random variable, then

$$\begin{aligned}\mathrm{Cov}[f(\xi_1,\xi_2,\ldots,\xi_k),g(\xi_1,\xi_2,\ldots,\xi_k)]&=E[f_{x_1}(\xi_1+\eta_1,\xi_2,\ldots,\xi_k)g_{x_1}(\xi_1+\zeta_1,\xi_2,\ldots,\xi_k)]\\&\quad+\mathrm{Cov}_{\xi_2,\ldots,\xi_k}(E_{\xi_1}f(\xi_1,\ldots,\xi_k),E_{\xi_1}(g(\xi_1,\ldots,\xi_k))\end{aligned} \tag{2.2}$$

for functions f and g with f_{x_1} and g_{x_1} finite almost everywhere. This can be seen by following the above arguments using conditioning on $(\xi_2,\ldots,\xi_k)$ and the fact that ξ_1 is a standard exponential random variable.

Special cases: (i) Let $f(x_1,\dots,x_k)=f_0(x_1+\cdots+x_k)$ and $g(x_1,\dots,x_k)=g_0(x_1+\cdots+x_k)$ where f_0 and g_0 are differentiable almost everywhere. Then $f_{x_1}(x_1,\dots,x_k)=f_0'(x_1+\cdots+x_k)$ and $g_{x_1}(x_1,\dots,x_k)=g_0'(x_1+\cdots+x_k)$ where f_0' and g_0' denote the derivatives of f_0 and g_0, respectively. Hence

$$\begin{aligned}&\mathrm{Cov}[f_0(\xi_1+\cdots+\xi_k),g_0(\xi_1+\cdots+\xi_k)]\\&\quad=E[f_0'(\xi_1+\eta_1+\xi_2+\cdots+\xi_k)g_0'(\xi_1+\zeta_1+\xi_2+\cdots+\xi_k)]\\&\qquad+\mathrm{Cov}_{\xi_2,\dots,\xi_k}(E_{\xi_1}f_0(\xi_1+\cdots+\xi_k),E_{\xi_1}g_0(\xi_1+\cdots+\xi_k)),\end{aligned} \tag{2.3}$$

whenever ξ_1, η_1 and ζ_1 are i.i.d. standard exponential random variables, ξ_1 is independent of $(\xi_2,\dots,\xi_k)$ and f_0 and g_0 are absolutely continuous functions with $E|f_0(\xi_1+\cdots+\xi_k)|^2<\infty$ and $E|g_0(\xi_1+\cdots+\xi_k)|^2<\infty$.

(ii) Let $f(x_1,\dots,x_k)=\sum_{i=1}^k x_i$ and $g(.)$ as in (i). Then $f_{x_1}\equiv 1$ and applying (2.2), we have

$$\begin{aligned}\mathrm{Cov}\left[\sum_{i=1}^k\xi_i,g(\xi_1,\dots,\xi_k)\right]&=E[g_{x_1}(\xi_1+\zeta_1,\xi_2,\dots,\xi_k)]\\&\quad+\mathrm{Cov}_{\xi_2,\dots,\xi_k}\left(E_{\xi_1}\left(\sum_{i=1}^k\xi_i\right),E_{\xi_1}g(\xi_1,\dots,\xi_k)\right),\end{aligned} \tag{2.4}$$

whenever ξ_1 is independent of $(\xi_2,\dots,\xi_k)$, and ξ_1 and ζ_1 are independent standard exponential random variables and $g(x_1,\dots,x_k)$ is a function such that g_{x_1} exists almost everywhere. Hence

$$\begin{aligned}\mathrm{Cov}\left[\sum_{i=1}^k\xi_i,g(\xi_1,\dots,\xi_k)\right]&=E[g_{x_1}(\xi_1+\zeta_1,\xi_2,\dots,\xi_k)]\\&\quad+\mathrm{Cov}_{\xi_2,\dots,\xi_k}(1+\xi_2+\cdots+\xi_k,E_{\xi_1}g(\xi_1,\xi_2,\dots,\xi_k))\\&=E[g_{x_1}(\xi_1+\zeta_1,\xi_2,\dots,\xi_k)]\\&\quad+\sum_{j=2}^k\mathrm{Cov}_{\xi_2,\dots,\xi_k}(\xi_j,E_{\xi_1}g(\xi_1,\cdots,\xi_k)).\end{aligned} \tag{2.5}$$

