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An axiomatic approach to equivariant cohomology theories

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Abstract. This paper presents a translation of a theorem of Cartan into an equivariant setting. This work is largely based on the study of the homotopical algebra in the sense of Quillen of the category of simplicial objects over the category of rational O_G-vector spaces. The application is a solution to the equivariant commutative cochain problem. This solution is slightly better than the solution obtained earlier by Triantafillou in that the transformation group G need not be finite.

Keywords. Cohomology of G-simplicial sets; simplicial differential graded algebra; Cartan cohomology; closed model category; O_G-Eilenberg–MacLane complex.

1. Introduction

In [2] Cartan introduced the notion of a 'cohomology theory' and used it to generalize Sullivan's theory of rational de Rham complex to simplicial cochain algebra. Recall that a simplicial differential graded algebra A over a ring R with 1 is a simplicial object in the category **DGA/R** of differential graded algebras over R so that for each $p \geqslant 0$ we have a differential graded algebra

$$(A_p^*, \delta): A_p^0 \xrightarrow{\delta} A_p^1 \xrightarrow{\delta} A_p^2 \longrightarrow \dots$$

together with face and degeneracy maps $d_i: A_p^* \to A_{p-1}^*$ and $s_i: A_p^* \to A_{p+1}^*$ which are homomorphisms of differential graded algebras satisfying the usual simplicial identities. Then a cohomology theory in the sense of Cartan is a simplicial differential graded algebra A over R such that (1) each cochain complex (A_p^*, δ) is exact, and $Z^0 A = \ker (\delta: A_*^0 \to A_*^1)$ is a simplicially trivial R-algebra (simplicially trivial means all the d_i and s_i are isomorphisms), (2) the homotopy groups $\pi_i(A_*^n)$ of the simplicial set $A_*^n = \{A_p^n\}_{p \geqslant 0}$ are trivial whenever $i, n \geqslant 0$.

A cohomology theory A determines a contravariant functor from the category $\mathscr{S}$ of simplicial sets to **DGA/R** by sending $K \in \mathscr{S}$ to the differential graded algebra $A(K) = \{\mathrm{Hom}(K, A_*^n)\}_{n \geqslant 0}$, where $\mathrm{Hom}(K, A_*^n)$ is the R-module of simplicial maps $K \to A_*^n$, and differential and multiplication are induced from those of A. Then the theorem of Cartan is as follows:

Theorem 1.1. *If A is a cohomology theory, then there is a natural isomorphism*

$$H^*(A(K)) \cong H^*(K; R(A)),$$

on simplicial sets K, where $R(A)$ is the R-module $(Z^0 A)_0$.

The present paper is concerned with a generalization of this theorem in an equivariant set-up. Theorem 1.1 has its origin in the commutative cochain problem which was posed by Thom in 1957. A solution to this problem entails the construction of a contravariant functor A:**TOP**$\to$**CDGA/R** (the category of commutative differential graded algebras over R) so as to yield a de Rham type theorem which asserts that there is an isomorphism

$$H^*(A(X)) \cong H^*(X; R)$$

for every topological space X, where the cohomology on the right is the singular cohomology (note that without the commutativity requirement a cochain problem does not exist since the usual construction of cochains renders an automatic solution to it). For example, the classical de Rham theorem provides a solution for the subcategory of smooth manifolds where $A(X)$ is the commutative differential graded algebra over $\mathbb{R}$ of smooth differential forms on a manifold X. On the other hand, the commutative cochain problem has no solution over the integers, the cohomology operations (such as Steenrod squares, etc.) being the obstructions. This difficulty does not arise over the rationals $\mathbb{Q}$, or any field containing $\mathbb{Q}$. In [9], Quillen solved the rational commutative cochain problem in an abstract setting. Then Sullivan [10] gave another proof using his theory of minimal models and the de Rham complex $A(K)$ of rational polynomial forms on a simplicial set K. An independent proof, which is based on an earlier proof by Thom in the real case, was given by Swan [11] when the coefficient ring R is a field of characteristic zero. Finally, Cartan [2] formulated the main ideas of Swan in the form of axioms for a chomology theory, and proved Theorem 1.1 from which one can recover Sullivan's PL de Rham theorem for a suitable choice of cohomology theory (see [2], Example 3). The main features of both [11] and [2] is that they avoid integration of forms which is standard to proofs of de Rham type theorems.

