ANALYSIS OF THE DIFFERENTIAL IN LIFE EXPECTANCY BY DIFFERENTIALS IN MORTALITY AT INDIVIDUAL AGES

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SUMMARY. The conventional double-decrement table for a body of lives is generalised to include the case, where at certain ages, one of the decrements may be negative and therefore of the nature of an increment. Expressions derived from the generalised table enable us to assess the contributions of the differences in mortality at and above a certain age to the difference in the complete expectation of life at that age, as between two life tables. These experessions have been verified in respect of the excess of the female expectancy of life at birth over the male, in the Eastern Zone (India) Life Tables. Males and Females, 1951-60, as also of the gain in life expectancy at birth in the female table, caused by a supposed partial elimination of the causes of death associated with pregnancy and childbirth to about the same proportionate extent at all ages in the female reproductive period and consequent improvement of motality at these ages.

1. Introduction

It may be required to assess the contributions of the differences in mortality at and above a given age to the difference in the expectation of life at that age as between two life tables. Or again, when a specified cause of death in a certain community is intended to be wholly or partially eliminated by appropriate measures, it may be necessary to estimate the increased expectation of life under the improved conditions and to assess the contributions to such increase, of the reductions of mortality at individual ages, as also the significance of the relative contributions in the context of effecting further reductions of mortality. In this paper, an attempt is made to provide a solution to the above problems, and as will be seen, the solution rests on the application of what may be said to be the generalised theory of the double-decrement table, where the net force of decrement may be the excess of a force of decrement over a hypothetical force of increment, as an extension of the usual case, where the total force of decrement is the sum of two forces of decrement.

2. The generalised double-decrement table

2.1. Let there be two mortality tables, which together with their associated functions are designated by α and β respectively, and where the force of mortality of table (α) is greater than that of table (β) at all ages. That is,

$$\mu_x^\alpha > \mu_x^\beta, \text{ for all values of } x,$$

or

$$\mu_x^\alpha = \mu_x^\beta + \mu_x^\gamma,$$

where μ_x^γ is a third and complementary force of decrement γ. Considering the forces of decrements β and γ to be in operation together as in a double-decrement table, q_x^β and q_x^γ, the respective dependent probabilities can, by the employment of standard methods,* be obtained in terms of $(\alpha q)_x$ and $(\beta q)_x$, the annual rates of mortality respectively in life tables (α) and (β). The life table (α) which is a single-decrement table

*P.F. Hooker and L. H. Longley Cook (1957) : *Life and Other Contingencies*, **II**, Cambridge University Press, 30–32.

showing decrements $(\alpha d)_x$ (deaths at successive ages x) against corresponding values of l^{α}_{x}, can therefore be split into a double-decrement table, showing at each age x, the decrements ${}_{\alpha}d^{\beta}_{x}$ and ${}_{\alpha}d^{\gamma}_{x}$, where

$$
{}_{\alpha}d^{\beta}_{x} = l^{\alpha}_{x}.\, q^{\beta}_{x},\quad {}_{\alpha}d^{\gamma}_{x} = l^{\alpha}_{x}.\, q^{\gamma}_{x},
$$

$$
{}_{\alpha}d^{\beta}_{x}+{}_{\alpha}d^{\gamma}_{x} = l^{\alpha}_{x}.\,(q^{\beta}_{x}+q^{\gamma}_{x}) = l^{\alpha}_{x}.\,(\alpha q)_x = (\alpha d)_x
$$

and

$$
l^{\alpha}_{x+1} = l^{\alpha}_{x}-(\alpha d)_x, \text{ as in life table } (\alpha)
$$

2.2. An alternative approach to the case discussed above is to write

$$
\mu^{\beta}_{x} = \mu^{\alpha}_{x}-\mu^{\gamma}_{x},
$$

which would now represent the case of a body of lives, subject to a force of decrement α, concurrently with hypothetical force of increment γ, and thus to a net force of decrement β, being the excess of the former over the latter. Defining ${}_{t}(\gamma I)_x$ as the proportion in which the number of lives increases from age x to age $x+t$ under the force of increment μ^{γ}_{x}, (γi_x) as the annual rate or independent probability of increment γ at age x and i^{γ}_{x} as the dependent probability of increment γ at age x, we have, by suitable modification of standard methods, the following results :

$$
1-(\beta q)_x = [1-(\alpha q)_x]\cdot[1+(\gamma i)_x] \qquad \ldots \quad (1)
$$

$$
(\gamma i)_x = [(\alpha q)_x-(\beta q)_x]/[1-(\alpha q)_x] \qquad \ldots \quad (2)
$$

$$
(\beta q)_x = q^{\alpha}_{x}-i^{\gamma}_{x} \qquad \ldots \quad (3)
$$

$$
q^{\alpha}_{x} = (\alpha q)_x[1+\tfrac{1}{2}(\gamma i)_x] \qquad \ldots \quad (4)
$$

and

$$
i^{\gamma}_{x} = (\gamma i)_x\,.\,[1-\tfrac{1}{2}(\alpha q)_x]. \qquad \ldots \quad (5)
$$

From the above, we can express q^{α}_{x} and i^{γ}_{x} in terms of $(\alpha q)_x$ and $(\beta q)_x$, and are therefore in a position to write life table (β) in the form of a decrement-increment table, showing at each age x, the decrement ${}_{\beta}d^{\alpha}_{x}$ and increment ${}_{\beta}I^{\gamma}_{x}$, where

$$
{}_{\beta}d^{\alpha}_{x} = l^{\beta}_{x}.\, q^{\alpha}_{x},\quad {}_{\beta}I^{\gamma}_{x} = l^{\gamma}_{x}.\, i^{\gamma}_{x},
$$

$$
{}_{\beta}d^{\alpha}_{x}-{}_{\beta}I^{\gamma}_{x} = l^{\beta}_{x}(q^{\alpha}_{x}-i^{\gamma}_{x}) = l^{\beta}_{x}.\,(\beta q)_x = (\beta d)_x
$$

and

$$
l^{\beta}_{x+1} = l^{\beta}_{x}-(\beta d)_x, \text{ as in the life table } (\beta).
$$

2.3. If the two life tables be such that at certain ages x,

$$
\mu^{\alpha}_{x} > \mu^{\beta}_{x}
$$

or

$$
\mu^{\alpha}_{x} = \mu^{\beta}_{x}+\mu^{\gamma}_{x}.
$$

the life table (α) can be transformed into a double-decrement table as in Section 2.1, the numbers living at ages x, l^{α}_{x} remaining the same as in the life table. If at certain other ages y,

$$
\mu^{\alpha}_{y} < \mu^{\beta}_{y}
$$

or

$$
\mu^{\alpha}_{y} = \mu^{\beta}_{y}-\mu^{\gamma}_{y},
$$

the life table (α) can be presented in the form of a decrement-increment table as in Section 2.2, the numbers living at ages y, l^{α}_{y} again remaining the same as in the life table. The different sections of the new table where opposite conditions as above hold, are mutually exclusive. At some ages x, it is the conventional double-decrement table, while at some other ages y, it is a decrement-increment table, or on the whole it may be described as a generalised double-decrement table, where one of the decrements may at certain ages be negative and therefore is of the nature of an increment.

According to the alternative approach mentioned in Section 2.2, we may also write

$$\mu_x^\beta = \mu_x^\alpha - \mu_x^\gamma$$

and

$$\mu_y^\beta = \mu_y^\alpha + \mu_y^\gamma,$$

so that the life table (β) can also be expressed in the form of a generalised double-decrement table, where at ages x, it is a decrement-increment table, and at ages y, it is the conventional double-decrement table, i.e., opposite in character in each of these sectors to the generalised table as under the previous approach. It is to be noted that whereas in the previous approach, it is the life table (α) that is transformed into a generalised double-decrement table; in the alternative approach, it is the life table (β) that is so transformed.

