Journal of Statistical Planning and Inference 40 (1994) 81–95

Strategies admitting nonnegative unbiased variance estimators

V.R. Padmawar

Stat–Math Division, Indian Statistical Institute, 8th Mile, Mysore Road, Bangalore 560059, India

Abstract

In this paper we characterise linear unbiased strategies for estimating the population total, admitting uniformly nonnegative unbiased variance estimators. We then characterise strategies for which the 'natural' unbiased variance estimators are uniformly nonnegative. We proceed to derive various necessary and sufficient conditions for the nonnegativity of unbiased variance estimators vis-à-vis nonnegative definite matrices. We finally propose a set of sufficient conditions for uniform nonnegativity of unbiased variance estimators and an algorithm to verify those conditions. In this paper the variance estimators considered are necessarily quadratic estimators.

AMS Subject Classification: Primary 62D05; secondary 15A48.

Key words: Sampling strategies; variance estimators; unbiasedness; uniform nonnegativity; quadratic forms; definite matrices.

1. Introduction

Consider a finite population of size N. Let y be the study variate taking value $y_i \in \mathbb{R}$ on unit i, $1 \leqslant i \leqslant N$. One of the basic problems in *sampling theory* is to estimate the total $Y = \sum_{i=1}^{N} y_i$ of the variate y. A 'reasonable' strategy is arrived at, based on various considerations, to estimate the total Y. However, for certain 'reasonable' strategies, uniformly nonnegative unbiased variance estimators (NNUVE) are not always available. Rao and Vijayan (1977) studied the problem in some generality though more specifically for the strategy that consists of a Midzuno–Sen sampling scheme and the ratio estimator.

We typically have a linear unbiased strategy (p, t) for estimating the total Y, where p is a sampling design and t is a linear unbiased estimator. The variance of such

Correspondence to: Prof. V.R. Padmawar, Stat-Math Division, Indian Statistical Institute, 8th Mile, Mysore Road, Bangalore 560059, India.

a strategy may be written as

$$V(p,t)=\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}y_iy_j, \tag{1.1}$$

where a_{ij}, $1\leqslant i,j\leqslant N$, are known coefficients. We assume that $V(p,t)$ vanishes at a known vector $x=(x_1,x_2,\dots,x_N)$ such that $x\in\mathbb{R}^N$ and $x_i\neq 0$ $\forall$ $i=1,2,\dots,N$. For many well known strategies (p,t) this condition is satisfied. In fact, often both p and t depend on x.

Rao and Vijayan (1977) derived an alternative expression for $V(p,t)$ and deduced the necessary form of an NNUVE. They also proposed a set of sufficient conditions for the uniform nonnegativity of such an estimator. We list their findings for our future reference.

(i) The alternative expression for the variance,

$$V=V(p,t)=-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}x_ix_j\left(\frac{y_i}{x_i}-\frac{y_j}{x_j}\right)^2, \tag{1.2}$$

where a_{ij}, $1\leqslant i,j\leqslant N$, are as in (1.1).

(ii) The necessary form of an NNUVE,

$$v\equiv v_S=-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(s)x_ix_j\left(\frac{y_i}{x_i}-\frac{y_j}{x_j}\right)^2, \tag{1.3}$$

where $s\in S$, the collection of all samples of size n, further

$$\begin{aligned} &a_{ij}(s)=0 \quad \text{if } i\notin s \text{ or } j\notin s,\\ &\sum_{s\ni i,j} a_{ij}(s)p(s)=a_{ij}, \quad i\neq j=1,2,\dots,N, \end{aligned} \tag{1.4}$$

and finally,

(iii) A set of sufficient conditions for the uniform nonnegativity of the estimator or in (1.3),

$$a_{ij}(s)\leqslant 0 \quad \text{for all } i\neq j\in s, \quad s\in S. \tag{1.5}$$

In view of (1.4), however, it may be observed that the conditions (1.5) would cease to hold as soon as any of the a_{ij}'s, $i\neq j$, in (1.2) are positive.

In this paper we first characterise strategies admitting uniformly NNUVEs. We then characterise strategies for which the 'natural' estimators are uniformly nonnegative. We proceed to derive various necessary and sufficient conditions for the nonnegativity of variance estimators vis-à-vis nonnegative definite (NND) matrices. We finally propose a set of sufficient conditions for uniform nonnegativity of variance estimators and an algorithm to verify those conditions. In this paper we consider only the quadratic variance estimators.

2. Strategies admitting uniform NNUVEs

Let us rewrite the expression for the variance (1.1) at $\boldsymbol{y}'=(y_1,y_2,\dots,y_N)$ as

$$V(p,t)=\boldsymbol{y}'A\boldsymbol{y},$$

where A is the $N\times N$ matrix of elements a_{ij}, $1\leqslant i,j\leqslant N$, in (1.1). Now the variance is a nonnegative function and hence,

$$\boldsymbol{y}'A\boldsymbol{y}\geqslant 0\quad \forall \boldsymbol{y}\in\mathbb{R}^N.$$

This implies that A is NND.

Further by our assumption that $V(p,t)$ is zero at $\boldsymbol{x}'=(x_1,x_2,\dots,x_N)$, we have

$$\boldsymbol{x}'A\boldsymbol{x}=0\;\Rightarrow\;A\boldsymbol{x}=\boldsymbol{0}.$$

By defining $b_{ij}=a_{ij}x_ix_j$, $1\leqslant i,j\leqslant N$, and $y_i/x_i=z_i$, $1\leqslant i\leqslant N$, the variance at $\boldsymbol{y}\in\mathbb{R}^N$ may be written as

$$\boldsymbol{z}'B\boldsymbol{z},\tag{2.1}$$

where B is NND and $B\boldsymbol{e}=\boldsymbol{0}$, $\boldsymbol{e}'$ being given by $\boldsymbol{e}'=(1,1,\dots,1)$.

