Indian J. pure appl. Math., **25** (1&2); 231-237, January & February 1994

A NOTE ON DEBLURRING

SRIMANTA PAL

Computer and Statistical Service Centre, Indian Statistical Institute, 203 *B T Road, Calcutta* 700 035

AND

D. DUTTA MAJUMDER

Emeritus Scientists, CSIR National Centre for Knowledge Based Computing Electronics & Communication Sceince Unit, Indian Statistical Institute, 203 *B T Road, Calcutta* 700 035

(*Accepted* 10 *May* 1993)

Blur is an well-known model for image and singal degradation. Deblurring, an inverse of blur is an ill-posed problem and its approximation solution based on regularization exists in a restricted domain of Polynomials or functions. We have observed that
(1) Blurring operator is closed in our domain of definition, and
(2) area of deblurred output is approximately $2e\sqrt{\pi}$ for the deblurring kernerls in terms of Weber parabolic cylinder functions and the gaussian $N(0, \sqrt{2})$ blur.

1. INTRODUCTION

In image processing, blurring is a filter through which the original image is passed and produced a sampled output at an uniform interval but the deblurring (inverse of blurring) the sample data implies the reconstruction of images closer to be original image. Stark[4] reviewed the image deblurring techniques but in a recent work by Hummel *et al.*[2] it was observed that

(1) the process of deblurring is unstable.

(2) cannot be represented as a convolution filter in the spatial domain, and

(3) a convolution inverse does exist and provides the restriction on the space of allowable functions to polynomials of fixed finite degree.

Recently Martens[3] have observed that the work of Hummel *et al.*[2] involved a number of following important restrictions : (1) only the case of gaussian blur was considered; (2) the blurring kernel was assumed to be the product of gaussian and a polynomial of fixed degree; and (3) the deblurring problem was only analyzed for analong signals. In this context both Hummel *et al.*[2] and Martens[3] had adopted that the image can be locally represented by a polynomial.

As we know, vision begins with the transformation of a flux of photon particles into a set of intensity values at an array of sensors[5]. Again the solution of differential equation for harmonic oscillations of particles in quantum mechanics are of the form of parabolic cylinder functions[8].

In this note we are assuming that the image can be locally represented by a Weber parabolic cylinder function[1,5]. In Section 2, we show how that Weber parabolic cylinder function is related to Hermite polynomial and other related relations. Also we shall show that the blurring operation is closed in the domain of definitions.

2. Weber Parabolic Cylinder Function

The well-known definition of Hermite polynomial and Weber parabolic cylinder function are given below :

Definition 1 — The Hermite polynomial may be denoted by $H_n(x)$ of n degrees of polynomial of x, expressed as follows :

$$H_n(x) = n! \sum_{m=0}^{[n/z]} (-1)^m \frac{(2x)^{n-2m}}{m!\,(n-2m)!}$$

or, by the Rodrigues formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})$$

where $H_0(x) = 1$.

Definition 2 — The Weber parabolic cylinder function may be denoted by $D_n(x)$ of n degrees of polynomial of x, expressed as follows :

$$D_n(x) = e^{-x^2/4}\, n! \sum_{m=0}^{[n/z]} (-!)^m \frac{(\sqrt{2x})^{n-2m}}{m!\,(n-2m)!}$$

or,

$$D_n(x) = e^{x^2/4} (-1)^n (\sqrt{2})^n \frac{d^n}{dx^n} (e^{-x^2/2})$$

where $D_0(x) = e^{-x^2/2}$

2.1 Relationship between Hermite Polynomial and Weber Parabolic Cylinder der Function

The relationship between Hermite polynomial and Weber parabolic cylinder function may be expressed as follows.

$$D_n(x\sqrt{2c}) = e^{-cx^2/2} H_n(x\sqrt{c}).$$

Where

c = a constant, and

n = degree of the polynomial

2.2 Asymptotic Relationship

If $n \to \infty$ then the expression by $D_n(x)$ as $\sum_{n=0}^{\infty} \frac{t^n}{n!} D_n(x)$ converges [1] to an exponential function of x and t as

$$e^{-x^2/4 + xt - t^2/2}$$

that is

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} D_n(x) = e^{-x^2/4 + xt - t^2/2}.$$

2.3 Observations

Let Π be an operator of blurring then by definition of convolution we get

$$\Pi D_n(y) = \int_{-\infty}^{\infty} K(y-x)\, D_n(x)\, dx$$

where

$D_n(x)$ = Weber parabolic cylinder function

$K(x)$ = Kernel.

