INDIAN STATISTICAL INSTITUTE
1st Semester Examination
B. Stat. II year: 2012–2013
C & Data Structures

Date: 03. 09. 2012 Marks: 50 Time: 3 Hours

The paper is of 60 marks. Answer any part of any question. The maximum marks you can get is 50.

1. (a) Implement a function in C programming language that checks whether two positive integers m, n $(m > n)$ are co-prime.

   (b) Write a function in C that finds a given substring in a circular string. For example, the substring "pqr" is absent in the string "rxyzsdfgpq", but exists when the string is considered in a circular manner.

$5 + 5 = 10$

2. (a) Describe how a priority queue can be implemented using an array in C programming language.

   (b) Explain the data structure and the insertion and deletion strategies with the data set 122, 36, 7, 221, 81, 99, 77, 185.

$5 + 5 = 10$

3. (a) Explain how a polynomial can be implemented using linked list.

   (b) Implement a C function that will take two polynomials as input and provide the quotient and remainder as the output.

$2 + 8 = 10$

4. (a) Derive the worst case time complexity to search a key in a balanced tree containing n nodes?

   (b) Clearly explain the insertion algorithm in a balanced binary search tree.

   (c) Write down C routines for single and double rotations.

   (d) Provide specific examples (with at least 8 nodes) to demonstrate single and double rotations while inserting a node in a balanced binary search tree.

$5+5+5+5 = 20$

5. (a) Outline the algorithmic steps to generate a pseudo-random permutation of the integers $[0, \ldots, n-1]$ given a short seed.

   (b) Explain how your algorithm works choosing the last two digits of your roll number as the seed and taking $n = 8$.

$5+5 = 10$

BSTAT 2ND YEAR: (MIDTERM) ECONOMICS QUESTION PAPER, ~~AUGUST~~ September 2012

Dt: 5.9.12

Each of the following questions carries 5 marks. Answer only eight of these. You have a time of two hours.

(1) Show that utility maximization and expenditure minimization are equivalent. Write down the resultant duality conditions.
(2) Derive the Slutsky equation using the duality conditions. Explain the intuition.
(3) Show that a Giffen good must be an inferior good; graphically or otherwise.
(4) As a government policy maker, would you prefer excise tax to income tax. Explain your preference.
(5) Consider an individual facing $(p_1, p_2, m) = (1, 2, 10)$ and having a utility function $u(x_1, x_2) = 3x_1 + 2x_2$. What are the set of equilibrium demand vectors? How does this set change if p_1 changes to 4?
(6) Consider an individual with utility function $u(x_1, x_2) = x_1 x_2$, facing the vector $(p_1, p_2, m) = (10, 10, 50)$. Find the indirect utility of the consumer. How would this value change if the utility function is replaced by $v(x_1, x_2) = (x_1 x_2)^{\frac{1}{20}}$?
(7) Consider a function $f(x, a)$ such that $\forall\, a, \frac{\partial^2 f(x,a)}{\partial x^2} < 0$. State and prove the Envelope theorem in this context (maximization of $f(.)$ with respect to x).
(8) Consider yourself as a the only producer of cotton in a market. If the price elasticity of demand is less than 1 in the present market situation. Would you, as a revenue maximizer, be tempted to increase the price? Explain you decision.
(9) What is a quasi-convex function? Show that indirect utility function is quasi-convex.
(10) Consider a three person economy where the individual demand of each person i

$$D_i(p) = 20 - ip$$

Write down the aggregate demand curve. If possible, find out the point of unit price elasticity. If not, explain?
(11) Write down the axioms underlying the theory of consumer behaviour and explain the relevance of each.

Date: August 27, 2012.

Indian Statistical Institute

Mid-Semestral Examination: 2012-13

Course Name: B. Stat II, **Subject name:** Biology I

Date: 05.09.12 ***Maximum Marks:*** **40;** ***Duration: 2.0 hrs***

All questions carry equal marks, answer any four

1) A). How many ATPs would be generated when one molecule of acetyl-CoA is completely oxidized in TCA cycle? Mention the specific reactions/steps which are responsible for generation of NADH and $FADH_2$ in TCA cycle. [7]

   B). Why monoclonal antibodies, instead of polyclonal antibodies, are used in medical diagnostics. [3]

2). A). Explain why "a protein "X" might become positively charge at pH-7.0 whereas another protein "Y" might become negatively charge at pH-7.0"? [4]

B). How a mixture of two proteins (mol.wt. 8,000 kD and 15,000 kD, respectively) could be separated from each other without denaturating them? [6]

3). Write the metabolic steps in which genetic defects might lead to galactosemia, albinism, ketone bodies and phenylketonuria. [10]

4). How stearic and oleic acids, with the following chemical formula $CH_3(CH_2)_{16}COOH]$ and $[CH_3(CH_2)_7CH=CH(CH_2)_7COOH]$ respectively, differ in oxidation steps to generate ATP ? [10]

5). How one will determine whether a protein molecule of mol. wt. 42,000 kD consists of single polypeptide chain or subunits? If it consists of subunits then how to determine whether subunits are identical or different? [10]

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2012-2013
Course Name: BStat II
Subject Name: Physics I

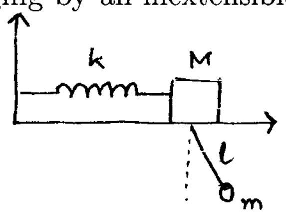
Date: 5 September 2012 **Maximum Marks: 40** **Duration: 1 hr 30 mins**

Note: You can attempt at most 6 questions. Each question carries 8 marks.

1. A ball is dropped from a height h. Considering the frictional force of air for small velocity, show that the height of the ball in time t is $y(t) = h - \frac{g}{\alpha}\left[t - \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)\right]$, where g is the acceleration due to gravity and α is a constant having inverse dimension of time.

2. Two masses m_1 and m_2 are suspended over a frictionless pulley by an inextensible string, constituting an Atwood's machine. Using Lagrangian method, show that the magnitude of acceleration of either mass is given by $a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g$, where g is the acceleration due to gravity.

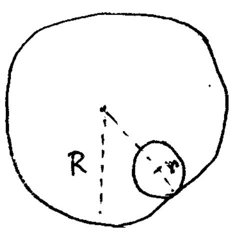
3. A spring of spring constant k is attached to a mass M from which a pendulum bob of mass m is hanging by an inextensible rope of length l, as shown in the figure. If



the system now starts osciallting, show that resonance will occur for the pendulum when $\frac{kl}{M} = g$, the acceleration due to gravity.

4. A planet revolves round the sun under the central force $F(r) \propto \frac{1}{r^2}$. Using the plane polar coordinate (r, θ), show that the angular velocity of the planet $\dot{\theta} \propto \frac{1}{r^2}$. Hence find out the expression for the effective potential of the system which helps in analyzing the motion without solving explicitly.

5. A solid homogeneous cylinder of radius r rolls without slipping inside a stationary large cylinder of radius R, as shown in the figure. Show that the period of oscillation



of the rolling cylinder about its equilibrium position is given by $T = \left[\frac{6\pi^2(R-r)}{g}\right]^{1/2}$.

6. A classical analogue of a linear triatomic molecule such as CO_2 is a central mass M connected with two masses (m each) by two springs of equal spring constant k, in a straight line. Using the principle of small oscillations, find out the possible frequencies of oscillation for such a molecule and show that there is one mode of oscillation where the central mass M remains stationary with the other two oscillating with identical amplitudes in opposite directions.

7. A parallel pendulum is constituted of two pendula hanging from a rigid support, with the bobs connected by a spring with spring constant k. Considering each bob has mass m and each string has length l, show that the system can have at least one mode of frequency matching the normal frequency of each individual pendulum $\sqrt{g/l}$.

INDIAN STATISTICAL INSTITUTE

Statistical Methods III

Mid Semester Examination

B II, Semester I, 2012-13

Date: 07.09.2012 Time: 3 hours

[Total points 120. Maximum you can score is 100]

1. Consider the two parameter Pareto distribution given by

$$f(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}, \quad x \geq 0.$$

A random sample of size 6 from this distribution contains the values 2000, 13515, 400000, 16437, 4993, 12000. Find the method of moments estimate of α. [10 points]

2. The prior distribution of Θ has pdf $\pi(\theta) = 4/\theta^5$ for $\theta > 1$. Given $\Theta = \theta$, the observations X follow a Pareto distribution with parameters $\alpha = 5$ and θ. A single observation from the distribution of X shows the number 7. Find the posterior probability of the event that Θ exceeds 3. [10 points]

3. Suppose that $X_1, \ldots, X_n$ form a random sample of size n from a distribution having the probability density function

$$f(x) = \frac{2x}{\theta^2}, \quad 0 \leq x \leq \theta.$$

   (a) Find the method of moments estimator of θ. Is it unbiased? If not, can you construct an unbiased estimator based on it?

   (b) Find the maximum likelihood estimator of θ. Is if unbiased? If not, can you construct an unbiased estimator based on it?

   (c) Among the unbiased estimators in parts 3a and 3b, which one has the smaller variance?

   (d) Show that the maximum likelihood estimator is a consistent estimator of θ.

[5 + 10 + 5 + 5 = 25 points]

4. Let $X_1, \ldots, X_n$ be an independently and identically distributed sample from a distribution having probability density function $f_\theta(x)$, where the support of the random variable X is independent of θ. Find the lower bound to the variance of all estimators of θ having bias $b(\theta)$, where $b(\theta)$ is a differentiable function of θ. [10 points]

5. Suppose that $I(\theta)$ is the Fisher information in the model about the parameter θ based on the density $f_\theta(x)$. Consider a one to one reparametrization $\eta = g(\theta)$ of the parameter. Let $I^*(\eta)$ be the Fisher information in the η formulation of the model. Show that

$$I^*(\eta) = \frac{I(\theta)}{[g'(\theta)]^2}.$$

[10 points]

6. Let $f_\theta(\cdot)$ be the joint density function of the observations $\mathbf{X} = (X_1, \ldots, X_n)$. Let us denote a particular realization of $\mathbf{X}$ by $\mathbf{x}$. Then $f_\theta(\cdot)$ is said to have a *monotone likelihood ratio* in the statistic $T(\mathbf{x})$ if for any choice θ_0 and θ_1 of parameter values such that $\theta_0 < \theta_1$, the likelihood ratio

$$\frac{f_{\theta_1}(\mathbf{x})}{f_{\theta_0}(\mathbf{x})}$$

depends on the data $\mathbf{x}$ only through the value of the statistic $T(\mathbf{x})$ and, in addition, this ratio is a monotone function of $T(\mathbf{x})$.

(a) Suppose that the family $f_\theta(\cdot)$ has monotone likelihood ratio in $T(\mathbf{x})$, and the ratio

$$\frac{f_{\theta_1}(\mathbf{x})}{f_{\theta_0}(\mathbf{x})}$$

is increasing in $T(\mathbf{x})$. For a given α in $(0, 1)$, suppose there exists a constant k_α such

$$P_{\theta_0}[T(\mathbf{X}) > k_\alpha] = \alpha.$$

Then show that the test ϕ defined by the nonrandomized critical function

$$\phi(\mathbf{x}) = \begin{cases} 0 & T(\mathbf{x}) \leq k_\alpha \\ 1 & T(\mathbf{x}) > k_\alpha \end{cases}$$

represents a most powerful test for the hypotheses $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ $(\theta_0 < \theta_1)$.

(b) Argue that the critical function ϕ in part 6a represents a most powerful test of level α for H_0 against each $\theta > \theta_0$, so that this test is a uniformly most powerful test for the hypotheses $H_0^* : \theta = \theta_0$ agaist $H_1^* : \theta > \theta_0$.

(c) Now consider testing the null hypothesis $H_0^{**} : \theta \leq \theta_0$ versus the alternative $H_1^{**} : \theta > \theta_0$. Notice that the null hypothesis is composite. A test for a composite null hypothesis is said to be of level α if the probability of Type I error at each point θ in the null is no greater than α. Show that the test ϕ in part 6a is a level α test for testing H_0^{**} against H_1^{**}. [Hint: All you have to show is that the size of the test for any $\theta \leq \theta_0$ is bounded above by α. This will be established if you can show that $P_\theta[T(\mathbf{X}) > k_\alpha]$ is an increasing function of θ].

(d) Finally argue that the test ϕ in 6a is the uniformly most powerful level α test for the hypotheses H_0^{**} versus H_1^{**}.