As an application of our main theorem (see Theorem 1.4 below), we propose a solution to the commutative cochain problem for the equivariant cohomology of a G-space in the spirit of Cartan's method. This problem has already been solved by Triantafillou [12], Theorem 4.9, using Sullivan's method when G is a finite group. Triantafillou needed the finiteness condition for the use of Bredon cohomology and, more importantly, for the construction of certain projective rational coefficient system (see [12], p. 515). In our formulation we do not require G to be finite. Throughout we let G be a discrete group, and we continue to suppose that R is a commutative ring with 1. Let $\mathbf{O}_G$ be the category of canonical orbits whose objects are left coset spaces G/H and morphisms are equivariant maps $\hat{g}: G/H \to G/H'$, corresponding to subconjugacy relations $g^{-1}Hg \subseteq H'$. Let $\mathbf{RO}_G$**-mod** denote the category of $\mathbf{O}_G$-R-modules, which are contravariant functors from $\mathbf{Q}_G$ to the category **R-mod** of R-modules.

In § 2 we construct for a G-simplicial set K and a coefficient system $\lambda \in \mathbf{RO}_G$**-mod** an equivariant cohomology $H_G^*(K; \lambda)$, which is a simplicial version of the Bredon–Illman cohomology (see [1,6]) in the sense that if X is a G-space, then the Bredon–Illman cohomology groups $\hat{H}_G^*(X; \lambda)$ are isomorphic to $H_G^*(SX; \lambda)$, where SX is the singular G-simplicial set associated to X.

DEFINITION 1.2

Let $\mathscr{C}_R$ be the category of cohomology theories over R in the sense of Cartan. Then a G-cohomology theory over R is a contravariant functor $A: \mathbf{O}_G \to \mathscr{C}_R$.

Thus for each $G/H \in \mathbf{O}_G$, $A(G/H)$ is a cohomology theory over R, and we have therefore a sequence of contravariant functors $A^n: \mathbf{O}_G \to$ **sR-mod** (the category of simplicial R-modules), $n \geqslant 0$, defined by $A^n(G/H) = A(G/H)^n_*$. We may therefore think of the functors A^n as simplicial objects in the abelian category $\mathbf{RO}_G$**-mod.** Given a G-simplicial set K, define a contravariant functor $\Phi K:\mathbf{O}_G \to \mathbf{sR\text{-}mod}$ by $\Phi K(G/H) = \mathrm{R}K^H$, whose set of p-simplexes $(RK^H)_p$ is the free R-module RK_p^H with basis the set of p-simplexes K_p^H of K^H. Again, we may think of ΦK as a simplicial object in $\mathbf{RO}_G\mathbf{\text{-}mod}$.

DEFINITION 1.3

Let A be a G-cohomology theory over R and $G\mathscr{S}$ the category of G-simplicial sets. We define a contravariant function $A_G:G\mathscr{S} \to \mathbf{DGA/R}$ by

$$A_G(K) = \{\mathrm{Hom}(\Phi K, A^n)\}_{n \geqslant 0},$$

where $\mathrm{Hom}(\Phi K, A^n)$ denotes the R-module of simplicial maps $\Phi K \to A^n$ in the category $\mathbf{sRO}_G\mathbf{\text{-}mod}$ of simplicial objects in $\mathbf{RO}_G\mathbf{\text{-}mod}$.

Also, we define $\lambda_A \in \mathbf{RO}_G\mathbf{\text{-}mod}$ by $\lambda_A(G/H) = (Z^0 A(G/H))_0$. Then our main theorem is the following

Theorem 1.4. *Let A be a G-cohomology theory over R. Then there is a natural isomorphism*

$$H_G^*(K;\lambda_A) \cong H^*(A_G(K)),$$

on G-simplicial sets K.

Given a G-cohomology theory A, we may define $\hat{A}_G:\mathbf{G\text{-}spaces} \to \mathbf{DGA/R}$ by $\hat{A}_G(X) = A_G(SX)$. Then we shall also prove the following

Theorem 1.5. *Let $\lambda \in \mathbf{RO}_G\mathbf{\text{-}mod}$. Then there exists a G-cohomology theory A with $\lambda_A = \lambda$ such that*

$$\hat{H}_G^*(X;\lambda) \cong H^*(\hat{A}_G(X)),$$

for every G-space X.

Here are some examples which will illustrate the ideas involved in the development of the proposed G-cohomology theory.

Example 1.6. Take $R = \mathbb{R}$, the field of real numbers, and Ω as the simplicial differential graded algebra where $\Omega_p^* = \Omega^*(\Delta^p)$ is the differential graded algebra of smooth differential forms on the standard p-simplex Δ^p in $\mathbb{R}^{p+1}$. Then Ω is a cohomology theory over $\mathbb{R}$ in the sense of Cartan with $R(\Omega) = \mathbb{R}$, and the constant functor $A:\mathbf{O}_G \to \mathscr{C}_{\mathbb{R}}$ defined by $A(G/H) = \Omega$ is a G-cohomology theory. Theorem 1.4 says that $H^*(A_G(K))$ is isomorphic to the Bredon–Illman cohomology $H_G^*(K;\lambda)$, where λ is the constant coefficient system $\lambda(G/H) = \mathbb{R}$, for every G-simplicial set K.