We would often identify the variance (2.1) with the matrix B and call it the 'associated matrix'.

Analogously letting $b_{ij}(s)=a_{ij}(s)x_ix_j$, $1\leqslant i,j\leqslant N$, $s\in S$, the estimator in (1.3) may be rewritten as

$$\boldsymbol{z}'(B(s)-\Delta B(s))\,\boldsymbol{z},\quad s\in S\tag{2.2}$$

where $B(s)$ is the $N\times N$ matrix of $b_{ij}(s)$, $1\leqslant i,j\leqslant N$, $s\in S$, and given any $m\times m$ matrix $C=((c_{ij}))$, ΔC is defined as

$$\Delta C=\operatorname{diag}(r_1(C),r_2(C),\dots,r_m(C)),\tag{2.3}$$

where $r_i(C)=\sum_{j=i}^{N}c_{ij}$, $1\leqslant i\leqslant m$.

We did not specify $a_{ii}(s)$, $i=1,2,\dots,N$; $s\in S$, in (1.4) as neither (1.3) nor its unbiasedness depends on them. (2.2) does not depend on the diagonal elements of $B(s)$, $s\in S$, either; as $C-\Delta C$ does not depend on the diagonal elements of C. We choose $b_{ii}(s)=-\sum_{j\neq i}^{N}b_{ij}(s)$, $1\leqslant i\leqslant N$, $s\in S$, so that the estimator (2.2) can be written as

$$\left.\begin{aligned}&\boldsymbol{z}'B(s)\boldsymbol{z},\quad s\in S,\\ &\text{where now}\\ &B(s)\boldsymbol{e}=\boldsymbol{0}\quad\forall\, s\in S\\ &\text{and}\\ &\sum_{s\ni i,j}b_{ij}(s)p(s)=b_{ij},\quad 1\leqslant i\neq j\leqslant N\end{aligned}\right\}.\tag{2.4}$$

Let us label the samples in S as $s_1, s_2, \ldots, s_M$; $M = \binom{N}{n}$, and associate an M-tuple of matrices with (2.4) as

$$\boldsymbol{B} = (B(s_1), B(s_2), \ldots, B(s_M)). \tag{2.5}$$

We would often identify an estimator in (2.4) with the 'associated tuple' (2.5). Clearly, an estimator in (2.4) is uniformly nonnegative if and only if $B(s_t)$ is NND for all $t = 1, 2, \ldots, M$ in the associated tuple.

We introduce a few more terms before characterising variance expressions $-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}(z_i - z_j)^2$ or equivalently the quadratic forms $z'Bz$ in (2.1) admitting a uniformly NNUE under a given design p.

Let $W_N = (C$: C is $N \times N$, $C' = C$ and $Ce = \mathbf{0})$ and $Z = \{\boldsymbol{C} = (C(s_1), C(s_2), \ldots, C(s_M))$: $C(s_t) \in W_N \ \forall\, t = 1, 2, \ldots, M$ and (i, j)th element of $C(s_t)$ is zero if $i \notin s_t$ or $j \notin s_t$, $1 \leqslant t \leqslant M\}$.

Define $H: Z \to W_N$ as $H(\boldsymbol{C}) = C$, where C is an $N \times N$ matrix with

$$c_{ij} = \sum_{t=1}^{M} c_{ij}(s_t) p(s_t), \quad 1 \leqslant i \neq j \leqslant N$$

and

$$c_{ii} = -\sum_{j \neq i}^{N} c_{ij}, \quad 1 \leqslant i \leqslant N.$$

Let

$$\mathscr{C} = \{H(\boldsymbol{C}) \colon \boldsymbol{C} \in Z \text{ and } C(s_t) \text{ is NND } \forall\, t = 1, 2, \ldots, M\}. \tag{2.6}$$

We now have the following theorem.

Theorem 2.1. *For a given design p, $V = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}(z_i - z_j)^2$ admits a uniformly NNUE if and only if the associated matrix B is in $\mathscr{C}$ of* (2.6).

Proof. Let V admit a uniformly NNUE, say v. Let $\boldsymbol{B} = (B(s_1), B(s_2), \ldots, B(s_M))$ be the associated tuple for v. Clearly, $B(s_t)\boldsymbol{e} = \mathbf{0}$ and $B(s_t)$ is NND $\forall\, t = 1, 2, \ldots, M$ as v is uniformly nonnegative. Further, since v is unbiased for V, $H(\boldsymbol{B}) = \boldsymbol{B}$. Hence $B \in \mathscr{C}$.

Conversely, if $B \in \mathscr{C}$ then $H(\boldsymbol{C}) = B$ for some $\boldsymbol{C} = (C(s_1), C(s_2), \ldots, C(s_M)) \in Z$ such that $C(s_t)$ is NND $\forall\, t = 1, 2, \ldots, M$. Define

$$\begin{aligned} w(s_t) &= z'C(s_t)z, \quad 1 \leqslant t \leqslant M \\ &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}(s_t)(z_i - z_j)^2, \quad 1 \leqslant t \leqslant M. \end{aligned}$$

This w would work as a uniformly NNUE. □

For a given design p, define

$$\pi_i = \sum_{s \ni i} p(s), \quad 1 \leqslant i \leqslant N,$$

$$\pi_{ij} = \sum_{s \ni i,j} p(s), \quad 1 \leqslant i \neq j \leqslant N.$$

Let p be such that $\pi_{ij} > 0 \ \forall\, i \neq j = 1, 2, \ldots, N$.