3. Analysis of the difference in the complete expectation of life at age x by two life tables

3.1. Considering $\mu_x^\alpha = \mu_x^\beta + \mu_x^\gamma$ for all values of x, the life table (α) being transformed into a double-decrement table involving decrements β and γ (Section 2.1), we have

$$\mathring{e}_x^{(\beta)} - \mathring{e}_x^{(\alpha)} = \int_0^\alpha [{}_t(\beta p)_x - {}_t(\alpha p)_x]dt, \quad {}_t(\alpha p)_x$$

where and ${}_t(\beta p)_x$ are the independent probabilities of survival from age x to age $x+t$ against decrements α and β respectively.

$$= \int_0^\alpha [{}_t(\beta p)_x - {}_t(\beta p)_x \cdot {}_t(\gamma p)_x]dt, \quad \text{since} \quad {}_t(\alpha p)_x = {}_t(\beta p)_x \cdot {}_t(\gamma p)_x$$

$$= \int_0^\alpha {}_t(\beta p)_x[1 - {}_t(\gamma p)_x]\, d$$

$$= \int_0^\alpha {}_t(\beta p)_x \cdot \int_0^t {}_r(\gamma p)_x \cdot \mu_{x+r}^\gamma dr\, dt$$

$$= \int_0^\alpha {}_t(\gamma p)_x \cdot \mu_{x+t}^\gamma \int_t^\alpha {}_r(\beta p)_x dr\, dt$$

$$\int_0^\alpha {}_t(\gamma p)_x \cdot \mu_{x+t}^\gamma \cdot {}_t(\beta p)_x \cdot \int_0^\alpha {}_r(\beta p)_{x+t}\, dr\, dt$$

$$= \int_0^\alpha {}_t(\alpha p)_x \cdot \mu_{x+t}^\gamma \cdot \mathring{e}_{x+t}^{(\beta)}\, dt$$

$$= \sum_{n=0}^\alpha \int_r^{n+1} {}_t(\alpha p)_x \cdot \mu_{x+t}^\gamma \cdot \mathring{e}_{x+t}^{(\beta)}\, dt$$

$$= \sum_{n=0}^\alpha {}_n(\alpha p)_x \int_0^1 {}_t(\alpha p)_{x+n} \cdot \mu_{x+n+t}^\gamma \cdot \mathring{e}_{x+n+t}^{(\beta)}\, dt.$$

$$\doteqdot \sum_{n=0}^\alpha {}_n(\alpha p)_x \int_0^1 {}_t(\alpha p)_{x+n} \cdot \mu_{x+n+t}^\gamma\, dt \cdot \int_0^1 \mathring{e}_{x+n+t}^{(\beta)}\, dt$$

$$\doteqdot \sum_{n=0}^\alpha {}_n(\alpha p)_x \cdot q_{x+n}^\gamma \cdot \mathring{e}_{x+n+\frac{1}{2}}^{(\beta)},$$ where q_{x+n}^γ is the dependent probability of decrement γ in the double-decrement table at age $x+n$

$$\doteqdot 1/l_x^\alpha \sum_{n=0}^\alpha l_{x+n}^\alpha \cdot q_{x+n}^\gamma \cdot \mathring{e}_{x+n+\frac{1}{2}}^{(\beta)}$$

$$\doteqdot 1/l_x^\alpha \cdot \sum_{n=0}^\alpha {}_\alpha d_{x+n}^\gamma \cdot \mathring{e}_{x+n+\frac{1}{2}}^{(\beta)} \qquad \ldots (6)$$

where ${}_{\alpha}d^{\gamma}_{x+n}$ = decrement γ in the double-decrement table at age $x+n$.

i.e., the elements ${}_{\alpha}d^{\gamma}_{x+n} \cdot \mathring{e}^{(\beta)}_{x+n+\frac{1}{2}}$ at individual ages, summed from the highest age in the table upwards upto age x and divided by l^{α}_{x} will, subject to the approximations as above, reproduce $\mathring{e}^{(\beta)}_{x} - \mathring{e}^{(\alpha)}_{x}$, to which the contribution by the improved mortality of life table (β) over life table (α) between ages $x+n$ and $x+n+1$ represented by ${}_{\alpha}d^{\gamma}_{a+n}\left(= \int_0^1 l^{\alpha}_{x+n+t} \cdot \mu^{\gamma}_{x+n+t}\, dt\right)$ may be taken as $1/l^{\alpha}_{x} \cdot {}_{\alpha}d^{\gamma}_{x+n} \cdot \mathring{e}^{(\beta)}_{x+n+\frac{1}{2}}$.

3.2. In the alternative approach $\mu^{\beta}_{x} = \mu^{\alpha}_{x} - \mu^{\gamma}_{x}$ for all values of x, the life table (β) is transformed into a decrement-increment table involving decrement α and increment γ (Section. 2.2) and we have

$$\mathring{e}^{(\beta)}_{x} - \mathring{e}^{(\alpha)}_{x} = \int_0^{\alpha} [{}_{t}(\beta p)_{x} - {}_{t}(\alpha p)_{x}]dt$$

$$= \int_0^{\alpha} {}_{t}(\alpha p)_{x} [{}_{t}(\gamma I)_{x} - 1], dt$$

since ${}_{t}(\beta p)_{x} = {}_{t}(\alpha p)_{x} \cdot {}_{t}(\gamma I)_{x}$

$$= \int_0^{\alpha} {}_{t}(\alpha p)_{x} \int_0^{t} {}_{r}(\gamma I)_{x} \cdot \mu^{\gamma}_{x+r} dr\, dt$$

$$= \int_0^{\alpha} {}_{t}(\gamma I)_{x} \cdot \mu^{\gamma}_{x+t} \cdot {}_{t}(\alpha p)_{x} \cdot \int_0^{\alpha} {}_{r}(\alpha p)_{x}\ {}_{t}\, dr\, dt$$

$$= \int_0^{\alpha} {}_{t}(\beta p)_{x} \cdot \mu^{\gamma}_{x+t} \cdot \mathring{e}^{(\alpha)}_{x+t} dt, \text{ which, proceeding on the same lines as before,}$$

$$\doteq 1/l^{\beta}_{x} \sum_{n=0}^{\alpha} {}_{\beta}I^{\gamma}_{x+n} \cdot \mathring{e}^{(\alpha)}_{x+n+\frac{1}{2}} \qquad \ldots \quad (7)$$

where ${}_{\beta}I^{\gamma}_{x+n}$ is the increment γ in the decrement-increment table at age $x+n$. Similarly as before, the superiority of life table (β) as compared to life table (α) in respect of mortality may be taken to be represented by ${}_{\beta}I^{\gamma}_{x+n}\left(= \int_0^1 l^{\beta}_{x+n+t} \cdot \mu^{\gamma}_{x+n+t}\, dt\right)$ between ages $x+n$ and $x+n+1$ and the contribution of the same to $\mathring{e}^{(\beta)}_{x} - \mathring{e}^{(\alpha)}_{x}$ as $1/l^{\beta}_{x} \cdot {}_{\beta}I^{\gamma}_{x+n} \cdot \mathring{e}^{(\alpha)}_{x+n+\frac{1}{2}}$. It will be noted however that

$$\int_0^{\alpha} {}_{t}(\alpha p)_{x} \cdot \mu^{\gamma}_{x+t} \cdot \mathring{e}^{(\beta)}_{x+t} dt = \int_0^{\alpha} {}_{t}(\beta p)_{x} \cdot \mu^{\gamma}_{x+t} \cdot \mathring{e}^{(\alpha)}_{x+t} dt$$