This result may be called the equivariant de Rham theorem, because it computes the G-cohomology of K from the de Rham complexes of various fixed point sets K^H.

Example 1.7. For an R-module M, consider the simplicial differential graded algebra $C(M)$ where $C_p^*(M) = \oplus_{n \geqslant 0} C^n(\Delta[p]; M)$ is the differential graded algebra of cochains of the contractible simplicial set $\Delta[p]$ with values in M, and a coefficient system $\lambda:\mathbf{O}_G \to \mathbf{R\text{-}mod}$. Define a contravariant functor $A:\mathbf{O}_G \to \mathscr{C}_R$ by $A(G/H) = C(\lambda(G/H))$. Then each $A(G/H)$ is a cohomology theory in the sense of Cartan, and A is a G-cohomology theory over R. Note that here $\lambda_A = \lambda$, and therefore, by Theorem 1.4, $H^*(A_G(K)) \cong H_G^*(K;\lambda)$.

Example 1.8. Let C be a cohomology theory over $\mathbb{Q}$ in the sence of Cartan, and $\lambda:\mathbf{O}_G \to \mathbb{Q}$**-mod** be the rational coefficient system defined as follows:

$$\lambda(G/H) = \mathrm{Hom}(\mathbb{Q}(G/H), \mathbb{Q}),$$

where $\mathbb{Q}(G/H)$ is the vector space over $\mathbb{Q}$ generated by the set G/H, and

$$\lambda(\hat{g}) = \mathrm{Hom}(\mathbb{Q}(\hat{g}), id),$$

where $\hat{g}$ is a morphism in $\mathbf{O}_G$. Then $A:\mathbf{O}_G \to \mathscr{C}_{\mathbb{Q}}$ defined by

$$A(G/H) = \lambda(G/H) \otimes C$$

is a G-cohomology theory, where $\lambda(G/H)$ is considered as a simplicial differential graded algebra concentrated in dimension zero. Then, as before, $H^*(A_G(K)) \cong H^*_G$ $(K; \lambda_A)$. Observe that here we have $\lambda_A = \lambda \otimes (Z^0 C)_0$.

The proofs of Theorem 1.4 and 1.5 appear in § 5. The method is based on a study of the homotopical algebra of the category **sRO**$_G$**-mod**, and on a classification theorem for equivariant cohomology. These prerequisites are presented in § 2 through 4. Finally, in § 6 we show that for a suitable choice of G-cohomology, where G is finite, Theorem 4.9 of Triantafillou [12] can be recovered from Theorem 1.5.

2. *G*-simplicial sets and equivariant cohomology

A G-simplicial set K is a simplicial set together with an action of G on K by simplicial maps, regarding G as a constant simplicial group $\underline{G}$ with $\underline{G}_n = G$ for all $n \geq 0$, and all the face and degeneracy maps the identity map of G. This makes each K_n a G-set, and the face and degeneracy maps commute with the action of G.

A simplicial version of the equivariant Bredon–Illman cohomology [1, 6] for K may be described as follows. Let $\lambda \in \mathbf{RO}_G$**-mod,** and $C^n(K;\lambda)$ be the R-module of functions c defined on n-simplexes $x \in K_n$ such that $c(x) \in \lambda(G/G_x)$, where G_x is the isotropy subgroup at x. The inclusion $G_x \subseteq G_{d_i x}$ gives rise to a morphism $G/G_x \to G/G_{d_i x}$ in $\mathbf{O}_G$, and hence a homomorphism of R-modules $\lambda(G/G_{d_i x}) \to \lambda(G/G_x)$ which we shall denote by $\lambda(d_i x \to x)$. Define homomorphism $d: C^n(K;\lambda) \to C^{n+1}(K;\lambda)$ by

$$d(c)(x) = \sum_{i=0}^{n+1} (-1)^i \lambda(d_i x \to x) c(d_i x).$$

This makes $C^*(K;\lambda)$ a cochain complex. Next define an action of G on $C^n(K;\lambda)$ by