[5 + 10 + 5 + 10 = 30 points]

7. Consider the problem of testing the null hypothesis $H_0 : \mu = 0$ versus $H_1 : \mu = 0.25$ based on the realizations $x_1, \ldots, x_{25}$ of a randomly drawn sample from the $N(\mu, 1)$ model. Find the power of the most powerful test of level $\alpha = 0.05$. [10 points]

8. Suppose that X is a discrete random variable having probability mass function $f_\theta(x)$. We are interested in testing the hypothesis $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. The probability distribution of the random variable for these two values of θ are given in the following table. Find a most powerful test of level $\alpha = 0.05$ for testing the above hypotheses. Is the test unique?

x	0	1	2	3	4	5
$f_0(x)$	0.50	0.05	0.10	0.10	0.05	0.20
$f_1(x)$	0.10	0.15	0.30	0.15	0.10	0.20

[15 points]

INDIAN STATISTICAL INSTITUTE

MID-TERM EXAMINATION (2012–13)

B. STAT. II YEAR

ANALYSIS III

Date : 10.09.2012 Maximum Marks : 80 Time : 3 hours

The question carries 90 marks. Maximum you can score is 80. Precisely justify all your steps. Carefully state all the results you are using.

1. Let $m > n$. Identify $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $\widehat{\boldsymbol{x}} = (x_1, x_2, \ldots, x_n, 0, \ldots, 0) \in \mathbb{R}^m$. Show that with this identification
   (a) Any closed subset of $\mathbb{R}^n$ remains closed in $\mathbb{R}^m$. [5]
   (b) But no nonempty open subset of $\mathbb{R}^n$ remains open in $\mathbb{R}^m$. [5]

2. Let $A \subseteq \mathbb{R}^k$ be nonempty. Define

$$d_A(x) = \inf\{\|x - a\| : a \in A\}, \quad x \in \mathbb{R}^k$$

   (a) Show that d_A is uniformly continuous. [5]
   (b) Show that $d_A(x) = 0$ if and only if $x \in \overline{A}$. [10]
   (c) If $A \subseteq \mathbb{R}^k$ is connected, then either A is singleton or uncountable. [10]
   (d) Show that $A \subseteq \mathbb{R}^k$ is compact if and only if every continuous $f : \mathbb{R}^k \to \mathbb{R}$ is bounded above on A. [10]

3. Let $S \subseteq \mathbb{R}^n$ be polygonally connected, that is, every pair of points in S can be joined by a continuous path that is finite union of straight line segments. Let $f : S \to \mathbb{R}^m$ be differentiable on S and $f'(x) = 0$ for all $x \in S$. Show that f is constant on S. [8]

4. Decide whether for any of the following vector fields on $\mathbb{R}^2$, there exists a scalar field φ on $\mathbb{R}^2$ such that $f = \nabla\varphi$. If it is, find such a φ : [7]
   (a) $f(x, y) = (3x^2y, x^3y)$
   (b) $f(x, y) = (3x^2y, x^3)$

5. Find the extreme values of $f(x, y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \leq 4$. [10]

6. A helix is a space curve parametrized by $\boldsymbol{r}(t) = (a\cos t, a\sin t, bt)$. Show that the curvature of a helix is constant. [10]

7. If a plane curve is described by the polar equation $r = f(\theta)$, where $a \leq \theta \leq b \leq a+2\pi$, prove that its arc length is [10]

$$\int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination – Semester I : 2012-2013
B.Stat. (Hons.) II Year
Probability Theory III

Date : 14.09.12 **Maximum Score : 80 pts** Time : $2\frac{1}{2}$ Hours

Note : This paper carries questions worth a **total** of **90 points**. Answer as much as you can. The **maximum** you can score is **80 points**.

1. Consider independent repetitions of a Bernoulli trial with success probability p. Let $A(s)$ denote the generating function of the sequence $\{a_n, n \geq 0\}$, where $a_0 = 1$, and, for $n \geq 1$, a_n denotes the probability that n trials result in an even number of successes.
(a) Using a recurrence among the a_n, show that $A(s) = \dfrac{1}{2[1-s]} + \dfrac{1}{2[1-(1-2p)s]}$.
(b) Deduce that $a_n = [1 + (1-2p)^n]/2$, for all $n \geq 0$. (12+6)=**[18]**

2. Let X and Y be independent random variables both having $Exp(\lambda)$ distribution. Denote $Z = \max\{X, Y\}$.
(a) Describe the conditional distribution of X given Z. [It will be enough if you just derive the conditional distribution function of X given $Z = z$.]
(b) Find $E[X \mid Z]$. (10+8)=**[18]**

3. Let $U_1 < U_2 < U_3 < U_4$ denote the order statistic of a sample of size four drawn from the density $f(x) = 2x$, $0 < x < 1$. Find the conditional distribution of the vector (U_1, U_2, U_4), given U_3, and hence find $E[U_1U_2U_4 \mid U_3]$. (10+8)=**[18]**

4. Let $\underset{\sim}{X} = (X_1, \ldots, X_n)'$ be normally distributed with mean vector $\underset{\sim}{\mu} = (\mu, \ldots, \mu)'$ and dispersion matrix Σ with all its diagonal entries equal to σ^2 and all off-diagonal entries equal to $\rho\sigma^2$ for some $\sigma^2 > 0$ and $-1 < \rho < 1$. Let $\underset{\sim}{Y} = A\underset{\sim}{X}$, where A is an orthogonal matrix whose first row is the vector $(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$.
(a) Show that $Y_1, \ldots, Y_n$ are independent.
(b) Show that $\overline{X}_n = (\sum_{j=1}^{n} X_j)/n$ and $S_n^2 = (\sum_{j=1}^{n}(X_j - \overline{X}_n)^2)/(n-1)$ are independent with $\overline{X}_n$ having a normal distribution with mean μ and $S_n^2/[(1-\rho)\sigma^2]$ having a χ^2_{n-1} distribution. (8+10)=**[18]**

5. (a) State and prove Fisher-Cochran theorem.
(b) Let $X_1, \ldots, X_n$ be i.i.d. $N(0,1)$ random variables. Consider the random variable $Y = \sum_i \sum_j a_{ij}X_iX_j$, where $A = ((a_{ij}))$ is a symmteric matrix. Show that Y has a χ^2-distrbution if and only if A is idempotent and, in that case, the degrees of freedom equal $trace(A)$. (10+8)=**[18]**

INDIAN STATISTICAL INSTITUTE

FIRST SEMESTRAL EXAMINATION (2012–13)

B. STAT. II YEAR

ANALYSIS III

Date : 16.11.2012 Maximum Marks : 100 Time : $3\frac{1}{2}$ hours

The question paper carries 115 marks. Maximum you can score is 100. Precisely justify all your steps. Carefully state all the results you are using.

1. Let $A \subseteq \mathbb{R}^n$ be a non-empty open set. Find all possible values of the dimension of the linear span of A. [10]

2. Define a connected set in $\mathbb{R}^n$. Let $A, B \subseteq \mathbb{R}^n$ be connected sets such that $A \cap B \neq \emptyset$. Show that $A \cup B$ is a connected set. [5 + 10 = 15]

3. Let S be an open connected subset of R^2. Let $f, g : S \to \mathbb{R}^2$ be continuously differentiable functions. Show that

$$\oint_C f\nabla g \cdot d\alpha = -\oint_C g\nabla f \cdot d\alpha$$

for every piecewise smooth Jordan curve C in S. [10]

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Show that f cannot be 1-1. [15]

[Hint : The function $g(x, y) = (f(x, y), y)$ has nonzero Jacobian at some point.]

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function. For each $x \in \mathbb{R}$, define $g_x : \mathbb{R} \to \mathbb{R}$ by $g_x(y) = f(x, y)$. Suppose that for each $x \in \mathbb{R}$, there exists a unique y such with $g'_x(y) = 0$; denote this y by $c(x)$. If $f_{yy}(x, y) \neq 0$ for all x, y, show that c is differentiable and [15]

$$c'(x) = -\frac{f_{xy}(x, c(x))}{f_{yy}(x, c(x))}.$$

6. Evaluate

$$\int_C ydx + zdy + xdz$$

where C is the intersection of two surfaces $z = |xy|$ and $x^2 + y^2 = 1$ traversed once in a direction that appears counterclockwise when viewed from high above the xy-plane. [15]

PTO

7. Sketch the region

$$S = \{(x,y) : x^2 \le y \le 2, -1 \le x \le 1\}$$

and express the double integral $\iint_S f(x,y)dxdy$ as an iterated integral in polar co-ordinates. [10]

8. Let C be a piecewise smooth Jordan curve and let T denote the union of C and its interior. Assume that $T \subseteq \mathbb{R}^2$. Let $r : T \to \mathbb{R}^3$ be a C^2 function given by

$$r(u,v) = (X(u,v), Y(u,v), Z(u,v))$$

so that $S = r(T)$ is a smooth parametric surface. Let $P : S \to \mathbb{R}$ be a C^1 function. Define $p(u,v) = P(r(u,v))$. Show that [10]

$$\iint_T \left(\frac{\partial p}{\partial Z}\frac{\partial(Z,X)}{\partial(u,v)} - \frac{\partial p}{\partial Y}\frac{\partial(X,Y)}{\partial(u,v)}\right)dudv = \oint_C p\frac{\partial X}{\partial u}du + p\frac{\partial X}{\partial v}dv.$$

9. Evaluate

$$\iiint_S \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) dxdydz,$$

where S is the solid bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. [15]

Indian Statistical Institute

First Semester Examination 2012-2013

Course Name: B Stat II Year

Subject: Economics I

Date: 20.11.12 Maximum Marks: 60 Duration: 2.5 Hours

1. a) Is it possible for a profit maximizing firm in a perfectly competitive market to produce and sell positive output while suffering a loss in the short-run? Explain.

b) Every individual producer of a good in a perfectly competitive market has the long-run cost function given by $C = q^2 + 1$, where q denotes output of an individual firm. The market demand function is given by $Q = 102 - P$, where Q and P denote market demand and price respectively. Derive the values of P, Q and the number of firms in the industry in the long-run industry equilibrium. [10+12=22]

2. a) Suppose a perfectly competitive industry consists of 100 identical firms each of which has the cost function $C = 0.1q^2 + q + 10$. Derive the aggregate supply function of the industry in the short run.

b) How will the supply function change if a specific tax at the rate $t = 0.5$ is imposed on the sale of the good? [16+6=22]

3. A monopolist can sell in the home market protected by law. The home market demand function is given by $P_h = 120 - (q_h/10)$, where P_h and q_h denote respectively price and quantity demanded in the home market. He can also sell in the perfectly competitive world market at a given price P_W. The cost function of the seller is $50q + (q^2/20) + 10$, where q denotes his total output. Derive the set of values of P_W for which the producer will sell (i) only in the home market, (ii) only in the world market. [22]

INDIAN STATISTICAL INSTITUTE
First Semester Examination: 2012-2013
Course Name: BStat II
Subject Name: Physics I

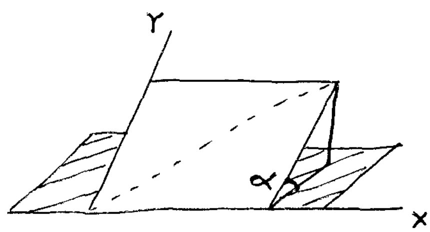
Date: 20 November 2012 Maximum Marks: 100 Duration: 3 hrs

Note: Answer as many questions as you can. Each question carries 20 marks.

1. (a) Derive Newton's law of motion from least action principle.
   (b) Using Lagrangian framework, prove that the total energy of an isolated system in 3 dimensions is conserved.

[10+10 =20]

2. (a) A bob of mass m is hanging from a rigid support by a spring of mass M having uniform linear density and spring constant k. Using Lagragian formalism, show that the time period of oscillation for such a system is given by $T = 2\pi\sqrt{\frac{m+M/3}{k}}$.
   (b) A particle is constrained to move in 2 dimensions (x, y) on a smooth inclined plane making an angle $\alpha < 90^0$ with the horizontal (see the figure).



Using Lagragian formalism, show the trajectory of the particle is a parabola with the vertex $\left(\frac{v_0^2}{2g\sin\alpha}, \frac{v_0^2}{4g\sin\alpha}\right)$. Here v_0 is the initial velocity of the particle and g is the standard acceleration due to gravity.