(iii) Suppose that $\xi_1,\dots,\xi_k$ are i.i.d. standard exponential random variables. Then $Z_k=\xi_1+\cdots+\xi_k$ has a gamma distribution with density

$$p_{Z_k}(z)=\frac{z^{k-1}\mathrm{e}^{-z}}{\Gamma(k)},\quad z\geqslant 0,\qquad p_{Z_k}(z)=0\quad\text{otherwise}. \tag{2.6}$$

Applying the result obtained in (i), we have

$$\mathrm{Cov}[f(Z_k),g(Z_k)]=E[f'(Z_k+\eta_1)g'(Z_k+\zeta_1)]+\mathrm{Cov}_{\xi_2,\dots,\xi_k}(E_{\xi_1}(f(Z_k)),E_{\xi_1}(g(Z_k)) \tag{2.7}$$

for any integer $k\geqslant 1$ where η_1 and ζ_1 are independent standard exponential random variables independent of ξ_1. Let $f(x)=\mathrm{e}^{itx}$ and $g(x)=\mathrm{e}^{isx}$. Then

$$\mathrm{Cov}[f(Z_k),g(Z_k)]=\phi_{Z_k}(t+s)-\phi_{Z_k}(t)\phi_{Z_k}(s),$$

$$E[f'(Z_k+\eta_1)g'(Z_k+\zeta_1)]=-ts\ \phi_{Z_k}(t+s)\phi_{\eta_1}(t)\phi_{\zeta_1}(s)$$

and

$$E_{\xi_1}[\mathrm{e}^{itZ_k}]=\mathrm{e}^{it(\xi_2+\cdots+\xi_k)}\phi_{\xi_1}(t).$$

Hence,

$$\mathrm{Cov}_{\xi_2,\dots,\xi_k}(E_{\xi_1}(f(Z_k)),E_{\xi_1}(g(Z_k)))=\phi_{\xi_1}(t)\phi_{\xi_1}(s)[\phi_{\xi_2+\cdots+\xi_k}(t+s)-\phi_{\xi_2+\cdots+\xi_k}(t)\phi_{\xi_2+\cdots+\xi_k}(s)]. \tag{2.8}$$

Identity (2.5) reduces to the equation

$$\phi_{Z_k}(t+s) - \phi_{Z_k}(t)\phi_{Z_k}(s) = -ts\ \phi_{Z_k}(t+s)\phi_{\eta_1}(t)\phi_{\zeta_1}(s) + \phi_{\xi_1}(t)\phi_{\xi_1}(s)[\phi_{\xi_2+\cdots+\xi_k}(t+s) - \phi_{\xi_2+\cdots+\xi_k}(t)\phi_{\xi_2+\cdots+\xi_k}(s)]. \tag{2.9}$$

Note that $\phi_{Z_k}(t) = (1-\mathrm{i}t)^{-k}, \phi_{\eta_1}(t) = (1-\mathrm{i}t)^{-1} = \phi_{\zeta_1}(t)$ and $\phi_{\xi_2+\cdots+\xi_k}(t) = (1-\mathrm{i}t)^{-k+1}$.

It is easy to see that the functional equation (2.9) is satisfied by the above solution which in turn gives an alternate proof for (2.7) by the bilinearity in f and g on both sides of (2.7) (cf. Bobkov and Houdre, 1997).

(iv) Suppose Z is a random variable such that $Z = \xi + W$ where ξ and W are independent random variables. Further suppose that the characteristic functions of Z, ξ and W satisfy the functional equation

$$\phi_Z(t+s) - \phi_Z(t)\phi_Z(s) = -ts\ \phi_Z(t+s)\phi_\xi(t)\phi_\xi(s) + \phi_\xi(t)\phi_\xi(s)[\phi_W(t+s) - \phi_W(t)\phi_W(s)] \tag{2.10}$$

for $-\infty < t,\ s < \infty$ where $\phi_\xi(t)$ denotes the characteristic function of ξ. Further suppose that the characteristic function of W is nonvanishing. It is easy to see that the functional equation (2.10) reduces to