or

$$1/l^{\alpha}_{x} \cdot \sum_{n=0}^{\alpha} {}_{\alpha}d^{\gamma}_{x+n} \cdot \mathring{e}^{(\beta)}_{x+n+\frac{1}{2}} = 1/l^{\beta}_{x} \cdot \sum_{n=0}^{\alpha} {}_{\beta}I^{\gamma}_{x+n} \cdot \mathring{e}^{(\alpha)}_{x+n+\frac{1}{2}},$$

but the individual elements $1/l^{\alpha}_{x} \cdot {}_{\alpha}d^{\gamma}_{x+n} \cdot \mathring{e}^{(\beta)}_{x+n+\frac{1}{2}}$ and $1/l^{\beta}_{x} \cdot {}_{\beta}I^{\gamma}_{x+n} \cdot \mathring{e}^{(\alpha)}_{x+n+\frac{1}{2}}$ are not the same. A simple and reasonable method would be to take the mean, $\frac{1}{2}[{}_{\alpha}d^{\gamma}_{x+n} \cdot \mathring{e}^{(\beta)}_{x+n+\frac{1}{2}}/$

$l_x^\alpha + {}_\beta l^\gamma_{x+n} \cdot \mathring{e}^{(\alpha)}_{x+n+\frac{1}{2}}/l^\beta_x]$ as the contribution of the improvement in the rate of mortality at age $x+n$, viz., $(\alpha q)_{x+n} - (\beta q)_{x+n}$ to the gain in the complete expectation of life at age x, $\mathring{e}^{(\beta)}_x - \mathring{e}^{(\alpha)}_x$.

3.3. Considering now that at certain ages x, $\mu^\alpha_x = \mu^\beta_x + \mu^\gamma_x$ and at certain other ages y, $\mu^\alpha_y = \mu^\beta_y - \mu^\gamma_y$, the life table (α) is transformed into a generalised double-decrement table over the whole range of ages (Section 2.3). To prove that a relation similar to (6) but consisting of both decrements at ages x and increments at ages y corresponding to μ^γ_x and μ^γ_y respectively, holds in this case as well, let $\mu^\alpha > \mu^\beta$ at all ages between x and $x+n$ and $\mu^\alpha < \mu^\beta$ at ages above $x+n$, for the purpose of demonstration. We have then,

$$\mathring{e}^{(\beta)}_x - \mathring{e}^{(\alpha)}_x = \int_0^n [{}_t(\beta p)_x - {}_t(\alpha p)_x]dt + \int_n^\alpha [{}_t(\beta p)_x - {}_t(\alpha p)_x]dt.$$

The first integral
$$= \int_0^n {}_t(\alpha p)_x \cdot \mu^\gamma_{x+t} \cdot \mathring{e}^{(\beta)}_{x+t}dt - \int_0^n {}_t(\alpha p)_x \cdot \mu^\gamma_{x+t} \cdot {}_{n-t}(\beta p)_{x+t} \cdot \mathring{e}^{(\beta)}_{x+n}dt.$$

The second term in above
$$= -\int_0^n {}_t(\beta p)_x \cdot {}_t(\gamma p)_x \cdot \mu^\gamma_{x+t} \cdot {}_{n-t}(\beta p)_{x+t} \cdot \mathring{e}^{(\beta)}_{x+n}dt$$
$$= -{}_n(\beta p)_x \cdot \mathring{e}^{(\beta)}_{x+n} \int_0^n {}_t(\gamma p)_x \cdot \mu^\gamma_{x+t}dt$$
$$= -{}_n(\beta p)_x \cdot [1 - {}_n(\gamma p)_x]\mathring{e}^{(\beta)}_{x+n}$$
$$= -[{}_n(\beta p)_x - {}_n(\alpha p)_x]\mathring{e}^{(\beta)}_{x+n}.$$

Therefore the first integral
$$= \int_0^n {}_t(\alpha p)_x \cdot \mu^\gamma_{x+t} \cdot \mathring{e}^{(\beta)}_{x+t}dt - [{}_n(\beta p)_x - {}_n(\alpha p)_x] \cdot \mathring{e}^{(\beta)}_{x+n}.$$

By similar methods, the second integral
$$= [{}_n(\beta p)_x - {}_n(\alpha p)_x]\mathring{e}^{(\beta)}_{x+n} - {}_n(\alpha p)_x \int_0^\alpha {}_t(\alpha p)_{x+n} \cdot \mu^\gamma_{x+n+t} \cdot \mathring{e}^{(\beta)}_{x+n+t}dt.$$

Hence,
$$\mathring{e}^{(\beta)}_x - \mathring{e}^{(\alpha)}_x$$
$$= \int_0^n {}_t(\alpha p)_x\mu^\gamma_{x+t}\mathring{e}^{(\beta)}_{x+t}dt - {}_n(\alpha p)_x \int_0^\alpha {}_t(\alpha p)_{x+n} \cdot \mu^\gamma_{x+n+t} \cdot \mathring{e}^{(\beta)}_{x+n+t}\, dt$$
$$= \int_0^n {}_t(\alpha p)_x\mu^\gamma_{x+t}\mathring{e}^{(\beta)}_{x+t}dt - \int_n^\alpha {}_t(\alpha p)_x\mu^\gamma_{x+t}\mathring{e}^{(\beta)}_{x+t}dt$$
$$\doteqdot 1/l^\alpha_x\Big[\sum_{k=0}^{n-1} {}_\alpha d^\gamma_{x+k} \cdot \mathring{e}^{(\beta)}_{x+k+\frac{1}{2}} - \sum_{k=n}^{\alpha} {}_\alpha I^\gamma_{x+k} \cdot \mathring{e}^{(\beta)}_{x+k+\frac{1}{2}}\Big]. \quad \ldots (8).$$

The above is of the same form as (6) and involves the same principle, viz. that at ages $x+k(k \geqslant n)$, the increments ${}_{\alpha}l^{\gamma}_{x+k}$ and the elements ${}_{\alpha}l^{\gamma}_{x+k} \cdot \mathring{e}^{(\beta)}_{x+k+\frac{1}{2}}$ being negative, and at ages $x+k(k < n)$, the decrements ${}_{\alpha}d_{x+k}$ and the elements ${}_{\alpha}d^{\gamma}_{x+k} \cdot \mathring{e}^{(\beta)}_{x+k+\frac{1}{2}}$ being positive, the cumulative sum of the elements with regard to sign from the limiting age of the table to age x, divided by 1^{α}_{x} is approximately equal to $\mathring{e}^{(\beta)}_{x}-\mathring{e}^{(\alpha)}_{x}$. This must also be true when $(\alpha q)_x-(\beta q)$ changes sign irregularly in the whole table.

Similarly, adopting the alternative approach (Section 2.3), $\mathring{e}^{(\beta)}_{x}-\mathring{e}^{(\alpha)}_{x}$ is also

$$\doteqdot 1/1^{\beta}_{x}\left[\sum_{k=0}^{n-1} {}_{\beta}l^{\gamma}_{x+k} \cdot \mathring{e}^{(\alpha)}_{x+k+\frac{1}{2}} - \sum_{k=n}^{\alpha} {}_{\beta}d^{\gamma}_{x+k} \cdot \mathring{e}^{(\alpha)}_{x+k+\frac{1}{2}}\right]. \qquad \dots \quad (9)$$

the increments and decrements taking the place respectively of decrements and increments of (8).

