Optimal circular fit to objects in two and three dimensions

B.B. CHAUDHURI

Electronics & Communication Sciences Unit, Indian Statistical Institute, 203 B.T. Road, Calcutta-700035, India

Abstract: This paper deals with the problem of optimal circular fit to simply connected objects in two and three dimensions. In two dimensions, the centroid of the object is chosen as the center of the fitting circle while the radius is chosen so that its area is equal to the area of the object. Similarly, in three dimensions, the centroid of the object surface is chosen as the center of the fitting sphere while its radius is chosen so that the surface area of the object is equal to that of the sphere. It is proved that these choices optimize a modified sum of squares objective function.

Key words: Image processing, pattern recognition, shape analysis, circular fit, spherical fit.

1. Introduction

The problem of finding an optimal circular fit to simply connected objects in two and three dimensions is slightly different from that of circular fit to a set of scattered points where the concepts of area and perimeter [or volume and surface area in three dimensions] are absent.

To make an optimal fit, an objective function should be optimized with respect to the parameters namely, the coordinates of the center and the radius of the fitting circle. If r_i denotes the magnitude of the radius vector of a border point (x_i, y_i) from the center (x_0, y_0) and if r_0 is the radius of the fitting circle, then an objective function of the form $\sum (r_i - r_0)^2$ can be used, where the summation extends over all border points. The partial derivatives of this function with respect to x_0, y_0 and r_0, when equated to zero, lead to three equations for the parameters of the fitting circle. Separation of these equations for closed form solution so that the left-hand side contains a parameter while the right-hand side contains an expression only in terms of the border point coordinates (x_i, y_i) is not known. As an alternative, Landau (1987) suggested an iterative algorithm for the parameter estimation of the fitting circle. A modified expression of the form $\sum (r_i^2 - r_0^2)^2$ leads to closed form expressions for the parameters of the circle (Thomas and Chan, 1989; Takiyama and Ono, 1989). While this approach can be applied to closed objects as well as scattered points, the expressions for the coordinates of the center and the magnitude of the radius are complicated and involve a fairly large amount of computation. Also, for closed objects this approach relies on data from the border only, making it dependent on digital grid resolution, border noise as well as border shape variation due to rotation and scaling.

We propose a method for closed objects, that leads to strikingly simple expressions for the center and radius of the fitting circle. The objective function in our case is similar to that due to (Thomas and Chan, 1989) and it is optimized with respect to the location coordinates of the center but instead of radius r_0, it is optimized with respect to r_0^2. The computation involves all points in the object rather than only the border in two dimensions, and hence it is less sensitive to border noise and scaling.

2. Two-dimensional circular fit

Instead of starting with an objective function and then finding the parameters of the fitting circle by the method of partial derivatives, we start with the solution and then show that the solution indeed optimizes an objective function. This approach leads to a better physical insight to the solution and makes the analysis simpler.

Thus, we propose that the coordinates of the center (x_0, y_0) and the radius r_0 of the fitting circle are given by

$$x_0 = \frac{1}{A} \sum_k x_k, \qquad y_0 = \frac{1}{A} \sum_k y_k, \tag{1}$$

$$r_0 = [A/\pi]^{1/2} \tag{2}$$

where A is the area of the object and the summation extends over all object points.

To find which objective function eqs. (1)–(2) optimize, consider a point Q inside the object from which all border points are visible. Draw a circle of radius r_0 obtained by eq. (2) with Q as center. Segment the space by equiangle strips at Q. The angular separation of the strip $\Delta\theta$ can be made arbitrarily small. Then, as shown in Figure 1, the area of mismatch between the circle and the object in the i-th strip is

$$(r_i^2 - r_0^2)\frac{\Delta\theta}{2}.$$

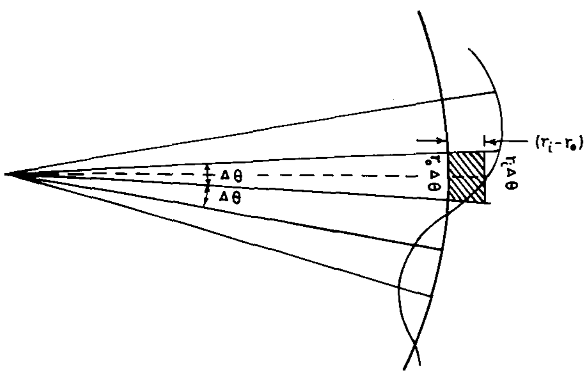

Figure 1.