[10+10 =20]

3. (a) An oscillating system can be described in terms of deviations (η_i) from equilibrium position, where i runs from 0 to n, the number of generalized coordinates. Show that for small oscillation of such a system, one can have an unique solution if $det|\mathbf{V} - \omega^2\mathbf{T}| = 0$, where $\mathbf{V}$ and $\mathbf{T}$ are $i \times j$ matrices representing the potential energy and kinetic energy matrix respectively and ω is the natural frequency of oscillation of the system.
   (b) A double pendulum is constituted of two pendula hanging one from the bottom of the other. Using the formalism of small oscillations, find out the modes of frequency of the system when both the pendula are made of bobs with same mass and massless inextensible strings with same length.

[10+10 =20]

4. (a) Using the standard Poisson bracket, prove that anything that commutes with the Hamiltonian of a system leads to a conserved quantity.

   (b) For a central force field the potential energy of the system is a function of the radial coordinate only. Using Hamiltonian formalism find out the equation of motion for a particle moving under such a field.

   (c) From the equation of motion of the above problem, Hence identify centrifugal force of the system.

[10+7+3 =20]

5. (a) Prove that both the quantities $\vec{\nabla} \times (\vec{\nabla}\phi)$ and $\vec{\nabla}.(\vec{\nabla} \times \vec{A})$ are zero for any arbitrary scalar ϕ and vector $\vec{A}$.

   (b) Using Stokes' theorem, prove that $\vec{\nabla} \times \vec{E} = 0$, where $\vec{E}$ is the electric field due to a distribution of charges.

   (c) Prove that the electric potential $V \propto \frac{1}{r}$ is a solution of the Laplace's equation $\vec{\nabla}^2 V = 0$.

[10+5+5 =20]

6. (a) Show that for any two vectors $\vec{A}_1$ and $\vec{A}_2$, the following relations hold good:
   (i) $\vec{\nabla}.(\vec{A}_1 \times \vec{A}_2) = \vec{A}_2.(\vec{\nabla} \times \vec{A}_1) - \vec{A}_1.(\vec{\nabla} \times \vec{A}_2)$
   (ii) $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}_1) = \vec{\nabla}(\vec{\nabla}.\vec{A}_1) - \vec{\nabla}^2\vec{A}_1$.

   (b) Derive Poisson's equation for magnetic field $\vec{\nabla}^2\vec{A} = -\mu_0\vec{J}$, where $\vec{A}$ and $\vec{J}$ are magnetic potential and volume current density respectively. In answering (b) you can take help of (a), if required.

[10+10 =20]

Indian Statistical Institute

Semester Examination (B. Stat-II, Biology I, Year-2012)

Answer any five; All questions carry equal marks; Full marks = 50; Time = 2.5 hours

Date: 20.11.12

1. (a) Distinguish between DNA and RNA with respect to their chemical structure and function. [5]

(b) In a couple, husband is albino and wife has an albino sibling. What is the risk of their baby to be an albino? (Albinism is an autosomal recessive disease and parents of the wife are normal). [5]

2. (a) Explain alleles, genotypes and traits using A B O blood group system in humans as examples? [5]

(b) Explain why two brothers, from the same parents, are not genetically identical. What is the chance that they would be genetically identical (monozygotic twins should be excluded)? [5]

3. If four babies are born on a given day. What are the chances that (a) number of boys and girls will be equal (b) all four will be girls and (c) at least one baby will be girl? What combination of boys and girls among four babies is most likely to occur? [10]

4. A man with X-linked color blindness marries a woman with no history of color blindness in her family. The daughter of this couple marries a normal man and their daughter also marries a normal man. What is the chance that the last couple will have a child with color blindness? If the last couple already had a child with color blindness, what is the chance that their next child will be color blind? Draw a pedigree and answer the questions. [10]

5. (A) In a random copolymer of $(AC)_n$ what will be the frequencies of different triplet codons if (a) A:C =1:1 and (b) A:C=5:1 in the copolymers? [5]

(B) Why primers are needed in PCR? What is annealing temperature for a primer and its importance in PCR? [2+2+1]

6. (a) Why multiple "origin of replication" is needed in the replication of human whole DNA? [2]

(b) Why transcription and translation occur simultaneously in bacterial cell but not in human cell? [3]

(c) Describe the features of genetic code. [5]

INDIAN STATISTICAL INSTITUTE

B Stat II, 1st Semester, 2012-13

Statistical Methods III

Semestral Examination

Date: 23.11.2012 Time: 3 hours

Total points 100. Answer all questions

1. Suppose that scores of men aged 21 to 25 years on the quantitative part of the National Assessment of Educational Progress (NAEP) test follow a normal distribution with an unknown mean and standard deviation $\sigma = 60$. Based on a sample of size n, you want to estimate mean score such that you are 90% confident that the estimate lies within 10 units of the true mean. How large a sample size must you choose? [10]

2. Suppose that $X_1, \ldots, X_n$ form a random sample from the Uniform $(\theta, \theta + 1)$ distribution.

   (a) Find the method of moments estimator of θ. Determine whether it is unbiased.

   (b) Find the maximum likelihood estimator of θ. Determine whether it is unbiased.

   (c) Find the mean square errors of the above two estimators and explain which one will be your preferred estimator.

   [4 + 4 + 7 = 15]

3. Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution with mean 0 and unknown standard deviation $\sigma > 0$. Find the asymptotic distribution of the statistic
$$\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right)^{-1}.$$
[10]

4. Let ϕ denote a test procedure such that the hypothesis $H_0 : f_0$, being tested against $H_1 : f_1$, where f_0 and f_1 are fixed densities, is rejected if $af_0(x) < bf_1(x)$, and we fail to reject if $af_0(x) > bf_1(x)$. Either decision may be taken when

P.T.O

$af_0(x) = bf_1(x)$. Here a and b are fixed positive constants. Show that for any other test procedure ϕ^*, we get

$$a\alpha(\phi) + b\beta(\phi) \leq a\alpha(\phi^*) + b\beta(\phi^*),$$

where $\alpha(\phi)$ and $\beta(\phi)$ represent the probabilities of Type I and Type II errors respectively of the test procedure ϕ. [10]

5. As a psychologist who works with people who have Down's syndrome, you design a study intended to determine which rewards are most effective for use in training your patients. You select four different, independent, groups of six patients and record the number of days it takes to teach them a particular task, with each group receiving one of four types of rewards: Reward 1, Reward 2, Reward 3, and Reward 4. the number of days are given in the following table.

Reward 1	Reward 2	Reward 3	Reward 4
3	6	9	12
5	7	10	13
6	9	15	15
2	7	12	18
1	11	11	15
2	6	10	13

Use the data above to conduct an appropriate one-way analysis of variance test at level $\alpha = 0.05$. State all your assumptions clearly. [15]

6. Suppose that the variables $X_1, \ldots, X_n$ form a random sample from a normal distribution for which both the mean μ and variance σ^2 are unknown. A t-test at a given level of significance α is to be carried out to test the hypothesis $H_0 : \mu = \mu_0$ versus $H_1 : \mu > \mu_0$. Show that the power of this test is equal at two different points (μ_1, σ_1^2) and (μ_2, σ_2^2), for which

$$\frac{\mu_1 - \mu_0}{\sigma_1} = \frac{\mu_2 - \mu_0}{\sigma_2},$$

where both μ_1 and μ_2 are larger than μ_0. [10]

7. Consider the contrived set of data generated by tossing a fair coin, obtained by letting $x_t = 1$ when a head is obtained and $x_t = -1$ when a tail is obtained. Construct y_t as

$$y_t = 5 + x_t - 0.7x_{t-1}.$$

(a) Is the process $\{y_t\}$ stationary? Justify your answer.

(b) Find the ACF function for all lags up to order 3.

(c) A sequence of 10 successive realizations of the $\{y_t\}$ series are given in the following table. Calculate all the sample autocorrelations up to order three from the given data, and compare them with the theoretical values. (Assume $x_0 = -1$.) [5+5+5=15]

t	1	2	3	4	5	6	7	8	9	10
Coin	H	H	T	H	T	T	T	H	T	H
x_t	1	1	-1	1	-1	-1	-1	1	-1	1
y_t	6.7	5.3	3.3	6.7	3.3	4.7	4.7	6.7	3.3	6.7

8. Consider the following data on revenue expenditure of Govt. of India in the following table for the indicated quarters/years. Find the seasonal indices by an appropriate method. [15]

Year	Quarter	Revenue Expenditure
1953-54	I	3,575
	II	4,342
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	II	4,404
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	II	5,327
	III	5,811
	IV	10,350
1956-57	I	4,693
	II	5,640
	III	5,957
	IV	12,961

INDIAN STATISTICAL INSTITUTE

Semestral Examination – Semester I : 2012-2013
B.Stat. (Hons.) II Year
Probability Theory III

Date : 27.11.12 **Total Marks : 120 pts** Time : 3 Hours

Note : This paper carries questions worth a **total** of **130 points**. Answer as much as you can. The **maximum** you can score is **120 points**.

1. (a) State Fisher-Cochran Theorem.
(b) Let $\underset{\sim}{X}$ be an n-dimensional random vector having a non-degenerate $N_n(\underset{\sim}{\mu}, \Sigma)$ distribution. Show that $Y = \frac{1}{2}(\underset{\sim}{X} - \underset{\sim}{\mu})'\Sigma^{-1}(\underset{\sim}{X} - \underset{\sim}{\mu})$ has a χ^2-distribution with n degrees of freedom. (4+10)=**[14]**

2. Let $(X_{(1)}, \dots, X_{(n)})$ be the order statistics of a random sample of size n from an exponential distribution with parameter $\lambda > 0$.
(a) Find the probability density function of $T = (n-1)X_{(1)}/[\sum_{i=2}^{n}(X_{(i)} - X_{(1)})$.
(b) Show that each $X_{(i)}$ can be expressed as $X_{(i)} = \sum_{j=1}^{i} Y_j/(n-j+1)$, where $Y_1, \dots, Y_n$ are i.i.d. exponential random variables with parameter λ.
(c) Find the dispersion matrix of $\underset{\sim}{X} = (X_{(1)}, \dots, X_{(n)})'$. (10+10+6)=**[26]**

3. (a) Find the characteristic function of the random variable having density function given by $f(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$.
(b) Using the known characteristic function of $N(0,1)$ distribution, find the characteristic function of XY where X and Y are i.i.d. $N(0,1)$ random variables.
(c) If X_1, X_2, X_3, X_4 are i.i.d. $N(0,1)$ random variables, find the distribution of the random variable $X_1X_2 - X_3X_4$. (10+10+6)=**[26]**

4. Show directly from definition that, if $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, then (i) $X_n + Y_n \xrightarrow{P} X + Y$, (ii) $X_nY_n \xrightarrow{P} XY$, and (iii) $\cos X_n^2 \xrightarrow{P} \cos X^2$. (8+8+8)=**[24]**

5. For this problem, you may use $\frac{\lambda}{1+\lambda^2}e^{-\lambda^2/2} < \int_\lambda^\infty e^{-u^2/2}\,du < \frac{1}{\lambda}e^{-\lambda^2/2}$ $\forall\ \lambda > 0$.
(a) Show that, if $\{X_n\}$ is a sequence of i.i.d. $N(0,1)$ random variables, then the probability $P[X_n > c\sqrt{2\log n}$ for infinitely many $n]$ equals 0 or 1 according as $c > 1$ or $c \le 1$.
(b) Deduce that for an i.i.d. sequence $\{X_n\}$ of random variables with $N(\mu, \sigma^2)$ distribution, $\limsup_{n\to\infty} \frac{X_n - \mu}{\sqrt{2\log n}}$ equals σ with probability 1. (10+10)=**[20]**

6. (a) Let X_n, $n \ge 1$ and X be random variables with distribution functions F_n, $n \ge 1$ and F respectively. Show that $X_n \xrightarrow{d} X$ if and only if $F_n(x) \to F(x)$ for all $x \in D$ where D is some dense subset of $\mathbb{R}$.
(b) Using appropriate results from probability, prove that the following limit exists and find it: $\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 \sin\left[\frac{\pi(x_1 + \cdots + x_n)}{n}\right] dx_1 \cdots dx_n$. (10+10)=**[20]**

INDIAN STATISTICAL INSTITUTE
1st Semestral Examination
B. Stat. II year : 2012–2013
C & Data Structures

Date : 30.11. 2012 Marks : 100 Time : 3 Hours

Answer any 5 questions. Each question is of 20 marks. Please try to write all the part answers of a question at the same place.