$$\phi_\xi(t+s) - \phi_\xi(t)\phi_\xi(s) = -ts\ \phi_\xi(t+s)\phi_\xi(t)\phi_\xi(s) \tag{2.11}$$

for $-\infty < t,\ s < \infty$ which characterizes the standard exponential distribution for ξ by the results obtained in Section 1. It can be checked that the functional equation (2.10) holds if and only if for every two absolutely continuous functions f and g such that $E|f(Z)|^2 < \infty$ and $E|g(Z)|^2 < \infty$,

$$\operatorname{Cov}(f(Z), g(Z)) = E[f'(Z+\eta)g'(Z+\zeta)] + \operatorname{Cov}_W(E_\xi(f(\xi+W)), E_\xi(g(\xi+W)), \tag{2.12}$$

where ξ, η and ζ are i.i.d. standard exponential random variables and $Z = \xi + W$.

3. Covariance identity for the geometric distribution

Suppose X is a discrete random varible with the geometric distribution $P(X=k) = pq^{k-1}$, $k \geqslant 1$, $q = 1-p$, $0 < p < 1$. It is easy to check that

$$p^2 \operatorname{Cov}[f(X), g(X)] = qE[(f(X+Y) - f(X+Y-1))(g(X+Z) - g(X+Z-1))] \tag{3.1}$$

for any two functions f and g such that $E|f(X)|^2 < \infty$ and $E|g(X)|^2 < \infty$ where X, Y and Z are i.i.d. as X. This can be seen by checking the identity (3.1) for functions of the type $f(x) = \mathrm{e}^{\mathrm{i}tx}$ and $g(x) = \mathrm{e}^{\mathrm{i}sx}$ where t and s are arbitrary real numbers and then using the bilinearity (cf. Bobkov and Houdre, 1997). For such functions, we have the functional equation

$$p^2[\phi_X(t+s) - \phi_X(t)\phi_X(s)] = q(1-\mathrm{e}^{-\mathrm{i}t})(1-\mathrm{e}^{-\mathrm{i}s})\phi_X(t+s)\phi_X(t)\phi_X(s) \tag{3.2}$$

and it can be easily checked that

$$\phi_X(t) = p\mathrm{e}^{\mathrm{i}t}(1-q\mathrm{e}^{\mathrm{i}t})^{-1}, \quad -\infty < t < \infty$$

is a solution of (3.2).

Let us now suppose that X, Y and Z are i.i.d. nonnegative integer valued random variables such that the identity (3.1) holds. Let $f(x) = t^x$ and $g(x) = s^x$ where t and s are real. Then the identity (3.1) reduces to

$$p^2[m(ts) - m(t)m(s)] = q(t-1)(s-1)(ts)^{-1}m(ts)m(t)m(s), \quad 0 < t < \infty, \tag{3.3}$$

where $m(t)$ is the probability generating function of X. It is easy to see that $m(t)$ is nonzero for all t. Define $\beta(t) = t\ m(t)^{-1}$, $0 < t < \infty$ and $\gamma(u) = \beta(\mathrm{e}^u)$, $-\infty < u < \infty$. Then Eq. (3.3) can be written in the form

$$\gamma(u+v) = \gamma(u)\gamma(v) - \lambda(\mathrm{e}^u - 1)(\mathrm{e}^v - 1), \quad -\infty < u,\ v < \infty \tag{3.4}$$

with $\gamma(0)=1$ and $\lambda = qp^{-2} > 0$, $q = 1-p$, $0 < p < 1$. It is clear that

$$\gamma(u) = (1 - q\mathrm{e}^{-u})p^{-1}, \quad -\infty < u < \infty \tag{3.5}$$

is a solution of (3.4) and hence

$$m(t) = pt(1-qt)^{-1}, \quad 0 < t < \infty$$

is a solution of (3.3) which is the probability generating function of the geometric distribution with parameter p. The problem that it is the only solution of (3.3) remains open. We conjecture that it is the only solution following the analogy of the characterization of the standard exponential distribution discussed in Section 1.

References

Aczel, J., 1966. Lectures on Functional Equations. Academic Press, New York.
Bobkov, S.G., Houdre, C., 1997. Converse Poincare-type inequalities for convex functions. Statist. Probab. Lett. 34, 37–42.