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THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF NUCLEAR SPACE-VALUED STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY POISSON RANDOM MEASURES

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In this paper, we study stochastic differential equations (SDE's) on duals of nuclear spaces driven by Poisson random measures. The existence of a weak solution is obtained by the Galerkin method. For uniqueness, a class of ℓ^2-valued processes which are called Good processes are introduced. An equivalence relation is established between SDE's driven by Poisson random measures and those by Good processes. The uniqueness is established by extending the Yamada-Watanabe argument to the SDE's driven by Good processes. This is an extension to discontinuous infinite dimensional SDE's of work done by G. Kallianpur, I. Mitoma and R. Wolpert for nuclear space valued diffusions.

KEY WORDS: Nuclear space, martingale problem, weak solution, strong solution, Good process.

AMS CLASSIFICATION: Primary 60H10; secondary 60B10.

1 INTRODUCTION

Stochastic differential equations (SDE's) on infinite dimensional spaces arise from such diverse fields as nonlinear filtering, infinite particle systems, neurophysiology, etc.

Some of the earliest examples come from nonlinear filtering theory in which the conditional distribution of the signal process satisfies a nonlinear measure-valued stochastic differential equation (SDE) obtained by Kushner [16] and studied by Fujisaki, Kallianpur and Kunita [1]. A more easily handled equation is the SPDE for the "unnormalized conditional density" derived by Zakai in [19].

K. Itô [7] and M. Hitsuda and I. Mitoma [5] considered the limit behavior of the empirical measure of interacting diffusion processes and characterized the limit processes by SDE's on the duals of nuclear spaces. The uniqueness of the solution

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for such SDE's was proved by I. Mitoma [21] and deduced by G. Kallianpur and V. Perez-Abreu [12] from a general result.

This paper is motivated by the applications to neurophysiology, specifically, to the behavior of voltage potentials of spatially extended neurons.

In the absence of stimuli, the voltage potential $V(x, t)$ at time t and at a point x of a spatially extended neuron satisfies a partial differential equation (PDE) which is called the cable equation. The stimuli received by the neuron can be modeled by a Poisson random measure or its limit case, a Gaussian white noise. Hence, with stimuli, the voltage potential is governed by a partial differential equation subject to random perturbations, i.e. a stochastic partial differential equation (SPDE).

SPDE's and infinite dimensional SDE's are closely related in the sense that a solution $u(t, x)$ of a SPDE may either be regarded as a random field in (t, x) or as a process $u(t, \cdot)$ taking values in a suitable function space, e.g. the Banach space of continuous functions $C[a, b]$. However, a formally written SPDE may have a solution only in a space of distributions (See Walsh [18]).

Treated as infinite dimensional SDE's, linear models for voltage potentials of spatially extended neurons have been studied by Walsh [18], G. Kallianpur and R. L. Wolpert [14]. We refer the reader to these papers for details.

More realistic problems of neuronal behavior, such as reversal potential problems lead to more complicated kinds to stochastic models. Suppose that the impulses arise from various types of ions with different equilibrium potentials passing through the neuron membrane. Each of them arrives according to independent Poisson processes. The change of voltage potential is determined not only by its magnitude but also by the difference between its equilibrium potential and that neuron's voltage potential at that moment.

The SDE's corresponding to the reversal potential problem are no longer linear. G. Kallianpur and R. L. Wolpert [13] studied this problem when the neuron can be looked as a single point and pointed out the importance of this problem for spatially extended neurons. There is an essential difference between a spatially extended neuron and a point neuron as the latter corresponds to a real valued SDE while the SDE corresponding to the former is infinite dimensional.

A Banach space valued SDE with non-linear coefficients and driven by a semimartingale (including a compensated random measure) has been studied by Gyöngy [2]. Both his paper and ours rely on the Galerkin method but there are several differences. The conditions imposed on the coefficients in [2] (especially the coercivity assumption) are not the same as ours and seem to be dictated by the choice of the solution space. In addition, our approach differs from that of [2] in an important respect. Gyöngy, following the method of Krylov and Rozovskii [3], directly aims for a unique strong solution. In this paper we first obtain a weak solution via the solution to a martingale problem. Up to this step, the monotonicity condition is not involved. The existence of a unique strong solution is then established by a separate argument that used a monotonicity condition on the coefficients.

A brief explanation is needed to point out the relevance of nuclear-valued SDE's instead of Banach space or Hilbert space valued SDE's. First of all, when we regard the solution $V(x, t)$ of a SPDE as the solution V_t of an infinite dimensional SDE, it is natural to consider V_t as distribution-valued and determined by the values of $V_t[\phi] = \int V(x, t)\phi(x)dx$ for all "smooth" functions ϕ. The set of all "smooth" functions usually turns out to be a nuclear space Φ (the simplest example is the space of all rapidly decreasing functions) and hence, V_t is a Φ'-valued process.

Next, the solution considered is for all $t \geqslant 0$ and not for t restricted to a fixed interval $[0, T]$. As we will see at the end of Section 3, for $t \in [0, T]$, we obtain a solution taking values in a Hilbert space $H_{-p_1(T)}$. But in general, there is no Hilbert space in which the solution lies for all $t \geqslant 0$.

Finally, even if we are only interested in a finite interval, using Φ' still has some technical advantages. Mitoma's paper [17] about the weak convergence of measures on $\mathbb{D}([0, T], \Phi')$ provides a powerful tool for establishing a solution in Φ'. After we obtain this solution, the regularity of the process is decided by finding the Hilbert space in which its paths lie.

Non-linear nuclear space-valued SDE's driven by Wiener processes have been studied by Kallianpur, Mitoma and Wolpert [11]. In this paper, we study the equations driven by Poisson random measures. Namely, we consider the following SDE's

$$X_t = X_0 + \int_0^t A(s, X_s)\,ds + \int_0^t \int_U G(s, X_{s-}, u)\tilde{N}(duds) \tag{1.1}$$

on the duals of a countably Hilbertian nuclear spaces Φ, where A: $R_+ \times \Phi' \to \Phi'$, G: $R_+ \times \Phi' \times U \to \Phi'$, $(U, \mathscr{U}, \mu)$ is a σ-finite measure space, $N(duds)$ is a Poisson random measure on $\mathbb{R}_+ \times U$ with intensity measure $\mu(du)ds$ and $\tilde{N}(duds)$ is the compensated random measure of $N(duds)$.

To begin with, in Section 2, we consider a sequence of Φ'-valued processes $\{X^n\}$ which are the solutions of a sequence of SDE's of the form (1.1) with coefficients (A^n, G^n) and fixed intensity measure μ. We first prove the tightness of this sequence under suitable conditions. Then we show that any cluster point of the distribution sequence of $\{X^n\}$ has to be a weak solution of (1.1) while the coefficients (A, G) is the limit of the sequence (A^n, G^n). Martingale methods are employed to provide a connecting idea in passing to the limit.

In Section 3, the existence of a weak solution for the SDE (1.1) is established under the continuity, coercivity and growth condition of Section 2 by making use of the results of Section 2 twice. First, we prove the existence of the weak solution when Φ is finite dimensional. Second, we project the SDE (1.1) to a sequence of finite dimensional spaces and apply the results of Section 2 to this sequence.

In Section 4, the unique strong solution of (1.1) is obtained under an additional monotonicity condition by introducing the "Good" processes to implement the Yamada-Watanabe argument in this setup.

Because of the limitations of space, the application to reversal potential models and the derivation of diffusion approximations for Φ'-valued SDE will be deferred to another paper. Some of the results of this paper have been announced in [15] by the first two authors.

2 WEAK CONVERGENCE THEOREMS

We begin this section by giving some facts about nuclear spaces and their duals.

DEFINITION 2.1 Φ is called a countably Hilbertian nuclear space, if Φ is a separable Fréchet space, whose topology is given by an increasing sequence of Hilbertian norms $\|\cdot\|_n, n \geq 0$, such that the following is satisfied: If H_n is the completion of Φ with respect to the norm $\|\cdot\|_n$, then for each n there exists $m > n$, such that the canonical injection $H_m \to H_n$ is Hilbert-Schmidt.

Let H_{-n} and Φ' denote the duals of H_n and Φ respectively. Then identifying H_0 with its dual H_0', we have the following sequence of canonical injections:

$$\Phi \to \cdots \to H_2 \to H_1 \to H_0 \equiv H_0' \to H_{-1} \to H_{-2} \to \cdots \to \Phi'. \tag{2.1}$$

It is well known that

$$\Phi = \bigcap_{n=1}^{\infty} H_n \quad \text{and} \quad \Phi' = \bigcup_{n=1}^{\infty} H_{-n}. \tag{2.2}$$

The following assumptions will be made throughout this paper: There exists a sequence (h_i) of elements in Φ, such that (h_i) is a complete orthonormal system (CONS) in H_0 and is a complete orthogonal system (COS) in each space $H_n, n \in \mathbb{Z}$.

The following notation will be used throughout the paper:

(1) $h_i^n \equiv h_i \|h_i\|_n^{-1}$, $n \in \mathbb{Z}$, $i \in \mathbb{N}^+$. It is easy to see that (h_i^n) is a CONS in H_n.
(2) For $v \in \Phi'$ and $\phi \in \Phi$ define $v[\phi]$ = the value of the continuous linear functional v at the point ϕ.
(3) $\forall p \equiv \mathbb{N}^+, \theta_p$ will denote the surjective linear isometry $H_{-p} \to H_p$ given by

$$\theta_p\left(\sum_{i=1}^{\infty} \alpha_i h_i^{-p}\right) \equiv \sum_{i=1}^{\infty} \alpha_i h_i^p. \tag{2.3}$$

It is easy to see that $\theta_p \phi \in \Phi$ for any $p \in \mathbb{N}^+$ and $\phi \in \Phi$.
(4) $\forall p \in \mathbb{N}^+, M > 0$, let

$$A_M^p \equiv \{Z \in \mathbb{D}([0, T], H_{-p}): \sup_{0 \leq t \leq T} \|Z_t\|_{-p} \leq M\}. \tag{2.4}$$

The following basic proposition can be demonstrated by standard Hilbert space techniques.

PROPOSITION 2.1 (a) $\forall m \in \mathbb{N}$ *and* $i \in \mathbb{N}^+$, *we have that* $\|h_i\|_m \|h_i\|_{-m} = 1$;
(b) $\forall m, i \in \mathbb{N}^+$ *and* $z \in H_{-m}$, *we have* $\langle z, h_i^{-m} \rangle_{-m} = z[h_i^m]$;
(c) $\forall m \in \mathbb{N}^+$ *and* $v \in \Phi'$, *we have* $v = \sum_{i=1}^{\infty} v[h_i^m] h_i^{-m}$, *where the expansion is in the strong topology of* Φ'.

To study the SDE (1.1) we need the following definition of a weak solution.

DEFINITION 2.2 A probability measure Q on $\mathbb{D}([0, T], \Phi')$ is called a *weak solution* on $[0, T]$ of the SDE (1.1) with initial distribution Q_0 on the Borel sets of Φ' if there exists a stochastic basis $(\Omega, \mathscr{F}, P, (\mathscr{F}_t))$ and a Poisson random measure N with σ-finite intensity measure μ, a Φ'-valued process X defined on it such that Q and Q_0 are the distributions of X and X_0 respectively i.e., $PX^{-1}=Q$ and $PX_0^{-1}=Q_0$. Further, for any $\phi\in\Phi$, $t\in[0,T]$, we have

$$X_t[\phi]=X_0[\phi]+\int_0^t A(s,X_s)[\phi]\,ds+\int_0^t\int_U G(s,X_{s-},u)[\phi]\tilde{N}(duds)\quad \text{P-a.s.} \tag{2.5}$$

If $[0,T]$ can be changed to $[0,\infty]$ and (2.5) hold for any $t\geqslant 0$, then we call Q on $\mathbb{D}([0,\infty],\Phi')$ a *weak solution* of SDE (1.1).

To show the existence of a weak solution of (1.1), we impose the following assumption (I) for (A,G,μ): $\forall T>0,\exists p_0=p_0(T)\in\mathbb{N}^+$, such that, $\forall p\geqslant p_0,\exists q\geqslant p$ and a constant $K=K(p,q,T)$ such that

(I1) (Continuity) $\forall t\in[0,T], A(t,\cdot)$: $H_{-p}\to H_{-q}$ is continuous; $\forall t\in[0,T]$ and $v\in H_{-p}, G(t,v,\cdot)\in L^2(U,\mu;H_{-p})$ and, for t fixed, the map $v\to G(t,v,\cdot)$ is continuous from H_{-p} to $L^2(U,\mu;H_{-p})$.

(I2) (Coercivity) $\forall t\in[0,T]$ and $\phi\in\Phi$,

$$2A(t,\phi)[\theta_p(\phi)]\leqslant K(1+\|\phi\|_{-p}^2); \tag{2.6}$$

(13) (Growth) $\forall t\in[0,T]$ and $v\in H_{-p}$,

$$\|A(t,v)\|_{-q}^2\leqslant K(1+\|v\|_{-p}^2)\quad\text{and}\quad\int_U\|G(t,v,u)\|_{-p}^2\mu(du)\leqslant K(1+\|v\|_{-p}^2). \tag{2.7}$$

Now, let $T>0$ be fixed, we consider the limit behaviour of a sequence of SDE's of the form of (1.1) on $[0,T]$:

$$X_t^n=X_0^n+\int_0^t A^n(s,X_s^n)\,ds+\int_0^t\int_U G^n(s,X_{s-}^n,u)\tilde{N}^n(duds) \tag{2.8}$$

under the following conditions:

(A1) (1°) The assumption (I) is satisfied by (A^n,G^n,μ^n) for each n. Furthermore, the continuity in (I1) is uniform in n, the indexes p,q,p_0 and the constant K in (I) are independent of n.

(2°) For each $n\geqslant 1$, the SDE (2.8) has a weak solution Q^n on $[0,T]$ with initial distribution Q_0^n. Let X^n be a Φ'-valued process on a stochastic basis $(\Omega^n,\mathscr{F}^n,P^n,(\mathscr{F}_t^n))$ corresponding to the weak solution Q^n. We further assume that there exists an index $p=p(T)\geqslant p_0(T)$ and a constant $\tilde{K}>0$ independent of n such that $X^n_\cdot(\omega^n)\in\mathbb{D}([0,T],H_{-p(T)})$ P^n-almost surely and

$$E^{P^n}\sup_{0\leqslant t\leqslant T}\|X_t^n\|_{-p(T)}^2\leqslant\tilde{K}. \tag{2.9}$$

(A2) (1°) $\mu^n=\mu$;

(2°) $\forall t\in[0,T), v\in H_{-p}$ and $\phi\in\Phi$, we have $A^n(t,v)[\phi]\to A(t,v)[\phi]$;

(3°) $\forall t\in[0,T], v\in H_{-p}, u\in U, \int_U\|G^n(t,v,u)-G(t,v,u)\|_{-p}^2\mu(du)\to 0$;

(4°) $\{Q_0^n\}$ converges to Q_0 weakly.