1. Write a C program to search, insert, delete and update records in a file. You may consider a record with employee number (integer, distinct for each employee), employee name, employee address (both character strings including blank spaces) and salary (floating point number). 20

2. (a) Describe the implementation of a linked list using C programming language. You should consider search, insertion and deletion of a node.
   (b) Explain how a stack and a queue can be implemented using a linked list.
   (c) Given a linked list with only one sided link, how can you reverse the list efficiently?

7+7+6 = 20

3. (a) Write a C program that accepts the pre-order traversal data (you may consider integers in an array) of a binary search tree as input and outputs the tree itself.
   (b) Explain what happens when the following codes are executed.
      i. `char *p, *q; while (*p++ = *q++);`
      ii.
```
int i, k = 1, n = 5;
for (i = 0; k < n+1; i = k-i)
        { printf("%d\n", k); k = k+i; }
```

10+(5+5) = 20

4. (a) Explain three hashing strategies with clear description of collision resolution.
   (b) Select a specific hashing strategy among the above three and implement a function in C programming language that can manage search and insertion in a hash table.
   (c) Provide an analysis of average search/insertion time complexity for double hashing.

9+6+5 = 20

5. (a) Write down the insertion algorithm in a balanced binary search tree.
   (b) Explain single and double rotations with suitable examples.

10+10 = 20

6. (a) Define a B-tree of order m.

   (b) Why a B-tree of order 3 is called a 2-3 tree?

   (c) Write the algorithm for insertion in a B-tree of order m.

   (d) Construct a B-tree of order 3 using your insertion algorithm with the keys 29, 21, 10, 15, 26, 39, 36, 38, 46, 50, 37, 5, 18, and 12.

$2+3+10+5 = 20$

7. (a) Derive the worst case time complexity to search a key in a height balanced binary tree containing n nodes?

   (b) What could be the maximum possible height of a B-tree of order m having N many keys?

$10+10 = 20$

8. Given positive integers a, b, n, the integer b is called the inverse of a modulo n if $ab - 1$ is divisible by n.

   (a) Given a, n, write a function in C that finds the inverse of a modulo n.

   (b) Execute your function with (i) $a = 7, n = 51$ and (ii) $a = 9, n = 39$.

   (c) Describe how one can efficiently check whether an integer is a prime.

$6+4+10 = 20$

Indian Statistical Institute
Back Paper Examination
Course Name: B Stat II Year (2012-2013)
Subject Name: Economics I

Date: 26.12.12

Maximum marks: 100

Duration 3 Hours

Answer all of the following questions

1. Consider a commodity bundle x' belonging to the consumption set X. Define the upper contour set Cx' and the lower contour set Lx'. State the continuity axiom.

What is the lexicographic preference rule? Explain with an example that it does not satisfy the continuity axiom.

Show that under continuity axiom, Lx' and Cx' are closed and share a common boundary.

[6+6+13=25]

2.(a) Suppose $q = f(x)$; x = the input vector, $f' > 0$ and $f(0)=0$, is the production function of a perfectly competitive firm and it is strictly concave. Show that the cost function $c = c(q)$ that the above production function yields has the following properties (i) $c(q) > 0$, (ii) $c = c(q)$ is strictly increasing in q and (iii) $c = c(q)$ is strictly convex in q.

(b) The cost function of a firm is given by $c(q) = (q^2/2)+50$. At what output is its average cost minimized? [20+5=25]

3. (a) The cost function faced by a competitive firm is given by

$$C(q) = TVC(q) + F$$

where F is a constant and $TVC(q)$ (total variable cost) is increasing and strictly convex in q with $TVC(0) = 0$. Now consider two cases. In Case 1, $C(0) = 0$ and in Case 2, $C(0) = F$. Draw the AVC, AC and MC schedules in a diagram and explain. Hence derive a competitive firm's supply curves in the above mentioned two cases.

(b) Supply function of a firm is given by $S(p) = 4p$. Suppose the price p changes from 10 to 15. What is the change in the producer's surplus? [15+10=25]

4. (a) A monopolist successfully sells his products in two different markets at two different prices. The demand curves in the two different markets are $P = 40 - 0.5 Q_1$ and

$P = 18 - 0.25Q_2$ respectively. The monopolist's marginal cost of production is 10 units. Calculate the profit maximizing levels of output and prices in the two different markets.

(b) A monopolist faces the market demand curve $P=100-5Q$, while the total cost of the monopolist is $TC = 300 + 20Q$. The monopolist practices perfect price discrimination. What will the monopolist's output and profit be? [13+12=25]

INDIAN STATISTICAL INSTITUTE

Backpaper Examination – Semester I : 2012-2013
B.Stat. (Hons.) II Year
Probability Theory III

Date : 27.12.12 | **Total Marks : 100** | Time : 3 Hours

[Note : This paper contains 5 questions. Each question carries equal marks.]

1. Let X be a non-negative interger-valued random variable with p.g.f. $P(s)$. Denote by $Q(s)$ the generating function of the sequence $a_n = P(X > n)$.
(a) Show that $\lim_{s\uparrow 1} P'(s) = \lim_{s\uparrow 1} Q(s)$, in the sense that, either both limits are finite and equal or both are infinite. Using this or otherwise, show that the left-hand derivative of P at $s = 1$ exists, if and only if $\lim_{s\uparrow 1} P'(s)$ is finite, if and only if X has finite expecation, and in that case, they are all equal.
(b) Suppose X has finite expectation. Defining $P'(1)$ as the left-hand derivative, consider the function $P'(s)$on $(-1, 1]$. Show that the left-hand derivative of P' at $s = 1$ exists, if and only if, $\lim_{s\uparrow 1} P''(s)$ is finite, if and only if, X has finite second moment, and in that case, the first two are equal, and $V(X) = P''(1) + P'(1) - (P'(1))^2$.

2. (a) Prove: $V(Y) = E[V(Y\,|\,X)] + V[E(Y\,|\,X)$ and $Cov(X, Y) = Cov(X, E(Y\,|\,X))$.
(b) Let X be a discrete random variable with p.m.f. $p(x) = 1/[x(x-1)]$, $x = 2, 3, 4, \ldots$ and given $X = x$, the conditional distrbution of Y is $U(x-2, x+2)$. Find the marginal distribution of Y and the conditional expectation $E(X\,|\,Y)$.

3. (a) Let n random points $(X_1, Y_1), \ldots, (X_n, Y_n)$ be picked from the unit square $(0,1) \times (0,1)$, independently of one another. A point (X_i, Y_i) is called a "boundary point" if, for all $j \le n$, *either* $X_j \le X_i$ *or* $Y_j \le Y_i$. Find the expected number of boundary points.
(b) Let $(X_{(1)}, \ldots, X_{(n)})$ be the order statistics of a random sample of size n from a distribution with a continuous distribution function F. Show that the vector $(F(X_{(1)}), F(X_{(2)}) - F(X_{(1)}), \ldots, F(X_{(n)}) - F(X_{(n-1)}))'$ has an appropriate Dirichlet distribution and find the dispersion matrix of $(F(X_{(1)}), \ldots, F(X_{(n)}))'$.

4. (a) Let $a > 0$. Find the characteristic function of the random variable X having density function $f(x) = (a - |x|)^+/a^2$, $x \in \mathbb{R}$) and use this to deduce that the function $\varphi(t) = (1 - |t|)^+$ is a characteristic function.
(b) Let $\{\xi_n\}$ be a sequence of independent random variables with $P(\xi_n = 2^{-n}) = P(\xi_n = -2^{-n}) = 1/2$. Denoting $S_n = \sum_{k=1}^{n} \xi_k$, find the characteristic function of S_n and use this to show that $S_n \xrightarrow{d} U(-1, 1)$.

5. (a) Show directly from definitions that, if $X_n \xrightarrow{P} X$, then $e^{2X_n} \xrightarrow{P} e^{2X}$.
(b) Let X_n, $n \ge 1$ be independent random variables with distribution functions F_n, $n \ge 1$ respectively. Show that $P(\sup_n X_n < \infty) = 1$ if and only if for some real x, $\sum_n (1 - F_n(x)) < \infty$, and in that case, the distribution function of $X = \sup_n X_n$ is given by $F(x) = \prod_n F_n(x)$.

INDIAN STATISTICAL INSTITUTE

First Semester Backpaper Examination (2012–13)

B. Stat. II Year

Analysis III

Date : 28.12.2012 | Maximum Marks : 100 | Time : 3 hours

Precisely justify all your steps. Carefully state all the results you are using.

1. Show that a nonempty connected subset of $\mathbb{R}^n$ is either a singleton or uncountable. [10]

2. Find and classify the critical points (if any) of the function $f(x, y) = y^2 - x^3$. [10]

3. Let S be an open connected subset of R^2. Let $f, g : S \to \mathbb{R}^2$ be continuously differentiable functions. Show that

$$\oint_C f\nabla g \cdot d\alpha = -\oint_C g\nabla f \cdot d\alpha$$

for every piecewise smooth Jordan curve C in S. [10]

4. Identify the space of all $n \times n$ matrices with $\mathbb{R}^{n^2}$. Show that the map $A \mapsto A^2$ is invertible on some open set containing the identity matrix. [15]

5. Let $A \subseteq \mathbb{R}^n$ be an open set and $f : A \to \mathbb{R}^n$ is continuously differentiable 1-1 function such that $\det f'(x) \neq 0$ for all $x \in A$. Show that $f(A)$ is an open set and $f^{-1} : f(A) \to A$ is differentiable. Show also that $f(B)$ is open for any open set $B \subseteq A$. [10]

6. Let $S = \mathbb{R}^2 \setminus \{(0, 0)\}$ and let $f = (f_1, f_2)$ be a vector field defined on S by the equation

$$f(x, y) = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}.$$

   (a) Show that $D_1 f_2 = D_2 f_1$ everywhere on S but f is not a gradient on S. [5]

   (b) Show that f is a gradient on the set [10]

$$T = \mathbb{R}^2 \setminus \{(x, y) : y = 0, x \leq 0\}.$$

7. Show that the formula in Green's theorem is invariant under co-ordinate changes. That is, if the theorem holds for a bounded open connected set $U \subseteq \mathbb{R}^2$ with piecewise smooth boundary and if F is a smooth function that maps U 1-1 onto another such domain V and that maps the boundary of U 1-1 smoothly onto the boundary of V, then Green's theorem holds for V also. [10]

PTO

8. Sketch the region

$$S = \{(x, y) : x^2 \leq y \leq 2, -1 \leq x \leq 1\}$$

and express the double integral $\iint_S f(x, y) dx dy$ as an iterated integral in polar co-ordinates. [8]

9. Evaluate

$$\iiint_S \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dx dy dz,$$

where S is the solid bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$ [12]

INDIAN STATISTICAL INSTITUTE

B Stat II, 1st Semester, 2012-13

Statistical Methods III

Semestral Examination

Date: 23.11.2012 Time: 3 hours

Total points 100. Answer all questions

1. Suppose that scores of men aged 21 to 25 years on the quantitative part of the National Assessment of Educational Progress (NAEP) test follow a normal distribution with an unknown mean and standard deviation $\sigma = 60$. Based on a sample of size n, you want to estimate mean score such that you are 90% confident that the estimate lies within 10 units of the true mean. How large a sample size must you choose? [10]

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   [4 + 4 + 7 = 15]

3. Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution with mean 0 and unknown standard deviation $\sigma > 0$. Find the asymptotic distribution of the statistic
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P.T.O

$af_0(x) = bf_1(x)$. Here a and b are fixed positive constants. Show that for any other test procedure ϕ^*, we get

$$a\alpha(\phi) + b\beta(\phi) \leq a\alpha(\phi^*) + b\beta(\phi^*),$$

where $\alpha(\phi)$ and $\beta(\phi)$ represent the probabilities of Type I and Type II errors respectively of the test procedure ϕ. [10]

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$$\frac{\mu_1 - \mu_0}{\sigma_1} = \frac{\mu_2 - \mu_0}{\sigma_2},$$

where both μ_1 and μ_2 are larger than μ_0. [10]

7. Consider the contrived set of data generated by tossing a fair coin, obtained by letting $x_t = 1$ when a head is obtained and $x_t = -1$ when a tail is obtained. Construct y_t as

$$y_t = 5 + x_t - 0.7x_{t-1}.$$

(a) Is the process $\{y_t\}$ stationary? Justify your answer.

(b) Find the ACF function for all lags up to order 3.