For estimating $V = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}(z_i - z_j)^2$ the two most 'natural' unbiased estimators correspond to

$$\left.\begin{array}{ll} b_{ij}(s) = \dfrac{b_{ij}}{M_2 p(s)}, & i \neq j \in s, \\ b_{ii}(s) = -\sum\limits_{j \neq i \in s} b_{ij}(s), & i \in s, \\ b_{ij}(s) = 0, & \text{if } i \notin s \text{ or } j \notin s, \end{array}\right\} \quad s \in S, \tag{2.7}$$

where $M_2 = \binom{N-2}{n-2}$ and

$$\left.\begin{array}{ll} b_{ij}(s) = \dfrac{b_{ij}}{\pi_{ij}}, & i \neq j \in s, \\ b_{ii}(s) = -\sum\limits_{j \neq i \in s} b_{ij}(s), & i \in s, \\ b_{ij}(s) = 0, & \text{if } i \notin s \text{ or } j \notin s, \end{array}\right\} \quad s \in S. \tag{2.8}$$

We now characterise variance expressions admitting (2.7) and (2.8) as uniformly NNUEs.

Define $T_1 : W_N \to Z$ as

$$T_1(B) = (B(s_1) - \Delta B(s_1), B(s_2) - \Delta B(s_2), \ldots, B(s_M) - \Delta B(s_M)),$$

where $B(s_t)$ is an $N \times N$ matrix that agrees with B for all elements (i,j) for which both i and j are in s_t, and the rest of the elements of $B(s_t)$ are all zero, $1 \leqslant t \leqslant M$.

Clearly, T_1 is one to one. Let $\mathscr{R}_1(T_1)$ be the range of T_1 and

$$\mathscr{R}_1^+(T_1) = \{C = (C(s_1), C(s_2), \ldots, C(s_M)) \colon C \in \mathscr{R}_1(T_1) \text{ and } C(s_t) \text{ is NND } \forall\, t = 1, 2, \ldots, M\}$$

also let $\mathscr{C}_* = \{T_1^{-1}(C) \colon C \in \mathscr{R}_1^+(T_1)\}$.

Theorem 2.2. *For a given design p the estimator (2.7) is a NNUE for estimating the variance $V = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}(z_i - z_j)^2$ if and only if the associated matrix is in $\mathscr{C}_*$.*

Proof. We omit the proof as it is implicit in the preamble to the theorem. □

We now obtain an analogous characterisation for the estimator (2.8). Given a design p with $\pi_{ij}>0$, $1\leqslant i\neq j\leqslant N$, let Π be an $N\times N$ matrix with (i,j)th entry as $1/\pi_{ij}$, $1\leqslant i\neq j\leqslant N$, and (i,i)th entry as $1/\pi_i$, $1\leqslant i\leqslant N$.

Define $T_2: W_N\to Z$ as

$$T_2(B)=((B\circ\Pi)(s_1)-\Delta(B\circ\Pi)(s_1),(B\circ\Pi)(s_2)-\Delta(B\circ\Pi)(s_2),\ldots,$$
$$(B\circ\Pi)(s_M)-\Delta(B\circ\Pi)(s_M)),$$

where $B\circ\Pi$ denotes the Hadamard product of matrices B and Π, $(B\circ\Pi)(s_t)$ agrees with $B\circ\Pi$ for all entries (i,j) for which both i and j are in s_t rest of the elements of $(B\circ\Pi)(s_t)$ are all zero, $1\leqslant t\leqslant M$. Since Π is a fixed $N\times N$ matrix T_2 is also one to one. Further, let $\mathscr{R}_2(T_2)$ be the range of T_2 and

$$\mathscr{R}_2^+(T_2)=\{C=(C(s_1),C(s_2),\ldots,C(s_M)):\ C\in\mathscr{R}_2(T_2)\text{ and}$$
$$C(s_t)\text{ is NND }\forall t=1,2,\ldots,M\}.$$

Finally, let $\mathscr{C}_{**}=\{T_2^{-1}(C):\ C\in\mathscr{R}_2^+(T_2)\}$. We now state the following theorem.

Theorem 2.3. *For a given design p the estimator (2.8) is uniformly NNUE for estimating* $V=-\frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N b_{ij}(z_i-z_j)^2$ *if and only if the associated matrix B* is in $\mathscr{C}_{**}$.

Proof. The proof is similar to that of Theorem 2.2. □

Remark 2.1. For a given design, Theorems 2.1–2.3 characterise variance expressions that admit (i) uniformly NNUE, (ii) (2.7) as uniformly NNUE and (iii) (2.8) as uniformly NNUE, respectively. Similar results can be obtained for any unbiased variance estimator that can be defined without reference to any specific associated matrix.

Having observed that the nonnegativity of an unbiased variance estimator depends on whether certain matrices of the type $Be=\mathbf{0}$ are nonnegative definite or not, we try to characterise such matrices.

Let

$$\mathscr{B}=\{B:\ B=B',\ B\text{ is NND and }Be=\mathbf{0}\}. \tag{2.9}$$

Although there are various characterisations of NND matrices in the literature, we would like to give a few equivalent descriptions of $\mathscr{B}$ using the additional condition $Be=\mathbf{0}$.