It follows from the assumption (A1) that Q^n are supported on $\mathbb{D}([0,T],H_{-p(T)})$. Let $p_1(T)\geqslant p(T)$ be an index such that the canonical injection from $H_{-p(T)}$ into $H_{-p_1(T)}$ is a Hilbert-Schmidt operator. As $H_{-p(T)}\subset H_{-p_1(T)}$, Q^n can be regarded as probability measures on $\mathbb{D}([0,T),H_{-p(T)})$.

LEMMA 2.1 *Under the assumption* (A1), $\{Q^n\}$ *is tight in* $\mathbb{D}([0,T],H_{-p_1(T)})$.

Proof For any $\phi\in\Phi$, let

$$C_t^n=\int_0^t A^n(s,X_s^n)[\phi]\,ds \quad\text{and}\quad M_t^n=\int_0^t\int_U G^n(s,X_{s-}^n,u)[\phi]\tilde{N}^n(duds). \tag{2.10}$$

Note that, $\forall\varepsilon>0,\exists\delta_0,\forall 0<\delta<\delta_0$, we have

$$\begin{aligned}
&\sup_n P^n\Big(\sup_{0<\beta-\alpha<\delta}|C_\alpha^n-C_\beta^n|>\varepsilon\Big)\\
&=\sup_n P^n\Big(\sup_{0<\beta-\alpha<\delta}\Big|\int_\alpha^\beta A^n(s,X_s^n)[\phi]\,ds\Big|>\varepsilon\Big)\\
&\leqslant\sup_n\frac{1}{\varepsilon^2}E^{P^n}\Big(\delta^2\sup_{0<s<T}|A^n(s,X_s^n)[\phi]|^2\Big)\\
&\leqslant\sup_n\Big(\frac{\delta}{\varepsilon}\Big)^2E^{P^n}\Big(K\Big(1+\sup_{0\leqslant s\leqslant T}\|X_s^n\|_{-p(T)}^2\Big)\|\phi\|_{q(T)}^2\Big)\\
&\leqslant K\delta^2\|\phi\|_{q(T)}^2(1+\tilde{K})/\varepsilon^2<\varepsilon.
\end{aligned} \tag{2.11}$$

i.e. $\{C^n\}$ is $\mathbb{C}$-tight. Similarly we can prove the $\mathbb{C}$-tightness for $\{\langle M^n\rangle\}$. Hence, it follows from ([8], p. 317, Corollary 3.33 and p. 322, Theorem 4.13) and the assumption (A2)(4°) that, $\forall\phi\in\Phi$, the sequence of semimartingales $X_t^n[\phi]=X_0^n[\phi]+C_t^n+M_t^n$ is tight in $\mathbb{D}([0,T],\mathbb{R})$. So, it follows from Mitoma's argument ([17]) that $\{Q^n\}$ is tight in $\mathbb{D}([0,T],\Phi')$.

Making use of the assumption (A1)(2°) and by the same arguments as in (2.11) we have that, $\forall\varepsilon>0,\rho>0,\exists\delta>0$ such that, for any $n\geqslant1$, $\phi\in\Phi$, then $\|\delta\|_{p(T)}\leqslant\delta$, implies

$$Q^n\Big\{Z\in\mathbb{D}([0,T],\Phi'):\sup_{0\leqslant t\leqslant T}|Z_t[\phi]|>\varepsilon\Big\}\leqslant\rho. \tag{2.12}$$

i.e. $\{Q^n\}$ is uniformly $p(T)$-continuous (see [17]) and hence, $\{Q^n\}$ is tight in $\mathbb{D}([0,T],H_{-p_1(T)})$.

Let Q^* be a cluster point of $\{Q^n\}$ in $\mathbb{D}([0,T],H_{-p_1(T)})$. To characterize Q^*, we neea a connecting idea which is the martingale problem formulated below. Let

$$\mathscr{D}_0^\infty(\Phi')=\{F:\Phi'\to\mathbb{R}/\exists h\in C_0^\infty(\mathbb{R})\quad\text{and}\quad\phi\in\Phi\text{ s.t. }F(v)=h(v(\phi))\} \tag{2.13}$$

and, for $F\in\mathscr{D}_0^\infty(\Phi')$, consider the operator $\mathscr{L}_s F:\Phi'\to\mathbb{R}$ defined by

$$\mathscr{L}_s F(v)=A(s,v)[\phi]h'(v[\phi])+\int_U\{h(v[\phi]+G(s,v,u)[\phi]) - h(v[\phi])-G(s,v,u)[\phi]h'(v[\phi])\}\mu(du). \tag{2.14}$$

For $Z\in\mathbb{D}([0,T],\Phi')$, let

$$M^F(Z)_t=F(Z(t))-F(Z(0))-\int_0^t\mathscr{L}_sF(Z(s))\,ds. \tag{2.15}$$

DEFINITION 2.3 A probability measure Q on $\mathbb{D}([0,T],\Phi')$ is called a solution on $[0,T]$ of the $\mathscr{L}$-martingale problem with initial distribution Q_0 if, $\forall F\in\mathscr{D}_0^\infty(\Phi')$, $\{M^F(Z)_t, 0\leqslant t\leqslant T\}$ is a Q-martingale and $Q\circ Z(0)^{-1}=Q_0$. If Q is a probability measure on $\mathbb{D}([0,\infty),\Phi')$ such that $\forall F\in\mathscr{D}_0^\infty(\Phi')$, $\{M^F(Z)_t, 0\leqslant t\leqslant\infty\}$ is a Q-martingale and $Q\circ Z(0)^{-1}=Q_0$, we call Q a solution of the $\mathscr{L}$-martingale problem with initial distribution Q_0.

Now, we proceed to prove that $\{M^F(Z)_t, 0\leqslant t\leqslant T\}$ is a Q^*-martingale for every $F\in\mathscr{D}_0^\infty(\Phi')$. Let $M_n^F(Z)_t$ be defined similarly. From the assumption (A1), it is easy to see that $\{M_n^F(Z)_t, 0\leqslant t\leqslant T\}$ is a Q^n-martingale. To pass to the limit, we need the following Lemmas.

LEMMA 2.2 *Under assumption* (A1), *we have*

$$E^{Q^n}|M_n^F(Z)_t|^2\leqslant\|h'\|_\infty^2K\|\phi\|_{p(T)}^2(\tilde{K}+1)T, \forall F\in\mathscr{D}_0^\infty(\Phi') \tag{2.16}$$

where $\|h'\|_\infty=\sup_{x\in\mathbb{R}}|h'(x)|$.

Proof Applying Itô's formula ([8], p. 57, Theorem 4.57) to (2.5), we have

$$h(X_t^n[\phi])-h(X_0^n[\phi])-\int_0^t\mathscr{L}_s^nF(X_s^n)\,ds = \int_0^t\int_U\{h(X_{s-}^n[\phi]+G^n(s,X_{s-}^n,u)[\phi])-h(X_{s-}^n[\phi])\}\tilde{N}^n(du\,ds). \tag{2.17}$$

Hence

$$\begin{aligned}E^{Q^n}|M_n^F(Z)_t|^2&=E^{P^n}\int_0^t\int_U|h(X_{s-}^n[\phi]+G^n(s,X_s^n,u)[\phi])-h(X_{s-}^n[\phi])|^2\mu^n(du)\,ds\\&\leqslant\|h'\|_\infty^2E^{P^n}\int_0^t\int_U|G^n(s,X_s^n,u)[\phi]|^2\mu^n(du)\,ds\\&\leqslant\|h'\|_\infty^2E^{P^n}\int_0^t\int_U\|G^n(s,X_{s-}^n,u)\|_{-p(T)}^2\|\phi\|_{p(T)}^2\mu^n(du)\,ds\\&\leqslant\|h'\|_\infty^2\|\phi\|_{p(T)}^2(\tilde{K}+1)T.\end{aligned} \tag{2.18}$$

LEMMA 2.3 *Under assumption* (A1), *we have*

$$E^{Q^*} \sup_{0 \leq t \leq T} \| Z_t \|^2_{-p_1(T)} \leq \tilde{K}. \tag{2.19}$$

Proof As Q^* is a cluster point of $\{Q^n\}$, without loss of generality, we may assume that Q^n converges to Q^* weakly. By Skorohod's Theorem ([6], p. 9, Theorem 2.7), there exists a probability space $(\Omega, \mathscr{F}, P)$ and $\mathbb{D}([0,T], H_{-p_1(T)})$-valued random variables ξ^n and ξ on it, such that ξ^n and ξ have distributions Q^n and Q^* respectively, and ξ^n converges to ξ almost surely. It follows from (A1) that

$$E \sup_{0 \leq t \leq T} \| \xi^n_t \|^2_{-p_1(T)} \leq E \sup_{0 \leq t \leq T} \| \xi^n_t \|^2_{-p(T)} \leq \tilde{K}. \tag{2.20}$$

Let $n \to \infty$, using Fatou's Lemma, we have

$$E^{Q^*} \sup_{0 \leq t \leq T} \| Z_t \|^2_{-p_1(T)} = E_0 \sup_{0 \leq t \leq T} \| \xi_t \|^2_{-p_1(T)}$$

$$= E \lim_{n \to \infty} \sup_{0 \leq t \leq T} \| \xi^n_t \|^2_{-p_1(T)} \leq \varliminf_{n \to \infty} E_0 \sup_{0 \leq t \leq T} \| \xi^n_t \|^2_{-p_1(T)} \leq \tilde{K}. \tag{2.21}$$

The following Lemma 2.4–Lemma 2.6 are elementary and we leave their proofs to the reader.

LEMMA 2.4 *Let C be a compact subset of $\mathbb{D}([0,T], H_{-p_1(T)})$. Then, there exists a compact subset C_0 of $H_{-p_1(T)}$ such that*

$$C \subset \{ Z \in \mathbb{D}([0,T], H_{-p_1(T)}) : Z_s \in C_0 \text{ for } s \in [0,T] \}. \tag{2.22}$$

LEMMA 2.5 *For $h \in C_0^\infty(\mathbb{R})$, let*

$$H(x,y) = h(x+y) - h(x) - h'(x)y, \quad \forall x, y \in \mathbb{R}. \tag{2.23}$$

Then, for any x, y, x_1, x_2, y_1 and $y_2 \in \mathbb{R}$, we have the following inequalities:

$$|H(x,y)| \leq \| h'' \|_\infty y^2; \tag{2.24}$$

$$|H(x_1, y) - H(x_2, y)| \leq \| h''' \|_\infty y^2 |x_1 - x_2|; \tag{2.25}$$

$$|H(x, y_1) - H(x, y_2)| \leq \| h'' \|_\infty (|y_1| + |y_2|)|y_1 - y_2|. \tag{2.26}$$

LEMMA 2.6 *Let C_0 be a compact subset of $H_{-p_1(T)}$. Under the assumption* (A2), *we have*

$$\sup_{v \in C_0} \| A^n(s,v) - A(s,v) \|_{-q_1(T)} \to 0; \tag{2.27}$$

$$\sup_{v \in C_0} \int_U \| G^n(s,v,u) - G(s,v,u) \|^2_{-p_1(T)} \mu(du) \to 0 \tag{2.28}$$

where $q_1(T)$ is obtained from $p_1(T)$ from assumption (I).

The following Lemma is the major step in passing to the limit.

LEMMA 2.7 *Suppose* (A, G, μ) *satisfies assumption* (I) *and* $\{(A^n, G^n, \mu^n)\}$ *satisfies the assumptions* (A1) *and* (A2). *Let* ξ^n *and* ξ *be* $\mathbb{D}([0,T], H_{-p_1(T)})$*-valued random variables on a probability space* $(\Omega, \mathscr{F}, P)$ *such that* ξ^n *converges to* ξ *almost surely.*

Then, for $F \in \mathscr{D}_0^\infty(\Phi')$ and $t \in [0,T] \setminus \mathscr{N}$, $M_n^F(\xi^n)_t$ converges to $M^F(\xi)_t$ in probability, where $\mathscr{N} = \{t: P(\omega: \xi_t \neq \xi_{t-}) > 0\}$.