(c) A sequence of 10 successive realizations of the $\{y_t\}$ series are given in the following table. Calculate all the sample autocorrelations up to order three from the given data, and compare them with the theoretical values. (Assume $x_0 = -1$.) [5+5+5=15]

t	1	2	3	4	5	6	7	8	9	10
Coin	H	H	T	H	T	T	T	H	T	H
x_t	1	1	−1	1	−1	−1	−1	1	−1	1
y_t	6.7	5.3	3.3	6.7	3.3	4.7	4.7	6.7	3.3	6.7

8. Consider the following data on revenue expenditure of Govt. of India in the following table for the indicated quarters/years. Find the seasonal indices by an appropriate method. [15]

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INDIAN STATISTICAL INSTITUTE

First Mid-Semestral Examination: 2012-13

Subject Name : **Elements of Algebraic Structure** DATE : 19.0[illegible]

Course Name : B.Stat. II yr. Maximum Score: 30 Duration: 3 Hours

Note: Attempt all questions. Marks are given in brackets. Total score is 35.

State the results which you want to use. Use **separate page** for each question.

Unless mentioned explicitly, a ring will always mean a **non-zero ring with identity**.

Problem 1. Let G be a group of size p where p is a prime. Show that G is isomorphic to $\mathbb{Z}_p$ modulo addition. Find a group of size p^2 which is not isomorphic to $\mathbb{Z}_{p^2}$. [4+4 = 8]

Problem 2. Let R be the ring of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where a and b are real numbers. Compute the inverse of the above matrix when $(a, b) \neq (0, 0)$. Prove that R is isomorphic (as a ring) to $\mathbb{C}$, the field of complex numbers. [2+4=6]

Problem 3. Let R_1 and R_2 be commutative rings and let $R = R_1 \times R_2$ be the direct product ring (coordinate wise addition and multiplication). Show that every ideal I of R is of the form $I = I_1 \times I_2$ with I_i an ideal of R_i for $i = 1, 2$. [5]

Problem 4. Show that if I is an ideal of a commutative ring R such that $1 + a$ is a unit in R for all $a \in I$, then I is contained in every maximal ideal of R. [5]

Problem 5. Let $z_0 \in \mathbb{C}$ be a complex number such that there is no irreducible polynomial $f(x) \in_{irr} \mathbb{Q}[x]$ with $f(z_0) = 0$. Let $\mathbb{Q}[z_0] = \{a_0 + a_1 z_0 + \ldots + a_n z_0^n, n \geq 1, a_i \in \mathbb{Q}\}$ be the set of all complex numbers which can be represented as a polynomial in z_0 with rational coefficients. Show that $\mathbb{Q}[z_0]$ is not a field. [5]

Problem 6. Let $R = \mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{3}i, a, b \in \mathbb{Z}\}$. Show that 2 is irreducible but not prime. [3+3 =6]

B Stat (II)

Economic Statistics and Official Statistics

Maximum marks 50

(All questions are compulsory)

DATE - 22.02.13

1. Name an index number and show that it satisfies factor reversal test 10

2. A price index number series was started with 1992 as base. By 1996 it rose by 25%.The link relative for 1997 was 95.In this year a new series was started. This new series rose by 15% in the next year(1998).During the following years the rise was as the following:

2002-5% higher than 2000

2000-8% higher than 1998

Splice the two series and calculate the index nos for various years by shifting the bas to 1998 **15**

3. Explain true cost of living index 10

4. a) Explain how the index of industrial production of a country is calculated with special reference to the adjustments done

b)what are the difficulties in calculating the Index no of agricultural commodities ?

c) Write a note on the Human Development Index of UNDP 15

Indian Statistical Institute

Mid-Semester Examination: 2012-13
Course Name: B Stat II Year
Subject Name: Economics II

Date: 25·02·13 **Maximum Marks: 40** **Duration: 2.5 Hours**

1. The following are the data for a hypothetical economy for 2012 in crores of rupees

Net rental income of persons	22.0
Depreciation	669.1
Wages and salaries	3780.4
Personal consumption expenditure	4378.2
Indirect taxes	525.3
Business transfer payments	28.7
Gross investment	882.0
Exports of goods and services	659.1
Subsidies of government to business	9.0
Government purchases of goods and services	1148.4
Imports of goods and services	724.3
Net interest paid to households	399.5
Proprietors' income (net)	441.6
Corporate profits (net)	485.8
Net factor income from rest of world	5.7

a. Compute GDP and NDP using the spending approach.
b. Compute national income in two ways. [7+13=20]

2. a. Consider the following information regarding a firm in the domestic economy in a given period

Revenue earned	Rs. 80,000
Unsold output	2,000
Raw materials purchased from other firms	30,500
Unused part of the raw materials purchased	2,500
Labour hired from households	12,000
Labour supplied by labour contractors	10,000
Rent paid to households	500
Interest paid to banks	500
Interest paid on bonds sold to households	1,000
Dividend paid to households of which $1/5^{th}$ goes to foreigners	5,000
Net subsidies received	-1,500
Depreciation	3,300
Corporate profit tax	5,000

What are the firm's contributions to GDP, NI and personal income? Does the firm make any contribution to the final expenditure?

2. b. A household buys from a firm in the current year a house for Rs.40 lakh. Half of the construction of the house was completed in the previous year. What is the contribution of this transaction, to households' investment and firms' investment in the current year?

[15+5=20]

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination: (2012 – 13)

B. Stat II Year

Biology II

Date 25-02-13 Maximum Marks 30 Duration Three hours .

(Attempt any Three questions)

(Number of copies of the question paper required Six)

1. Define Moisture Availability Index? Draw a suitable rice calendar with the following data. 2+8

Week No.	***22***	***23***	***24***	***25***	***26***	***27***	***28***	***29***	***30***	***31***	***32***	***33***	***34***	***35***	***36***	***37***	***38***	***39***	***40***
Rainfall (mm) at 0.5 Prob.	***25***	***0***	***15***	***30***	***29***	***37***	***28***	***35***	***18***	***90***	***98***	***142***	***95***	***80***	***15***	***32***	***2***	***0***	***0***
PET (mm)	***31***	***42***	***30***	***27***	***23***	***23***	***22***	***20***	***22***	***20***	***19***	***17***	***19***	***20***	***25***	***28***	***31***	***33***	***34***

2. Name the different weather parameters that are related to crop production? Write down the names of the apparatus used for measuring/estimating different weather components with their units. 3+7

3. Define drought. Write about different types of drought. Write in details about any one of the drought indices 2+4+4

4. Write short notes on any five of the following: 2 x 5

   a. Monsoon onset
   b. Microclimate
   c. Cup counter anemometer
   d. Monsoon withdrawal
   e. Phytoclimate
   f. climate

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination 2012-13

Course Name: B.Stat II Year

Subject Name: Physics II

Date: 25-02-13

Maximum marks: 50

Duration: 2 hours

THERMODYNAMICS

Answer any two questions. All questions carry 10 marks.

1) (a) Calculate the variation of temperature of atmospheric gas with height above the sea level. Assume that the expansion of the gas, as it rises by convention current to low pressure upper zone, to be adiabatic. M is the average molecular weight of the gas.

(b) Calculate the entropy of an ideal gas with constant volume specific heat C_v, at T, V.

(c) Consider a transformation where an amount of heat ΔQ is flowing spontaneously from a system at temperature T_2 to a system at temperature T_1. Show that the entropy of the combined system increases.

(5+2+3=10)

2) (a) Derive the work done by a reversible Carnot cycle operating between temperatures T_1, T_2 with $T_2 > T_1$ and find its efficiency. Assume that the cycle is performed by one mole of ideal gas.

(b) Consider T, V to be the independent variables. Find the expression of dS (S being entropy) for an arbitrary transformation.

(c) From the above expression, construct an equation by application of the fact the S is a state function.

(5+2+3=10)

3) (a) Derive Clapeyron's equation with explanation of the parameters involved.

(b) Find an approximate solution for the saturated vapor pressure by considering a transformation from liquid to gas. For simplicity assume that the saturated vapor obeys the ideal gas equation of state.

(6+4=10)

Answer any one short note of 5 marks each.

(a) Critical Isothermal in liquid-vapor phase diagram

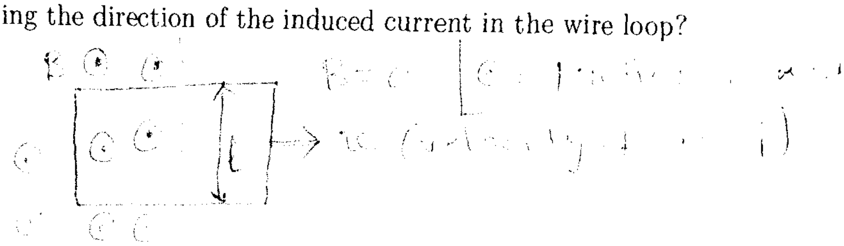
(b) Idea of Corresponding States in Van der Waal's equation

(c) Stability condition for isolated system from consideration of entropy

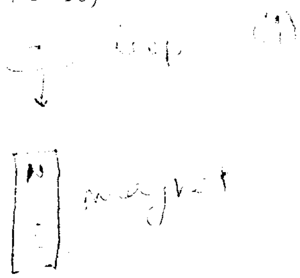
ELECTRODYNAMICS

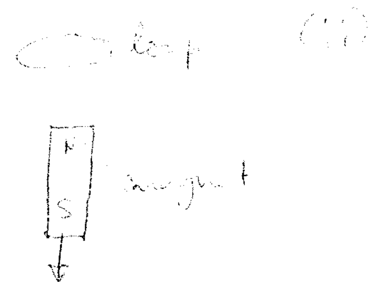
Answer any two questions. All questions carry 10 marks.

1) (a) Calculate the EMF induced in a rectangular closed wire loop as it moves with uniform velocity in a uniform static magnetic field perpendicular to the plane of the wire loop (see figure). Derive the EMF also from Faraday's law. How does Lenz's law help in determining the direction of the induced current in the wire loop?



(b) Consider a bar magnet and a closed wire loop above the magnet (see figure). Find the direction of the induced current in the wire loop as (i) the loop moves towards the magnet as shown in figure; (ii) the bar magnet moves away from the wire loop as shown in figure. (6+4=10)





2) (a) Explain the significance of the term introduced by Maxwell in Ampere's law.

(b) Express the Maxwell's equations in terms of electric and magnetic potentials.

(4+6=10)

(3) (a) Explain the idea of gauge invariance in electrodynamics.

(b) Give two examples of popular gauge choices. What is the equation satisfied by the electric potential in Coulomb gauge.

(c) What is the significance of Poynting vector in the conservation principle of energy in electrodynamics.

(3+3+4=10)

Answer any one short note of 5 marks each.

(a) Coefficient of Mutual induction for a system of two arbitrary wire loops

(b) Velocity of electromagnetic wave in vacuum

(c) Prove the identities (f is a scalar function and $\vec{A}$ is a vector function):

$$(i)\ \vec{\nabla}.(f\vec{A}) = f.(\vec{\nabla}.\vec{A}) + \vec{A}.(\vec{\nabla}f)$$

$$(ii)\ \vec{\nabla}X(f\vec{A}) = f(\vec{\nabla}X\vec{A}) - \vec{A}X\vec{\nabla}f)$$

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination: 2012-2013

B. Stat. (Hons.) 2nd Year. 2md Semester

Statistical Methods IV

Date: March 01, 2013 **Maximum Marks: 60** **Duration: 2 and 1/2 hours**

- This question paper carries 70 points. Answer as much as you can. However, the maximum you can score is 60.
- You should provide as much details as possible while answering a question. You must state clearly any result stated (and proved) in class you may need in order to answer a particular question.

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1. Let $\boldsymbol{X} \sim \mathrm{N}_p(\mathbf{0}, \boldsymbol{\Sigma})$. Let

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix},$$

where $\boldsymbol{X}_i$ $(i = 1, 2)$ is $p_i \times 1$, and $\boldsymbol{\Sigma}$ is a $p \times p$ positive definite matrix. Moreover, $\boldsymbol{\Sigma}_{ij}$ $(i, j = 1, 2)$ is $p_i \times p_j$.

(a) Find a matrix $\mathbf{A}$ such that $\boldsymbol{X}_1$ and $\boldsymbol{X}_2 - \mathbf{A}\boldsymbol{X}_1$ are independent.