Note that the quantity can be negative or positive depending on whether $r_i < r_0$ or not. If there are m strips so that $m\Delta\theta = 2\pi$, then it can be proved that eq. (2) implies

$$\sum_{i=1}^{m} (r_i^2 - r_0^2)\frac{\Delta\theta}{2} = 0, \tag{3}$$

i.e., the areas of mismatch outside and inside the fitting circle are equal.

Proof. If $\Delta\theta$ is sufficiently small, then $r_i^2 \Delta\theta/2$ is the area of the object in the i-th strip. Hence

$$\sum_{i=1}^{m} r_i^2 \frac{\Delta\theta}{2} = \text{area of the object} = A.$$

Also,

$$\sum_{i=1}^{m} r_0^2 \frac{\Delta\theta}{2} = r_0^2 \sum_{i=1}^{m} \frac{\Delta\theta}{2} = \pi r_0^2.$$

Then, eq. (2) implies eq. (3), and vice versa. □

Dropping the constant $\Delta\theta/2$, we see that eq. (3) can be obtained by differentiating

$$J_1 = \sum_{i=1}^{m} (r_i^2 - r_0^2)^2 \tag{4}$$

with respect to r_0^2 and equating the result to zero. Thus, J_1 is the objective function that is optimized by the choice of r_0 using eq. (2).

We show that J_1 is optimized by eq. (1) as well. To prove this, we note that

$$J_1 = \sum_{j=1}^{m} (r_i^4 - 2r_0^2 r_i^2 + r_0^4)$$

$$= \sum_{i=1}^{m} r_i^4 - 2m r_0^4 + m r_0^4$$

$$= \sum_{i=1}^{m} r_i^4 - m r_0^4$$

$$= \sum_{i=1}^{m} r_i^4 - K_1 \tag{5}$$

where K_1 is a positive contant. Now consider Figure 2 where a small segment of length $r\Delta\theta$ and width $\mathrm{d}r$ is at a distance r from the center. It is easy

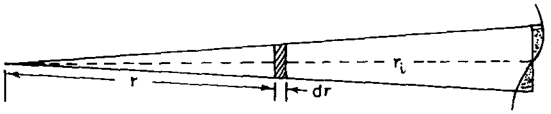

Figure 2.

to see that

$$r_i^4 = 4 \int_0^{r_i} r^2 (r \Delta\theta \cdot dr) \tag{6}$$

where the term in parentheses is the area of the segment. Since the number of points within the segment is proportional to the area, the term under the integral is the sum of squares of distances of all points in the segment from the center. Thus, the right-hand side of eq. (6) is proportional to the sum of distances of all points in the i-th strip of the object from the center, and we can write

$$r_i^4 = K_2 \sum_{k_i} r_{k_i}^2 \tag{7}$$

where the summation extends over all points in the i-th strip of the object and K_2 is a positive constant. Adding contributions from all strips, we can write eq. (5) as

$$J_1 = K_2 \sum_k r_k^2 - K_1 \tag{8}$$

where the summation extends over all points in the object. Now,

$$r_k^2 = (x_k - x_0)^2 + (y_k - y_0)^2. \tag{9}$$

Differentiating J_1 partially with respect to x_0 and y_0 and setting the results equal to zero, we arrive at eq. (1).

Note that the discrepancy due to the dotted region in Figure 2 can be made arbitrarily small if $\Delta\theta$ is made arbitrarily small.

3. Three-dimensional circular fit

In three dimensions also, we first propose a solution and then prove that the solution indeed optimizes an objective function. Let A_s denote the surface area of the three-dimensional object. Then, the center (x_0, y_0, z_0) and radius r_0 of the fitting sphere are proposed as

$$x_0 = \frac{1}{A_s} \sum_k x_k, \qquad y_0 = \frac{1}{A_s} \sum_k y_k,$$

$$z_0 = \frac{1}{A_s} \sum_k x_k, \tag{10}$$

$$r_0 = (A_s / 4\pi)^{1/2} \tag{11}$$

where the summations extend over all points (x_k, y_k, z_k) on the surface of the object.