Proof As ξ^n converges to ξ, then, for any $\varepsilon > 0$, there exists a compact subset C of $\mathbb{D}([0,T], H_{-p_1(T)})$ such that

$$P(\omega: \xi^n \in C) > 1 - \varepsilon \quad \text{and} \quad P(\omega: \xi \in C) > 1 - \varepsilon. \tag{2.29}$$

Let C_0 be the compact subset of $H_{-p_1(T)}$ given by Lemma 2.4 and let $M > 0$ be such that

$$C_0 \subset \{x \in H_{-p_1(T)}: \|x\|_{-p_1(T)} \leqslant M\}. \tag{2.30}$$

For $F \in \mathscr{D}_0^\infty(\Phi')$, there exist $h \in C_0^\infty(\mathbb{R})$ and $\phi \in \Phi$ such that $F(v) = h(v[\phi])$ for any $v \in \Phi'$. By the definition of $M_n^F(Z)_t$ and $M^F(Z)_t$, for ω such that $\xi^n(\omega)$ and $\xi(\omega) \in C$, we have (suppressing ω for convenience)

$$\begin{aligned}
&|M_n^F(\xi^n)_t - M^F(\xi)_t| \\
&\leqslant |h(\xi_t^n[\phi]) - h(\xi_t[\phi]) - h(\xi_0^n[\phi]) + h(\xi_0[\phi])| \\
&\quad + \int_0^t |A^n(s, \xi_s^n)[\phi] h'(\xi_s^n[\phi]) - A(s, \xi_s)[\phi] h'(\xi_s[\phi])| \, ds \\
&\quad + \int_0^t \int_U |H(\xi_s^n[\phi], G^n(s, \xi_s^n, u)[\phi]) - H(\xi_s[\phi], G(s, \xi_s, u)[\phi])| \mu(du) \, ds \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{2.31}$$

Note that

$$\begin{aligned}
I_3 \leqslant &\int_0^t \int_U |H(\xi_s^n[\phi], G^n(s, \xi_s^n, u)[\phi]) - H(\xi_s[\phi], G^n(s, \xi_s^n, u)[\phi])| \mu(du) \, ds \\
&+ \int_0^t \int_U |H(\xi_s[\phi], G^n(s, \xi_s^n, u)[\phi]) - H(\xi_s[\phi], G(s, \xi_s^n, u)[\phi])| \mu(du) \, ds \\
&+ \int_0^t \int_U |H(\xi_s[\phi], G(s, \xi_s^n, u)[\phi]) - H(\xi_s[\phi], G(s, \xi_s, u)[\phi])| \mu(du) \, ds
\end{aligned}$$

$$\leq \int_0^t \int_U \|h'''\|_\infty |G^n(s,\xi_s^n,u)[\phi]|^2 |\xi_s^n[\phi] - \xi_s[\phi]| \mu(du)\,ds$$

$$+ \int_0^t \int_U \|h''\|_\infty (|G^n(s,\xi_s^n,u)[\phi]| + |G(s,\xi_s^n,u)[\phi]|) |G^n(s,\xi_s^n,u)[\phi]$$

$$- G(s,\xi_s^n,u)[\phi]| \mu(du)\,ds + \int_0^t \int_U \|h'''\|_\infty (|G(s,\xi_s^n,u)[\phi]|$$

$$+ |G(s,\xi_s,u)[\phi]|) |G(s,\xi_s^n,u)[\phi] - G(s,\xi_s,u)[\phi]| \mu(du)\,ds$$

$$= I_{31} + I_{32} + I_{33}, \quad \text{say,} \tag{2.32}$$

where the second inequality follows from (2.25) and (2.26). For ω such that $\xi^n(\omega)$ and $\xi(\omega) \in C$, we have (again suppressing ω),

$$I_{31} \leq \|h'''\|_\infty K(1+M^2) \|\phi\|_{p_1}^2 \int_0^t |\xi_s^n[\phi] - \xi_s[\phi]| ds \to 0, \quad \text{a.s.;} \tag{2.33}$$

$$I_{32}^2 \leq \|h''\|_\infty \int_0^t \int_U (|G^n(s,\xi_s^n,u)[\phi]| + |G(s,\xi_s^n,u)[\phi]|)^2 \mu(du)\,ds$$

$$\int_0^t \int_U |G^n(s,\xi_s^n,u)[\phi] - G(s,\xi_s^n,u)[\phi]|^2 \mu(du)\,ds$$

$$\leq \|h''\|_\infty 4KT(1+M^2) \|\phi\|_{p_1}^4$$

$$\int_0^t \sup_{v \in C_0} \int_U \|G^n(s,v,u) - G(s,v,u)\|_{-p_1(T)}^2 \mu(du)\,ds \to 0; \tag{2.34}$$

and

$$I_{33}^2 \leq \|h''\|_\infty 4KT(1+M^2) \|\phi\|_{p_1}^4 \int_0^t \int_U \|G(s,\xi_s^n,u)$$

$$- G(s,\xi_s,u)\|_{-p_1(T)}^2 \mu(du)\,ds \to 0. \tag{2.35}$$

Hence, for ω such that $\xi^n(\omega)$ and $\xi(\omega) \in C$, we have $I_3 \to 0$. The some arguments yield that $I_2 \to 0$. It is easy to see that, $t \notin \mathcal{N}$, we have that $I_1 \to 0$ almost surely. So, combining with (2.29), we see that, for $t \notin \mathcal{N}$, $M_n^F(\xi^n)_t$ converges to $M^F(\xi)_t$ in probability.

The following Theorem characterizes Q^*.

THEOREM 2.1 *Suppose* (A, G, μ) *satisfies the assumption* (I) *and* $\{(A^n, G^n, \mu^n)\}$ *satisfies the assumptions* (A1) *and* (A2). *Then* Q^* *is a solution on* $[0, T]$ *of the* $\mathcal{L}$*-martingale problem.*

Proof Let ξ_n and ξ be as given in the proof of the Lemma 2.3. By Lemma 2.2, for fixed t, we can easily see that $\{M_n^F(\xi^n)_t\}_{n\in\mathbb{N}}$ are uniformly integrable. Hence, for any bounded continuous $\mathscr{B}_s$-measurable function f on $\mathbb{D}([0,T],H_{-p_1(T)})$, we have that $\{f(\xi^n)M_n^F(\xi^n)_t\}_{n\in\mathbb{N}}$ are uniformly integrable, So, by Lemma 2.7, for $t, s\notin\mathscr{N}$ and $s<t$, we have

$$E^{Q^*}M^F(Z)_t f(Z) = EM^F(\xi)_t f(\xi) = \lim_n EM_n^F(\xi^n)_t f(\xi^n)$$

$$= \lim_n E^{Q^n}M_n^F(Z)_t f(Z) = \lim_n E^{Q^n}M_n^F(Z)_s f(Z) = \lim_n EM_n^F(\xi^n)_s f(\xi^n)$$

$$= EM^F(\xi)_s f(\xi) = E^{Q^*}M^F(Z)_s f(Z). \tag{2.36}$$

i.e.

$$E^{Q^*}M^F(Z)_t f(Z) = E^{Q^*}M^F(Z)_s f(Z). \tag{2.37}$$

For general $s<t$, as $\mathscr{N}$ is at most countable, we can find two sequences s_n and t_n decreasing to s and t respectively and such that $s_n<t_n$. Then, (2.37) still holds with (s,t) replaced by (s_n,t_n) as f is also $\mathscr{B}_{s_n}$-measurable. By the right continuity and the uniform integrability of $M^F(Z)_{t_n}f(Z)$ and $M^F(Z)_{s_n}f(Z)$, passing to the limit, we see that (2.37) still holds for any $t>s$. Define two signed measures on $\mathscr{B}_s$ by

$$\mathscr{V}_t(A) = E^{Q^*}M^F(Z)t\,1_A(Z) \quad \text{and} \quad \mathscr{V}_s(A) = E^{Q^*}M^F(Z)s\,1_A(Z). \tag{2.38}$$

Then, from above, we see that the integrals of f with respect to signed measures $\mathscr{V}_t$ and $\mathscr{V}_s$ coincide for any bounded continuous $\mathscr{B}_s$-measurable functions f. Hence $\mathscr{V}_t=\mathscr{V}_s$ on $\mathscr{B}_s$. i.e. $\{M^F(Z)_t\}$ is a Q^*-martingale.

It remains to prove Q^* is a weak solution on $[0,T]$ of the SDE (1.1). The idea is to show that the martingale $M_\phi(t,Z)$, defined to Lemma 2.9 below, can be represented as a stochastic integral with respect to a Poisson random measure. We do this by proving that $M_\phi(t,Z)$ is purely-discontinuous in Theorem 2.2 and characterizing the jump process $\Delta M_\phi(t,Z)$ in Lemma 2.11.

The proof of the following Lemma is left to the reader.

LEMMA 2.8 *There exist two sequences of real functions $\{\rho_m\}$, $\{g_m\}$ on $\mathbb{R}$ and a constant L such that, $\forall m\in\mathbb{N}$, $\rho_m\in C_0^\infty(\mathbb{R})$ and*

(1) *$\rho_m(x)=x$ when $|x|\leqslant m-1$ and $|\rho_m(x)|\leqslant L|x|$ for any $x\in\mathbb{R}$;*
(2) *$\|\rho_m'\|_\infty\leqslant L$, $\|\rho_m''\|_\infty\leqslant L/m$, and $\|\rho_m\rho_m''\|_\infty\leqslant L$;*
(3) *$g_m(x)$ are nonnegative smooth functions that increase to $|x|$ as m tends to ∞.*

Furthermore, there exist two sequences of positive numbers $\{d_m\}$ and $\{D_m\}$ such that $g_m(x)=0$ when $|x|\leqslant d_m$ or $|x|\geqslant D_m$.

LEMMA 2.9 *$\forall\phi\in\Phi$, let*

$$M_\phi(t,Z) = Z_t[\phi] - Z_0[\phi] - \int_0^t A(s,Z_s)[\phi]\,ds. \tag{2.39}$$

Under the conditions of Theorem 2.1, $\{M_\phi(t,Z)\}_{t\leqslant T}$ is a Q^*-square integrable martingale.

Proof Let ρ_m be given by Lemma 2.8. Let $F_m \in \mathscr{D}_0^\infty(\Phi')$ be given by $F_m(v)=\rho_m(v[\phi])$. Then, for $Z\in A^{p_1(T)}_{(m-1)\|\phi\|^{-1}_{p_1(T)}}$, we have $|Z_s[\phi]|\leqslant m-1$ and hence,

$$M^{F_m}(Z)_t = M_\phi(t,Z) - \int_0^t \int_U H_m(Z_s[\phi], G(s,Z_s,u)[\phi])\mu(du)\,ds, \qquad (2.40)$$

where H_m is defined as in Lemma 2.5 with h replaced by ρ_m and $A^{p_1(T)}_{(m-1)\|\phi\|^{-1}_{p_1(T)}}$, denoted by $A^{p_1(T)}$, is given by (2.4). Hence, by (2.40), Lemma 2.5, assumption (13) and (2.19), we have

$$\begin{aligned} E^{Q^*}|M^{F_m}(Z)_t - M_\phi(t,Z)1_{A^{p_1(T)}}(Z) &\leqslant E^{Q^*}\int_0^t\int_U \|\rho''_m\|_\infty |G(s,Z_s,u)[\phi]|^2\mu(du)\,ds \\ &\leqslant (L/m)tK(1+\tilde{K})\|\phi\|^2_{p_1(T)} \to 0 \quad \text{as} \quad m\to\infty. \qquad (2.41)\end{aligned}$$

On the other hand,

$$\begin{aligned} &Q^*((A^{p_1(T)})^c) \\ &\quad = Q^*(Z\in\mathbb{D}([0,T],H_{-p_1(T)}): \sup_{0\leqslant t\leqslant T}\|Z_t\|_{-p_1(T)} \geqslant (m-1)\|\phi\|^{-1}_{p_1(T)}) \\ &\quad \leqslant \frac{1}{(m-1)^2\|\phi\|^{-2}_{p_1(T)}} E^{Q^*}\sup_{0\leqslant t\leqslant T}\|Z_t\|^2_{-p_1(T)} \leqslant \frac{\|\phi\|^2_{p_1(T)}}{(m-1)^2}\tilde{K}\to 0, \quad \text{as} \quad m\to\infty. \qquad (2.42)\end{aligned}$$

So, $\forall\varepsilon>0$, we have

$$\begin{aligned} &Q^*(Z\in\mathbb{D}([0,T],H_{-p_1(T)}): |M^{F_m}(Z)_t - M_\phi(t,Z)|>\varepsilon) \\ &\quad \leqslant Q^*((A^{p_1(T)})^c) + (1/\varepsilon)E^{Q^*}|M^{F_m}(Z)_t - M_\phi(t,Z)1_{A^{p_1(T)}}(Z)\to 0. \qquad (2.43)\end{aligned}$$

i.e.

$$M^{F_m}(Z)_t \to M_\phi(t,Z) \text{ in } Q^* \text{ probability}. \qquad (2.44)$$

Next, by assumption (I) and the properties of ρ_m, it is easy to show that there exists a constant C' independent of m such that

$$M^{F_m}(Z)_t| \leqslant C'\left(1+\sup_{0\leqslant t\leqslant T}\|Z_t\|^2_{-p_1(T)}\right). \qquad (2.45)$$

Hence, by Lemma 2.3, the left hand side of (2.45) is integrable with respect to Q^* uniformly in m. Then, by (2.44),

$$E^{Q^*}|M^{F_m}(Z)_t - M_\phi(t,Z)|\to 0. \qquad (2.46)$$

But $\{M^{F_m}(Z)_t\}$ are Q^*-martingales, so $\{M_\phi(t,Z)\}$ is a Q^*-martingale. Finally, by assumption (I), it is easy to see that there exists a constant C'' such that

$$|M_\phi(t,Z)|^2 \leqslant C''\left(1+\sup_{0\leqslant t\leqslant T}\|Z_t\|^2_{-p_1(T)}\right). \tag{2.47}$$

Hence, by Lemma 2.3, $\{M_\phi(t,Z)\}$ is a Q^*-square-integrable-martingale.

LEMMA 2.10 *Let $\langle M_\phi\rangle(t,Z)$ be the quadratic characteristic of the square integrable martingale M_ϕ. Under the conditions of Theorem 2.1, we have*

$$\langle M_\phi\rangle(t,Z)=\int_0^t\int_U (G(s,Z_s,u)[\phi])^2\,\mu(du)\,ds. \tag{2.48}$$

Proof $\forall\phi\in\Phi$, let

$$N_\phi(t,Z)=Z_t[\phi]^2-Z_0[\phi]^2-2\int_0^t A(s,Z_s)[\phi]Z_s[\phi]\,ds$$
$$-\int_0^t\int_U (G(s,Z_s,u)[\phi])^2\,\mu(du)\,ds. \tag{2.49}$$

Then, by a similar argument as in the proof of Lemma 2.9, $\{N_\phi(t,Z)\}_{t\leqslant T}$ is a Q^*-martingale. By the definition of M_ϕ, it can easily be seen that

$$\Delta Z_s[\phi]=\Delta M_\phi(s,Z)\quad\text{and}\quad \langle M_\phi^c\rangle_t=\langle Z[\phi]^c\rangle_t, \tag{2.50}$$

where M_ϕ^c and $Z[\phi]^c$ are the continuous parts of the semimartingales M_ϕ and $Z[\phi]$ respectively. By Theorem 4.52 of ([8], p55), we get

$$[Z[\phi]]_t=\sum_{s\leqslant t}(\Delta Z_s[\phi])^2+\langle Z[\phi]^c\rangle_t=[M_\phi]_t, \tag{2.51}$$

where $[Z[\phi]]$ and $[M_\phi]$ are the quadratic variation processes of the semimartingales $Z[\phi]$ and M_ϕ respectively. By (2.50) and (2.51), it is easy to show that

$$Z_t[\phi]^2=Z_0[\phi]^2+2\int_0^t A(s,Z_s)[\phi]Z_s[\phi]\,ds$$
$$+2\int_0^t Z_{s-}[\phi]\,dM_\phi(s)+[Z[\phi]]_t. \tag{2.52}$$

Hence, by the definition of $N_\phi(t,Z)$ and (2.52), we have

$$N_\phi(t,Z)=2\int_0^t Z_{s-}[\phi]\,dM_\phi(s)+[Z[\phi]]_t-\int_0^t\int_U (G(s,Z_s,u)[\phi])^2\,\mu(du)\,ds$$
$$=2\int_0^t Z_{s-}[\phi]\,dM_\phi(s)+[M_\phi]_t-\int_0^t\int_U (G(s,Z_s,u)[\phi])^2\,\mu(du)\,ds. \tag{2.53}$$

Hence

$$\langle M_\phi\rangle(t,Z)-\int_0^t\int_U (G(s,Z_s,u)[\phi])^2\mu(du)\,ds$$

$$=(\langle M_\phi\rangle(t,Z)-[M_\phi]_t)+N_\phi(t,Z)-2\int_0^t Z_{s-}[\phi]\,dM_\phi(s) \qquad (2.54)$$

is a local martingale as all three terms on the right hand side of (2.54) are local martingales. On the other hand, it easy to see from the left hand side of (2.54) that it is a predictable process which has finite variation on any finite time interval. Hence, by Corollary 3.16 of ([8], p32), we have (2.48).