Combining (8) and (9). $\mathring{e}^{(\beta)}_{x}-\mathring{e}^{(\alpha)}_{x}$

$$\doteqdot \sum_{k=0}^{n-1} \frac{1}{2}\,[{}_{\alpha}d^{\gamma}_{x+k} \cdot \mathring{e}^{(\beta)}_{x+k+\frac{1}{2}}/1^{\alpha}_{x}+{}_{\beta}l^{\gamma}_{x+k} \cdot \mathring{e}^{(\alpha)}_{x+k+\frac{1}{2}}/1^{\beta}_{x}]$$

$$-\sum_{k=n}^{\alpha} \frac{1}{2}\,[{}_{\alpha}l^{\gamma}_{x+k} \cdot \mathring{e}^{(\beta)}_{x+k+\frac{1}{2}}/1^{\alpha}_{x}+{}_{\beta}d^{\gamma}_{x+k} \cdot \mathring{e}^{(\alpha)}_{x+k+\frac{1}{2}}/1^{\beta}_{x}].$$

and the contribution to $\mathring{e}^{(\beta)}_{x}-\mathring{e}^{(\alpha)}_{x}$ by the difference in the mortality rates at age $x+k(k < n) = \frac{1}{2}[{}_{\alpha}d^{\gamma}_{x+k} \cdot \mathring{e}^{(\beta)}_{x+k+\frac{1}{2}}/1^{\alpha}_{x}+{}_{\beta}l^{\gamma}_{x+k} \cdot \mathring{e}^{(\alpha)}_{x+k+\frac{1}{2}}/1^{\beta}_{x}]$ or more generally at any age $x+k$, where $(\alpha q)_x-(\beta q)_x$ is positive, and that at age $x+k(k \geqslant n) = -\frac{1}{2}[\,{}_{\alpha}l^{\gamma}_{x+k} \cdot \mathring{e}^{(\beta)}_{x+k+\frac{1}{2}}/ 1^{\alpha}_{x}+{}_{\beta}d^{\gamma}_{x+k} \cdot \mathring{e}^{(\alpha)}_{x+k+\frac{1}{2}}/1^{\beta}_{x}]$, and also whenever, $(\alpha q)_x-(\beta q)_x$ is negative.

4. Improvement in life expectancy by complete or partial removal of a specified cause of death, and its analysis by the reductions of mortality at various ages

This involves the same principle as applied before. But here we have to determine the expectations of life at various ages under conditions of reduced mortality caused by the complete or partial elimination of a specified cause of death.

4.1. It is supposed that we have a life table representing the mortality rates of the community before steps are taken to eliminate completely or partially the specified cause of death. If μ_x, μ^{γ}_{x} and μ^{β}_{x} are respectively the prevailing forces of mortality at exact age x, under all causes, the specified cause and the remainder of all causes, and the specified cause is independent of all other causes, we have

$$\mu_x = \mu^{\beta}_{x}+\mu^{\gamma}_{x}.$$

and assuming x to be integral and that a fraction f_x of μ^{γ}_{x+t} for all values of t between 0 and 1, were to be eliminated in the future. μ^{β}_{x+t} remaining unaffected,

$$\mu_{x+t} = [\mu^{\beta}_{x+t}+(1-f_x)\mu^{\gamma}_{x+t}]+f_x \cdot \mu^{\gamma}_{x+t}.$$

If further we assume that $\mu^{?}_{x+t} = k_x \cdot \mu_{x+t}$ for all values of t between 0 and 1, we have

$$\mu^{\theta}_{x+t} = (1-k_x) \cdot \mu_{x+t}$$

and
$$\mu_{x+t} = (1-k_x \cdot f_x) \cdot \mu_{x+t} + k_x \cdot f_x \cdot \mu_{x+t} \qquad \dots \quad (10)$$

so that at all ages between x and $x+1$, of the total force of mortality prevailing at present (the basis of the life table), a fraction $k_x \cdot f_x$ being supposed eliminated, the remainder $1-k_x \cdot f_x$ would remain in operation as the basis of the reduced future mortality. In the case of the complete elimination of the specified cause of death, $f_x = 1$, and the above fractions would of course be k_x and $1-k_x$ respectively.

4.2. While f_x would necessarily depend upon the intensity and appropriateness of the measures adopted, the value of k_x under present conditions may be obtained from an analysis by causes of current statistics of deaths. For instance, if θ_x and $\theta^{?}_x$ be the calendar year deaths due to all causes and the specified cause respectively, at age x last birthday, we have

$$\theta^{?}_x = \int_0^1 \int_0^1 P^r_{x+t} \cdot \mu^{?}_{x+t} \cdot dt\, dr,$$

where $P^r_{x+t} \cdot dr$ = number of lives attaining exact age $x+t$ between times r and $r+dr$ from the beginning of the calender year

$$= k_x \int_0^1 \int_0^1 P^r_{x+t} \cdot \mu_{x+t} \cdot dt\, dr$$

$$= k_x \cdot \theta_x$$

$$\therefore \qquad k_x = \theta^{?}_x / \theta_x.$$

Further, the calendar year deaths due to the other causes, $\theta^{\theta}_x = (1-k_x) \cdot \theta_x$, and the future elimination of a part of the mortality due to the specified cause as described above would mean that if it is applied to the present conditions otherwise unaffected (including the values of P^r_{x+t}), the calendar year deaths eliminated would be $f_x \cdot \theta^{?}_x$ or $f_x \cdot k_x \cdot \theta_x$ and the balance representing the total of θ^{θ}_x and the remainder of $\theta^{?}_x$ would be $(1-f_x \cdot k_x) \cdot \theta_x$ to which θ_x should have been reduced (compare equation (10)).

If the death statistics are provided as usual in age-groups, say in quinquennial age-groups, x to $x+4$ last birthday, we may assume that values of k_x (and also of f_x) are the same between integral ages x and $x+5$.

4.3. Obtaining the value of k_x (Section 4.2) and assuming a value for f_x, equation (10) can be written as

$$\mu_{x+t} = r_x \cdot \mu_{x+t} + s_x \cdot \mu_{x+t}$$

for all values of t between 0 and 1, where $s_x = k_x \cdot f_x$ and $r_x + s_x = 1$

$$= \mu'_{x+t} + \mu''_{x+t}, \qquad \dots \quad (11)$$

where $\mu'_{x+t} = r_x \cdot \mu_{x+t}$, $\mu''_{x+t} = s_x \cdot \mu_{x+t}$ and μ'_{x+t} is the force of mortality at age $x+t$, to which the prevailing force of mortality μ_{x+t} would be reduced after the elimination of a part of it $= \mu''_{x+t}$.

Applying the above procedure to all integral ages x and above, the life table referred to above may be presented as a double-decrement table (Section 2.1), the life table deaths d_y being subdivided into deaths d'_y and d''_y,

where $$d'_y = l_y \cdot \int_0^1 {}_tp_y \cdot \mu'_{y+t}\, dt = r_y \cdot l_y \int_0^1 {}_tp_y \cdot \mu_{y+t}\, dt$$

$$= r_y \cdot l_y \cdot q_y = r_y \cdot d_y$$

and similarly, $$d''_y = s_y \cdot d_y, \text{ so that } d'_y + d''_y = (r_y + s_y) \cdot d_y = d_y.$$

If $\mathring{e}_x$ and $\mathring{e}'_x$ be the complete expectations of life at age x according to the prevailing (life table) mortality and reduced mortality respectively, we have by equation (6) (Section 3.1)

$$\mathring{e}'_x - \mathring{e}_x \doteq 1/l_x \cdot \sum_{n=0}^{\infty} d''_{x+n} \cdot \mathring{e}'_{x+n+\frac{1}{2}} \qquad \ldots \quad (12)$$

or $$l_x(\mathring{e}'_x - \mathring{e}_x) \doteq \sum_{n=0}^{\infty} d''_{x+n} \cdot \mathring{e}'_{x+n+\frac{1}{2}},$$

which, remembering that d_{x+n} may be assumed to occur at the middle of the year of age i.e., at age $x+n+\frac{1}{2}$, at which the complete expectation of life under the reduced mortality $= \mathring{e}'_{x+n+\frac{1}{2}}$, may, subject to the approximation in equation (12), be interpreted as follows :

The increase in the number of years lived after age x by a group of lives aged that would be caused by a partial elimination of a specified cause of death, is equal to the total of the numbers of years that the deaths occurring to them at successive ages under the conditions of the original mortality and corresponding to that part of it intended to be eliminated, but supposed to have a fresh lease of life on such elimination, would live under the conditions of the improved mortality.