$$(gc)(x) = \lambda(\hat{g})(c(g^{-1}x)),$$

where $c \in C^n(K;\lambda)$, $x \in K_n$, and $\lambda(\hat{g}):\lambda(G/G_{g^{-1}x}) \to \lambda(G/G_x)$ is the isomorphism induced by the conjugacy relation $g^{-1}G_x g = G_{g^{-1}x}$. Let $C^n_G(K;\lambda)$ denote the submodule of G-invariant cochains $(C^n(K;\lambda))^G$. It is easily verified that $d(C^n_G(K;\lambda)) \subseteq C^{n+1}_G(K;\lambda)$. Define the equivariant cohomology of K with coefficient system λ by

$$H^n_G(K;\lambda) = H_n(C^*_G(K;\lambda)).$$

To complete the definition of the cohomology theory, let us note that a G-simplicial map $f: K \to L$ between G-simplicial sets induces a cochain map $f^{\#}: C^*_G(L;\lambda) \to C^*_G(K;\lambda)$ defined by $f^{\#}(c)(x) = \lambda(fx \to x)c(fx)$, where $\lambda(fx \to x):\lambda(G/G_{fx}) \to \lambda(G/G_x)$ is the homomorphism induced by the inclusion $G_x \subseteq G_{fx}$. Therefore we have a homomorphism

$$f^*: H^*_G(L;\lambda) \to H^*_G(K;\lambda).$$

The following theorem relates the Bredon–Illman cohomology $\hat{H}^*_G(X;\lambda)$ of a G-space X [6] with the cohomology $H^*_G(SX;\lambda)$, of the associated singular G-simplicial set SX.

Theorem 2.1. *Let X be a G-space with G discrete, and $\lambda \in$ **RO**$_G$**-mod**. Then there is an isomorphism*

$$\hat{H}^*_G(X;\lambda) \cong H^*_G(SX;\lambda),$$

which is functorial with respect to X.

Proof. Note that $\hat{H}^*_G(X;\lambda)$ is the homology of a cochain complex $\hat{S}^*_G(X;\lambda)$, where $\hat{S}^n_G(X;\lambda)$ is the R-module of functions c on equivariant singular n-simplexes $T:\Delta^n \times G/H \to X$ satisfying $c(T) \in \lambda(G/H)$, and certain compatibility condition (see [6], Ch. 1, Def. 4.3). We shall exhibit an isomorphism of cochain complexes

$$C^*_G(SX;\lambda) \cong \hat{S}^*_G(X;\lambda).$$

Let $T:\Delta^n \times G/H \to X$ be an equivariant singular n-simplex in X. Then $\sigma_T:\Delta^n \to X$ with $\sigma_T(x) = T(x, eH)$ is a singular n-simplex in X; that is, a simplex of the singular G-simplicial set SX. Moreover $H \subseteq G_{\sigma_T}$, for if $h \in H$ then $(h\sigma_T)(x) = h\sigma_T(x) = hT(x, eH) = T(x, eH) = \sigma_T(x)$. Thus we have a homomorphism $\lambda(G_{\sigma_T} \to H):\lambda(G/G_{\sigma_T}) \to \lambda(G/H)$. Now define $\alpha:C^n_G(SX;\lambda) \to \hat{S}^n_G(X;\lambda)$ by setting $\alpha(c)(T) = \lambda(G_{\sigma_T} \to H)c(\sigma_T)$. Next, define a homomorphism $\alpha':\hat{S}^n_G(X;\lambda) \to C^n_G(SX;\lambda)$ as follows. Let $\sigma:\Delta^n \to X$ be a singular n-simplex in X. Then $T_\sigma:\Delta^n \times G/G_\sigma \to X$ given by $T_\sigma(x, gG_\sigma) = g\sigma(x)$ is an equivariant singular n-simplex in X. We set $\alpha'(c)(\sigma) = c(T_\sigma)$. One sees easily that α and α' are well-defined cochain maps inverse to one another. ■

The following theorem, which we shall use in § 5, provides an alternative description of the cohomology groups $H^*_G(K;\lambda)$ (cf. Bredon [1], ch. I, § 9). Given a G-simplicial set K, define $\underline{C}_n(K) \in$ **RO**$_G$**-mod**, for each integer $n \geqslant 0$, in the following way

$$\underline{C}_n(K)(G/H) = C_n(K^H; R),$$

where $C_n(K^H; R)$ denotes the free R-module generated by the n-simplexes of K^H, and, for a G-map $\hat{g}:G/H \to G/H'$ induced by a subconjugacy relation $g^{-1}Hg \subseteq H'$,

$$\underline{C}_n(K)(\hat{g}) = g*,$$

where g^* is the chain map induced by the left translation $g:K^{H'} \to K^H$. Clearly this gives a chain complex $\underline{C}_*(K)$ (where the boundary $\partial:\underline{C}_n(K;R) \to \underline{C}_{n-1}(K;R)$ is coming from $\partial(G/H):C_n(K^H;R) \to C_{n-1}(K^H;R)$) in the abelian category **RO**$_G$**-mod**, and if $\lambda \in$ **RO**$_G$**-mod**, then $\mathrm{Hom}(\underline{C}_*(K), \lambda)$, which is the R-module of natural transformations $\underline{C}_*(K) \to \lambda$, becomes a cochain complex of R-modules.