THEOREM 2.2 *Under the conditions of Theorem* 2.1, $M_\phi(t,Z)$ *is purely-discontinuous.*

Proof Let $C_0^1(\mathbb{R})$ be non-negative and such that $g(x)=0$ when $|x|\leqslant a$ for some $a>0$. Let Y^n and F^n be functionals defined on $\mathbb{D}([0,T],H_{-p_1(T)})$ by

$$Y^n(Z)=\int_0^t\int_U g((G^n(s,Z_s,u)[\phi])^2)\mu(du)\,ds \qquad (2.55)$$

and

$$F^n(Z)=\sum_{0<s\leqslant t} g((\Delta Z_s[\phi])^2)-Y^n(Z) \qquad (2.56)$$

Similarly, we define functionals Y and F on $\mathbb{D}([0,T],H_{-p_1(T)})$. Let ξ^n and ξ be as given in the proof of Lemma 2.3. By the same arguments as in the proof of Lemma 2.7 it follows that $Y^n(\xi^n)$ converges to $Y(\xi)$ in probability. As

$$\sum_{0<s\leqslant t} g((\Delta\xi_s^n[\phi])^2)\to\sum_{0<s\leqslant t} g((\Delta\xi_s[\phi])^2) \quad \text{a.s.}, \qquad (2.57)$$

$F^n(\xi^n)$ converges to $F(\xi)$ in probability.

On the other hand, from

$$X_t^n[\phi]=X_0^n[\phi]+\int_0^t A^n(s,X_s^n)[\phi]\,ds$$

$$+\int_0^t\int_U G^n(s,X_{s-}^n,u)[\phi]\tilde{N}^n(du\,ds) \qquad (2.58)$$

we have

$$\Delta X_s^n[\phi]=G^n(s,X_{s-}^n,p^n(s))[\phi]1_{D^n}(s) \qquad (2.59)$$

where $p^n(\cdot), D^n$ are the point processes and jumping sets corresponding to the Poisson random measures N^n. Hence

$$\begin{aligned}\sum_{0<s\leqslant t} g((\Delta X_s^n[\phi])^2) &= \sum_{0<s\leqslant t} g((G^n(s, X_{s-}^n, p^n(s))[\phi]1_{D^n}(s))^2)\\ &= \sum_{0<s\leqslant t} g((G^n(s, X_{s-}^n, p^n(s))[\phi])^2)1_{D^n}(s)\\ &= \int_0^t\int_U g((G^n(s, X_{s-}^n, u)[\phi])^2)N^n(duds).\end{aligned} \tag{2.60}$$

So

$$F^n(X^n) = \int_0^t\int_U g((G^n(s, X_{s-}^n, u)[\phi])^2)\tilde{N}^n(duds). \tag{2.61}$$

Hence

$$E(F^n(\xi^n)) = E^{P^n}(F^n(X^n)) = 0 \tag{2.62}$$

and

$$\begin{aligned}E(F^n(\xi^n))^2 = E^{P^n}(F^n(X^n))^2 &= E^{P^n}\int_0^t\int_U g^2((G^n(s, X_{s-}^n, u)[\phi])^2)\mu(du)ds\\ &\leqslant E^{P^n}\int_0^t\int_U K_g(G^n(s, X_s^n, u)[\phi])^2\mu(du)ds\\ &\leqslant K_gE^{P^n}\int_0^t\int_U \|G^n(s, X_s^n, u)\|_{-p_1(T)}^2\|\phi\|_{p_1(T)}^2\mu(du)ds\\ &\leqslant K_g\|\phi\|_{p_1(T)}^2K(1+\tilde{K})T,\end{aligned} \tag{2.63}$$

where K_g is a constant such that $|g^2(x)| \leqslant K_g|x|$. So, $\{F^n(\xi^n)\}$ are uniformly integrable and, passing to the limit, we have $E(F(\xi)) = 0$. i.e.

$$E\sum_{0<s\leqslant t} g((\Delta\xi_s[\phi])^2) = E\int_0^t\int_U g((G(s, \xi_s, u)[\phi])^2)\mu(du)ds. \tag{2.64}$$

So

$$E^{Q^*}\sum_{0<s\leqslant t} g((\Delta Z_s[\phi])^2) = E^{Q^*}\int_0^t\int_U g((G(s, Z_{s-}, u)[\phi])^2)\mu(du)ds. \tag{2.65}$$

Let g_m be given by Lemma 2.8, then $g_m \in C_0^1(\mathbb{R})$ non-negative and vanish in a neighborhood of 0. (2.65) still holds with g replaced by g_m. As $g_m(x) \uparrow |x|$ when $m \uparrow \infty$, we get

$$E^{Q^*} \sum_{0<s\leqslant t} (\Delta Z_s[\phi])^2 = E^{Q^*} \int_0^t \int_U (G(s, Z_s, u)[\phi])^2 \mu(du)\, ds$$

(By Monotone Convergence Theorem)

$$= E^{Q^*} \langle M_\phi \rangle (t, Z) \qquad \text{(By Lemma 2.10)}$$

$$= E^{Q^*} [M_\phi](t, Z) = E^{Q^*} [Z[\phi]]_t. \qquad \text{(By (2.51))} \quad (2.66)$$

Hence

$$E^{Q^*} \langle M_\phi^c \rangle (t, Z) = 0 \tag{2.67}$$

i.e. $\forall t$, $\langle M_\phi^c \rangle (t, Z) = 0$ a.s.. Then, by the continuity of $\langle M_\phi^c \rangle (t, Z)$ in t, we get $\langle M_\phi^c \rangle (t, Z) = 0\ \forall t$, a.s.. This proves that $\langle M_\phi \rangle (t, Z)$ is purely-discontinuous.

We next identify the compensator of the point process ΔZ_s.

LEMMA 2.11 *Let*

$$\Gamma = \left\{ A \in \mathscr{B}(H_{-p_1(T)} \setminus \{0\}) \colon E^{Q^*} \sum_{0<s\leqslant t} 1_A(\Delta Z_s) < \infty, \quad \forall 0 < t \leqslant T \right\}. \tag{2.68}$$

Then, for $A \in \Gamma$,

$$\sum_{0<s\leqslant t} 1_A(\Delta Z_s) - \int_0^t \int_U 1_A(G(s, Z_s, u)) \mu(du)\, ds \tag{2.69}$$

is a Q^*-martingale.

Proof Let h be a bounded non-negative continuous $\mathscr{B}_s$-measurable function on $\mathbb{D}([0, T], H_{-p_1(T)})$ and f on $\mathbb{R}_+$ be given by

$$f(x) = \exp\left(\frac{x}{x-1} \right) \qquad 0 \leqslant x \leqslant 1$$
$$= 0 \qquad x \geqslant 1.$$

For $0 < a < a'$, let

$$S_{a,a'} = \{ x \in H_{-p_1(T)} \colon a \leqslant |x|_{-p_1(T)} \leqslant a' \} \tag{2.70}$$

and, for any closed subset F of $H_{-p_1(T)}$ contained in $S_{a,a'}$ and $k \geqslant 3$, define

$$g_k(x) = f(k\rho(x, F)/a) \tag{2.71}$$

where $\rho(x, F)$ is the distance from x to set F in $H_{-p_1(T)}$. Then $g_k(x) \neq 0$ iff $\|x\|_{-p_1(T)} \leqslant \frac{a}{k}$. Let $\{X^n\}$, $\{\xi^n\}$ and ξ be as defined in the proof of Lemma 2.3 and $F^n_{k,t}$ be functionals on $\mathbb{D}([0, T], H_{-p_1(T)})$ defined by

$$F^n_{k,t}(Z) = \sum_{0<s\leqslant t} g_k(\Delta Z_s) - \int_0^t \int_U g_k(G^n(s, Z_s, u))\mu(du)\,ds. \tag{2.72}$$

Define the functional $F_{k,t}$ similarly. Then, for fixed k,

$$\begin{aligned}|F^n_{k,t}(\xi^n) - F_{k,t}(\xi)| \leqslant{}& \left|\sum_{0<s\leqslant t} g_k(\Delta\xi^n_s) - \sum_{0<s\leqslant t} g_k(\Delta\xi_s)\right| \\ &+ \left|\int_0^t \int_U g_k(G^n(s, \xi^n_s, u)) - g_k(G(s, \xi_s, u))\mu(du)\,ds\right|.\end{aligned} \tag{2.73}$$

The first term converges to 0 almost surely and, for the second term, let $b^n = \rho(G^n(s, \xi^n_s, u), F)$, $b = \rho(G(s, \xi_s, u), F)$ and let $\tilde{f}$ on $\mathbb{R}_+$ be defined by $\tilde{f}(t) = f(\sqrt{t})$. Then we have

$$\begin{aligned}&\left|\int_0^t \int_U g_k(G^n(s, \xi^n_s, u)) - g_k(G^n(s, \xi_s, u))\mu(ds)\right| \\ &\leqslant \left|\int_0^t \int_U (g_k(G^n(s, \xi^n_s, u)) - g_k(G(s, \xi_s, u)))1_{b^n\leqslant a/2, b\leqslant a/2}\,\mu(du)\,ds\right| \\ &\quad + \left|\int_0^t \int_U g_k(G^n(s, \xi^n_s, u))1_{b^n\leqslant a/2, b>a/2}\,\mu(du)\,ds\right| \\ &\quad + \left|\int_0^t \int_U g_k(G(s, \xi_s, u))1_{b^n>a/2, b\leqslant a/2}\,\mu(du)\,ds\right| \\ &\leqslant \|\tilde{f}'\|_\infty \left(\frac{k}{a'}\right)^2 \int_0^t \int_U |\rho(G^n(s, \xi^n_s, u), F)^2 \\ &\quad - \rho(G(s, \xi_s, u), F)^2|1_{b^n\leqslant a/2, b\leqslant a/2}\,\mu(du)\,ds \\ &\quad + \left|\int_0^t \int_U g_k(G^n(s, \xi^n_s, u))1_{b^n\leqslant a/k, b>a/2}\,\mu(du)\,ds\right| \\ &\quad + \left|\int_0^t \int_U g_k(G(s, \xi_s, u))1_{b^n>a/2, b\leqslant a/k}\,\mu(du)\,ds\right|\end{aligned}$$

$$\leqslant \|\tilde{f}'\|_\infty\left(\frac{k}{a'}\right)^2\int_0^t\int_U \|G^n(s,\xi_s^n,u)$$

$$-G(s,\xi_s,u)\|_{-p_1(T)}(b^n+b)1_{b^n\leqslant a/2,\,b\leqslant a/2}\mu(du)\,ds$$

$$+2\int_0^t \mu\left\{u:|b^n-b|>\left(\frac{1}{2}-\frac{1}{k}\right)a\right\}ds$$

$$\leqslant \|\tilde{f}'\|_\infty\left(\frac{k}{a'}\right)^2\int_0^t\int_U \|G^n(s,\xi_s^n,u)-G(s,\xi_s,u)\|_{-p_1(T)}(\|G^n(s,\xi_s^n,u)\|_{-p_1(T)}$$

$$+\|G(s,\xi_s,u)\|_{-p_1(T)})\mu(du)\,ds$$

$$+\frac{8k^2}{(k-2)^2}\int_0^t\int_U \|G^n(s,\xi_s^n,u)-G(s,\xi_s,u)\|^2_{-p_1(T)}\mu(du)\,ds \tag{2.74}$$

which converges to 0 in probability by the same arguments as in the proof of Lemma 2.7. It follows as in the proof of Theorem 2.1 that, for fixed k and t, $\{F^n_{k,t}(\xi^n)\}$ are uniformly integrable and

$$Eh(\xi^n)(F^n_{k,t}(\xi^n)-F^n_{k,s}(\xi^n))=0. \tag{2.75}$$

Let n tend to ∞, we get

$$Eh(\xi)(F^n_{k,t}(\xi)-F^n_{k,s}(\xi))=0 \tag{2.76}$$

Hence, we have

$$E^{Q^*}h(Z)\left(\sum_{s<r\leqslant t} g_k(\Delta Z_r)-\int_s^t\int_U g_k(G(r,Z_r,u))\mu(du)\,dr\right)=0. \tag{2.77}$$

Since g_k decreases to 1_F as $k\to\infty$, by the monotone convergence theorem, we have

$$E^{Q^*}h(Z)\sum_{s<r\leqslant t}1_F(\Delta Z_r)=E^{Q^*}h(Z)\int_s^t\int_U 1_F(G(r,Z_r,u))\mu(du)\,dr \tag{2.78}$$

for any closed subset F of $S_{a,a'}$. As both sides of (2.78) define two measures on $S_{a,a'}$ and coincide for all closed sets, (2.78) holds for any Borel subset of $S_{a,a'}$. Letting $a\to 0$ and $a'\to\infty$, (2.78) holds for any Borel subset of $H_{-p_1(T)}$. This proves the lemma.

THEOREM 2.3 *Under the conditions of Theorem* 2.1, Q^* *is a weak solution on* $[0, T]$ *of the* SDE (1.1).