Theorem 2.4. *The following classes of matrices are equal to* $\mathscr{B}$.

(i) $\mathscr{B}_1=\{B:\ B=B'$ *and* $Bx=\lambda x\Rightarrow\lambda\geqslant0$ *and* $\lambda\langle x,e\rangle=0\}$, *where* $\langle .\,,.\rangle$ *denotes the inner product.*

(ii) $\mathscr{B}_2=\{B: B=P'DP, (D,P)\in\mathscr{G}\}$, *where*

$$\mathscr{G}=\{(\Lambda,P): \Lambda=\operatorname{diag}(\lambda_1,\lambda_2,\dots,\lambda_n),\ \lambda_i\geqslant 0,\ 1\leqslant i\leqslant n,$$

λ_i is zero for at least one index say j, $1\leqslant j\leqslant n$. P is an orthogonal matrix with $(1/\sqrt{n})e'$ as its jth row.$\}$

Proof. (i) Let $B\in\mathscr{B}$. $\boldsymbol{e}$ is an eigenvector of B corresponding to the eigenvalue 0. Since B is NND $B\boldsymbol{x}=\lambda\boldsymbol{x} \Rightarrow \lambda\geqslant 0$.

Further, if $\boldsymbol{x}$ is an eigenvector corresponding to $\lambda\neq 0$ then $\langle \boldsymbol{x},\boldsymbol{e}\rangle=0$. Thus, $B\boldsymbol{x}=\lambda\boldsymbol{x} \Rightarrow \lambda\geqslant 0$ and $\lambda\langle \boldsymbol{x},\boldsymbol{e}\rangle=0$. Hence, $B\in\mathscr{B}_1$. Conversely, if $B\in\mathscr{B}_1$ then for $\lambda>0$ if $B\boldsymbol{x}=\lambda\boldsymbol{x}$ then $\langle \boldsymbol{x},\boldsymbol{e}\rangle=0$.

Thus, $\boldsymbol{e}$ is orthogonal to all eigenvectors corresponding to nonzero eigenvalues. Hence, $\boldsymbol{e}$ must be an eigenvector corresponding to the eigenvalue 0, i.e. $B\boldsymbol{e}=\boldsymbol{0}$. Thus, $B\in\mathscr{B}$. Therefore, $\mathscr{B}_1=\mathscr{B}$.

(ii) Let $B\in\mathscr{B}$. For any real symmetric matrix B, there exits an orthogonal matrix Q with its rows as eigenvectors of B such that

$$B=Q'DQ,$$

where D is the diagonal matrix of the eigenvalues of B. Since $B\in\mathscr{B}$, $\boldsymbol{e}$ is an eigenvector of B corresponding to the eigenvalue 0. Thus, $(D,Q)\in\mathscr{G}$ and $B\in\mathscr{B}_2$. On the other hand, if $B\in\mathscr{B}_2$ then as $B=P'\Lambda P$ for some $(\Lambda,P)\in\mathscr{G}$ all the eigenvalues of B are nonnegative. Hence, B is NND.

Further, for $(\Lambda,P)\in\mathscr{G}$, $\Lambda P\boldsymbol{e}=0 \Rightarrow B\boldsymbol{e}=P'\Lambda P\boldsymbol{e}=\boldsymbol{0}$. Hence , $B\in\mathscr{B}$. Thus, $\mathscr{B}_2=\mathscr{B}$.

We now give a set of necessary and sufficient conditions for the uniform non-positivity of

$$Q(B,z)=\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}(z_i-z_j)^2,\quad z\in\mathbb{R}^n, \tag{2.10}$$

where $B=((b_{ij}))$ is an $n\times n$ symmetric matrix. Let $s=\{1,2,\dots,n\}$ and $J=\{J_1,J_2,\dots,J_m\}$, $1\leqslant m\leqslant n$, be a partition of s. Further, let $c_{pq}=\sum_{i\in J_p}\sum_{j\in J_q}b_{ij}$ and $C=((c_{pq}))$, $1\leqslant p$, $q\leqslant m$. Let $Q_m(C(J),\boldsymbol{\alpha})=\sum_{p=1}^{m}\sum_{q=1}^{m}c_{pq}(\alpha_p-\alpha_q)^2$, $\boldsymbol{\alpha}'=(\alpha_1,\alpha_2,\dots,\alpha_m)$, $\boldsymbol{\alpha}\in\mathbb{R}^m$. In this set-up we have the following theorem.

Theorem 2.5. *$Q(B,z)\leqslant 0$ $\forall z\in\mathbb{R}^n$ if and only if $Q_m(C(J),\boldsymbol{\alpha})\leqslant 0$ $\forall\boldsymbol{\alpha}\in\mathbb{R}^m$ and $\forall$ partition J of s.*

Proof. As sufficiency is obvious, we only prove the necessity. Let $Q(B,z)\leqslant 0$ $\forall z\in\mathbb{R}^n$. For a partition J of s define $z^J\in\mathbb{R}^n$ as

$$z_i=\alpha_p\quad \forall i\in J_p,\quad 1\leqslant p\leqslant m,\qquad \boldsymbol{\alpha}'=(\alpha_1,\alpha_2,\dots,\alpha_m),\quad \boldsymbol{\alpha}\in\mathbb{R}^m.$$

Now

$$\begin{aligned} Q(B, z^J) &= \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}(z_i - z_j)^2 \\ &= \sum_{p=1}^{m} \sum_{q=1}^{m} \sum_{i \in J_p} \sum_{j \in J_p} b_{ij}(z_i - z_j)^2 \\ &= \sum_{p=1}^{m} \sum_{q=1}^{m} c_{pq}(\alpha_p - \alpha_q)^2 \\ &= Q_m(C(J), \boldsymbol{\alpha}). \end{aligned}$$