(b) Let $Q := \boldsymbol{X}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{X} - \boldsymbol{X}_1^{\mathrm{T}}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{X}_1$. Show that $Q \sim \chi^2_{p_2}$. [3+8=11]

2. Suppose $\boldsymbol{X} \sim \mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let $\mathbf{A}$ be a $p \times p$ symmetric matrix. Show that $\mathrm{Var}(\boldsymbol{X}^{\mathrm{T}}\mathbf{A}\boldsymbol{X}) = 2\,\mathrm{trace}(\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}) + 4\boldsymbol{\mu}^{\mathrm{T}}\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\mu}$. [11]

3. Suppose $\mathbf{M} \sim \mathrm{W}_p(\boldsymbol{\Sigma}, n)$, $n \geq p + 2$, where $\boldsymbol{\Sigma}$ is a $p \times p$ positive definite matrix. Show that $\mathrm{E}(\mathbf{M}^{-1}) = \boldsymbol{\Sigma}^{-1}/(n - p - 1)$. [10]

4. Suppose $(X_{i,1}, X_{i,2})^{\mathrm{T}}$, $i = 1, \ldots, n$, are independent and identically distributed (i.i.d.) $\mathrm{N}_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ variables, where $\boldsymbol{\mu} \in \mathbb{R}^2$ and $\boldsymbol{\Sigma} = ((\sigma_{ij}))$ is positive definite. Let $\rho := \sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}}$. Let $\bar{X}_k := \sum_{i=1}^n X_{i,k}/n$, $k = 1, 2$, $S_{ij} = \sum_{i=1}^n (X_{i,1} - \bar{X}_1)(X_{i,2} - \bar{X}_2)$. Also, define $r = S_{12}/\sqrt{S_{11}S_{22}}$. It is known that a suitable test-statistic for testing $\mathrm{H}_0 : \rho = 0$ against $\mathrm{H}_1 : \rho \neq 0$ is given by $T := \sqrt{n-2}\, r/\sqrt{1 - r^2}$, large values of $|\, T \,|$ being significant, noting that $|\, T \,|$ is an increasing function of $|\, r \,|$. Find the null distribution of T from first principles. [13]

5. Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n, \boldsymbol{X}_{n+1}$ be i.i.d. $\mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ $(n \geq p + 1)$ variables, where $\boldsymbol{\Sigma}$ is positive definite. For $j = n, n+1$, let $\bar{\boldsymbol{X}}_j$ and $\mathbf{S}_j$ be defined by $j\bar{\boldsymbol{X}}_j := \sum_{i=1}^j \boldsymbol{X}_i$, $\mathbf{S}_j := \sum_{i=1}^j (\boldsymbol{X}_i - \bar{\boldsymbol{X}}_j)(\boldsymbol{X}_i - \bar{\boldsymbol{X}}_j)^{\mathrm{T}}$.

[Please turn over.]

(a) Show that $\mathbf{S}_{n+1} = \mathbf{S}_n + n(\boldsymbol{X}_{n+1} - \bar{\boldsymbol{X}}_n)(\boldsymbol{X}_{n+1} - \bar{\boldsymbol{X}}_n)^{\mathrm{T}}/(n+1)$.

(b) Let $T := (\boldsymbol{X}_{n+1} - \bar{\boldsymbol{X}}_{n+1})^{\mathrm{T}}\mathbf{S}_{n+1}^{-1}(\boldsymbol{X}_{n+1} - \bar{\boldsymbol{X}}_{n+1})$, $T_1 := (\boldsymbol{X}_{n+1} - \bar{\boldsymbol{X}}_n)^{\mathrm{T}}\mathbf{S}_n^{-1}(\boldsymbol{X}_{n+1} - \bar{\boldsymbol{X}}_n)$. Show that T is an increasing function of T_1.

(c) Show that $(n+1)T/n$ follows a Beta distribution, with parameters to be obtained by you. [4+5+4=13]

6. Suppose $\mathbf{X} = (\boldsymbol{X}_1 \boldsymbol{X}_2 \cdots \boldsymbol{X}_n)^{\mathrm{T}}$ is an $n \times p$ $(n \geq p+1)$ data matrix from $\mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. It is known that $\boldsymbol{\mu} = k\boldsymbol{\mu}_0$ for some unknown constant k, where $\boldsymbol{\mu}_0$ is a known $p \times 1$ non-null vector, and $\boldsymbol{\Sigma}$ is an unknown positive definite matrix.

(a) Find the maximum likelihood estimator (MLE) of $(k, \boldsymbol{\Sigma})$.

(b) Denote the MLE of $(k, \boldsymbol{\Sigma})$ by $(\hat{k}, \hat{\boldsymbol{\Sigma}})$. Show that $\hat{k}$ is unbiased for k. [8+4=12]

* * * * ** *Best of Luck!* * * * * **

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination : 2012-2013

B. Stat. (Hons.) II Year

Subject : Demography and SQC & OR

Full Marks: 100 **Time: 3hours** **Date of Examination:**

Note: 1. Begin each group on a fresh answer script

Group A: Demography

Total Marks: 50

(Answer all questions.)

1) Write short notes on the following.

a) Population Momentum

b) Effect of change in age structure on population

c) Age heaping and age shifting.

d) U.N. Joint Score

[2+3+3+4 =12]

2) a) Explain the logic behind the construction of the Whipple Index.

b) Construct the table for calculation of the Myers' Index.

[5+5 = 10]

3) State the assumption behind the use of Sex Ratio Score for measuring accuracy of data on population age distribution. Write mathematical expressions for Sex Ratio Score along explanations of the symbols.

[2+4 = 6]

4) Explain fully how a Y-transformation is used in smoothing and adjustment of an age distribution in comparison to a standard age distribution.

[10]

5) a) State the formula, used in Carrier-Farrag Ratio Method, along with all necessary explanations.

b) Explain in detail a method of rectifying an erroneously noted survey data for a particular age group.

[5+7 = 12]

Group B: SQC & OR

Total Marks: 50

Note: 1. Begin this group on a fresh answer script.

1. A process producing coaxial cables is being monitored by a "number of defects per unit" chart. The process average has been calculated as 0.10 defects per unit. Three sigma control limits are employed and samples of size 200 are taken on a daily basis.

a) Calculate the upper and lower control limits for the chart.

b) If the process mean were to shift suddenly to 0.15 per unit, what is the probability that this shift would be detected at the 5^{th} subsequent day?

c) What is the expected number of samples until an out-of-control signal is received?

[4+5+6=15]

2. Consider the following plans, where the symbols have their standard meanings:

Plan	*N*	*n*	*c*
A	1000	240	2
B	1000	170	1
C	1000	100	0

The AQL has been fixed at 1%. The customer is willing to accept 2.2% defective lot.

a) As a customer which plan would you prefer?
b) As a supplier which plan would you prefer?

[9+8=17]

3. Answer the following questions briefly:

a) Define Quality costs.
b) How does the ISO define Quality?
c) Define a sampling scheme.

[2*3=6]

4. Who is/are credited with the development of

(i) acceptance sampling plans
(ii) control charts
(iii) the seven tools of QC
(iv) loss related definition of Quality
(v) histogram
(vi) PDCA wheel.

[6]

5. State whether the following statements are ***True*** or ***False***. You need not copy the statements.

a) A form, in either diagram or table format, that is prepared in advance for recording data is known as a flowchart.

b) A process that is in statistical control will never have a point beyond a three-sigma limit on $\bar{\bar{X}} - R$ control chart.

c) In practice the proportion of non-conforming items coming from a manufacturing process that produces millions of items per day can be successfully monitored with a p- chart.

d) The principal function of a control chart is a tool for process adjustment.

e) The chance of getting a false out-of-control signal above the 3σ limit is the same for both an $\bar{X}$ and an $R-$ chart.

f) Quality is directly proportional to variability.

[6]

INDIAN STATISTICAL INSTITUTE

B.Stat. 2^{nd} year End Sem. Exam. 2012-13 DATE: 29.

Subject Name : **Elements of Algebraic Structures** Score: 100 Duration: 3 Hours

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Note: Attempt all questions. Marks are given in brackets. Total score is 110, Maximum you can score 100. **State the results** which you want to use. Use **separate page** for each question. Unless mentioned explicitly, a ring will always mean a **non-zero ring with the multiplicative identity** e. We always mean p **to be prime and** $i^2 = -1$.

Problem 1. Let $G := \{x \in \mathbb{R} : x \geq 1\}$. For $a, b \in G$, we define $a * b = a^{\log_2 b}$. Prove or disprove that G is a group? [10]

Problem 2. Let $x, y \in E/F$ be two transcendental elements over F. Show that if x is algebraic over $F(y)$ then y is also algebraic over $F(x)$. [10]

Problem 3. (a) Compute $[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}]$.
(b) Prove that $x^4 - 2$ is irreducible over $\mathbb{Q}(i)$. [8 + 12 = 20]

Problem 4. Suppose $m \in \mathbb{Z}$ is not a square and $m \equiv c^2 \mod p$ for some $c \in \mathbb{Z}$. Prove that $I_p = \{a + b\sqrt{m} : a, b \in \mathbb{Z}, p|a, p|b\}$ is an ideal of $\mathbb{Z}[\sqrt{m}]$ which is **not** maximal. [5 + 10= 15]

Problem 5. Let $\mathbb{F}$ be an **infinite** field with characteristic p and E be the splitting field of the polynomial $x^m - e$ over $\mathbb{F}$, where $(m, p) = 1$. Let $G = \{x \in E : x^m = e\}$.

(a) Show that $|G| = m$.
(b) Show that G is a cyclic group under field multiplication. [10+10=20]

Problem 6. Let G be a finite group of size $n := mp^t$ and $S = \{M \subseteq G : |M| = p^t\}$.

(a) Show that for all $M \in S$, the stabilizer $st(M) = \{g \in G : gM = M\}$ is a subgroup of G.
(b) Define $M \sim M'$ if $\exists g \in G$ with $gM = M'$. Prove that $\sim$ is an equivalence relation on S.
(c) Let the partition induced by the above equivalence relation be $\{[M_1], \ldots, [M_r]\}$ for some $M_i \in S$, where $[M_i]$ denotes the class containing M_i. Prove that

$$\binom{mp^t}{p^t} = \sum_{i=1}^{r} [G : st(M_i)].$$ [5+5+10= 20]

Problem 7. Let R be a commutative ring in which $a^2 = 0$ only if $a = 0$. Show that if $q(x) = a_0 + a_1 x + \ldots + a_n x^n \in R[x]$ is a zero divisor of $R[x]$ then there exists a non-zero $b \in R$ such that $ba_0 = \cdots = ba_n = 0$. [15]

[redacted]

Semestral Examination : 2012-13

B .Stat(2^{nd} year)

Subject Name : Economic Statistics and Official Statistics

Date : 02.04.13 Maximum Marks :100 Duration: 3 hrs

You can answer any question. The maximum you can score is 100.

1.Explain the notion of compensated demand curve with a specific utility function. How does it differ from ordinary demand curve ? 6+4=10

2.Derive the Cournot aggregation condition. What is its importance? 7+3=10

3. What is an Engel Curve? State some of the standard forms of Engel Curves. Classify different goods in terms of Engel elasticity. 3+4+3=10

4. Derive Slutsky's equation in two commodity framework and state its importance. 8+2=10

5. Explain the concept of dynamic stability in a market model with lagged adjustment. 10

6. Write short note on: a)Shephard's Lemma, b)Cobb Douglas Production function,

c) Isocost curve. 4+3+3=10

7. Explain mathematically how elasticity of substitution may be positive. 10

8.Compute price and quantity index nos for 1999 with 1998 as base year from the data given below by using the following index number formulas a)Laspeyre's b)Paasche's c)Marshall Edgeworth d)Fisher's AND check which of the index numbers satisfy the factor reversal test. 5+5=10

Year	Article 1		Article2		Article3		Article4	
	price	quantity	price	Quantity	price	quantity	price	quantity
1998	5	5	7.75	6	9.63	4	12.5	9
1999	6.5	4	8.80	10	7.75	6	12.75	9

9.Derive the necessary and sufficient conditions for constrained cost minimisation by a producing firm. 10

10.a)What are the five organisations of World Bank? b) State the millennium Development goals of the United Nations. 4+6=10

11.a)What are the types of data collected by NSSO and CSO ?

b)Explain the Sample Registration Scheme of the Indian Population Census. 6+4=10

12. a) Interpret a parameter of the Pareto distribution in terms of inequality.

b) Derive the equation of Lorenz curve of lognormal distribution. 5+5=10

13. State and interpret the significance of any three important axioms for poverty measurement . 10

INDIAN STATISTICAL INSTITUTE
Second Semestral Examination: 2012–2013
B.Stat. (Hons.) 2nd Year. 2nd Semester
Statistical Methods IV

Date: May 06, 2013 **Maximum Marks: 75** **Duration: 4 hours**

- This question paper carries 85 points. Answer as much as you can. However, the maximum you can score is 75.
- You should provide as much details as possible while answering a question. You must state clearly any result stated (and proved) in class you may need in order to answer a particular question.