Proof From Lemma 2.11 we know that the point process ΔZ_s has compensator

$$q(t, E, \omega) = \mu\{u: G(t, Z_{t-}, u) \in E\} \tag{2.79}$$

then, using Theorem 7.4 of ([6], p93), there exists an extension $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}, \tilde{\mathscr{F}}_t)$ of the stochastic basis

$$(\mathbb{D}([0, T], H_{-p_1(T)}), \mathscr{B}(\mathbb{D}([0, T], H_{-p_1(T)})), Q^*, \mathscr{B}_t)$$

and a stationary $\tilde{\mathscr{F}}_t$-Poisson point process N on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}, \tilde{\mathscr{F}}_t)$ with characteristic measures $\mu(du)$, such that

$$\#\{s \leqslant t: \Delta Z_s \in E\} = \int_0^t \int_U 1_E(G(s, Z_{s-}, u)) N(du ds). \tag{2.80}$$

By the definition of M_ϕ and (2.80), we have

$$\Delta M_\phi(s) = \Delta Z(s) = G(s, Z_{s-}, p(s))[\phi] 1_D(s) \tag{2.81}$$

where $p(\cdot)$, D are the point processes and jumping sets corresponding to the Poisson random measure N. But M_ϕ is a purely-discontinuous martingale, so that, by definition 1.27 of ([10], p72), we see that

$$M_\phi(t) = \int_0^t \int_U g(s, Z_{s-}, u)[\phi] \tilde{N}(du ds), \tag{2.82}$$

and hence

$$Z(t) = Z(0) + \int_0^t A(s, Z_s)\, ds + \int_0^t \int_U g(s, Z_{s-}, u) \tilde{N}(du ds). \tag{2.83}$$

3 EXISTENCE OF A WEAK SOLUTION

In this section, we use the basic results of the last section to derive the existence of a weak solution of the SDE (1.1). The idea is as follows: first, we prove the existence of the weak solution on $[0, T]$ of (1.1) when the nuclear space Φ is finite dimensional, say $\mathbb{R}^d$. Then, employing the Galerkin method, we project the coefficients of the equation (1.1) to a sequence of finite dimensional subspaces and consider the corresponding SDE on these subspaces. We get the desired existence by proving that this sequence of equations satisfies the assumptions (A1) and (A2) of §2. Applying the results to the intervals $[0, T]$, $[2T, 3T]$, ..., we get a sequence of solutions of (1.1) in these intervals and, connecting them, we obtain a solution on the interval $[0, \infty]$.

First of all, let us consider (1.1) when $\Phi = \mathbb{R}^d$. In this case, $H_p = \mathbb{R}^d$ for all p. The SDE (1.1) can be rewritten as

$$x_t = \xi + \int_0^t a(s, x_s)\,ds + \int_0^t \int_U c(s, x_{s-}, u)\tilde{N}(du\,ds) \tag{3.1}$$

where $a: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $c: \mathbb{R}_+ \times \mathbb{R}^d \times U \to \mathbb{R}^d$ are two measurable mappings, N is a Poisson random measure on $\mathbb{R}_+ \times U$ with respect to a stochastic base $(\Omega, \mathscr{F}, P, (\mathscr{F}_t))$ and ξ is a $\mathscr{F}_0$-measurable $\mathbb{R}^d$-valued random variable.

In the present setup, we make the following assumption (F): $\forall T > 0$, there exist constants K_1 and K_2 such that

(F1) (Continuity) $\forall t \in [0, T], a(t, \cdot): \mathbb{R}^d \to \mathbb{R}^d$ is continuous; $\forall t \in [0, T]$ and $x \in \mathbb{R}^d$, $c(t, x, \cdot) \in L^2(U, \mu; \mathbb{R}^d)$ and, for t fixed, the map $x \to c(t, x, \cdot)$ is $L^2(U, \mu, \mathbb{R}^d)$ continuous.
(F2) (Coercivity) $\forall t \in [0, T]$,

$$2\langle a(t, x), x\rangle \leqslant K_1(1 + |x|^2); \tag{3.2}$$

(F3) (Growth) $\forall t \in [0, T]$ and $x \in \mathbb{R}^d$,

$$|a(t, x)|^2 \leqslant K_2(1 + |x|^2) \quad \text{and} \quad \int_U |c(t, x, u)|^2 \mu(du) \leqslant K_1(1 + |x|^2) \tag{3.3}$$

where $\langle \cdot, \cdot \rangle$ and $|\cdot|$ are the inner product and norm in $\mathbb{R}^d$ respectively.

Remark 3.1 If we replace K_1 and K_2 by $K = \max(K_1, K_2)$, the assumption (F) is just a re-statement of the assumption (I) of §2 in the present setup. We distinguish K_1 and K_2 for technical reasons which will become clear later on.

Even in the finite dimensional situation, to solve the SDE (3.1), we follow [3] and assume an additional monotonicity condition (FM): There exists a constant $L > 0$, such that

$$2\langle x - y, a(t, x) - a(t, y)\rangle + \int_U |c(t, x, u) - c(t, y, u)|^2 \mu(du) \leqslant L|x - y|^2, \quad \forall x, y \in \mathscr{R}^d. \tag{3.4}$$

It is one of the major points made in this paper that the assumption (F) is needed for the existence of the solution of (3.1) and the role played by (FM) is for the uniqueness of the solution. But, to make use of the existing results, we still impose the condition (FM) to solve (3.1) and remove it in Theorem 3.1. The estimate (3.5) given below is of crucial importance for this paper.

LEMMA 3.1 *Under the assumptions* (F) *and* (FM), the SDE (3.1) *has a unique solution. Furthermore, if* $E|\xi|^2 < \infty$, *then, there exists a constant* $\tilde{K} = \tilde{K}(K_1, T, E|\xi|^2)$ *such that*

$$E \sup_{0 \leqslant t \leqslant T} |x_t|^2 \leqslant \tilde{K}(K_1, T, E|\xi|^2) < \infty. \tag{3.5}$$

Proof The existence and uniqueness of the solution of (3.1) is a special case of the Theorem 1 of ([3], p. 5). So, we only need to prove the estimate (3.5). Applying Itô's formula to (3.1), we get

$$|x_t|^2 = |\xi|^2 + 2\int_0^t \langle x_s, a(s,x_s)\rangle ds + \int_0^t \int_U |c(s,x_s,u)|^2 \mu(du)\, ds$$
$$+ \int_0^t \int_U \{|c(s,x_{s-},u)|^2 + 2\langle x_{s-}, c(s,x_{s-},u)\rangle\} \tilde{N}(duds). \tag{3.6}$$

Let $\tau_m = \inf\{t \leqslant T : |x_t| > m\}$ be a sequence of increasing stopping times. Hence, by (3.6), we have

$$|x_{t\wedge\tau_m}|^2 \leqslant |\xi|^2 + 2K_1 \int_0^{t\wedge\tau_m} (1 + |x_s|^2)\, ds$$
$$+ \int_0^{t\wedge\tau_m} \int_U \{|c(s,x_{s-},u)|^2 + 2\langle x_{s-}, c(s,x_{s-},u)\rangle\} \tilde{N}(duds). \tag{3.7}$$

Let

$$f^m(t) = E \sup_{r\leqslant t\wedge\tau_m} |x_r|^2 \quad \text{and} \quad M_t = \int_0^t \int_U \langle x_{s-}, c(s,x_{s-},u)\rangle \tilde{N}(duds), \tag{3.8}$$

then

$$f^m(t) \leqslant E|\xi|^2 + 2K_1 t + 2K_1 E\int_0^{t\wedge\tau_m} |x_s|^2\, ds + 2E \sup_{r\leqslant t\wedge\tau_m} M_r$$
$$+ E \sup_{r\leqslant t\wedge\tau_m} \left\{ \int_0^r \int_U |c(s,x_{s-},u)|^2 \tilde{N}(duds) \right\}. \tag{3.9}$$

Note that

$$E \sup_{r\leqslant t\wedge\tau_m} \left\{ \int_0^r \int_U |c(s,x_{s-},u)|^2 \tilde{N}(duds) \right\}$$
$$\leqslant E \sup_{r\leqslant t\wedge\tau_m} \left\{ \int_0^r \int_U |c(s,x_{s-},u)|^2 N(duds) + \int_0^r \int_U |c(s,x_{s-},u)|^2 \mu(du)\, ds \right\}$$
$$= 2E\int_0^{t\wedge\tau_m} \int_U |c(s,x_s,u)|^2 \mu(du)\, ds \leqslant 2K_1 t + 2K_1 E\int_0^{t\wedge\tau_m} |x_s|^2\, ds. \tag{3.10}$$

On the other hand, M, defined in (3.8) is a local martingale with quadratic variation process

$$[M]_t = \int_0^t \int_U \langle x_{s-}, c(s,x_{s-},u)\rangle^2 N(duds) \tag{3.11}$$

by the same arguments as in (2.60). It follows from the Burkholder-Davis-Gundy inequality that

$$
\begin{aligned}
2E \sup_{r \leqslant t \wedge \tau_m} M_r \leqslant 8E[M]^{1/2}_{t \wedge \tau_m} &= 8E\int_0^{t \wedge \tau_m} \int_U \langle x_s, c(s, x_s, u)\rangle^2 N(du\,ds)\}^{1/2} \\
&\leqslant 8E\left\{\int_0^{t \wedge \tau_m} \int_U |x_s|^2 |c(s, x_s, u)|^2 N(du\,ds)\right\}^{1/2} \\
&\leqslant 8E\left(\sup_{r \leqslant t \wedge \tau_m} |x_r| \left\{\int_0^{t \wedge \tau_m} \int_U |c(s, x_s, u)|^2 N(du\,ds)\right\}^{1/2}\right) \\
&\leqslant \frac{1}{2} E \sup_{r \leqslant t \wedge \tau_m} |x_r|^2 + 8E\int_0^{t \wedge \tau_m} \int_U |c(s, x_s, u)|^2 N(du\,ds) \\
&\leqslant \frac{1}{2} E \sup_{r \leqslant t \wedge \tau_m} |x_r|^2 + 8K_1 t + 8K_1 E\int_0^{t \wedge \tau_m} |x_s|^2\, ds. \qquad (3.12)
\end{aligned}
$$

Hence, by (3.9), (3.10) and (3.12), we have

$$
f^m(t) \leqslant E|\xi|^2 + 12K_1 t + 12K_1 E\int_0^{t \wedge \tau_m} |x_s|^2\, ds + \frac{1}{2} f^m(t). \qquad (3.13)
$$

Hence

$$
f^m(t) \leqslant 2\left(E|\xi|^2 + 12K_1 t + 12K_1 \int_0^t f^m(s)\, ds\right) \qquad (3.14)
$$

and so

$$
f^m(t) \leqslant 2(E|\xi|^2 + 12K_1 T) + \int_0^T 2(E|\xi|^2 + 12K_1 s) e^{12K(T-s)}\, ds = \tilde{K}(K_1, T, E|\xi|^2) < \infty. \qquad (3.15)
$$

Letting $m \to \infty$, we get our estimate.

The following Theorem yields the existence of a weak solution on $[0, T]$ of the SDE (3.1) without the monotonicity condition (FM).

THEOREM 3.1 *Under assumption* (F) *and* $E|\xi|^2 < \infty$, *the* SDE (3.1) *has a weak solution* x *on* $[0, T]$ *and*

$$
E \sup_{0 \leqslant t \leqslant T} |x_t|^2 \leqslant \tilde{K}(K_1, T, E|\xi|^2) < \infty. \qquad (3.16)
$$

Proof Let J be the Friedrichs mollifier and

$$\begin{aligned} a^n(t,x) &= \int a(t, x - n^{-1}z)J(z)\,dz \quad \text{for} \quad |x| \leqslant n \\ &= a^n(t, nx/|x|) \qquad\qquad \text{for} \quad |x| > n \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} c^n(t,x,u) &= \int c(t, x - n^{-1}z, u)J(z)\,dz \quad \text{for} \quad |x| \leqslant n \\ &= c^n(t, nx/|x|, u) \qquad\qquad \text{for} \quad |x| > n. \end{aligned} \tag{3.18}$$

It is easy to verify that, for each n, (a^n, c^n, μ) satisfies the assumptions (F) and (FM) with

$$K_1^n = 3K_1 + 4\sqrt{K_2}, \quad K_2^n = 3K_2 \tag{3.19}$$

and

$$L = 8(1 + (n+1)^2)n^2 \int J(z)\frac{z^2}{(1-|z|^2)^4}\,dz \max(K_1, K_2). \tag{3.20}$$

Hence, by Lemma 3.1, the SDE

$$x_t^n = \xi + \int_0^t a^n(s, x_s^n)\,ds + \int_0^t \int_U c^n(s, x_{s-}^n, u)\tilde{N}(du\,ds) \tag{3.21}$$

has a unique solution x^n and

$$E \sup_{0 \leqslant t \leqslant T} |x_t^n|^2 \leqslant \tilde{K}(3K_1 + 4\sqrt{K_2}, T, E|\xi|^2) < \infty. \tag{3.22}$$

This proves that the sequence $\{(a^n, c^n, \mu)\}$ satisfies the assumption (A1) with

$$K = \max(3K_1 + 4\sqrt{K_2}, 3K_2) \quad \text{and} \quad \tilde{K} = \tilde{K}(3K_1 + 4\sqrt{K_2}, T, E|\xi|^2). \tag{3.23}$$

The assumption (A2) is easy to check. Hence, by Theorem 2.3, the SDE (3.1) has a weak solution x on $[0, T]$. Applying the proof of Lemma 3.1 to this weak solution we obtained the estimate (3.16).

Now, we come back to our original problem and project the SDE (1.1) onto a sequence of finite dimensional subspaces. Let a^d: $\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and g^d: $\mathbb{R}_+ \times \mathbb{R}^d \times U \to \mathbb{R}^d$ be defined by

$$\begin{aligned} a^d(s,x)_k &= A\left(s, \sum_{j=1}^d x_j h_j^{-p(T)}\right)[h_k^{p(T)}], \\ g^d(s,x,u)_k &= G\left(s, \sum_{j=1}^d x_j h_j^{-p(T)}, u\right)[h_k^{p(T)}] \end{aligned} \tag{3.24}$$

where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Let Q_0 be a probability measure on H_{-r_0} such that

$$E^{Q_0} \| v \|^2_{-r_0} < \infty. \tag{3.25}$$

Let $p(T) = \max(p_0(T), r_0)$ and $\pi\colon H_{-p(T)} \to \mathbb{R}^d$ be a mapping given by

$$\pi(v)_k = v[h_k^{p(T)}], \quad k = 1, 2, \ldots, d \tag{3.26}$$

and let $Q_0^d \equiv Q_0 \circ \pi^{-1}$ be the induced measure of Q_0 on $\mathbb{R}^d$.

LEMMA 2.2.2 *Under the assumptions* (I) *and* (3.25), *the* SDE

$$x_t^d = x_0^d + \int_0^t a^d(s, x_s^d)\,ds + \int_0^t \int_U g^d(s, x_{s-}^d, u)\tilde{N}(du\,ds) \tag{3.27}$$

on $\mathbb{R}^d$ *with initial measure* Q_0^d *has a weak solution* x^d *on a stochastic basis* $(\Omega^d, \mathscr{F}^d, P^d, (\mathscr{F}_t^d))$ *and*

$$E^{P^d} \sup_{0 \leqslant t \leqslant T} |x_t^d|^2 \leqslant \tilde{K}(K, T, E^{Q_0} \| v \|^2_{-p(T)}) < \infty. \tag{3.28}$$

Proof For each d, it is easy to see that the assumption (F) is satisfied by (a^d, g^d, μ) with

$$K_1^d = K \quad \text{and} \quad K_2^d = \max(\| h_k \|^2_{q(T)} \| h_k \|^{-2}_{p(T)} : 1 \leqslant k \leqslant d) K. \tag{3.29}$$

The assertion of the Lemma follows from Theorem 3.1.