Therefore, $Q_m(C(J), \boldsymbol{\alpha}) \leqslant 0\ \forall \boldsymbol{\alpha} \in \mathbb{R}^m$ and $\forall$ partition J of s. Hence the theorem. □

3. Sufficient conditions for a uniformly NNUVE

So far we have given different characterisations of uniformly NNUVEs. The problem of deciding whether a given estimator is an NNUVE or not still remains. In this section we propose a set of sufficient conditions for the uniform nonnegativity of a given estimator and also give an algorithm to verify the conditions. However, we first observe that given an $n \times n$ symmetric matrix B with $B\boldsymbol{e} = \boldsymbol{0}$ the problem of checking whether it is nonnegative-definite can easily be reduced to that of an $(n-1) \times (n-1)$ matrix as follows.

Let

$$B = \begin{bmatrix} b_{11} & \boldsymbol{b}' \\ \boldsymbol{b} & B_1 \end{bmatrix}$$

be such that $B\boldsymbol{e} = \boldsymbol{0}$, where $\boldsymbol{b}' = (b_{12}, b_{13}, \ldots, b_{1n})$ and $B_1 = ((b_{ij}))$, $i, j = 2, 3, \ldots, n$ and

$$E = \begin{bmatrix} 1 & \boldsymbol{e}' \\ \boldsymbol{e} & F \end{bmatrix}$$

be an $n \times n$ matrix such that F is an $(n-1) \times (n-1)$ lower triangular matrix $((f_{ij}))$ with

$$f_{ij} = \begin{cases} 0 & \text{if } i < j, \\ -i & \text{if } i = j, \\ 1 & \text{if } i > j. \end{cases}$$

Clearly, the rows of E are mutually orthogonal.

Let C be an $(n-1) \times (n-1)$ matrix defined as

$$C = b_{11}\boldsymbol{e}\boldsymbol{e}' + F\boldsymbol{b}\boldsymbol{e}' + \boldsymbol{e}\boldsymbol{b}'F' + FB_1F'. \tag{3.1}$$

We now have the following theorem.

Theorem 3.1. *In the above set-up the matrix B is NND if and only if the matrix C is NND.*

Proof. The proof is simple as the rows of E are mutually orthogonal and

$$EBE' = \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix},$$

where C is given by (3.1).

Remark 3.1. In view of Theorem 3.1, to check whether B is nonnegative-definite, one may now use any of the various standard conditions to check this property for the matrix C, which is of smaller size.

We now give a set of sufficient conditions for the nonpositivity of $Q(B, z)$ defined in (2.10).

Let B be any $n \times n$ symmetric matrix. Define $\Omega = ((\omega_{ij}))$ as

$$\omega_{ij} = 1 \quad \text{if } b_{ij} > 0,\ i \neq j = 1, 2, \ldots, n,$$
$$\omega_{ij} = 0 \quad \text{if } b_{ij} \leqslant 0,\ i \neq j = 1, 2, \ldots, n,$$
$$\omega_{ii} = 1, \quad i = 1, 2, \ldots, n.$$

Let P be a permutation matrix such that

$$P'\Omega P = \Omega_1 \oplus \Omega_2 \oplus \cdots \oplus \Omega_m, \quad 1 \leqslant m \leqslant n,$$

where Ω_p is an $n_p \times n_p$ irreducible matrix, $1 \leqslant p \leqslant m$, $\sum_{p=1}^m n_p = n$. Let $\{J_1, J_2, \ldots, J_m\}$ be the partition of $s = \{1, 2, \ldots, n\}$ induced by $\Omega_1, \Omega_2, \ldots, \Omega_m$. Define $C = P'BP$, $G_p = \{(i,j)$: $c_{ij} > 0,\ i \neq j \in J_p\}$, $G = \bigcup_{p=1}^m G_p$ and $c_p = \sum_{(i,j) \in G_p} c_{ij}$. Clearly, $Q(B, z) \leqslant 0\ \forall z \in \mathbb{R}^n$ if and only if $Q(C, z) \leqslant 0\ \forall z \in \mathbb{R}^n$.

We are now equipped to state the following theorem.

Theorem 3.2. *If there exist subsets $K_1, K_2, \ldots, K_m$ of s such that*

(i) $J_p \cap K_p = \emptyset\ \forall p = 1, 2, \ldots, m$,

(ii) *If* $J_p \cap K_q \neq \emptyset$ *then*

$$J_q \cap K_p = \emptyset \ \forall p \neq q = 1, 2, \ldots, m, \tag{3.2}$$

and

(iii) $c_p + \sum_{k \in K_p} \max(c_{i_k}, c_{j_k}) \leqslant 0\ \forall\ i \neq j \in J_p,\ 1 \leqslant p \leqslant m$,

then $Q(C, z) \leqslant 0\ \forall z \in \mathbb{R}^n$.