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1. Suppose $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are i.i.d. $\mathrm{N}_p(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} \in \mathbb{R}^p$, Σ is a $p \times p$ positive definite matrix, and $n > p$. Let $\boldsymbol{\mu}$ be unknown. We wish to test the hypothesis $\mathrm{H}_0 : \Sigma$ *is diagonal* against $\mathrm{H}_1 : \mathrm{H}_0$ *is false*. Denote by $\mathbf{R}$, the sample correlation matrix. Let λ denote the likelihood ratio test (LRT) statistic for testing H_0 against H_1.

(a) Show that $-2\log\lambda = -n\log(\det \mathbf{R})$.

(b) Argue that under H_0, $-2\log\lambda \xrightarrow{d} \chi^2_{p(p-1)/2}$ as $n \to \infty$. [7+5 = 12]

2. Suppose $\boldsymbol{X} = (X_1, \ldots, X_p)^{\mathrm{T}} \sim \mathrm{N}_p(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} \in \mathbb{R}^p$ and Σ are both unknown. Denote the multiple correlation coefficient between X_1 and $(X_2, \ldots, X_p)^{\mathrm{T}}$ by $\bar{R}$. We wish to test the hypothesis $\mathrm{H}_0 : \bar{R} = 0$ against $\mathrm{H}_1 : \bar{R} > 0$ based on n i.i.d. realizations of $\boldsymbol{X}$, denoted by $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$, where $n > p$. Denote by R, the sample multiple correlation coefficient. Let λ denote the LRT statistic for testing H_0 against H_1.

(a) Show that $\lambda = (1 - R^2)^{n/2}$.

(b) Show that under H_0, $[R^2/(1-R^2)] \cdot [(n-p)/(p-1)] \sim F_{p-1,n-p}$. [8+8 = 16]

3. Suppose $Y_{i1}, \ldots, Y_{in_i}$ are i.i.d. $\mathrm{N}(\theta_i, \sigma^2)$, $i = 1, \ldots, k$, $n_i \geq 2$ for every i; where $\theta_1, \ldots, \theta_k \in \mathbb{R}$. We wish to test the hypothesis $\mathrm{H}_0 : \theta_1 = \cdots = \theta_k$ against $\mathrm{H}_1 : \mathrm{H}_0$ *is false*.

(a) Discuss how the problem of testing H_0 against H_1 can be formulated as test for significance of regression in a suitable multiple linear regression model.

(b) Obtain the LRT for this testing problem under this formulation.

[P.T.O.]

(c) Obtain the non-null distribution of a suitable equivalent test statistic of the LRT statistic. [3+8+7 = 18]

4. Suppose $X_1, X_2, \ldots$ are i.i.d. $N(\theta, 1)$ variables, where $\theta \in \mathbb{R}$. Consider the problem of estimating the parametric function $\psi(\theta) := P_\theta(X_1 \leq c_0)$, where c_0 is a known constant.

(a) Find the MLE of $\psi(\theta)$, denoted by $\hat{\psi}_n$, based on $X_1, \ldots, X_n$.

(b) Show that for suitable $\mu \in \mathbb{R}$ and $\sigma > 0$, to be obtained by you, the asymptotic distribution of $n^{1/2}(\hat{\psi}_n - \mu)$, as $n \to \infty$, is normal with mean zero and variance σ^2.

[3+7 = 10]

5. Consider n i.i.d observations, denoted $X_1, \ldots, X_n$, from an exponential distribution with location parameter θ and scale parameter σ, $\theta \in \mathbb{R}, \sigma > 0$. Denote the order statistics by $X_{(1)} \leq \cdots \leq X_{(n)}$. Suppose now that we have a Type II censored sample where only the first r order statistics $X_{(1)} \leq \cdots \leq X_{(r)}$ are observed.

(a) Find the MLE of (θ, σ), denoted by $(\hat{\theta},\ \hat{\sigma})$.

(b) Show that $W_1 := 2n(\hat{\theta} - \theta)/\sigma \sim \chi^2_2$, $W_2 := 2r\hat{\sigma}/\sigma \sim \chi^2_{2r-2}$, and that W_1 and W_2 are independent.

(c) Discuss how you can obtain a $100(1-\alpha)\%$ confidence interval for θ. [6+8+4 = 18]

[**Note.** You may assume the following result to hold, and use it, if required.

Let $E_{(1)} \leq E_{(2)} \leq \cdots \leq E_{(n)}$ denote the order statlstlcs corresponding to a sequence of i.i.d. $Exp(0,1)$ random varlables $E_1, \ldots, E_n$. If we define $E_{(0)} = 0$, the normalized exponentlal spaclngs $(n-i+1)(E_{(i)} - E_{(i-1)})$, $i = 1, \ldots, n$, are i.i.d. $Exp(0,1)$ random varlables.]

6. Suppose $X_1, X_2, \ldots$ are i.i.d. Cauchy(θ, σ) variables, where $\theta \in \mathbb{R}$ and $\sigma > 0$ are unknown.

(a) Discuss how the sample semi-interquartile range R_n, based on first n observations, can be used to estimate σ.

(b) Stating clearly the assumptions you need, find the asymptotic distribution of your estimate in (a). [4+7 = 11]

****** *Best of Luck!* ******

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination: (2012 – 2013)

B. Stat II Year

Biology II

Date 09.05.13 Maximum Marks 50 Duration 3:00 hours .

(Attempt any five questions)

(Number of copies of the question paper required ______)

1. Write in brief about different types of rice. Briefly describe the cultural practices associated with rainfed direct seeded rice cultivation. 3+7

2. Briefly mention the criteria for essentiality of plant nutrients. Calculate the quantity of VC, Urea, SSP and KCL required for 1 hectare rice crop to supply the nutrients requirement of 160 kg N, 80 kg P_2O_5 and 80 kg K_2O per hectare. 50% of required N should be given through VC. 3+7

3. Describe the suitable agro-techniques for rice nursery bed preparation. Estimate the expected yield of rice grain in t/ha from the following data:
(i) Spacing - 20x10cm, (ii) Average no. of tillers/hill –15, (iii) Average no. of effective tillers/hill –13, (iv) Average no. of grain/panicle –217, (v) Average panicle length -15 cm, (vi) Average no. of unfilled grain/panicle –30, (vi) Test weight -24 g. 4+6

4. Write the differences between: 2.5 x 4
   a) Manures and Fertilizers
   b) Intercropping and mixed cropping
   c) Soil texture and soil structure
   d) Macro and micro nutrients

5. Write short notes on <u>any five</u> of the following : 2 x 5
   a) Monsoon onset
   b) Potential Evapo-transpiration
   c) Cup counter anemometer
   d) Reproductive stages in rice
   e) Permanent wilting point
   f) Field capacity
   g) Capillary water

6. Briefly discuss about <u>any two</u> of the following : 5X2
   a) Variation in yield of winter (Aman) rice and summer (<u>Boro</u>) rice,
   b) System of rice intensification,
   c) Weather forecasting.

7. Let N and P are two inter-crops. The dynamics of the inter-crops follow the following systems of differential equations

$$dN/dt = N\,(r - a_{11}N - a_{12}P)$$

$$dP/dt = P\,(S - a_{21}N - a_{22}P)$$

(Notations have their usual meanings)

Discuss the stability and equilibrium of the above dynamical system.

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination : 2012-13

Course Name : B. STAT. II YEAR

Subject Name : PHYSICS II

Date : 09.05.13

Maximum Marks : 50, Duration: 2.5 hours

Answer question 7 ($2 \times 5 = 10$) **and any four questions from the rest** ($4 \times 10 = 40$)**:**

1.(i) Using Maxwell's equation in vacuum, without charge and currents, derive the wave equation satisfied by the electric field. What is the velocity of wave propagation?

(ii) Derive the equation for electromagnetic wave propagation in a conducting medium. What is meant by attenuation of the wave?

(3+2)+(3+2)=10

2.(i) Derive the boundary conditions satisfied by electric and magnetic fields in an intersection of two linear non-conducting media having permittivity ϵ_1, ϵ_2 and permeability μ_1, μ_2 respectively.

(ii) Consider normal incidence of electromagnetic wave on the boundary and compute the reflected and transmitted amplitudes of electric field, in terms of the incident electric field amplitude and relevant parameters.

4+6=10

3.(i) Consider the monochromatic electric field plane wave $\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(kx-\omega t)}$, traveling along x-axis in vacuum (with no charge and current) and $\vec{E}_0$ is a constant vector. Use Maxwell's equations to derive $\vec{B} = \frac{k}{\omega}(\hat{\imath} \times \vec{E})$ where $\vec{B}$ is the magnetic field and $\hat{\imath}$ is the unit vector along x-axis.

(ii) Find the Poynting vector and energy density for this electromagnetic field.

6+(2+2)=10

4.(i) Consider two macroscopic systems in thermal contact, each system having a fixed volume. Find an expression for temperature from equilibrium condition of the systems.

(ii) A macroscopic system is in thermal contact with a heat reservoir. Derive the Boltzmann distribution, *i.e.* the probability of the system being in a particular state r.

5+5=10

5.(i) A system consists of N_0 spin 1/2 magnetic atoms in external magnetic field $\vec{H}$, in equilibrium. Calculate the magnetic susceptibility χ.

(ii) Give a schematic graph of total average magnetic moment versus inverse of temperature and explain the low temperature behavior of total average magnetic moment.

6+(2+2)=10

6.(i) Calculate the partition function for a classical ideal gas (a system of non-interacting particles). Derive its equation of state from the partition function.

(ii) Compute the partition function of a quantum harmonic oscillator and find its average energy. What is the significance of the high temperature limit of the average energy?

(3+1)+(5+1)=10

7. **Write short notes on any two of the following topics** $(2 \times 5 = 10)$:

(i) Skin depth

(ii) velocity of electromagnetic wave in non-conducting medium

(iii) Fermi-Dirac distribution function with schematic plots of the energy versus distribution function at zero temperature and at a small non-zero temperature

(iv) Bose-Einstein distribution function and from it the photon distribution function

Indian Statistical Institute
Second Semester Examination 2012-13
Course Name: BStat Second Year
Subject: Economics II (Macroeconomics)

Date - 07.05.13 **Maximum Marks: 60** **Duration: 2.5 Hours**

Answer the following questions

1. a) Suppose in a simple Keynesian model for an open economy without government, consumption, investment and import functions are given respectively by $C = 100 + 0.9Y$, $I = 50 + 0.5Y$ and $M = 10 + 0.2Y$. By how much will Y change following an increase in the amount of autonomous expenditure by 10 units? Is your answer consistent with the way producers are usually assumed to behave? Explain.

b) Suppose a simple Keynesian model for an economy without government is given by the following set of functions: $C = 7 + 0.75Y, I = 7, \overline{X} = 14$ and $M = 7 + mY, 0 < m < 1$. Following an increase in autonomous import by 5 units, equilibrium value of planned consumption is found to decline by 7.5 units. Derive the value of marginal propensity to spend on domestically produced goods with respect to a change in Y.