Remark 3.2 That K_2^d in (3.29) depends on d is the reason that we do not like the estimate (3.16) depending on K_2 and we distinguish K_1 and K_2 in the assumption (F).

For the weak solution x^d, we define the corresponding $H_{-p(T)}$-valued cadlag process X^d by

$$X_t^d = \sum_{k=1}^d (x_t^d)_k h_k^{-p(T)}. \tag{3.30}$$

Then

$$\sup_d E \sup_{0 \leqslant t \leqslant T} \| X_t^d \|^2_{-p(T)} \leqslant \tilde{K}(K, T, E \| X_0 \|^2_{-p(T)}). \tag{3.31}$$

Let $A^d\colon \mathbb{R}_+ \times \Phi' \to \Phi'$ and $G^d\colon \mathbb{R}_+ \times \Phi' \times U \to \Phi'$ be two sequences of measurable mappings given by

$$\begin{aligned} A^d(s, v) &= \sum_{k=1}^d A\left(s, \sum_{j=1}^d v[h_j^{p(T)}] h_j^{-p(T)}\right)[h_k^{p(T)}] h_k^{-p(T)} \\ G^d(s, v, u) &= \sum_{k=1}^d G\left(s, \sum_{j=1}^d v[h_j^{p(T)}] h_j^{-p(T)}, u\right)[h_k^{p(T)}] h_k^{-p(T)}. \end{aligned} \tag{3.32}$$

Let $\gamma: H_{-p(T)} \to H_{-p(T)}$ be a mapping given by

$$\gamma(v) = \sum_{k=1}^{d} v[h_k^{p(T)}]h_k^{-p(T)} \tag{3.33}$$

and let $\tilde{Q}_0^d \equiv Q_0 \circ \gamma^{-1}$ be the induced measure of Q_0 on $H_{-p(T)}$. Then X^d is a solution of the SDE

$$X_t^d = X_0^d + \int_0^t A^d(s, X_s^d)\,ds + \int_0^t \int_U G^d(s, X_{s-}^d, u)\tilde{N}(duds) \tag{3.34}$$

on the stochastic basis $(\Omega^d, \mathscr{F}^d, P^d, (\mathscr{F}_t^d))$ with initial measure $\tilde{Q}_0^d$.

THEOREM 3.2 *Under the assumptions* (I) *and* (3.25), *the* SDE (1.1) *has a weak solution* X *on* $[0, T]$ *with initial distribution* Q_0 *and*

$$E \sup_{0 \leqslant t \leqslant T} \|X_t\|_{-p_1(T)}^2 \leqslant \tilde{K}(K, T, E^{Q_0}\|v\|_{-p(T)}^2). \tag{3.35}$$

Proof By Theorem 2.3, we only need to check that (A^d, G^d, μ) satisfies the assumptions (A1) and (A2). By the continuity of $A(t, \cdot)$ on $H_{-p}, \forall w \in H_{-p}$, $\forall \varepsilon > 0$, $\exists \tilde{\delta}(w), \forall w' \in H_{-p}$ with $\|w - w'\|_{-p} < \tilde{\delta}(w)$, we have $\|A(t, w) - A(t, w')\|_{-q} < \varepsilon$. For fixed $v_0 \in H_{-p}$, let

$$C = \left\{ \sum_{j=1}^{d} v_0[h_j^{-p}]h_j^{-p} : d \in \mathbb{N} \right\} \cup \{v_0\} \text{ and } S(w, \tilde{\delta}(w)) = \{w' \in H_{-p} : \|w - w'\|_{-p} < \tilde{\delta}(w)\}. \tag{3.36}$$

As C is a compact subset of H_{-p} and $\{S(w, \tilde{\delta}(w)/2 : w \in C\}$ is an open covering of C, there exist $w_1, \ldots, w_n \in C$ such that

$$C \subset \bigcup_{k=1}^{n} S(w_k, \tilde{\delta}(w_k)/2). \tag{3.37}$$

Let $2\delta = \min\{\tilde{\delta}(w_k): k = 1, \ldots, n\}$. If $w \in C$ and $w' \in H_{-p}$, $\|w - w'\|_{-p} < \delta$, we have a k such that $\|w - w_k\|_{-p} < \tilde{\delta}(w_k)/2$, and hence

$$\|w_k - w'\|_{-p} \leqslant \|w - w_k\|_{-p} + \|w - w'\|_{-p} < \tilde{\delta}(w_k) \tag{3.38}$$

so that

$$\|A(t, w) - A(t, w')\|_{-q} \leqslant \|A(t, w) - A(t, w_k)\|_{-q} + \|A(t, w_k) - A(t, w')\|_{-q} < 2\varepsilon. \tag{3.39}$$

Hence, $\forall \varepsilon > 0, \exists \delta > 0, \forall v \in H_{-p}, \| v - v_0 \|_{-p} < \delta$, let $w = \gamma(v_0)$ and $w' = \gamma(v)$, then $w \in C$ and $\| w - w' \|_{-p} < \delta$. Hence, by (3.39),

$$\begin{aligned} \| A^d(t,v) - A^d(t,v_0) \|_{-q}^2 &= \| \sum_{k=1}^{d} (A(t,w') - A(t,w)[h_k^q] h_k^{-q} \|_{-q}^2 \\ &\leqslant \| A(t,w') - A(t,w) \|_{-q}^2 < 4\varepsilon^2. \end{aligned} \tag{3.40}$$

This proves that $A^d(t,v)$ are continuous in v uniformly in d and assumption (A1) is satisfied. Assumption (A2) is verified similarly.

Finally, we construct a weak solution on $[0,\infty]$ for (1.1). First of all, let us construct a sequence of measures Q_n on $\mathbb{D}^n = \mathbb{D}([0,nT], H_{-p_1(nT)})$ by induction. If $n = 1$, take $Q_1 = Q^*$. Suppose that Q_n on $\mathbb{D}^n$ has been constructed, we now construct Q_{n+1} on $\mathbb{D}^{n+1}$.

For $0 \leqslant t \leqslant T$, $v \leqslant \Phi'$ and $u \in U$, let

$$\tilde{A}(t,v) = A(t+nT,v) \quad \text{and} \quad \tilde{G}(t,v,u) = G(t+nT,v,u). \tag{3.41}$$

Then $\tilde{A}$ and $\tilde{G}$ satisfy the assumption (I) with $p_0(T)$ and $K(p,q,T)$ replaced by $p_0((n+1)T)$ and $K(p,q,(n+1)T)$ respectively. With initial distribution $\tilde{Q}_0 = Q_n \circ Z_{nT}^{-1}$, the SDE

$$X_t = X_0 + \int_0^t \tilde{A}(s,X_s)\,ds + \int_0^t \int_U \tilde{G}(s,X_{s-},u)\tilde{N}(duds) \tag{3.42}$$

has an $H_{-p_1((n+1)T)}$-valued weak solution $\tilde{Q}_n^*$ on $[0,T]$. As $\mathbb{D}^{1,n+1} = \mathbb{D}([0,T], H_{-p_1((n+1)T)})$ is a Polish space, the regular conditional probability measure

$$\tilde{Q}_{z_0}^*(\cdot) = E^{\tilde{Q}_n^*}(Z \in \cdot \,|\, Z_0 = z_0) \tag{3.43}$$

exists. Let

$$\pi : \mathscr{D}(\pi) \subset \mathbb{D}^n \times \mathbb{D}^{1,n+1} \to \mathbb{D}^{n+1} \tag{3.44}$$

be given by

$$\pi(Z^1, Z^2)_t = \begin{cases} Z_t^1 & \text{as} \quad 0 \leqslant t \leqslant nT \\ Z_{t-nT}^2 & \text{as} \quad nT \leqslant t \leqslant (n+1)T \end{cases} \tag{3.45}$$

where $\mathscr{D}(\pi) = \{(Z^1,Z^2) \in \mathbb{D}^n \times \mathbb{D}^{1,n+1} : Z_{nT}^1 = Z_0^2\}$.

Define a measure Q_{n+1}^* on $\mathbb{D}^n \times \mathbb{D}^{1,n+1}$ by

$$Q_{n+1}^*(A \times B) = \int_A \tilde{Q}_{Z_{nT}^1}^*(B) Q_n(dZ^1) \tag{3.46}$$

for $A \in \mathbb{D}^n$ and $B \in \mathbb{D}^{1,n+1}$. It is easy to show that $Q_{n+1}^*(\mathscr{D}(\pi)) = 1$, and hence, Q_{n+1}^* induces a measure $Q_{n+1} = Q_{n+1}^* \circ \pi^{-1}$ on $\mathbb{D}^{n+1}$.

$\{Q_n\}$ can be regarded as probability measures on $\mathbb{D}([0,\infty),\Phi')$ and

$$Q_{n+1}|\mathscr{B}_{nT}=Q_n \tag{3.47}$$

where $\mathscr{B}_{NT}$ is the natural σ-algebra on $\mathbb{D}([0,\infty],\Phi')$ upto time nT. Hence, the following set function

$$Q(B)=Q_n(B) \quad \text{for} \quad B\in\mathscr{B}_{nT}. \tag{3.48}$$

on the algebra $\cup_n\mathscr{B}_{nT}$ is well-defined and σ-additive. Hence Q can be extended to a probability measure on $\vee_n\mathscr{B}_{nT}=\mathscr{B}$. Denote this extension also by Q, we have

$$Q|\mathscr{B}_{nT}=Q_n. \tag{3.49}$$

LEMMA 3.3 *Q is a solution of the $\mathscr{L}$-martingale problem.*

Proof We only need to show that, for any $F\in\mathscr{D}_0^\infty(\Phi')$, $0\leqslant s<t<\infty$ and $B\in\mathscr{B}_s$, we have

$$\int_B (M^F(Z)_t-M^F(Z)_s)Q(dZ)=0. \tag{3.50}$$

The proof is by induction. If $t\leqslant T$, (3.50) follows from Lemma 2.1. Suppose (3.50) holds when $t\leqslant nT$, we prove it still holds when $t\leqslant(n+1)T$.

First, we assume that $nT\leqslant s<t\leqslant(n+1)T$. Let $\tilde{\mathscr{L}}$ and $\tilde{M}^F$ be defined by (2.15) with A and G replaced by $\tilde{A}$ and $\tilde{G}$ of (3.41) respectively. As $B\in\mathscr{B}_s$, $\pi^{-1}(B\cap\mathbb{D}^{n+1})\in\mathscr{B}^1_{nT}\times\mathscr{B}^2_{s-nT}$, it follows from standard arguments of measure theory that we may assume $\pi^{-1}(B\cap\mathbb{D}^{n+1})=C\times D$ with $C\in\mathscr{B}^1_{nT}$ and $D\in\mathscr{B}^2_{s-nT}$ in the following calculations:

$$\begin{aligned}
&\int_B (M^F(Z)_t-M^F(Z)_s)Q(dZ)\\
&=\int_{B\cap\mathbb{D}^{n+1}}(M^F(Z)_t-M^F(Z)_s)Q_{n+1}(dZ)\\
&=\int_{\pi^{-1}(B\cap\mathbb{D}^n)}(\tilde{M}^F(Z^2)_{t-nT}-\tilde{M}^F(Z^2)_{s-nT})\tilde{Q}^*_{Z^1_{nT}}(dZ^2)Q_n(dZ^1)\\
&=\int_C Q_n(dZ^1)E^{\tilde{Q}^*_\cdot}((\tilde{M}^F(Z^2)_{t-nT}-\tilde{M}^F(Z^2)_{s-nT}))1_D(Z^2)|Z^2_0=Z^1_{nT})\\
&=\int_C Q_n(dZ^1)E^{\tilde{Q}^*_\cdot}(E^{\tilde{Q}^*_\cdot}((\tilde{M}^F(Z^2)_{t-nT}\\
&\quad-\tilde{M}^F(Z^2)_{s-nT}))1_D(Z^2)|\mathscr{B}^2_{s-nT})Z^2_0=Z^1_{nT})=0.
\end{aligned} \tag{3.51}$$

Finally, if $s \leqslant nT < t \leqslant (n+1)T$, then

$$E^Q(M^F(Z)_t | \mathscr{B}_s) = E^Q(E^Q(M^F(Z)_t | \mathscr{B}_{nT} | \mathscr{B}_s) = E^Q(M^F(Z)_{nT} | \mathscr{B}_s) = M^F(Z)_s Q\text{-a.s.}$$

Similar arguments yield the following Lemma.

LEMMA 3.4 (1°) *For any $\phi \in \Phi$, $\{M_\phi(t, Z)\}_{t \geqslant 0}$ given by Lemma 2.9 is a Q-square integrable purely-discontinuous martingale.*

(2°) *Let*

$$\Gamma = \left\{ A \in \mathscr{B}(\Phi' \setminus \{0\}) : E^Q \sum_{0 < s \leqslant t} 1_A(\Delta Z_s) < \infty, \quad \forall t > 0 \right\}. \tag{3.53}$$

Then, for $A \in \Gamma$, we have

$$\sum_{0 < s \leqslant t} 1_A(\Delta Z_s) - \int_0^t \int_U 1_A(G(s, Z_s, u)) \mu(du)\, ds \tag{3.54}$$

is a Q-martingale on $[0, \infty]$.

THEOREM 3.3 *Let assumption* (I) *hold and $\forall \phi \in \Phi$, let $E|X_0[\phi]|^2 < \infty$. Then* (1.1) *has a Φ'-valued weak solution such that, $\forall T > 0, \exists p_1(T)$ and we have*

$$E \sup_{0 \leqslant t \leqslant T} \| X_t \|^2_{-p_1(T)} \leqslant \tilde{K}(K, T, E \| X_0 \|^2_{-p(T)}). \tag{3.55}$$

Proof Let

$$V(\phi) = (E|X_0[\phi]|^2)^{1/2}. \tag{3.56}$$

Then, it is easy to check the conditions of Lemma 2.2 of ([9], p15). We have an index r such that, $\forall \phi \in \Phi$, $V(\phi) \leqslant \theta \| \phi \|_r$. i.e.

$$E[X_0[\phi]|^2 \leqslant \theta^2 \| \phi \|_r^2. \tag{3.57}$$

By the definition of nuclear space, there exists an index $r_0 > r$ such that $\sum_k \| h_k^{r_0} \|_r^2 < \infty$. Hence, by (3.57), we have

$$E \| X_0 \|^2_{-r_0} = \sum_k E|X_0[h_k^{r_0}]|^2 \leqslant \sum_k \theta^2 \| h_k^{r_0} \|_r^2 < \infty. \tag{3.58}$$

The rest is exactly the same as in the proof of Theorem 2.3.