Proof.

$$\begin{aligned} Q(C,z) &= \sum_{i=1}^n \sum_{j=1}^n c_{ij}(z_i - z_j)^2 \\ &= \sum_{p=1}^m \left\{ \sum_{(i,j) \in G_p} c_{ij}(z_i - z_j)^2 + \sum_{i \neq j \in J_p,\, c_{ij} \leqslant 0} c_{ij}(z_i - z_j)^2 \right\} \\ &\quad + \sum_{p=1}^m \sum_{q \neq p = 1}^m \sum_{i \in J_p} \sum_{j \in J_q} c_{ij}(z_i - z_j)^2 \\ &\leqslant \sum_{p=1}^m c_p (z_{ip} - z_{jp})^2 \\ &\quad + \sum_{p=1}^m \sum_{q \neq p = 1}^m \sum_{i \in J_p} \sum_{j \in J_q} c_{ij}(z_i - z_j)^2 + \text{nonpositive terms}, \end{aligned}$$

where

$$(z_{ip}-z_{jp})^2 = \max_{(i,j)\in G_p} (z_i-z_j)^2.$$

From (iii)

$$\begin{aligned} c_p(z_{ip}-z_{jp})^2 &\leqslant -\sum_{k\in K_p} \max(c_{ipk}, c_{jpk})(z_{ip}-z_{jp})^2 \\ &\leqslant -2\sum_{k\in K_p} \max(c_{ipk}, c_{jpk})\{(z_{ip}-z_k)^2+(z_{jp}-z_k)^2\} \end{aligned}$$

as c_{ipk}'s are negative and for any $a,b,c\in\mathbb{R}$,

$$(a-b)^2 \leqslant 2\{(a-c)^2+(b-c)^2\}.$$

Therefore,

$$c_p(z_{ip}-z_{jp})^2 \leqslant -2\left\{\sum_{k\in K_p} c_{ipk}(z_{ip}-z_k)^2 + \sum_{k\in K_p} c_{jpk}(z_{jp}-z_k)^2\right\}.$$

Thus,

$$\begin{aligned} Q(C,z) &\leqslant -2\sum_{p=1}^{m}\left\{\sum_{k\in K_p} c_{ipk}(z_{ip}-z_k)^2 + \sum_{k\in K_p} c_{jpk}(z_{jp}-z_k)^2\right\} \\ &\quad + \sum_{p=1}^{m}\sum_{q\neq p=1}^{m}\sum_{i\in J_p}\sum_{j\in J_q} c_{ij}(z_i-z_j)^2 \\ &\leqslant 0 \end{aligned} \tag{3.3}$$

as each $c_{ij}(z_i-z_j)^2$ in the second summation appears at most once in the first summation and those are the only terms in the first summation of the right-hand side of (3.3). Thus,

$$Q(C,z) \leqslant 0 \quad \forall z\in\mathbb{R}^n. \quad \square$$

Remark 3.2. Condition (ii) of (3.2) is to ensure that the coefficients in the second term of the right-hand side of (3.3) are used at most once each. Condition (iii) of (3.2) may be replaced by a condition that is much simpler to verify though marginally restrictive.

(iii)* $$c_p + \sum_{k\in K_p} \max_{i\in J_p} c_{ik} \leqslant 0, \quad 1\leqslant p\leqslant m.$$

Remark 3.3. Conditions (1.5) due to Rao and Vijayan (1977), in this set-up, would be

$$c_{ij} \leqslant 0 \quad \forall\, i\neq j = 1,2,\ldots,n.$$

As this would clearly imply (3.2), conditions (3.2) are much less stringent than those proposed by Rao and Vijayan (1977).

Remark 3.4. One can now use Theorem 3.2 to give a set of sufficient conditions for the uniform nonnegativity of an unbiased estimator (2.4) by simply demanding the analogue of conditions (3.2) to be true for each of the entries in the 'associated tuple' (2.5).

Theorem 3.2 may also be used to check whether a given symmetric matrix is nonnegative definite.

Corollary 3.1. *In the above set-up any symmetric matrix C is NND if*
(i) *conditions* (3.2) *of Theorem* 3.2 *hold and*
(ii) $\sum_{j=1}^{n} c_{ij} \geqslant 0 \ \forall\, i=1,2,\dots,n$.

Proof. (i) implies that $Q(C,z) \leqslant 0 \ \forall z \in \mathbb{R}^n$ which is equivalent to $C - \Delta C$ being NND.
(ii) implies that ΔC is NND.
Therefore, C being the sum of two NND matrices, i.e. $C=(C-\Delta C)+\Delta C$, C is NND. □

We conclude this paper by proposing an algorithm to construct sets $K_1, K_2, \dots, K_m$ of Theorem 3.2.

Algorithm 3.1. In the framework of Theorem 3.2, we assume, w.l.o.g., that

$$c_1 \geqslant c_2 \geqslant \cdots \geqslant c_m.$$

If $c_1=0$ then $K_p=\emptyset \ \forall p=1,2,\dots,m$; otherwise let

$$\beta_{11}=\min_{q\neq 1} \max_{(i,j),\, i\neq j\in J_1} \textstyle\sum_{k\in J_q} \max(c_{ik}, c_{jk}).$$

If $c_1+\beta_{11}\leqslant 0$ then $K_1=J_{q_{11}}$, where q_{11} attains β_{11}; otherwise let

$$\beta_{12}=\min_{q\neq 1,\, q_{11}} \max_{(i,j),\, i\neq j\in J_1} \sum_{k\in J_q\cup J_{q_{11}}} \max(c_{ik}, c_{jk}).$$

If $c_1+\beta_{12}\leqslant 0$ then $K_1=J_{q_{11}}\cup J_{q_{12}}$ where q_{12} attains β_{12}, and so on.

Note that if $c_1+\max_{(i,j),\, i\neq j\in J_1} \sum_{q=2}^{m} \sum_{k\in J_q} \max(c_{ik}, c_{jk})>0$ then K_1 does not exist. Having obtained K_1, one similarly obtains K_2.