Thus, from equation (12), the contribution to $\mathring{e}'_x - \mathring{e}_x$ by the reduction in mortality between ages $x+n$ and $x+n+1$ may be taken as $1/l_x \cdot d''_{x+n} \cdot \mathring{e}'_{x+n+\frac{1}{2}}$.

4.4. To evaluate these contributions, we have to obtain the values of $\mathring{e}'_y$ for all integral values of $y \geqslant x$. We have

$$\log_e p'_y = -\int_0^1 \mu'_{y+t} \cdot dt$$

$$= -r_y \cdot \int_0^1 \mu_{y+t}\, dt$$

$$= r_y \cdot \log_e p_y,$$

from which, the values of r_y and p_y being known at all integral values of y, the corresponding values of p'_y may be obtained and used to build up the l'_y column of the life table based on the reduced mortality (taking a suitable value for l'_x) and hence L'_y and T'_y columns, whence $e'_y = T'_y/l'_y$.

4.5. Adopting the alternative approach (Section 2.2), equation (11) may be written as

$$\mu'_{x+t} = \mu_{x+t} - \mu^{\bullet}_{x+t},$$

and the life table based on the reduced mortality constructed as above and showing values of l'_y and d'_y at successive ages may be presented as a decrement-increment table, where

$$\text{the decrement } d_y = l'_y \int_0^1 {}_tp'_y \cdot \mu_{y+t} \cdot dt$$

$$= l'_y \cdot 1/r_y \int_0^1 {}_tp'_y \cdot \mu'_{y+t} \cdot dt$$

$$= 1/r_y \cdot l'_y \cdot q'_y = 1/r_y \cdot d'_y$$

$$\text{and the increment } I^{\bullet}_y = l'_y \int_0^1 {}_tp'_y \cdot \mu^{\bullet}_{y+t} \cdot dt$$

$$= l'_y \cdot s_y/r_y \int_0^1 {}_tp'_y \cdot \mu'_{y+t} \cdot dt$$

$$= s_y/r_y \cdot l'_y \cdot q'_y = s_y/r_y \cdot d'_y,$$

so that

$$d_y - I^{\bullet}_y = (1/r_y - s_y/r_y) \cdot d'_y = (1-s_y)/r_y \cdot d'_y = d'_y,$$

and $I^{\bullet}_y = d_y - d'_y$, may be looked upon as the overstatement of deaths on the basis of the original mortality as against the reduced mortality, i.e., to the extent of the part of the original mortality eliminated. Further, by equation (7) (Section 3.2),

$$\mathring{e}'_x - \mathring{e}_x \doteqdot 1/l'_x \sum_{n=0}^{\infty} I^{\bullet}_{x+n}\mathring{e}_{x+n+\frac{1}{2}} \qquad \dots \quad (13)$$

or

$$l'_x(\mathring{e}'_x - \mathring{e}_x) \doteqdot \sum_{n=0}^{\infty} I^{\bullet}_{x+n} \cdot \mathring{e}_{x+n+\frac{1}{2}},$$

a verbal interpretation of which, subject to the approximation in (13), may be as follows :

The increase in the number of years lived after age x by a group of lives aged x, currently experiencing mortality reduced by a partial elimination of a specified cause of death, is the total of the numbers of years that would be lived under conditions of the original mortality, by the excess of lives reckoned as or the overstatement of deaths at successive ages on the basis of the original as against the reduced mortality.

Thus from equation (13), the contribution to $\mathring{e}'_x - \mathring{e}_x$ by the reduction in mortality between ages $x+n$ and $x+n+1$

$$= 1/l'_x \,.\, I'_{x+n} \,.\, \mathring{e}_{x+n+\frac{1}{2}}$$

and finally, considering the other value of the same (Section 4.3) and taking the mean, the contribution

$$= \frac{1}{2}[d'_{x+n} \,.\, \mathring{e}'_{x+n+\frac{1}{2}}/l_x + I'_{x+n} \,.\, e_{x+n+\frac{1}{2}}/l'_x]$$

or

$$= \frac{1}{2}[d'_{x+n} \,.\, \frac{1}{2}(\mathring{e}'_{x+n} + \mathring{e}'_{x+n+1})/l_x + I'_{x+n} \,.\, \frac{1}{2}(\mathring{e}_{x+n} + \mathring{e}_{x+n+1})/l'_x].$$

5. Applications

5.1. For verification of the theory leading to formulae (8) and (9), we have selected the Eastern Zone (India) Life Tables, Males and Females, 1951-60, which show opposite differentials between the respective mortality rates at various groups of ages. In contrast with the developed countries, where female mortality is consistently lighter than male at all ages and life expectancy of females is higher than that of males at every age, in India, the complete expectation of life at birth for females has been consistently smaller than for males, although the position is reversed from after middle life, on account of female mortality being lighter at the corresponding ages. This is also true for the different population zones into which it has been customary to divide the whole of India since the 1951 census for the purpose of drawing up the zonal life tables, with the exception of the Eastern Zone in 1961, where the complete expectation of life at birth for females is fractionally higher than that for males, while at other ages, the differential shows several changes of sign, being in favour of females from a comparatively early age. The reproduction of these differentials at the initial points of the distinctive groups of ages by the application of formulae (8) and (9) are shown in Table 1, as also the analysis of the differential at age 0 by the differentials in mortality rates in those groups of ages.

5.2. From the fact that the excess of the life expectancy at birth of females over that of males as by the above tables is prevented from being anything more than the small value (see note under Table 1), mainly because of the heavier mortality of females at ages comprising the major portion of the reproductive span of life, it would be of interest to see to what extent a partial removal of a cause of death specifically related to pregnancy and childbirth would alter the picture of comparative mortality and life expectancy at various ages.

We have from *Vital Statistics of India* (Registrar General of India, 1960) the proportions of female deaths (1960) in West Bengal in various quinquennial age-groups, due to a specified cause of death, viz., deliveries and complications of pregnancy, childbirth and the puerperium. Assuming that about half of the force of mortality due to the specified cause would be removed by appropriate measures, we have the values of k_x, s_x and r_x in the following table.

