ON A CHARACTERISATION OF THE MULTIVARIATE NORMAL DISTRIBUTION

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1. Introduction

In recent years considerable work has been done on the characterisations of the normal distribution. Bernstein (1941), Frechet (1951), Darmois (1951) and Basu (1951) have proved some theorems on the stochastic independence of linear functions of independent chance variables. It has been established that if $x_1, x_2, \ldots, x_n$ are independent chance variables such that $a_1x_1+a_2x_2+\ldots+a_nx_n$ is independent of $b_1x_1+b_2x_2+\ldots+b_nx_n$ then each x_i is normally distributed, provided that $a_ib_i \neq 0$ $(i = 1, 2, \ldots, n)$. Geary (1936) and Lukacs (1942) have proved that if the sample mean is distributed independently of the sample variance in the case of a sample from a population with finite variance, then the population is normal. Basu and Laha (1954) extended Geary's theorem by proving that if the sample mean is distributed independently of any sample k-statistic (as defined by Fisher) then the population is normal. The author (1953) has made another extension of Geary's theorem, proving that if $x_1, x_2, \ldots, x_n$ are identically distributed independent random variables with a finite variance σ^2 and if the conditional expectation of any unbiased quadratic estimate of σ^2 for the given sum $x_1+x_2+\ldots+x_n$ be independent of the latter, then the distribution of each x_i is normal. The theorem on the stochastic independence of symmetric and homogeneous linear and quadratic statistics proved by Lukacs (1952) follows as a simple corollary from the author's theorem above. The object of the present note is to establish the multivariate analogue of this theorem.

Theorem : *Let $X_{(p\times n)}$ represent a sample of size n drawn independently at random from any p-variate distribution with a dispersion matrix $\Sigma = (\sigma_{\alpha\beta})$ $(\alpha, \beta = 1, 2, \ldots, p)$. If the conditional expectation of any unbiased quadratic estimate of Σ for the fixed sample means does not contain the latter, then the joint distribution of the variates is p-variate normal.*

Proof: Let XAX' denote any unbiased quadratic estimate of Σ such that

$$E(XAX') = \Sigma. \qquad \ldots \quad (2.1)$$

Let $\underline{x}_\alpha = (x_{\alpha 1}, x_{\alpha 2}, \ldots, x_{\alpha n})$ $(\alpha = 1, 2, \ldots, p)$; then it follows evidently from (2.1) that

$$E(\underline{x}_\alpha A \underline{x}'_\beta) = \sigma_{\alpha\beta} \qquad (\alpha, \beta = 1, 2, \ldots, p). \qquad \ldots \quad (2.2)$$

On simplification, (2.2) yields

$$(\sigma_{\alpha\beta}+m_\alpha m_\beta)\sum_{r=1}^{n} a_{rr}+m_\alpha m_\beta \sum_{\substack{r,s \\ r\neq s}} a_{rs} = \sigma_{\alpha\beta} \qquad . \quad (2.3)$$

where $E(x_{\alpha j}) = m_\alpha$.

Hence the condition of unbiasedness leads to the relations

$$\sum_{r=1}^{n} a_{rr} = 1 \quad \text{and} \quad \sum_{r,s=1}^{n} a_{rs} = 0. \qquad \dots \quad (2.4)$$

By the condition of the theorem, it follows evidently that

$$E\{(\underline{x}_\alpha A \underline{x}'_\beta).e^{i(t_1\Sigma x_1+t_2\Sigma x_2+\dots+t_p\Sigma x_p)}\}$$
$$= E(\underline{x}_\alpha A \underline{x}'_\beta).E(e^{i(t_1\Sigma x_1+t_2\Sigma x_2+\dots+t_p\Sigma x_p)}). \qquad \dots \quad (2.5)$$

After some algebraic simplifications, (2.5) reduces to the form

$$\frac{\partial^2\phi}{\partial t_\alpha \partial t_\beta}\cdot\frac{1}{\phi}\sum_{r=1}^{n} a_{rr}+\frac{\partial\phi}{\partial t_\alpha}\cdot\frac{\partial\phi}{\partial t_\beta}\cdot\frac{1}{\phi^2}\sum_{\substack{r,s\\ r\neq s}} a_{rs} = -\sigma_{\alpha\beta} \qquad \dots \quad (2.6)$$

where ϕ denotes the characteristic function of the above p-variate distribution.

On writing $\psi = ln\phi$, we have

$$\frac{\partial\psi}{\partial t_\alpha} = \frac{\partial\phi}{\partial t_\alpha}\cdot\frac{1}{\phi} \quad \text{and} \quad \frac{\partial^2\psi}{\partial t_\alpha\partial t_\beta} = \frac{\partial^2\phi}{\partial t_\alpha\partial t_\beta}\Big/\phi - \frac{\partial\phi}{\partial t_\alpha}\cdot\frac{\partial\phi}{\partial t_\beta}\Big/\phi^2. \qquad \dots \quad (2.7)$$

Now substituting the relations of (2.7) in (2.6) and then using the conditions of (2.4), we have at once

$$\frac{\partial^2\psi}{\partial t_\alpha\partial t_\beta} = -\sigma_{\alpha\beta}. \qquad \dots \quad (2.8)$$

Since the above relation (2.8) holds for all $\alpha, \beta = 1, 2, \dots, p$, $\psi(t_1, t_2, \dots, t_p)$ is a quadratic form in $t_1, t_2, \dots, t_p$, which proves the theorem.

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