Define $\Gamma_r=\{r+1, r+2, \dots, m\} \cup \{p\colon p<r \text{ and } J_r\cap K_p=\emptyset\}$, $1<r<m$. If $c_2=0$ then $K_p=\emptyset \ \forall p=2,3,\dots,m$; otherwise let

$$\beta_{21}=\min_{q\in \Gamma_2} \max_{(i,j),\, i\neq j\in J_2} \sum_{k\in J_q} \max(c_{ik}, c_{jk}).$$

If $c_2+\beta_{21}\leqslant 0$ then $K_2=J_{q_{21}}$ where q_{21} attains β_{21}; otherwise let

$$\beta_{22}=\min_{q\neq q_{21},\, q\in \Gamma_2} \max_{(i,j),\, i\neq j\in J_2} \sum_{k\in J_q\cup J_{q_{21}}} \max(c_{ik}, c_{jk}).$$

If $c_2+\beta_{22}\leqslant 0$ then $K_2=J_{q_{21}}\cup J_{q_{22}}$ where q_{22} attains β_{22}, and so on. Note that if

$$c_2+\max_{(i,j),i\neq j\in J_2}\sum_{q\neq 2}\sum_{k\in J_q}\max(c_{ik},c_{jk})>0,$$

then K_2 cannot be obtained, i.e. if

$$c_2+\max_{(i,j),i\neq j\in J_2}\sum_{q\in\Gamma_2}\sum_{k\in J_q}\max(c_{ik},c_{jk})>0 \quad\text{and}\quad J_2\cap K_1=\emptyset,$$

then K_2 does not exist.

If

$$c_2+\max_{(i,j),i\neq j\in J_2}\sum_{q\in\Gamma_2}\sum_{k\in J_q}\max(c_{ik},c_{jk})>0 \quad\text{and}\quad J_2\cap K_1\neq\emptyset,$$

then the sufficient conditions may perhaps be satisfied provided

$$c_2+\max_{(i,j),i\neq j\in J_2}\sum_{q\neq 2}\sum_{k\in J_q}\max(c_{ik},c_{jk})\leqslant 0.$$

The step to be taken at this stage is to start with a different choice of K_1, obtained as before but with an additional constraint, namely $K_1\cap J_2=\emptyset$ and then to repeat the steps above to get K_2.

Having obtained $K_1,K_2,\ldots,K_{r-1}$, $r\leqslant m$, one similarly obtains K_r. If $c_r=0$ then $K_p=\emptyset$, $\forall p=r,r+1,\ldots,m$; otherwise let

$$\beta_{r1}=\min_{q\in\Gamma_r}\max_{(i,j),i\neq j\in J_r}\sum_{k\in J_q}\max(c_{ik},c_{jk}).$$

If $c_r+\beta_{r1}\leqslant 0$ then $K_r=J_{q_{r1}}$ where q_{r1} attains β_{r1}; otherwise let

$$\beta_{r2}=\min_{q\neq q_{r1},q\in\Gamma_r}\max_{(i,j),i\neq j\in J_r}\sum_{k\in J_q\cup J_{q_{r1}}}\max(c_{ik},c_{jk}).$$

If $c_r+\beta_{r2}\leqslant 0$ then $K_r=J_{q_{r1}}\cup J_{q_{r2}}$ where q_{r2} attains β_{r2}, and so on. However, if this sequence of steps does not produce K_r then one has to go beyond the set Γ_r. Note that if

$$c_r+\max_{(i,j),i\neq j\in J_r}\sum_{q\neq r}^{m}\sum_{k\in J_q}\max(c_{ik},c_{jk})>0,$$

then K_r does not exist. Otherwise, obtain K_r as follows. Let

$$\gamma_{r1}=\max_{(i,j),i\neq j\in J_r}\sum_{k\in H_r\cup J_{q_1}}\max(c_{ik},c_{jk}),$$

where $H_r=\bigcup_{q\in\Gamma_r}J_q$ and $q_1=\max_q\{q\colon q<r \text{ and } J_r\cap K_q\neq\emptyset\}$.

Now if $c_r+\gamma_{r1}\leqslant 0$ then $K_r=H_r\cup J_{q_1}$; otherwise let

$$\gamma_{r2}=\max_{(i,j),i\neq j\in J_r}\sum_{k\in H_r\cup J_{q_1}\cup J_{q_2}}\max(c_{ik},c_{jk})$$

where $q_2=\max_q\{q\colon q<q_1 \text{ and } J_r\cap K_q\neq\emptyset\}$. If $c_r+\gamma_{r2}\leqslant 0$ then $K_r=H_r\cup J_{q_2}$ and so on.

Having obtained K_r this way it becomes necessary to obtain new K_{q_1}, K_{q_2}, etc., using the earlier steps with an additional restriction that they do not intersect with J_r. In turn, it might be necessary to change some of the subsequent K's.

We continue to follow these steps until either we get all the required $K_1, K_2, \ldots, K_m$ or get into a loop only to conclude that the required K's do not exist.

Let us now illustrate the use of Algorithm 3.1.