Before proceeding further, we return to the point made in the Introduction that in infinite dimensional SDE's for which one seeks a Hilbert space valued solution one might encounter situations where there exists no Hilbert space in which the solution X_t

will lie (almost surely) for all $t \geqslant 0$. The following example, essentially due to Kallianpur and Ramaswamy ([10]), lends support to this view.

Example Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, $\{\lambda_j\}$ a sequence of positive numbers and $\{h_j\}$ a CONS of H. Let

$$\Phi = \{\phi \in H: \|\phi\|_r < \infty \ \forall r \in R\} \tag{3.59}$$

where

$$\|\phi\|_r^2 = \sum_j \langle\phi, h_j\rangle^2 (1+\lambda_j)^{2r}. \tag{3.60}$$

Suppose we have $r_1 > 0$ such that

$$\sum_j (1+\lambda_j)^{-2r_1} < \infty \tag{3.61}$$

then Φ is a countably Hilbertian nuclear space. Let (U,μ) be a measure space with $\mu(U) = 1$ and define mappings A and G by

$$A(t,v) = 0 \quad \text{and} \quad G(t,v,u)[\phi] \equiv f(t)[\phi] = \sum_j \langle\phi, h_j\rangle (1+\lambda_j)^t, \tag{3.62}$$

it can be shown that SDE (1.1) with coefficients given by (3.62) has a unique solution

$$X_t = \int_0^t \int_U f(s)\tilde{N}(dsdu) \tag{3.63}$$

and there is no p such that $X_t \in H_{-p}$ for all $t \geqslant 0$.

4 EXISTENCE AND UNIQUENESS OF THE STRONG SOLUTION

In this section, we shall impose an additional condition to ensure that the SDE (1.1) has a unique strong solution. This will be achieved by establishing pathwise uniqueness and extending Yamada-Watanabe argument ([6]) to this setup.

To implement the Yamada-Watanabe argument, we need to realize the driving processes (the Poisson random measures in our case) in a common space. This space is to be chosen such that the regular conditional probability measures exist for any probability measures on it. Unfortunately, this property is not enjoyed by the space of all measures on $\mathbb{R}_+ \times U$. Based on these considerations, we shall establish an equivalence relation between the SDE (1.1) and another kind of SDE driven by an ℓ^2-valued martingale which will be called a *Good process*. As the Good processes can be realized on the Polish space $\mathbb{D}([0,T],\ell^2)$, the Yamada-Watanabe argument is applicable and we obtain the uniqueness of the solution for the new equation. Hence, by the equivalence, we get the uniqueness of the solution for the SDE (1.1).

We first state some basic definitions.

DEFINITION 4.1 Let $(\Omega, \mathscr{F}, P, \mathbb{F})$ be a stochastic basis and $\tilde{N}(duds)$ a compensated Poisson random measure on $[0, T] \times U$. Suppose that X_0 is a H_{-p}-valued random variable such that $E\|X_0\|_{-p}^2 < \infty$. Then by an H_{-p} valued strong solution on Ω to the SDE (1.1) we mean a process X_t defined on Ω such that

(a) X_t is an H_{-p}-valued $\mathscr{F}_t$-measurable random variable;
(b) $X \in \mathbb{D}([0, T], H_{-p})$;
(c) There exists a sequence (σ_n) of stopping times on Ω increasing to infinity, such that, $\forall n$

$$E\int_0^{T\wedge\sigma_n}\int_0 \|G(s, X_s, u)\|_{-p}^2 \mu(du)\,ds < \infty, \tag{4.1}$$

$$E\int_0^{T\wedge\sigma_n} \|A(s, X_s)\|_{-p}^2 \mu(du)\,ds < \infty; \tag{4.2}$$

(d) The SDE (1.1) is satisfied for all $t \in [0, T]$ and almost all $\omega \in \Omega$.

DEFINITION 4.2 (pathwise uniqueness) We say that pathwise uniqueness of the H_{-p}-valued solution for the SDE (1.1) holds if X and X' are two H_{-p}-valued solutions defined on the same probability space $(\Omega, \mathscr{F}, P)$ with respect to the same Poisson random measure N and starting from the same initial point $X_0 \in H_{-p}$, then the path of X and X' coincide for almost all $\omega \in \Omega$.

Now, we impose the monotonicity condition (M): $\forall t \in [0, T], v_1, v_2 \in H_{-p}$, we have that

$$\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_{-q} + \int_U \|G(t, v_1, u) - G(t, v_2, u)\|_{-q}^2 \mu(du) \leqslant K\|v_1 - v_2\|_{-q}^2 \tag{4.3}$$

where q is introduced in assumption (I).

LEMMA 4.1 *Under the assumptions* (I) *and* (M), SDE (1.1) *satisfies the pathwise uniqueness property.*

Proof Let X and X' be two H_{-p}-valued solutions. Without loss of generality, suppose the same sequence (σ_n) of stopping times such that (c) of the Definition 4.1 holds for X and X'. For $\phi \in \Phi$, we have

$$(X_t - X'_t)[\phi] = \int_0^t (A(s, X_s) - A(s, X'_s))[\phi]\,ds + \int_0^t\int_U (G(s, X_{s-}, u) - G(s, X'_{s-}, u))[\phi]\tilde{N}(duds). \tag{4.4}$$

By Itô's formula, we have that

$$Ee^{-K(t\wedge\sigma_n)}[(X_t - X'_t)[\phi]]^2$$

$$= 2E\int_0^{t\wedge\sigma_n} e^{-Ks}(X_s - X'_s)[\phi](A(s,X_s) - A(s,X'_s))[\phi]\,ds$$

$$- E\int_0^{t\wedge\sigma_n} Ke^{-Ks}((X_s - X'_s)[\phi])^2\,ds + E\int_0^{t\wedge\sigma_n}\int_U e^{-Ks}((G(s,X_s,u)$$

$$- G(s,X'_s,u))[\phi])^2\,\mu(du)\,ds. \tag{4.5}$$

Let $\phi = h_k^q, k\in\mathbb{N}$ and adding, we have

$$Ee^{-K(t\wedge\sigma_n)}\|X_t - X'_t\|_{-q}^2$$

$$= 2E\int_0^{t\wedge\sigma_n} e^{-Ks}\langle X_s - X'_s, A(s,X_s) - A(s,X'_s)\rangle_{-q}\,ds - E\int_0^{t\wedge\sigma_n} Ke^{-Ks}\|X_s - X'_s\|_{-q}^2\,ds$$

$$E + \int_0^{t\wedge\sigma_n}\int_U e^{-Ks}\|G(s,X_s,u) - G(s,X'_s,u)\|_{-q}^2\,\mu(du)\,ds \leqslant 0, \tag{4.6}$$

Hence, by the right continuity of X and X' and (4.6), $X = X'$ almost surely. ■

DEFINITION 4.3 (Uniqueness in law) Uniqueness in law holds for (1.1) if, for any two stochastic bases $(\Omega^k, \mathscr{F}^k, P^k, \mathbb{F}^k)$, two Poisson random measures N^k on $\mathbb{R}\times U$ with characteristic measures μ and two H_{-p}-valued solutions X, X' of (1.1) with the same initial distribution on H_{-p}, $(k=1,2)$, we have that X and X' have the same probability distribution on $\mathbb{D}([0,T], H_{-p})$.

The following assumption will be made throughout the rest of the paper: $(U, \mathscr{Y}, \mu)$ is a separable measure space.

Now, we introduce the Good processes which will play an essential role in the implementation of the Yamada-Watanabe argument.

DEFINITION 4.4 Let $(\Omega, \mathscr{F}, P, \mathbb{F})$ be a stochastic basis. An ℓ^2-valued process H_t on $(\Omega, \mathscr{F}, P, \mathbb{F})$ is called a *Good process with respect to a* CONS (ϕ_n) *of* $L^2(U, \mathscr{Y}, \mu)$ if $\exists$ a Poisson random measure $N(duds)$ on $\mathbb{R}_+\times U$ with intensity measure μ such that

$$H_t = \sum_{n=1}^{\infty}\frac{1}{n}\int_0^t\int_U \phi_n(u)\tilde{N}(duds)\,e_n \tag{4.7}$$

where $e_n = (0,\ldots,0,1,0,\ldots)\in\ell^2$.

It is easy to see that the series in (4.7) converges and, with respect to the same CONS(ϕ_n) of $L^2(U,\mathscr{Y},\mu)$, all Good processes have the same distribution on $(\mathbb{D}([0,T],\ell^2),\mathscr{B}\{\mathbb{D}([0,T],\ell^2)\})$ which will be denoted by P_G and called the Good measure.

For any $s\in[0,T]$ and $v\in H_{-p_1}$, define the linear operator $\psi(s,v)$ from ℓ^2 to H_{-p_1}, by

$$\psi(s,v)e_n = n\int_U G(s,v,u)\phi_n(u)\mu(du). \tag{4.8}$$

Let X be an H_{-p_1}-valued cadlag process on the stochastic basis $(\Omega,\mathscr{F},P,\mathbb{F})$, then it is easy to see that

$$\int_0^t\int_U G(s,X_{s-},u)\tilde{N}(duds) = \int_0^t \psi(s,X_{s-})\,dH_s. \tag{4.9}$$

Hence, the SDE (1.1) can be written in a different form

$$X_t = X_0 + \int_0^t A(s,X_s)\,ds + \int_0^t \psi(s,X_{s-})\,dH_s. \tag{4.10}$$

Now, we demonstrate how to couple two solutions of (1.1) and discuss some properties of the coupled process.

Suppose X' and X'' are two solution of the SDE (1.1) on stochastic bases $(\Omega',\mathscr{F}',P',\mathbb{F}')$ and $(\Omega'',\mathscr{F}'',P'',\mathbb{F}'')$ with initial random variables X_0' and X_0'' (having the same distribution λ on $H_{-p_1(T)}$) and Poisson random measures N' and N'' (having the same intensity measure μ on U) respectively. Let H' and H'' be defined in terms of (4.7) with respect to the same CONS (ϕ_n) of $L^2(U,\mathscr{Y},\mu)$ with N replaced by N' and N'' respectively. Then (X',H',X_0') and (X'',H'',X_0'') are two solutions of the SDE (4.10) on the stochastic bases $(\Omega',\mathscr{F}',P',\mathbb{F}')$ and $(\Omega'',\mathscr{F}'',P'',\mathbb{F}'')$ respectively. Let Q' and Q'' be the probability measures on $\mathbb{D}([0,T],H_{-p_1(T)})\times\mathbb{D}([0,T],\ell^2)\times H_{-p_1(T)}$ with product Borel σ-field induced by (X',H',X_0') and (X'',H'',X_0'') respectively. Define a mapping

$$\pi\colon \mathbb{D}([0,T],H_{-p_1(T)})\times\mathbb{D}([0,T],\ell^2)\times H_{-p_1(T)}\to\mathbb{D}([0,T],\ell^2)\times H_{-p_1(T)} \tag{4.11}$$

by $\pi(w_1,w_2,x)=(w_2,x)$. Then, $Q'\circ\pi^{-1}=Q''\circ\pi^{-1}=P_G\otimes\lambda$.

Let $Q'^{w_2,x}(dw_1)$ and $Q''^{w_2,x}(dw_1)$ be the regular conditional Probability of w_1 given w_2 and x with respect to Q' and Q'' respectively. This is possible since $\mathbb{D}([0,T],H_{-p_1(T)})$ is a Polish space. On the space

$$\Omega = \mathbb{D}([0,T],H_{-p_1(T)})\times\mathbb{D}([0,T],H_{-p_1(T)})\times\mathbb{D}([0,T],\ell^2)\times H_{-p_1(T)} \tag{4.12}$$

define a Borel probability measure Q by

$$Q(A) = \int\!\!\int\Big(\int\!\!\int 1_A(w_1,w_2,w_3,x)\,Q'^{w_3,x}(dw_1)\,Q''^{w_3,x}(dw_2)\Big)P_G(dw_3)\lambda(dx) \tag{4.13}$$

for $A \in \mathscr{B}(\Omega)$, where $\mathscr{B}(\Omega)$ is the topological σ-field of Ω. Then, it is easy to show that (w_1, w_3, x) and (X', H', X'_0) have the same distribution and as does (w_2, w_3, x) and (X'', H'', X''_0).

LEMMA 4.2 *For any $A \in \mathscr{B}_t(\mathbb{D}([0,T], H_{-p_1(T)}))$, we define two functions f_1 and f_2*

$$f_1(w,x) = Q'^{w,x}(A) \quad and \quad f_2(w,x) = Q''^{w,x}(A). \tag{4.14}$$

Then, f_1 and f_2 are measurable with respect to the completion of the σ-field $\mathscr{B}_t(\mathbb{D}([0,T], \ell^2)) \times \mathscr{B}(H_{-p_1(T)})$ under the probability measure $P_G \otimes \lambda$.