TABLE 1. VERIFICATION OF FORMULAE (8) AND (9) BY EASTERN ZONE (INDIA) LIFE TABLE, MALES, 1951-60 (α) AND EASTERN ZONE (INDIA) LIFE TABLE, FEMALES, 1951-60 (β) AND CONTRIBUTIONS BY THE DIFFERENCES IN MORTALITY IN AGES/AGE GROUPS TO $\mathring{e}_0^{(\beta)}-\mathring{e}_0^{(\alpha)}$

age x	sign of $(\alpha q)_x-(\beta q)_x$	μ_x^α = total/net force of decrement: $\Sigma_\alpha d_x^\gamma\cdot\mathring{e}_{x+\frac{1}{2}}^{(\beta)}(+)$ $\Sigma_\alpha l_x^\gamma\cdot\mathring{e}_{x+\frac{1}{2}}^{(\beta)}(-)$	$\sum_x^\infty(3)$	$\mathring{e}_x^{(\beta)}-\mathring{e}_x^{(\alpha)}$: $(4)/l_x^\alpha$	$\mathring{e}_x^{(\beta)}-\mathring{e}_x^{(\alpha)}$: from life tables	μ_x^β = net/total force of decrement: $\Sigma_\beta l_x^\gamma\cdot\mathring{e}_{x+\frac{1}{2}}^{(\alpha)}(+)$ $\Sigma_{\beta'} l_x^\gamma\cdot\mathring{e}_{x+\frac{1}{2}}^{(\alpha)}(-)$	$\sum_x^\infty(7)$	$(8)/l_x^\beta$	age/ age-group	mean of of (3) and (7)	contribution to $\mathring{e}_0^{(\beta)}-\mathring{e}_0^{(\alpha)}$ = (11)/ 100,000 ($l_0^\alpha = l_0^\beta$ = 100,000)	percentage contribution
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(3)
0	+	+ 34,929	+27,287	+ .273	+ .272	+ 35,018	+27,417	+ .274	0	+34,974	+ .3497	+128
1 : 10	−	− 22,516	− 7,642	− .088	− .090	− 22,623	− 7,601	− .087	1—10	−22,570	− .2257	− 83
11			+14,784	+ .201	+ .201		+15,022	+ .202				
: 26	+	+ 25,762				+ 25,948			11—26	325,855	+ .2586	+ 95
27 : 45	−	−104,286	−10,888	− .163	− .163	− 99,086	−10,926	− .161	27—45	−101,686	−1.0169	−372
46 : :	+	+ 93,398	+93,398	+1.919	+1.917	+ 88,160	+88,160	+1.921	46—	+90,779	+ .9078	+332
									total	+27,352	+ .2735	100

Notes on Table 1

(1) The values at age x in cols. (5) and (9) are close to one another and also to $\mathring{e}_x^{(\beta)}-\mathring{e}_x^{(\alpha)}$ in col. (6), being the same up to the second place of decimals.

(2) The percentage contributions in col. (13) are to be taken with the assumption that the individual contributions in col. (12) are reduced in the same proportion so as to add up to +.272 of col. (6). The smallness of the latter value is primarily due to the much heavier mortality of females in the major portion of the female reproductive period, the negative contribution of which is so much as 372%. This may be in part due to more than usual mortality at childbirth, a point strengthened by the fact that from the end of that period to the limit of life, female mortality is consistently lighter, the positive contribution of which is so much as 332%.

TABLE 2. VALUES OF k_x, s_x AND r_x, DERIVED FROM WEST BENGAL FEMALE DEATHS (1960), ($f_x \doteqdot \frac{1}{2}$)

age-group (last birthday)	k_x	$s_x = f_x.k_x$	r_x
(1)	(2)	(3)	(4)
10–14	.044	.02	.98
15–19	.471	.25	.75
20–24	.341	.17	.83
25–29	.280	.14	.86
30–34	.288	.15	.85
35–39	.241	.12	.88
40–44	.070	.04	.96
45–49	.016	.01	.99

Assuming that the distribution of female deaths according to cause of death (West Bengal, 1960) held for East Zone (1951-60) (East Zone comprising of West Bengal and other adjacent states) and applying the above values of r_x to Eastern Zone Life Table, Females, 1951-60, the life table on the basis of the reduced mortality is built up (taking $l'_x = l_x$ for all integral values of x from 0 to 10), and the values of $\mathring{e}'_x$ for all values of x from 0 to 49 and the respective gains in life expectancy obtained ($\mathring{e}'_x = \mathring{e}_x$ for $x \geqslant 50$). The verification of the latter by formulae (8) and (9) at age 0 and every quinquennial age from age 10, and the analysis of the gain at age 0 by the reductions in mortality over the quinquennial age-goups are shown in Table 3.

5.2.1. The partial removal of the specified cause of death as described above, improving as it does, the female mortality of the Eastern Zone at every age from 10 to 49, makes it still lighter than the male mortality of the same zone, where it was already lighter, such as at ages 11-26 and 46 and above (col. (2), Table 1), and substantially reduces its excess over male mortality where it was heavier, such as at age 10 and ages 27-45. In fact, in the latter case, female mortality at ages 10, 27 and 28 becomes actually lighter, while at age 29 it becomes practically the same as male mortality. Considering however, the complete expectation of life, it becomes greater for females than for males at all ages from birth to end of life, as shown in Table 4 at selected ages, instead of the difference showing changes of sign over the ages, as it did by the original life tables. The superior life expectancy of females at all ages will of course be further strengthened, if the elimination of the specific cause of death is more complete, i.e., if the values of s_x (Table 2) are made greater, in which case, female mortality rates also could be lighter than male at all ages in the reproductive period. If at any age x, ${}_1\mathring{e}'_x$ is the enhanced complete expectation of life of females, it can be shown that adopting the approach of Section. 4.3, the contribution to ${}_1\mathring{e}'_x - \mathring{e}'_x$ of the further reduction of mortality and increase in life expectancy at age $y (y > x)$

$$= 1/l_x[\Delta d''_y \cdot \mathring{e}'_{y+\frac{1}{2}} + {}_1d''_y \cdot \Delta \mathring{e}'_{y+\frac{1}{2}}], \qquad \dots \quad (14)$$

TABLE 3. ANALYSIS OF THE GAIN IN COMPLETE EXPECTATION OF LIFE AT BIRTH BY A PARTIAL ELIMINIATION OF A SPECIFIED CAUSE OF DEATH, BASED ON THE EASTERN ZONE (INDIA) LIFE TABLE, FEMALES, 1951-60, WHERE μ_x, THE FORCE OF MORTALITY AT AGE $x = \mu_x' + \mu_x''$, μ_x' AND μ_x'' BEING RESPECTIVELY THE REDUCED AND ELIMINATED FORCES OF MORTALITY AT AGE x.

age x	μ_x = total force of decrement: $\Sigma d_x' \cdot \overset{\circ\prime}{e}_{x+\frac{1}{2}}$	μ_x = total force of decrement: $\sum_x^{\infty}(2)$	$(3)/l_x$	$\overset{\circ\prime}{e}_x - \overset{\circ}{e}_x$ from the life tables	$\Sigma l_x' \cdot \overset{\circ}{e}_{x+\frac{1}{2}}$	μ_x' = net force of decrement: $\sum_x^{\infty}(6)$	$(7)/l_x'$	age-group	mean of (2) and (6)	contribution to $\overset{\circ\prime}{e}_0 - \overset{\circ}{e}_0$ = (10)/100,000 ($l_0' = l_0$ = 100,000)	percentage contribution
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
0 : 9	...	91,867	.919	.921	...	91,822	.918	0– 9	...	...	...
10 : 14	1,296	91,867	1.231	1.235	1,262	91,822	1.231	10–14	1,279	.013	1.4
15 : 19	20,083	90,571	1.239	1.242	19,546	90,560	1.239	15–19	19,815	.198	21.6
20 : 24	12,382	70,488	.994	.994	12,152	71,014	.993	20–24	12,267	.123	13.4
25 : 29	11,790	58,106	.845	.845	11,664	58,862	.844	25–29	11,727	.117	12.7
30 : 34	20,039	46,316	.702	.702	20,091	47,198	.702	30–34	20,065	.201	21.9
35 : 39	19,384	26,277	.432	.431	19,890	27,107	.432	35–39	19,637	.196	21.4
40 : 44	5,687	6,893	.128	.126	5,948	7,217	.128	40–44	5,818	.058	6.3
45 : 49	1,206	1,206	.026	.025	1,269	1,269	.026	45–49	1,238	.012	1.3
50 : :	...				...			50–	...	...	...
								total	91,846	.918	100.0