Assume

$$K(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2 \right\}, -\infty \le x \le \infty.$$

Let us choose $\sigma = \sqrt{2}$ and $m = 0$ then

$$K(x) = \frac{1}{\sqrt{4\pi}} \exp\{ x^2/4 \},\ -\infty \le x \le \infty.$$

Lemma 1 — Π is closed on the domain of $D_n(x)$.

PROOF : Since $\Pi\ D_n(y) = \int_{-\infty}^{\infty} K(y-x)\ D_n(x)\ dx$

$$\Pi\ D_n(y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi}} \exp\{ -(y-x)^2/4 \} *$$

$$e^{x^2/4} (-1)^n (\sqrt{2})^n \frac{d^n}{dx^n} (e^{-x^2/2})\ dx$$

$$= \frac{e^{-y^2/4} (-1)^n (\sqrt{2})^n}{\sqrt{4\pi}} \int_{-}^{\infty} e^{(y/2)x} \frac{d^n}{dx^n} (e^{-x^2/2})\ dx$$

$$= (y/\sqrt{2}) \frac{e^{-y^2/4} (-1)^{n-1} (\sqrt{2})^{n-1}}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{(y/2)x} \frac{d^{n-1}}{dx^{n-1}} (e^{-x^2/2}) \, dx$$

[Integration by parts]

$$= (y/\sqrt{2}) \, \Pi D_{n-1} (y)$$

$$= (y/\sqrt{2})^{n-1} \, \Pi D_1 (y)$$

$$= (y/\sqrt{2})^{n-1} \frac{1}{\sqrt{2}} (y/\sqrt{2}) \, e^{-y^2/8}$$

[Since $\Pi D_1 (y) = \frac{1}{\sqrt{2}} (y/\sqrt{2}) \, e^{-y^2/8}$]

$$= \frac{1}{\sqrt{2}} (y/\sqrt{2})^n \, e^{-y^2/8}$$

Property 1 — $\Pi [D_i(x) + D_j(x)] = \Pi [D_i(x)] + \Pi [D_j(x)]$.

PROOF : By definition we can write

$$\Pi [D_i(x) + D_j(x)] = \int_{-\infty}^{\infty} K(x-y) \, [D_i(y) + D_j(y)] \, dy$$

$$= \int_{-\infty}^{\infty} K(x-y) \, D_i(y) \, dy + \int_{-\infty}^{\infty} K(x-y) \, D_j(y) \, dy$$

$$= \Pi [D_i(x)] + \Pi [D_j(x)].$$

Lemma 2 — An image may be expressed by the combinations of $D_n(y)$ as $\sum_{i=0}^{N} \frac{t^i}{i!} D_i(y)$ where $N \to \infty$ and its blurring also converges to a fixed function. In that case blurring operation is closed in the domain of definition.

PROOF : By Lemma 1 we get

$$\Pi \, D_n(y) = (y/\sqrt{2})^n \left(\frac{1}{\sqrt{2}} e^{-y^2/8} \right)$$

Let $\quad \gamma = 1/\sqrt{2}$ and $\psi = \frac{1}{\sqrt{2}} e^{-y^2/8}$ then

$$\Pi \left(\frac{a_i}{\gamma_i} D_i(y) \right) = \psi \, a_i \, y^i, \text{ where } a_i = \text{constant}.$$

Again by property 1 we can write for finite N

$$\Pi\left(\sum_{i=0}^{N}\frac{a_i}{\gamma_i}D_i(y)\right)=\sum_{i=0}^{N}\psi\, a_i y^i.$$

Assume $a_i = 1/i!$ and $\gamma = 1/t = 1/\sqrt{2}$.

Therefore

$$\Pi\left(\sum_{i=0}^{N}\frac{t^i}{i!}D_i(y)\right)=\psi\sum_{i=0}^{N}y^i/i!.$$

As $N \to \infty$ then

$$\Pi\,[e^{-y^2/4+yt-t^2/2}]=\psi\; e^y=\frac{1}{\sqrt{2}}\,e^{-y^2/8+y}.$$

By Lemma 2 we have seen that

$$\Pi\left(\sum_{i=0}^{\infty}\frac{t^i}{i!}D_i(y)\right)=\frac{1}{\sqrt{2}}e^{-y^2/8+y}.$$

So, we shall construct Π^{-1} which is inverse transform of Π.