Theorem 2.2. *There is an isomorphism*

$$C^*_G(K;\lambda) \cong \mathrm{Hom}(\underline{C}_*(K), \lambda)$$

of cochain complexes.

Proof. Associate with each $c \in C^n_G(K;\lambda)$ a natural transformation

$$\varphi(c):\underline{C}_n(K) \to \lambda$$

as follows. If $x \in K_n^H$, then $H \subseteq G_x$, and this induces a homomorphism $\lambda(G_x \to H)$: $\lambda(G/G_x) \to \lambda(G/H)$. Then $\varphi(c)(G/H): C_n(K^H; R) \to \lambda(G/H)$ is the homomorphism

$$\varphi(c)(G/H)(x) = \lambda(G_x \to H)c(x).$$

This gives $\varphi: C_G^n(K; \lambda) \to \mathrm{Hom}(\underline{C}_n(K), \lambda)$. Next, define its inverse $\varphi': \mathrm{Hom}(\underline{C}_n(K), \lambda) \to C_G^n(K; \lambda)$ as follows. If $T: \underline{C}_n(K) \to \lambda$ is a natural transformation and $x \in K_n$, then

$$\varphi'(T)(x) = T(G/G_x)(x).$$

It can be checked without difficulty that φ and φ' are well-defined cochain maps inverse to each other. ∎

3. Closed model structure of $\mathbf{sRO}_G$-mod

First note that the category $\mathbf{sRO}_G$**-mod** may be identified with the category of contravariant functors $\mathbf{O}_G \to$ **sR-mod.** Then, by a result of Dwyer and Kan [3, 4], $\mathbf{sRO}_G$**-mod** is a closed model category in the sense of Quillen [8] with the following structures: A morphism $f: T \to S$ is a fibration (resp. weak equivalence) if for every $G/H \in \mathbf{O}_G$ the simplicial map $f(G/H): T(G/H) \to S(G/H)$ is a fibration (resp. weak equivalence), and f is a cofibration if it satisfies LLP (left lifting property) with respect to trivial fibrations. Also it follows easily that

Lemma 3.1. *Every object in* $\mathbf{sRO}_G$**-mod** *is fibrant as well as cofibrant.* ∎

Note that an object $T \in \mathbf{sRO}_G$**-mod** is fibrant (resp. cofibrant) if the morphism $T \to \underline{0}$ (resp. $\underline{0} \to T$) is a fibration (resp. cofibration), where $\underline{0}$ is the initial object in $\mathbf{sRO}_G$**-mod.**

We shall now briefly discuss the homotopy theory in $\mathbf{sRO}_G$**-mod.** There are two notions of homotopy in $\mathbf{sRO}_G$**-mod**: (i) the left homotopy coming from its closed model structure, and (ii) the abstract homotopy coming from combinatorial considerations as described in [7], § 5. We shall show that the two notions are essentially the same.

First let us look at the abstract notion of homotopy in $\mathbf{sRO}_G$**-mod.** Let $\underline{RI}: \mathbf{O}_G \to$ **sR-mod** be the contravariant functor defined by $\underline{RI}(G/H) = RI$ and $\underline{RI}(\hat{g}) = id$, where RI is the free simplicial R-module generated by $I = \Delta[1]$. Note that if $\{e_0, e_1, \ldots, e_{n+1}\}$ is the basis of the R-module $(RI)_n$, where $e_k = (0, 0, \ldots, 0, 1, \ldots, 1)$ (with $n - k + 1$ zeros and k ones) $\in \Delta[1]_n$, then

$$(RI)_n = Re_0 \oplus \cdots \oplus Re_{n+1}.$$

If $T: \mathbf{O}_G \to$ **sR-mod** is another contravariant functor, define a contravariant functor $T \otimes \underline{RI}: \mathbf{O}_G \to$ **sR-mod** by $T \otimes \underline{RI}(G/H) = T(G/H) \otimes RI$. Also, define natural transformations $i_0, i_1: T \to T \otimes \underline{RI}$ by

$$i_0(G/H)(x) = x \otimes (e_0, 0, \ldots, 0) \quad \text{and} \quad i_1(G/H)(x) = x \otimes (0, \ldots, e_{n+1}).$$