TABLE 4. CHANGE IN SEX DIFFERENTIAL IN LIFE EXPECTANCY AT VARIOUS AGE BY REMOVAL OF ABOUT 50% OF A SPECIFIED CAUSE OF DEATH OF FEMALES, EASTERN ZONE, 1951-60

age x	excess of life expectancy of females over that of males (original) $\mathring{e}_x^{(f)}-\mathring{e}_x^{(m)}$	gain in life expectancy of females by the partial removal of the specified cause of death $\mathring{e}_x^{'(f)}-\mathring{e}_x^{(f)}$	excess of life expectancy of females over that of males (final) $\mathring{e}_x^{'(f)}-\mathring{e}_x^{(m)}$ (2)+(3)
(1)	(2)	(3)	(4)
0	+ .272	+ .921	+1.193
1	− .090	+1.057	+ .967
5	+ .098	+1.183	+1.281
10	+ .200	+1.235	+1.435
15	+ .156	+1.242	+1.398
20	+ .050	+ .994	+1.044
25	− .112	+ .845	+ .733
30	− .097	+ .702	+ .605
35	+ .567	+ .431	+ .998
40	+1.434	+ .126	+1.560
45	+1.852	+ .025	+1.877
50	+2.063	...	+2.063
60	+1.966	...	+1.966
70	+1.713	...	+1.713
80	+1.172	...	+1.172
90	+ .320	...	+ .320

where Δd_y^* is the increase in the sub-divided decrement $d_y^*(={}_1d_y^*-d_y^*)$ due to the increase in s_y and $\Delta\mathring{e}'_{y+\frac{1}{2}}$ is the increase in the value of $\mathring{e}'_{y+\frac{1}{2}}(={}_1\mathring{e}'_{y+\frac{1}{2}}-\mathring{e}'_{y+\frac{1}{2}})$ due to the reduction in mortlity rates at and above age y.

Assuming that there is a further reduction of mortality at age y, but at no other ages above and below y, the contribution to the increase in life expectancy at age x, from any age z above $y=0$, since in equation (14), $\Delta d_z^* = \Delta\mathring{e}'_{z+\frac{1}{2}} = 0$, and that from age y, assuming that s_y is increased to $(1+g)\,.\,s_y$, which is still less than k_y (Table 2)

$$=1/l_x[g\,.\,d_y^*\,.\,\mathring{e}'_{y+\frac{1}{2}}+(1+g)\,.\,d_y^*\,.\,\Delta\mathring{e}'_{y+\frac{1}{2}}].$$

In above, $\Delta\overset{\circ}{e}'_{y+\frac{1}{2}}$

$$= \frac{1}{2}\Delta(\overset{\circ}{e}'_y+\overset{\circ}{e}'_{y+1}) = \frac{1}{2}\Delta\overset{\circ}{e}'_y, \text{ since } \Delta\overset{\circ}{e}'_{y+1} = 0$$

$$= \frac{1}{2}\Delta[\overset{\circ}{e}'_{y:\overline{1}|}+p'_y \cdot \overset{\circ}{e}'_{y+1}]$$

$$= \frac{1}{2}[\Delta_{\frac{1}{2}}p'_y+\Delta p'_y \cdot \overset{\circ}{e}'_{y+1}]$$

$$= 1/2l'_y[\Delta l'_{y+\frac{1}{2}}+\Delta l'_{y+1} \cdot \overset{\circ}{e}'_{y+1}]$$

$$= 1/2l'_y\left[\Delta\left(l'_y-\frac{1}{2}d'_y\right)+\Delta(l'_y-d'_y) \cdot \overset{\circ}{e}'_{y+1}\right]$$

$$= 1/2l'_y\left[-\frac{1}{2}\Delta d'_y-\Delta d'_y \cdot \overset{\circ}{e}'_{y+1}\right]$$

$$= -1/2l'_y\left[\Delta d'_y\left(\frac{1}{2}+\overset{\circ}{e}'_{y+1}\right)\right]$$

$$= -\frac{1}{2}\Delta q'_y \cdot \left(\frac{1}{2}+\overset{\circ}{e}'_{y+1}\right)$$

$$= \frac{1}{2}(q'_y-{}'q'_y) \cdot \left(\frac{1}{2}+\overset{\circ}{e}'_{y+1}\right),$$

where ${}'q'_y$ is the reduced rate of mortality at age y.

Therefore contribution from age y

$$= 1/l_x\left[g \cdot d'_y \cdot \overset{\circ}{e}'_{y+\frac{1}{2}}+(1+g) \cdot d'_y \cdot \frac{1}{2}(q'_y-{}'q'_y) \cdot \left(\frac{1}{2}+\overset{\circ}{e}'_{y+1}\right)\right]$$

$$= 1/l_x \cdot d'_y \cdot \overset{\circ}{e}'_{y+\frac{1}{2}}\left[g+(1+g) \cdot \frac{1}{2}(q'_y-{}'q'_y) \cdot \left(\frac{1}{2}+\overset{\circ}{e}'_{y+1}\right)\Big/\overset{\circ}{e}'_{y+\frac{1}{2}}\right].$$

Similarly, from equation (14), the contribution from any age h below y and above x

$$= 1/l_x \cdot d'_h \cdot \Delta\overset{\circ}{e}'_{h+\frac{1}{2}}$$

$$= 1/l_x \cdot d'_h \cdot \Delta(\overset{\circ}{e}'_{h+\frac{1}{2}:\overline{y-h-\frac{1}{2}}|}+{}_{y-h-\frac{1}{2}}p'_{h+\frac{1}{2}} \cdot \overset{\circ}{e}'_y)$$

$$= 1/l_x \cdot d'_h \cdot {}_{y-h-\frac{1}{2}}p'_{h+\frac{1}{2}} \cdot \Delta\overset{\circ}{e}'_y$$

$$= 1/l_x \cdot d'_h \cdot {}_{y-h-\frac{1}{2}}p'_{h+\frac{1}{2}} \cdot (q'_y-{}'q'_y) \cdot \left(\frac{1}{2}+\overset{\circ}{e}'_{y+1}\right)$$

$$= 1/l_x \cdot d'_h \cdot \overset{\circ}{e}'_{h+\frac{1}{2}}\left[{}_{y-h-\frac{1}{2}}p'_{h+\frac{1}{2}} \cdot (q'_y-{}'q'_y) \cdot \left(\frac{1}{2}+\overset{\circ}{e}'_{y+1}\right)\Big/\overset{\circ}{e}'_{h+\frac{1}{2}}\right],$$ which is small,

considering the smallness of the quantity $(q'_y - {}'q'_y)$ and in any case, small compared to the contribution from age y, taken for the same reason, approximately as $1/l_x \cdot g \cdot d'_y \cdot \mathring{e}'_{y+\frac{1}{2}}$. This being therefore the main contribution from all ages, x and above, the increase in life expectancy at age x, ${}_1\mathring{e}'_x - \mathring{e}'_x$ may be taken approximately as $g/l_x \cdot d'_y \cdot \mathring{e}'_{y+\frac{1}{2}}$. From col. (2), Table 3, the values of $\Sigma d'_y \cdot \mathring{e}'_{y+\frac{1}{2}}$ are comparatively large in the age-groups, 15-19, 30-34 and 35-39, corresponding to larger values in cols. (11) and (12). Since female mortality is already lighter than male in the age-group, 15-19, in the Eastern Zone Life Tables, it should be a good strategy to attempt a further elimination of the specified cause of death especially in the age-groups, 30-34 and 35-39 to produce a comparatively large increase in the complete expectation of life at birth of females, and hence in its excess over that of males. This would also improve female mortality at these ages, possibly to the extent of removing its excess over or making it even lighter than male mortality.