Following the same arguments of Section 2, it can be shown that eq. (11) leads to

$$\sum_{i,j}^{m,n} (r_{ij}^2 - r_0^2)\, \Delta\theta\, \Delta\phi = 0 \tag{12}$$

where the object and the fitting sphere are partitioned into strips of arbitrarily small solid angle $\Delta\theta\,\Delta\phi$ at the centre and r_{ij} corresponds to the surface point of the (i,j)-th strip. The physical significance of eq. (12) is that the choice of radius using eq. (11) makes the surface areas of mismatch outside and inside the sphere equal. Again, dropping the constant term $\Delta\theta\,\Delta\phi$, we note that eq. (12) can be obtained by differentiating the objective function

$$J_2 = \sum_{i,j} (r_{ij}^2 - r_0^2)^2 \tag{13}$$

with respect to r_0^2 and setting the result equal to zero. Now we have to show that J_2 is optimized by eq. (10) as well.

Again using derivations similar to eq. (5) we can write eq. (13) as

$$J_2 = \sum_{i,j} r_{ij}^4 - K_3 \tag{14}$$

where K_3 is a positive constant. Consider the surface of the object on the (i,j)-th solid angle strip. If $\Delta\theta$ and $\Delta\phi$ are very small, the area of surface in this strip is

$$(r_{i,j} \Delta\theta)(r_{ij} \Delta\phi) = r_{ij}^2\, \Delta\theta\, \Delta\phi.$$

The distance of points on this surface from the center is r_{ij}. Then, in this surface segment, we can write

$$r_{ij}^4 = \frac{1}{\Delta\theta\,\Delta\phi}(r_{ij}^2\,\Delta\theta\,\Delta\phi)(r_{ij}^2)$$
$$= K \cdot \text{(area of the surface segment)} \cdot \text{(the distance of the surface from the center)}^2$$
$$= K_4 \sum_{k_{ij}} r_{k_{ij}}^2 \tag{15}$$

where $r_{k_{ij}}$ is the distance of a surface point in the segment from the center and the summation is made over the points of the segment. Also K and K_4 are positive constants. Thus, we can write

$$J_2 = K_4 \sum_k r_k^2 - K_3 \tag{16}$$

where the summation extends over all points on the surface of the object. Differentiating J_2 partially with respect to x_0, y_0 and z_0 and making the partial derivatives equal to zero, we arrive at eq. (11). To find the partial derivatives, the relation

$$r_k^2 = (x_k - x_0)^2 + (y_k - y_0)^2 + (z_k - z_0)^2 \tag{17}$$

is used.

4. Conclusion

Closed form expressions for the location of center and the magnitude of radius for the optimum circular fit to objects in two and three dimensions are proposed. The expressions are simple and easy to compute and their physical interpretation can be clearly understood.

Future work in this area may be directed towards finding similar closed form expressions for the parameters of an optimum elliptic fit in two and three dimensions.

Acknowledgement

The author wishes to thank Prof. D. Dutta Majumder for his interest in the work. Secretarial help rendered by Smt. S. De. Bhowmick is acknowledged with thanks.

References

Bookstein, F.L. (1979). Fitting conic sections to scattered data. *Computer Graphics and Image Processing* 9, 56–71.

Haralick, R.M. (1974). A measure of circularity of digital figures. *IEEE Trans. Syst. Man Cybernet.* 4, 394–396.

Landau, U.M. (1987). Estimation of a circular arc center and its radius. *Computer Vision, Graphics and Image Processing* 38, 317–326.

Takiyama, R. and N. Ono (1989). A least square error estimation of the center and radii of concentric arcs. *Pattern Recognition Letters* 10, 237–242.

Thomas, S.M. and Y.T. Chan (1989). A simple approach for the estimation of circular arc center and its radius. *Computer Vision, Graphics and Image Processing* 45, 362–370.