We then obtain the following

Lemma 3.2. *Two simplicial maps* $f, g: T \to S$ *in* $\mathbf{sRO}_G$**-mod** *are homotopic (in the abstract sense) if and only if there exists a simplicial map*

$$F: T \otimes \underline{RI} \to S$$

with $F \circ i_0 = f$ *and* $F \circ i_1 = g$. ∎

Lemma 3.3. *Every homotopy equivalence in* **sRO$_G$-mod** *is a weak equivalence.* ■

Now we turn to the notion of left homotopy in a closed model category $\mathscr{C}$. Let $A \vee_{\varnothing} A$ be the push out of the diagram $A \leftarrow \varnothing \rightarrow A$ in $\mathscr{C}$ and

$$\nabla_A : A \vee A \rightarrow A$$

be the corresponding folding map. Recall from [8], ch. I, § 1 than in $\mathscr{C}$, a cylinder of an object A is an object IA together with morphisms $i_0, i_1 : A \rightarrow IA$ and $p: IA \rightarrow A$ such that $i_0 + i_1 : A \vee_{\varnothing} A \rightarrow IA$ is a cofibration, p is a weak equivalence, and $p \circ (i_0 + i_1) = \nabla_A$. Two morphisms $f_0, f_1 : A \rightarrow B$ in $\mathscr{C}$ are called left homotopic ($f_0 \sim_c f_1$) if there is a morphism $H: IA \rightarrow B$ such that $f_0 = H \circ i_0$ and $f_1 = H \circ i_1$. Quillen proved that if A is cofibrant then i_0 and i_1 are trivial cofibrations, and the left homotopy relation $\sim_c$ is an equivalence relation.

In the closed model category **sRO$_G$-mod** every object is cofibrant (Lemma 3.1), the initial object ϕ is just $\underline{0}$, and, for an object T, $T \vee_{\varnothing} T$ is simply $T \oplus T$ with the folding map $\nabla_T : T \oplus T \rightarrow T$ given by $\nabla_T(G/H)(x, x') = x + x'$. We define $IT = T \otimes \underline{RI}$, the natural transformations $i_0, i_1 : T \rightarrow IT$ as in Lemma 3.2, and $p: IT \rightarrow T$ by $p(G/H)(x \otimes u) = x$. Then the natural transformation $i_0 + i_1 : T \oplus T \rightarrow IT$ is given by

$$(i_0 + i_1)(G/H)(x, x') = i_0(G/H)(x) + i_1(G/H)(x'),$$

and we have $p \circ (i_0 + i_1) = \nabla_T$. Also, p is a homotopy equivalence in the abstract sense with homotopy inverse i_0. Therefore, by Lemma 3.3, p is a weak equivalence in **sRO$_G$-mod**.

Again $i_0 + i_1$ is a cofibration. To see this, consider a LLP of $i_0 + i_1$ with respect to a trivial fibration $q: U \rightarrow V$ in **sRO$_G$-mod**.

$$\begin{array}{ccc} T \oplus T & \xrightarrow{\alpha} & U \\ {\scriptstyle i_0 + i_1}\downarrow & & \downarrow{\scriptstyle q} \\ T \otimes \underline{RI} & \xrightarrow{\beta} & V \end{array}$$

We may identify $T \otimes \underline{RI}$ with $T \oplus T \oplus S$, where $S = \operatorname{coker}(i_0 + i_1)$, by means of a splitting of the exact sequence

$$\underline{0} \rightarrow T \oplus T \rightarrow T \otimes \underline{RI} \rightarrow S \rightarrow \underline{0},$$

(note that $i_0 + i_1$ is injective). Also, the LLP of the confibration $\underline{0} \rightarrow S$ with respect to the trivial fibration $q: U \rightarrow V$ has a solution $\gamma: S \rightarrow U$ such that $q \circ \gamma = \beta$. Then a solution to the LLP of $i_0 + i_1$ is given by $\alpha + \gamma : T \oplus T \oplus S \rightarrow U$. Thus we have proved the following lemma.