Example 3.1. We start for simplicity with a matrix C that, in the framework of the algorithm, satisfies $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_m$. We do not specify the diagonal entries of the matrix C, as they are irrelevant.

i \ j	1	2	3	4	5	6	7	8	9	10
1	*	2	2	−5	−4	−5	−2	−5	−4	−2
2	2	*	1	−4	−3	−1	−6	−3	−9	−11
3	2	1	*	−2	−3	−5	−7	0	−11	−7
4	−5	−4	−2	*	1	0	1	−2	−3	−3
5	−4	−3	−3	1	*	1	0	−6	−4	−2
6	−5	−1	−5	0	1	*	0	−3	−4	−5
7	−2	−6	−7	1	0	0	*	−2	−6	−4
8	−5	−3	0	−2	−6	−3	−2	*	1	−3
9	−4	−9	−11	−3	−4	−4	−6	1	*	−2
10	−2	−11	−7	−3	−2	−5	−4	−3	−2	*

For the above matrix, $n=10$, $s=\{1,2,\ldots,10\}$, $m=4$, $J_1=\{1,2,3\}$, $J_2=\{4,5,6,7\}$, $J_3=\{8,9\}$, $J_4=\{10\}$, $c_1=10$, $c_2=6$, $c_3=2$, $c_4=0$.

We construct an array from the submatrix of C specified by $J_1 \times (s-J_1)$ as follows

q	2				3		4	C_{12}	C_{13}	C_{14}
k	4	5	6	7	8	9	10			
(i,j)										
(1, 2)	−4	−3	−1	−2	−3	−4	−2	−10	−7	−2
(1, 3)	−2	−3	−5	−2	0	−4	−2	−12	−4	−2
(2, 3)	−2	−3	−1	−6	0	−9	−7	−12	−9	−7
$\mathcal{M}_1$								−10	−4	−2

where entries in the array correspond to $\max(c_{ik}, c_{jk})$, that in $\mathcal{M}_1$ to the maximum entries in the columns C_{iq} and that in C_{iq} to

$$\sum_{k \in J_q} \max(c_{ik}, c_{jk}), \quad q=2,3,4.$$

Looking at the entries in $\mathcal{M}_1$, we have, $\beta_{11}=-10$; $\beta_{11}+c_1=-10+10=0$; therefore, $q_{11}=2$ and $K_1=J_2=\{4,5,6,7\}$.

We construct an array from the submatrix of C specified by $J_2 \times (s - J_1 \bigcup J_2)$ as follows.

q	3		4	C_{23}	C_{24}	C_2
k	8	9	10			
(i,j)						
(4, 5)	−2	−3	−2	−5	−2	−7
(4, 6)	−2	−3	−3	−5	−2	−7
(4, 7)	−2	−3	−3	−5	−3	−8
(5, 6)	−3	−4	−2	−7	−2	−9
(5, 7)	−2	−4	−2	−6	−2	−8
(6, 7)	−2	−4	−4	−6	−4	−10
$\mathscr{M}_2$				−5	−2	−7

where the entries correspond to $\max(c_{ik}, c_{jk})$, that in $\mathscr{M}_2$ to the maximum entries in the columns and that in C_{2q} to $\sum_{k \in J_q} \max(c_{ik}, c_{jk})$, $q = 3, 4$ and finally the entries in C_2 correspond to $\sum_{q=3}^{4} \sum_{k \in J_q} \max(c_{ik}, c_{jk})$.

Looking at the entries in $\mathscr{M}_2$, we have $\beta_{21} = -5$, $q_{21} = 3$ and $c_2 + \beta_{21} = 6 - 5 = 1$, we have to look at $\beta_{22} = -7$, $q_{22} = 4$. As $c_2 + \beta_{22} = 6 - 7 = -1$, we have $K_2 = J_3 \cup J_4 = \{8, 9, 10\}$.

As K_3 is a subset of $s - J_2 \cup J_3$ we construct an array from the submatrix of C specified by $J_3 \times (s - J_2 \cup J_3)$ as follows.

q		1		4	C_{31}	C_{34}
k	1	2	3	10		
(i,j)						
(8, 9)	−4	−3	0	−2	−7	−2
$\mathscr{M}_3$					−7	−2

where entries in the array correspond to $\max(c_{ik}, c_{jk})$, that in $\mathscr{M}_3$ to the maximum entries in the columns C_{3q} and that in C_{3q} to $\sum_{k \in J_q} \max(c_{ik}, c_{jk})$, $q = 1, 4$. Looking at the entries in $\mathscr{M}_3$, we have

$$\beta_{31} = -7; \qquad \beta_{31} + c_3 = -7 + 2 = -5 < 0,$$

therefore,

$$q_{31} = 1 \quad \text{and} \quad K_3 = J_1 = \{1, 2, 3\}.$$

Finally, since $c_4 = 0$, K_4 may be taken to be the empty set $\emptyset$. Thus, for the given matrix C, we have

$$K_1 = \{4, 5, 6, 7\}, \qquad K_2 = \{8, 9, 10\}, \qquad K_3 = \{1, 2, 3\} \quad \text{and} \quad K_4 = \emptyset .$$

The problems of (a) comparing variances of different NNUVEs, (b) enhancing the chances of getting NNUVEs using sampling techniques like stratification, and finally (c) demonstrating reduction in mean squared error for biased variance estimators obtained from NNUVEs vis-à-vis a 'superpopulation model' are presently being studied.

Acknowledgement

The author is extremely thankful to Dr. Gadadhar Misra for many constructive suggestions during the preparation of this paper. Thanks are also due to the referee for his comments.

References

Rao, J.N.K. (1979). On deriving the mean square errors and their nonnegative unbiased estimators in finite population sampling. *J. Indian Statist. Assoc.* **17**, 125–136.

Rao, J.N.K. and K. Vijayan (1977). On estimating the variance in sampling with probability proportional to aggregate size. *J. Amer. Statist. Assoc.* **72**, 579–584.