[10+12]

2. Suppose in an IS-LM model all functions are linear. Following a change in the income tax rate, the equilibrium in the model is found to change from (Y=2050, r = 0.2) to (Y=2025, r = 0.1). **(i)** Illustrate this change in a diagram. **(ii)** It is given that the demand for real balance changes by 0.25 if Y changes by unity. Find out the change in the value of demand for real balance if r changes by unity. **[6+16]**

3. a) A farmer in a given year produced wheat and sold it to a miller for Rs.20,000. The farmer used Rs.2000 worth of seeds carried over from the previous year's production. The miller converted the wheat purchased from the farmer into flour and sold it to a baker for Rs.24000. The baker held one third of the flour bought in stock and converted the rest into bread, which he sold in the market for Rs.20000. Find the respective GVAs of the farmer, miller and the baker.

b)In a certain year firms in an economy produced GDP of Rs.35 crore. Out of this total production, households purchased goods of Rs. 20 crore. Firms on the other hand purchased goods of Rs.10 crore and added them to stock. The remaining part of GDP of Rs.5 crore could not be sold. What is the gross investment of firms? Derive the GDP of the economy by expenditure method.

c) A retired person lives in his own house and keeps his savings in post office savings bank account. Does he contribute anything to the GDP of the economy? **[9+7+6]**

Or

P.T.O

Consider an IS-LM model for a closed economy without government, where the equation of the IS curve is given by $Y = 480 - 2000r$. **(i)** Find out the state of the goods market at $(Y = 160, r = 0.2)$. **(ii)** Suppose interest-sensitivity of aggregate demand for goods and services is -500. Derive the horizontal and vertical shifts of the given IS curve following an increase in the autonomous expenditure by 5 units. **(iii)** Suppose, following an increase in the amount of real balance by 5 units, equilibrium value of Y in this model rises by 10 units. Derive the change in the equilibrium value of r. **[6+8+8]**

INDIAN STATISTICAL INSTITUTE
B. Stat. (Hons.) II Year 2012-2013

Second Semester Examination

Subject : Demography & SQC & OR

Date : 13th May 2013 Full Marks : 100 Duration : 3 hrs.

Instruction : ***Begin each group on a separate answer-script.***

Group B : SQC & OR (Maximum Marks: **50**)

This group carries 56 marks. You may answer as much as you can; but the maximum you can score is 50.

1. Consider the following constraints of a LPP written in standard form:

$$x_1 + x_2 + 4x_3 + 2x_4 + 3x_5 = 8$$

$$4x_1 + 2x_2 + 2x_3 + x_4 + 6x_5 = 4$$

$$x_i \geq 0, \qquad i = 1, \cdots, 5.$$

Identify

a) a basic feasible solution

b) a basic infeasible solution

c) infinity of solutions.

[2 X 3 = 6]

2. Write the following LP into a standard form:

$$\text{Max } z = x_1 + 2x_2 - x_3$$

subject to the following constraints:

$$x_1 + x_2 - x_3 \leq 5$$
$$-x_1 + 2x_2 + 3x_3 \geq -4$$
$$2x_1 + 3x_2 - 4x_3 \geq 3$$
$$x_1 + x_2 + x_3 = 2$$
$$x_1 \geq 0; \ x_2 \geq p; \ x_3 \text{ is unrestricted in sign.}$$

Mention the range of p in the standard LPP.

[8+2=10]

3. Piston rings for an automotive engine are produced by a forgoing process. The quality characteristic of interest is the inside diameter of the rings manufactured by this process. The production manager proposes to monitor this process using an appropriate control chart. He collects 25 preliminary samples of size 5 and arrives at the following results:

$$\sum_{i=1}^{25} \bar{X}_i = 1850.028; \quad \sum_{i=1}^{25} R_i = 0.581 .$$

The specification limits on the diameter are 74.000 $\pm$0.05 mm.

a) Find the trial control limits for the chart.

b) Assuming that the process is in control, estimate the process mean and the process standard deviation.

c) Estimate the fraction of non-conforming piston rings produced. Mention clearly any assumption that you make.

d) Find the probability of detecting a shift of two standard deviation units on the first sample following the shift.

e) Get point estimates of C_p and C_{pk} for the manufacturing process.

[5+2+5+4+4=20]

4. Suppose that a single sampling acceptance rectification plan with $n = 150$, $c = 1$ is being used for inspection where the vendor ships the products in lots of size $N = 3,000$. Find the AOQL for this plan.

[10]

5. Answer the following questions briefly, but as best as you can:
   a. What is meant by Consumer's risk is 10%?
   b. How do we know from the simplex tableau that the LPP has unbounded solution?
   c. Mention the name of a textbook on Operations Research. Also mention its author.
   d. Mention the name of a textbook on Statistical Quality Control. Also mention its author.
   e. Describe briefly the application of any one SQC tool by the Dean's Office of ISI.

[1+1+2+2+4=10]

INDIAN STATISTICAL INSTITTUE, KOLKATA

Second Semestral Examination

B.Stat (Hons.) II Year

Subject: Demography and SQC & OR

Full Marks: 100 Date of Examination : 13.05.13

*(**Instruction**: Answer Group A and Group B on separate answer scripts. Standard notations are followed.)*

Group A: Demography

Total Marks : 50

(You may answer all questions, but maximum you can score is 50)

1(a) Define the following.

i) Infant Mortality Rate

ii) Late Neo-natal Mortality Rate

iii)Standardised Mortality Ratio

iv)Maternal Mortality Ratio

(b) Explain step by step how a life table is constructed from the m-type mortality rates to obtain life expectancy at some age. (1.5+1.5+1.5+1.5+4 = 10)

2(a) Express $_nL_x$ in terms of l_x, $_na_x$ and $_nd_x$.

(b) Why q_x cannot be calculated directly? Show how it gets estimated.

(c) For age groups width n years, derive an equation that relates q-type mortality rate with m-type mortality rate. (2 + 1 + 4 + 3 = 10)

3. The data below refers to the male populations of three countries A, B and C in mid-1990s.

(a) Calculate the crude death rate for each country.

(b) Using the population of A as the standard, calculate the standardized death rates for B and C.

(c) Comment on your results.

Age Group (in years)	Country A		Country B		Country C	
	Population ('000)	No. of Deaths	Population ('000)	No. of Deaths	Population ('000)	No. of Deaths
0 - 4	1,767	11,832	1,857	5,179	150	860
5 - 14	3,062	1,390	3,372	2,300	286	132
15 - 24	2,430	2,816	3,123	6,646	243	322
25 - 44	4,101	9,690	3,724	12,702	294	614
45 - 64	2,755	36,581	1,587	15,441	134	925
65 +	1,129	70,138	478	27,034	51	2343

(3 + 5 + 2 = 10)

4. What is the purpose of El-Badry's procedure? Explain how the purpose is served. All symbols must be defined clearly.

(1 + 9 = 10)

5. The table below gives the parity progression ratios for a number of recent birth cohorts in a country. Assuming that no woman in any of these birth cohorts had a fifth child, calculate

(a) the proportion of women in each birth cohort who had exactly 0, 1, 2, 3 and 4 children,

(b) the total fertility rate for women in each birth cohort,

(c) Comment on your results.

Calendar years of birth	Parity Progression Ratios			
	0 -1	1 – 2	2 – 3	3 – 4
1931-33	0.861	0.804	0.555	0.518
1934-36	0.885	0.828	0.555	0.489
1937-39	0.886	0.847	0.543	0.455
1940-42	0.890	0.857	0.516	0.416
1943-45	0.892	0.854	0.458	0.378
1946-48	0.885	0.849	0.418	0.333

(6 + 3 + 1 = 10)

(a) Write how Newton's halving formula is used for tackling errors due to inaccurate age reports or faulty enumeration.

(b) Construct the table for estimation of TFR, GRR and NRR.

(5 + 5 = 10)

INDIAN STATISTICAL INSTITUTE

Backpaper Examination: 2012–2013

B.Stat. (Hons.) 2nd Year. 2nd Semester

Statistical Methods IV

Date: June 28, 2013 | Maximum Marks: 100 | Duration: 4 hours

- Answer all the questions.
- You should provide as much details as possible while answering a question. You must state clearly any result stated (and proved) in class you may need in order to answer a particular question.

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1. Let $V_1, \ldots, V_n, X_1, \ldots, X_n$ be independent and identically distributed $N(0,1)$ variables. Let $S := V_1^2 + \cdots + V_n^2$. Find the distribution of $T_1 := (V_1X_1 + \cdots + V_nX_n)/\sqrt{S}$. Hence, or otherwise, find the distribution of $T_2 := (V_1X_1 + \cdots + V_nX_n)/S^2$. [7+5 = 12]

2. Let $d \sim N_p(0, I_p), M \sim W_p(I_p, m)$, $m \geq p$ be independent. Show that

$$\frac{d^T M^{-1} d}{1 + d^T M^{-1} d} \sim \text{Beta}\left(\frac{p}{2}, \frac{m-p+1}{2}\right).$$ [12]

3. Suppose $\mathbf{X} = (X_1 X_2 \cdots X_n)^T$ is an $n \times p$ $(n \geq p+1)$ data matrix from $N_p(\mu, \Sigma)$. It is known that $A\mu = b$, where A is a known $q \times p$ matrix of rank$(A) = q$, b is a known $q \times 1$ vector such that $Ax = b$ is consistent and that Σ is an unknown positive definite (p.d.) matrix. Find the maximum likelihood estimate (MLE) of (μ, Σ). [12]

4. Suppose $\mathbf{X} = (X_1 X_2 \cdots X_n)^T$ is an $n \times p$ $(n \geq p+1)$ data matrix from $N_p(\mu, \Sigma)$, where μ is an unknown $p \times 1$ vector and Σ is an unknown $p \times p$ p.d. matrix. We wish to test $H_0 : A\mu = b$ against $H_1 : H_0$ *is false*, where A and b are as in question 3 above. Let $\bar{X}$ and S denote the sample mean vector and sample dispersion matrix, respectively. Also, let λ denote the likelihood ratio test (LRT) statistic for testing H_0 against H_1.

   (a) Show that $-2\log\lambda = n\log\left[1 + (A\bar{X} - b)^T(ASA^T)^{-1}(A\bar{X} - b)\right]$.

   (b) Show that $(n-1)(A\bar{X} - b)^T(ASA^T)^{-1}(A\bar{X} - b) \sim T^2(q, n-1)$ under H_0.

[8+6 = 14]

[P.T.O.]

5. Suppose $y \sim N_n(\mathbf{X}\beta, \sigma^2 \mathbf{I}_n)$, where $\mathbf{X}$ is an $n \times (p+1)$ matrix with fixed (non-random) entries and all entries in its first column are 1, $\beta \equiv (\beta_0, \beta_1, \beta_2, \ldots, \beta_p)^{\mathrm{T}} \in \mathbb{R}^{p+1}$, $\sigma > 0$. Assume that the rank of $\mathbf{X}$ equals $p+1$. Suppose we wish to test the hypothesis $\mathrm{H}_0 : \beta_1 = \cdots = \beta_p = 0$ against $\mathrm{H}_1 : \mathrm{H}_0$ *is false.* Let $\mathbf{J}_n$ denote the $n \times n$ matrix with all entries equal to 1, and let $\mathbf{H} := \mathbf{X}(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}$.

(a) Argue that a suitable test for testing H_0 against H_1 is the following: reject H_0 if F is large, where

$$F := \frac{y^{\mathrm{T}}(\mathbf{H} - n^{-1}\mathbf{J}_n)y/p}{y^{\mathrm{T}}(\mathbf{I}_n - \mathbf{H})y/(n-p-1)}.$$

(b) Find the non-null distribution of F. [10+10 = 20]

6. Suppose $X_1, X_2, \ldots$ are i.i.d. observations from a distribution whose semi-interquartile range is denoted by R. A natural estimate of R is its sample analogue R_n which is based on first n observations. Stating clearly the assumptions you need, find the asymptotic distribution of R_n. [12]

7. Suppose that the lifetimes $T_1, \ldots, T_n$ are independent and follow an exponential distribution with location parameter 0 and scale parameter σ, $\sigma > 0$. Let a Type 2 censoring scheme be employed where only the r smallest lifetimes $t_{(1)} \leq \cdots \leq t_{(r)}$ are observed, r being a specified integer between 1 and n.

(a) Show that the maximum likelihood estimate (MLE) of σ is given by $\hat{\sigma} = T/r$, where $T := \sum_{i=1}^{r} t_{(i)} + (n-r)t_{(r)}$.

(b) Show that $2T/\sigma \sim \chi^2_{2r}$.

(c) Discuss how you can obtain a $100(1-\alpha)\%$ confidence interval for σ which is based on T. [6+6+6 = 18]

[**Note.** You may assume the following result to hold, and use it, if required.

Let $E_{(1)} \leq E_{(2)} \leq \cdots \leq E_{(n)}$ denote the order statistics corresponding to a sequence of i.i.d. $Exp(0,1)$ random variables $E_1, \ldots, E_n$. If we define $E_{(0)} = 0$, the normalized exponential spacings $(n-i+1)(E_{(i+1)} - E_{(i)})$, $i = 1, \ldots, n$, are i.i.d. $Exp(0,1)$ random variables.]

****** *Best of Luck!* ******