Proof For fixed $t > 0$ and $A \in \mathscr{B}_t(\mathbb{D}([0,T], H_{-p_1(T)}))$, let $Q_t'^{w,x}(A)$ be defined as $Q'^{w,x}(A)$ with Q' replaced by its restriction to sub-σ-field $\mathscr{B}_t(\mathbb{D}([0,T], H_{-p_1(T)})) \times \mathscr{B}_t(\mathbb{D}([0,T], \ell^2) \times \mathscr{B}_t(H_{-p_1(T)})$, then $(w,x) \mapsto Q_t'^{w,x}(A)$ is measurable with respect to the σ-field $\mathscr{B}_t(\mathbb{D}([0,T], \ell^2)) \times \mathscr{B}(H_{-p_1(T)})$. Now, we only need to show that

$$Q_t'^{w,x}(A) = f_1(w,x) \quad \text{for} \quad P_G \otimes \lambda\text{-a.s}\,(w,x). \tag{4.15}$$

i.e. for any $C \in \mathscr{B}(\mathbb{D}([0,T], \ell^2)) \times \mathscr{B}(H_{-p})$, we have to show that

$$\int_C Q_t'^{w,x}(A) P_G(dw) \lambda(dx) = Q'(A \times C). \tag{4.16}$$

Consider a continuous mapping ρ: $\mathbb{D}([0,T], \ell^2) \times \mathbb{D}([0,T\text{-}t], \ell^2) \to \mathbb{D}([0,T], \ell^2$ given by

$$\begin{aligned} \rho(w^1, w^2)_s &= w_s^1 && \text{if } s < t \\ &= w_{s-t}^2 + w_t^1 && \text{if } s \geqslant t. \end{aligned} \tag{4.17}$$

From the definition of P_G, we have

$$P_G\{w \in \mathbb{D}([0,T], \ell^2) : w(t-) \neq w(t)\} = 0 \tag{4.18}$$

and hence, ρ has a continuous inverse ρ^{-1}. So, we only need to prove (4.16) for C of the form

$$C = \{w \in \mathbb{D}([0,T], \ell^2) : \rho^{-1} w \in A_1 \times A_2\} \times D, \tag{4.19}$$

where $A_1 \in \mathscr{B}(\mathbb{D}([0,t], \ell^2))$, $A_2 \in \mathscr{B}(\mathbb{D}([0,T-t], \ell^2))$ and $D \in \mathscr{B}(H_{-p_1(T)})$. As Good processes are of independent increments, $P_G \circ \rho = P_1 \otimes P_2$, where P_1 and P_2 are probability measures on $\mathbb{D}([0,T], \ell^2)$ and $\mathbb{D}([0,T-t], \ell^2)$ respectively. Furthermore, as $Q_t'^{w,x}(A)$ is $\mathscr{B}_t(\mathbb{D}([0,T], \ell^2)) \times \mathscr{B}(H_{-p_1(T)})$ measurable, we can find a measurable function g in $\mathbb{D}([0,T], \ell^2) \times H_{-p_1(T)}$ such that

$$Q_t'^{w,x}(A) = g(\rho^{-1}(w)^1, x). \tag{4.20}$$

Hence

$$\int_C Q_t'^{w,x}(A) P_G(dw)\lambda(dx)$$

$$= \int_{A_1 \times A_2 \times D} g(w^1, x) P_1(w^1) P_2(w^2)\lambda(dx)$$

$$= \int_{A_1 \times D} g(w^1, x) P_1(w^1)\lambda(dx) P_2(A_2)$$

$$= \int Q_t'^{w,x}(A) 1_{\rho^{-1}(W)^1 \in A_1} 1_D(x) P_G(w)\lambda(dx) P_2(A_2)$$

$$= Q'(A \times \{(\rho^{-1}w)^1 \in A_1\} \times D) P_2(A_2)$$

$$= P'(X' \in A, H'|_{[0,t]} \in A_1, X_0' \in D) P'(H'(t+\cdot) - H'(t) \in A_2)$$

$$= P'(X' \in A, H'|_{[0,t]} \in A_1, X_0' \in D, H'(t+\cdot) - H'(t) \in A_2)$$

$$= P'(X' \in A, (H', X_0') \in C) = Q'(A \times C). \tag{4.21}$$

LEMMA 4.3 *Let $\mathscr{B}_t'$ be the completion of*

$$\mathscr{B}_t(\mathbb{D}([0,T], H_{-p_1(T)})) \times \mathscr{B}_t(\mathbb{D}([0,T], H_{-p_1(T)})) \times \mathscr{B}_t(\mathbb{D}([0,T], \ell^2)) \times \mathscr{B}(H_{-p_1(T)}). \tag{4.22}$$

Then w_3 is a Good process on an extension $(\tilde{\Omega}, \tilde{\mathscr{B}}, \tilde{Q}, \tilde{\mathscr{B}}_t)$ of $(\Omega, \mathscr{B}', Q, \mathscr{B}_t')$.

Proof By the definition of P_G, there exist a stochastic basis $(\Omega, \mathscr{F}, P, \mathbb{F})$ and a Good process H on it such that P_G is the distribution of H. We prove our lemma in four steps.

Step 1. w_3 is an ℓ^2-valued Q-square-integrable martingale.

Let $A_1, A_2 \in \mathscr{B}_s(\mathbb{D}([0,T], H_{-p_1(T)}))$, $A_3 \in \mathscr{B}_s(\mathbb{D}([0,T], \ell^2))$, $A_4 \in \mathscr{B}(H_{-p_1(T)})$ and $a \in \ell^2$. Then we have

$$E^Q\{e^{i\langle a, w_3(t) - w_3(s)\rangle_{\ell^2}} 1_{A_1 \times A_2 \times A_3 \times A_4}\}$$

$$= \int_{A_3 \times A_4} e^{i\langle a, w_3(t) - w_3(s)\rangle_{\ell^2}} Q'^{w_3,x}(A_1) Q''^{w_3,x}(A_2) P_G(dw_3)\lambda(dx)$$

$$= \int e^{i\langle a, w_3(t) - w_3(s)\rangle_{\ell^2}} f_1(w_3, x) f_2(w_3, x) P_G(dw_3)\lambda(dx)$$

$$= E^Q e^{i\langle a, w_3(t) - w_3(s)\rangle_{\ell^2}} Q(A_1 \times A_2 \times A_3 \times A_4). \tag{4.23}$$

Hence, w_3 is of independent increments. It follows from

$$E^Q((w_3)_t) = E^P(H_t) = 0, \tag{4.24}$$

and

$$E^Q|(w_3)_t|^2 = E^P|H_t|^2 = \sum_{n=1}^{\infty} \frac{t}{n^2} < \infty \tag{4.25}$$

that w_3 is an ℓ^2-valued Q-square-integrable martingale.

Step 2. $\forall a \in \ell^2$, the quadratic variation of the square-integrable martingales $\langle w_3, a\rangle_{\ell^2}$ is given by

$$\langle w_3\rangle_t(a,a) = t \sum_n \frac{a_n^2}{n^2}. \tag{4.26}$$

We only need to prove that

$$R_t = \langle (w_3)_t, a\rangle_{\ell^2}^2 - t \sum_n \frac{a_n^2}{n^2} \tag{4.27}$$

is a Q-martingale. In fact,

$$E^Q(R_t - R_s | \mathscr{B}_s')$$

$$= E^Q((\langle (w_3)_t - (w_3)_s, a\rangle_{\ell^2}^2 + 2\langle (w_3)_t - (w_3)_s, a\rangle_{\ell^2}\langle (w_3)_s, a\rangle_{\ell^2} | \mathscr{B}_s') - (t-s)\sum_n \frac{a_n^2}{n^2}$$

$$= E^Q\langle (w_3)_t - (w_3)_s, a\rangle_{\ell^2}^2 - (t-s)\sum_n \frac{a_n^2}{n^2} = E^P\langle H_t - H_s, a\rangle_{\ell^2}^2 - (t-s)\sum_n \frac{a_n^2}{n^2} = 0. \tag{4.28}$$

Step 3. $\langle w_3, a\rangle_{\ell^2}$ is purely-discontinuous.

It is easy to see that the mapping

$$w_3 \to \sum_{s \leqslant t} |\Delta \langle (w_3)_s, a\rangle_{\ell^2}|^2 \tag{4.29}$$

from $\mathbb{D}([0,T], \ell^2)$ into $\mathbb{R}$ is measurable. Hence

$$E^Q \sum_{s \leqslant t} |\Delta\langle (w_3)_s, a\rangle_{\ell^2}|^2 = E^P \sum_{s\leqslant t} |\Delta\langle H_s, a\rangle_{\ell^2}|^2$$

$$= E^P \sum_{n,m=1}^{\infty} \frac{a_n a_m}{nm} \int_0^t \int_U \phi_n(u)\phi_m(u) N(du\,ds)$$

$$= \sum_{n=1}^{\infty} \frac{a_n^2}{n^2} t = E^Q\langle w_3\rangle_t(a,a). \tag{4.30}$$

So, it follows from the same argument as in the proof of Theorem 2.2 that $\langle w_3, a\rangle_{\ell^2}$ is purely-discontinuous.

Step 4. As w_3 and H have the same distribution, the point process $\Delta w_3(s)$ has the same compensator as the point process $\Delta H(s)$ which is

$$q(t,E,\omega)=\mu\left\{u: \sum_{n=1}^{\infty}\frac{1}{n}\phi_n(u)e_n\in E\right\} \qquad \forall E\in\mathscr{B}(\ell^2). \tag{4.31}$$

It follows from the same arguments as in the proof of the Theorem 2.3 that there exists a Poisson random measure M with intensity measure μ on an extension of $(\Omega,\mathscr{B}',Q,\mathscr{B}'_t)$ such that

$$(w_3)_t=\sum_{n=1}^{\infty}\frac{1}{n}\int_0^t\int_U \phi_n(u)\tilde{M}(duds)e_n. \tag{4.32}$$

Hence, w_3 is a Good process on an extension of $(\Omega,\mathscr{B}',Q,\mathscr{B}'_t)$.

We leave the proof of the following elementary lemma to the reader.

LEMMA 4.4 *Let P_1 and P_2 be two probability measures on a Polish space X with metric ρ. If $(P_1\times P_2)\{(x_1,x_2):x_1=x_2\}=1$, there exists a unique $x\in X$ such that $P_1=P_2=\delta_{\{x\}}$.*

THEOREM 4.1 *Under assumptions* (I) *and* (M), *uniqueness in law holds and the* SDE (1.1) *has a unique strong solution.*

Proof Let X' and X'' be two solutions of the SDE (1.1). From the arguments above, we see that (w_1,w_3,x) and (w_2,w_3,x) are two solutions of (4.10) on the same stochastic basis $(\tilde{\Omega},\tilde{\mathscr{B}},\tilde{Q},\tilde{\mathscr{B}}_t)$. Let N be the Poisson random measure on this stochastic basis corresponding to the Good process w_3. Then (w_1,N,x) and (w_2,N,x) are solutions of (1.1) on the same stochastic basis. By the pathwise uniqueness proved in Lemma 4.1, we have that $\tilde{Q}(w_2=w_1)=1$. Coming back to the original probability space, we have $Q(w_2=w_1)=1$. But, by (4.13),

$$Q(w_2=w_1)=\iint Q'^{w,x}\otimes Q''^{w,x}(w_2=w_2)P_G(dw)\lambda(dx), \tag{4.33}$$

so, for $P_G\otimes\lambda$-a.s. (w,x), we have

$$Q'^{w,x}\otimes Q''^{w,x}(w_1=w_2)=1. \tag{4.34}$$

By Lemma 4.4 and (4.34), we have a mapping

$$F:\mathbb{D}([0,T],\ell^2)\times H_{-p_1(T)}\to\mathbb{D}([0,T],H_{-p_1(T)}) \tag{4.35}$$

such that

$$Q'^{w,x}=Q''^{w,x}=\delta_{F(w,x)}. \tag{4.36}$$

For any $A \in \mathscr{B}_t(\mathbb{D}([0,T], H_{-p_1(T)}))$, by (4.36), Lemma 4.2 and

$$1_{F^{-1}(A)}(w,x) = Q'^{w,x}(A), \tag{4.37}$$

$F^{-1}(A)$ is in the completion of $\mathscr{B}_t(\mathbb{D}([0,T],\ell^2)) \times \mathscr{B}(H_{-p_1(T)})$ under $P_G \otimes \lambda$, and hence, $F(w,x)$ is adapted. Then, for any Poisson random measure N and initial $H_{-p_1(T)}$-valued random variable X_0, corresponding to a Good processes H with respect to a fixed CONS (ϕ_n) of $L^2(U,\mathscr{U},\mu)$, $F(H,X_0)$ is a strong solution of the SDE (1.1).

The uniqueness of the strong solution follows directly from the pathwise uniqueness of the SDE (1.1). The uniqueness in law follows from (4.36).

Finally, we consider the strong solution of (1.1) on $[0,\infty]$.

Definition 4.5 Let $(\Omega,\mathscr{F},P,\mathbb{F})$ be a stochastic basis and $\tilde{N}(duds)$ a compensated Poisson random measure on $\mathbb{R}_+ \times U$. X_0 is a Φ'-valued random variable. Then by a Φ'-valued strong solution on Ω to the SDE (1.1) we mean a process X_t defined on Ω such that

(a) X_t is Φ'-valued, $\mathscr{F}_t$-measurable;

(b) $X \in \mathbb{D}([0,\infty],\Phi')$;

(c) There exists a sequence (σ_n) of stopping times on Ω increasing to infinity and independent of ϕ such that, $\forall n \in \mathbb{N}$ and $\forall \phi \in \Phi$

$$E\int_0^{\sigma_n}\int_U |G(s,X_s,u)[\phi]|^2 \mu(du)\,ds < \infty, \tag{4.38}$$

$$E\int_0^{\sigma_n} |A(s,X_s)[\phi]|^2\,ds < \infty, \tag{4.39}$$

$$E|X_0[\phi]|^2 < \infty; \tag{4.40}$$

(d) $X_t[\phi] = X_0[\phi] + \int_0^t A(s,X_s)[\phi]\,ds + \int_0^t\int_U G(s,X_{s-},u)[\phi]\tilde{N}(duds)$, for each $t > 0$.

Theorem 4.2 *Under assumptions* (I) *and* (M), SDE (1.1) *has a unique* Φ'*-valued solution if* $\forall \phi \in \Phi$, *we have* $E|X_0[\phi]|^2 < \infty$.

Proof (1°) (existence) By the proof of Theorem 3.3, we have an r_0 such that X_0 lies in H_{-r_0} and $E\|X_0\|^2_{-r_0} < \infty$. For every $n \in \mathbb{N}$, by Theorem 4.1, there exists an $H_{-p_1(n)}$-valued solution X^n for the SDE (1.1) in $[0,n]$. As $p_1(n) \leqslant p_1(n+1)$, X^{n+1} and X^n are two $H_{-p_1(n+1)}$-valued solutions for the SDE (1.1) in $[0,n]$ and hence, by the uniqueness of $H_{-p_1(n+1)}$-valued solution in $[0,n]$ of Theorem 4.1, we see that $X^n_t = X^{n+1}_t$ for $t \leqslant n$. Let $\xi_t = X^n_t$ for $n-1 \leqslant t < n, n \in \mathbb{N}$, then it is easy to see that ξ is a Φ'-valued solution of the SDE (1.1) on $[0,\infty)$.

2° (uniqueness) Let X be any other Φ'-valued solution of SDE (1.1). By (c) of definition 4.5 we have

$$E \sup_{0 \leqslant t \leqslant n \wedge \sigma_n} (X_t[\phi])^2 < \infty. \tag{4.41}$$

It follows from the same arguments as in the proof of Theorem 3.3 that there exists an index p_n such that X_t lies in H_{-p_n} when $t \leqslant n \wedge \sigma_n$. By the proof of 1°, we may assume without restricting the generality that ξ_t also lies in H_{-p_n} when $t \leqslant n \wedge \sigma_n$. By the same arguments as in the proof of Lemma 4.1 we get our uniqueness.

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