That is

$$\sum_{i=0}^{\infty}\frac{t^i}{i!}D_i(y)=\Pi^{-1}\left(\frac{1}{\sqrt{2}}e^{-y^2/8+y}\right)$$

$$\text{or, }\sum_{i=0}^{\infty}\frac{t^i}{i!}D_i(y)=\Pi^{-1}\left(\frac{1}{\sqrt{2}}e^{-y^2/8}\sum_{i=0}^{N}\frac{y^i}{i!}\right)$$

If f be a function that belongs to the domain of Weber parabolic cylinder function, and g be another function can be generated by blurring transformation with f. That is

$$g=\Pi f$$

then

$$f=B_N\otimes g$$

where, B = deblurring kernel

From the above assumption we get as $N \to \infty$

$$f=\sum_{i=0}^{N}\frac{t^i}{i!}D_i(y) \longrightarrow e^{-y^2/4+yt-t^2/2}$$

and

$$g=\frac{1}{\sqrt{2}}\,e^{-y^2/8}\sum_{i=0}^{N}\frac{y^i}{i!}D_i(y)\to\frac{1}{\sqrt{2}}e^{-y^2/8+y}$$

Now we shall choose a kernel so that the inverse transformation Π^{-1} operation on g will give f. That is the kernel $B_{N \to \infty} \approx B(x)$ will satisfy the following relationship

$$e^{-y^2/4 + yt - t^2/2} \approx B(x) \otimes \left(\frac{1}{\sqrt{2}} e^{-y^2/8 + y} \right).$$

When $t = \sqrt{2}$, then the integral equation satisfy the following

$$e^{-(y^2 - 4\sqrt{2}y + 4)/4} \approx \int_{-\infty}^{\infty} B(y-x) \frac{1}{\sqrt{2}} e^{-x^2/8 + x} \, dx.$$

Integrating both side with respect to y we get

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B(y-x) \frac{1}{\sqrt{2}} e^{-x^2/8 + x} \, dx \, dy \approx \int_{-\infty}^{\infty} e^{-(y^2 - 4\sqrt{2y} + 4)/4} \, dy \approx 2e\sqrt{\pi}$$

This integral equation gives the deblurring kernel $B(x)$. So the area of blurring output with gaussian $N(0, \sqrt{2})$ kernel is $2e\sqrt{\pi}$ where the approximation of local image is representd by Weber parabolic cylinder function.

Again if $B(x)$ is the deblurring kernel then

$$B(x) \otimes \left\{ (x/\sqrt{2})^n \left(\frac{1}{\sqrt{2}} e^{-x^2/8} \right) \right\} (y) = D_n(y)$$

that is

$$\int_{-\infty}^{\infty} B(y-x) \left\{ (x/\sqrt{2})^n \left(\frac{1}{\sqrt{2}} e^{-x^2/8} \right) \right\} dx = D_n(y)$$

or

$$\int_{-\infty}^{\infty} B(x) \, \{ (y-x)/\sqrt{2} \}^n \, \frac{1}{\sqrt{2}} e^{-(y-x)^2/8} \, dx = D_n(y). \qquad \ldots (1)$$

The above integral equation is equivalent to standard Fredholm integral equation of the first kind. This kind of integral equation has been investigated by several authors[6,7] as ill-posed problem and solved by regularization technique.

3. Conclusions

In this note we find that deblurring problem can be solved with gaussian blurring kernel and the image is locally approximated by Weber parabolic cylinder function as Hermite polynomial approximation. We are trying to solve the integral equation (1) and to find the optimal deblurring kernel.

Acknowledgement

We would like to thank S. C. Kundu, Head, Computer and Statistical Service Centre, Indian Statistical Institute, Calcutta for his continuous encouragement to study this problem.

REFERENCES

1. H. Bateman, Higher Transcendental functions. Volume II, McGraw Hill (1953) p.119.
2. R. A. Hummel, B. Kimia and S. W. Zucker, Deblurring gaussian Blur, Computer vision, Graphics and Image Processing, 38(1987) 66-80.
3. J. B. Martens, Deblurring Digital Images By means of polynomial Transforms. Computer Vision, Graphics and Image Processing 50(1990) 157-176.
4. H. Stark, Image Recovery - Theory and Application. Academic Process, New York, (1987)
5. V. Torre and T. A. Poggio, On Edge Detection, IEEE PAMI 8(1986) 147-163.
6. J. P. Bulter, J. A. Reeds and J. V. Dawson, Estimating SOlutions of first integral equations with non-negative constraints and optimal smoothing. SIAM J. Numer. Anal. 18(1981) 381-397.
7. R. S. Rutman and L. M. Cabral, Descriptive regularization of Fredholm integral equations of the first kind. In "Treatment of integral equations by numerical methods," eds. C. T. H. Baker and G. F. Miller, Academic Press, London, (1982) 313-324.
8. S. Iyanaga and Y. kawada (ed), Encyclopaedic Dictionary of Mathematics by Mathematical Society of Japan. The MIT press, Cambridge, Massachusetts, (1954), 565.