Lemma 3.4. *In the category* **sRO$_G$-mod,** $T \otimes \underline{RI}$ *is a cylinder object for* T. ■

Theorem 3.5. *Two morphisms* $f, g: T \rightarrow S$ *in* **sRO$_G$-mod** *are left homotopic if and only if they are homotopic in the abstract sense. Consequently, the homotopy between morphisms in the abstract sense is an equivalence relation.* ■

4. A classification theorem in sRO$_G$-mod

In this section we shall show that if $\lambda \in$ **RO$_G$-mod**, then for any G-simplicial set K there is a bijection between the cohomology group $H^n_G(K;\lambda)$ and the set $[\Phi K, K(\lambda, n)]$ of homotopy classes of morphisms in **sRO$_G$-mod,** where ΦK is as in Definition 1.3, and $K(\lambda, n)$ is what we call an **O**$_G$-Eilenberg–MacLane complex of the type (λ, n). This a contravariant functor T:**O**$_G \to$ **sR-mod** such that

(1) $T(G/H)$ is an Eilenberg–MacLane complex $K(\lambda(G/H), n)$,
(2) $T(\hat{g})\colon T(G/H) \to T(G/H')$ is the unique simplicial homomorphism induced by the linear map $\lambda(\hat{g})\colon \lambda(G/H) \to \lambda(G/H')$, $g^{-1}H'g \subseteq H$.

The first condition means that

$$\underline{\pi}_n \circ T = \lambda, \quad \text{and} \quad \underline{\pi}_j \circ T = \underline{0} \quad \text{if} \quad j \neq n,$$

where $\underline{\pi}_n(K)$ is defined by $\underline{\pi}_n(K)(G/H) = \pi_n(K^H)$. It may be noted that each $K(\lambda(G/H), n)$ is minimal by definition, and that the Eilenberg–MacLane G-simplicial set may be obtained from $K(\lambda, n)$ by applying a functorial bar construction (see [5]), and as in ([5], Cor., p. 280) we have the following

Theorem 4.1. *Any two* **O**$_G$*-Eilenberg–MacLane complexes of type* (λ, n) *are naturally isomorphic.*

Proof. Suppose that T, S:**O**$_G \to$ **sR-mod** are two **O**$_G$-Eilenberg–MacLane complexes of type (λ, n). Define a natural transformation $\eta\colon T \to S$ by setting $\eta(G/H)\colon T(G/H) \to S(G/H)$ to be the unique simplicial homomorphism induced by $id\colon \lambda(G/H) \to \lambda(G/H)$. Since there is a bijection

$$[T(G/H), S(G/H)] \leftrightarrow \mathrm{Hom}(\lambda(G/H), \lambda(G/H)),$$

$\varphi(G/H)$ is a homotopy equivalence, and hence an isomorphism, because a homotopy equivalence between two minimal Kan complexes is an isomorphism. Now the following diagram commutes.

$$\begin{array}{ccc} T(G/H) & \xrightarrow{\varphi(G/H)} & S(G/H) \\ {\scriptstyle T(\hat{g})}\downarrow & & \downarrow{\scriptstyle S(\hat{g})} \\ T(G/H') & \xrightarrow{\varphi(G/H')} & S(G/H') \end{array}$$

This is so because $S(\hat{g}) \circ \varphi(G/H)$ and $\varphi(G/H') \circ T(\hat{g})$ are the unique simplicial maps induced by $\lambda(\hat{g}) \circ id_{\lambda(G/H)}$ and $id_{\lambda(G/H')} \circ \lambda(\hat{g})$ respectively. This proves the theorem. ∎

From now on we shall consider only normalized chain and cochain complexes (see May [7]). Fix $n \geqslant 0$, and define contravariant functors

$$L(\lambda, n+1), K(\lambda, n)\colon \mathbf{O}_G \to \textbf{sR-mod}$$

by setting

$$L(\lambda, n+1)(G/H)([q]) = C^n(\Delta[q]; \lambda(G/H)),$$

$$K(\lambda, n)(G/H)([q]) = Z^n(\Delta[q]; \lambda(G/H)),$$

where $[q]$ denotes the ordered set $\{0 < 1 \ldots < q\}$. Then $K(\lambda, n)$ is an $\mathbf{O}_G$-Eilenberg–MacLane complex of the type (λ, n). Define a map

$$\Lambda : \mathrm{Hom}(\Phi K, L(\lambda, n+1)) \to \mathrm{Hom}(\underline{C}_n(K), \lambda)$$

as follows. Let $f : \Phi K \to L(\lambda, n+1)$ be a natural transformation. Then $f(G/H) : RK^H \to L(\lambda, n+1)(G/H)$ is a simplicial map. If $x \in K_n^H$, then $f(G/H)(x) \in L(\lambda, n+1)(G/H)[n] = C^n(\Delta[n]; \lambda(G/H))$ is a cochain. Since $C^n(\Delta[n]; \lambda(G/H))$ is the R-module of all linear transformations from the R-module $R\Delta_n$ with basis Δ_n to $\lambda(G/H)$, we may identify it with $\lambda(G/H)$. Then define $\Lambda f : \underline{C}_n(K) \to \lambda$ by