5.2.2. It is believed that in the Eastern Zone, 1951-60, female mortality was heavier than male during much of the reproductive period of life, mainly because of excessive pregnancy and maternity hazards, a proper elimination of which, as we have seen, is able to reverse the position. As for ages 1-10, the heavier mortality of females as shown by the Life Tables may be called to question because of the approximate methods employed for constructing and consequent uncertainty in the derived mortality rates at these ages. Considering the whole range of ages, there is no reason to suppose that Eastern Zone or for that matter Indian females are any exception to the biological fact that females have a greater hold on life than males at all ages. It has been noticed that the complete expectation of life at birth of Eastern zone females, 1951-60 had already exceeded that of males and it is likely that with the gradual elimination of mortality hazards peculiar to females following improvement of social conditions and health measures, and with comparative freedom from such hazards at the earlier ages caused by the rising age at marriage and reductions in the proportions married at those ages, the relative mortality pattern of males and females in this country will ultimately be the same as in the developed countries of the world.

5.3. In the development of the theory of the methods applied above, the following assumptions have been made :

(i) *Uniform distribution of decrements in the year of age.* This is not applicable to mortality decrements or deaths at least in the first year of age. This means that the values of the elements ${}_\alpha d_0^\gamma \cdot \mathring{e}_1^{(\beta)}$ and ${}_\beta l_0^\gamma \cdot \mathring{e}_1^{(\alpha)}$ in cols. (3) and (7) respectively in Table 1 require modification, for the first of the above elements, $l_x^\alpha \cdot q_0^\gamma \cdot \mathring{e}_1^{(\beta)}$ has been evaluated with $q_0^\gamma = (\gamma q)_0 \left[1 - \frac{1}{2}(\beta q)_0\right]$, derived on the basis of uniform distribution of deaths in the year of life, and with $\mathring{e}_t^{(\beta)}$, taken as the value of $\int_0^1 \mathring{e}_t^{(\beta)}\, dt = \frac{1}{2}(\mathring{e}_0^{(\beta)} + \mathring{e}_1^{(\beta)})$, as a practical expedient, also approximately true on the same basis. To find the value on the correct basis of uneven distribution of deaths, we have

$$q_0'' \doteq \int_0^1 {}_t(\beta p)_0 dt \cdot \int_0^1 {}_t(\gamma p)_0 \cdot \mu_t^\gamma dt$$

$$= (\gamma q)_0 \int_0^1 [1 - {}_t(\beta q)_0] dt$$

$$= (\gamma q)_0 \left[1 - (\beta q)_0 \int_0^1 {}_t(\beta q)_0/(\beta q)_0 dt \right]$$

$$= (\gamma q)_0 [1 - (\beta q)_0 \cdot k],$$

where $k > \frac{1}{2}$, from the nature of the uneven distribution and hence of the function ${}_t(\beta q)_0/(\beta q)_0$, which increasing from 0 to 1 as t increases from 0 to 1, increases sharply at first and then slows down $\left(\text{with an even distribution of deaths, the function} = t, \text{ and } k = \frac{1}{2}\right)$, so that the value of q_0'' is less on the correct basis. It can be seen also that the nature of the function $\mathring{e}_t^{(\beta)}$ is similar to that of the function ${}_t(\beta q)_0/(\beta q)_0$, so that the correct value of $\int_0^1 \mathring{e}_t^{(\beta)} dt$ is greater than $\frac{1}{2}(\mathring{e}_0^{(\beta)} + \mathring{e}_1^{(\beta)})$. To see how far these two opposite effects on the value of the element should counter-balance one another, by reference to actual data, we have from *Vital Statistics of India*, 1962 (Registrar General of India, 1966, p. 75), proportions of annual deaths in West Bengal in the first year of life, during the first one month and in the first six months of age, from which, assuming that the above proportions apply to the life table (β), a smooth graph of ${}_tq_0/q_0$ is drawn. From the area of the graph, $\int_0^1 {}_tq_0/q_0 dt = .724$, which is taken as the value of k, applicable to the same life table. Again, from the graph of e_t, drawn with values of e_t for various values of t, calculated from the equation, $\mathring{e}_t = (\mathring{e}_0 - \mathring{e}_{0:\overline{t}|})/{}_tp_0$, where $\mathring{e}_{0:\overline{t}|} = \int_0^t {}_rp_0 dr = \int_0^t (1 - {}_rq_0) dr = t - q_0 \int_0^t {}_rq_0/q_0\, dr$, and ${}_tp_0 = 1 - {}_tq_0 = 1 - q_0 \cdot {}_tq_0/q_0$, the values of $\int_0^t {}_rq_0/q_0 dr$ and ${}_tq_0/q_0$ being obtained from the graph of ${}_tq_0/q_0$, and with $\mathring{e}_0$ taken as $\mathring{e}_0^{(\beta)} = 40.05$, $\int_0^1 \mathring{e}_t^{(\beta)} dt$ comes out to $b = 40.05 + 3.618 = 43.668$ approximately, as against 42.475, the arithmetic mean of $\mathring{e}_0^{(\beta)}$ and $\mathring{e}_1^{(\beta)}$. Using the above values, the value of the element comes out to be 34,803 (as against 34,929, col. (3), Table 1), so that the revised value of $\mathring{e}_0^{(\beta)} - \mathring{e}_0^{(\alpha)}$ becomes .272 (as against .273 in col. (5), Table 1), the value of the same directly from the life tables being also .272 (col. (6), Table 1).

Considering the other element and applying the same procedure as above on life table (α) the revised value of the element comes to be 34,675 (as against 35,018, col. (7), Table 1) and that of $\overset{\circ}{e}{}_0^{(\beta)} - \overset{\circ}{e}{}_0^{(\alpha)}$ becomes .271 (as against .274, col. (9), Table 1) and therefore closer to the value by the life tables.

It will therefore be seen that adjustments for the non-uniform distribution of deaths in the first year of life tend to produce results in closer agreement with the theory.

(ii) The value of the function k_x is the same for all ages between x and $x+1$ or between x and $x+5$ as the case may be, though in fact it is variable within the age interval in question. This objection may be met by considering the value used as the average in the age interval, For instance,

$$\int_0^5 k_{x+t} \cdot P^r_{x+t} \cdot \mu_{x+t} \, dt \doteq 1/5 \int_0^5 k_{x+t} \, dt \cdot \int_0^5 P^r_{x+t} \cdot \mu_{x+t} \, dt = \bar{k}_x \cdot \int_0^5 P^r_{x+t} \cdot \mu_{x+t} \, dt,$$

where $\bar{k}_x$ is the average value of the function k_{x+t} in the age interval.

As for the function f_x, its constancy in the age interval is simply a postulate in the scheme of operations for reduction of the force of mortality.

(iii) Mutual independence of a specified cause of death and other causes, though in fact a death seldom occurs by a single cause or causes of death are frequently inter-related. As a practical expedient, this point is met by classification of deaths by mutually exclusive and appropriately chosen groups of causes (the specified causes of death in this paper is really a group of causes), so that when a death occurs, it may be assigned to only one group and to no other. We can then write

$$\theta_x = \theta^\beta_x + \theta^\gamma_x$$

or

$$\mu_x = \mu^\beta_x + \mu^\gamma_x,$$

as when deaths and forces of mortality, (β) and (γ) are mutually independent and add up respectively to the total deaths and total force of mortality at age x.

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