$$(\Lambda f)(G/H)(x) = (f(G/H)(x))(\Delta_n).$$

It is straightforward to check that Λf is natural with respect to morphisms in $\mathbf{O}_G$.
Next define

$$\Lambda' : \mathrm{Hom}(\underline{C}_n(K), \lambda) \to \mathrm{Hom}(\Phi K, L(\lambda, n+1))$$

as follows. Let $T : \underline{C}_n(K) \to \lambda$ be a natural transformation. Then it is sufficient to define simplicial map

$$\Lambda'(T)(G/H) : RK^H \to L(\lambda, n+1)(G/H).$$

Let $x \in RK_q^H$. This induces a simplicial map $\bar{x} : R\Delta[q] \to RK^H$ with $\bar{x}(\Delta_q) = x$. Then

$$\bar{x}^* : C^n(RK^H; \lambda(G/H)) \to C^n(R\Delta[q]; \lambda(G/H))$$

is a cochain map. Observe that $C^n(R\Delta[q]; \lambda(G/H)) = L(\lambda, n+1)(G/H)([q])$. We then set

$$(\Lambda' T)(G/H)(x) = \bar{x}^*(T(G/H)).$$

It can be verified that $\Lambda' T$ is a natural transformation, and that Λ and Λ' are inverse to each other. We have thus proved the following

Lemma 4.2. *The map Λ is an isomorphism between the functors* $\mathrm{Hom}(\Phi K, L(\lambda, n+1))$ *and* $\mathrm{Hom}(\underline{C}_n(K), \lambda)$ *with inverse Λ'.* ■

Let $\varphi' : \mathrm{Hom}(\underline{C}_n(K), \lambda) \to C_G^n(K; \lambda)$ denote the isomorphism of Theorem 2.2, with inverse φ. Denote the composition

$$\varphi' \circ \Lambda : \mathrm{Hom}(\Phi K, L(\lambda, n+1)) \to C_G^n(K; \lambda)$$

by Γ. Then we obtain

Lemma 4.3. *The map Γ is an isomorphism between the cocycles $Z_G^n(K; \lambda)$ and* $\mathrm{Hom}(\Phi K, K(\lambda, n))$ *with inverse $\Gamma' = \Lambda' \circ \varphi$.*

Proof. Let $f \in Z_G^n(K; \lambda)$. We need to show that $(\varphi f)(G/H)(x) \in K(\lambda(G/H), n)$, for all $x \in K_q^H$ and $H \subseteq G$, that is, $(\varphi f)(G/H)(x) \in Z^n(\Delta[q]; \lambda(G/H))$. But this is true, because

$$\begin{aligned}\delta(\Gamma' T(G/H)(x)) &= \delta(\Lambda'(\varphi T)(G/H)(x)) \\ &= \delta(\theta_H(id \times \hat{e})^* \tilde{x}^*(\widehat{\varphi T})) \\ &= \delta(\theta_H(id \times \hat{e})^* \tilde{x}^* T) = 0.\end{aligned}$$

Conversely, suppose that $(\varphi f)(G/H)(y) \in Z^n(\Delta[q]; \lambda(G/H))$ for all $y \in K_q^H$. Then, if $x \in K_{n+1}$ is non-degenerate, we find after a simple computation that $\delta(\Gamma f)(x) = 0$. This proves the lemma. ■

Theorem 4.4. *Let $f_0, f_1 \in \mathrm{Hom}(\Phi K, K(\lambda, n))$. Then $f_0 \sim f_1$ if and only if Γf_0 and Γf_1 are cohomologous.*

Proof. Let $F: \Phi K \otimes RI \to K(\lambda, n)$ be a homotopy between f_0 and f_1 (see § 3). Therefore for each $H \subseteq G$, the simplicial maps $f_0(G/H)$ and $f_1(G/H)$ are homotopic by the homotopy $F(G/H)$. Define an element u_H in the n-cochain group $C^n(L(\lambda, n+1)(G/H); \lambda(G/H))$ by setting $u_H(c) = c(\Delta_n)$. This gives a homomorphism

$$f(G/G_x)^*: C^n(L(\lambda, n+1)(G/G_x); \lambda(G/G_x)) \to C^n(RK^{G_x}; \lambda(G/G_x))$$

such that

$$\begin{aligned} f(G/G_x)^*(u_{G_x})(x) &= u_{G_x}(f(G/G_x)(x)) \\ &= f(G/G_x)(x)(\Delta_n) \\ &= \Gamma f(x). \end{aligned}$$

Now since the simplicial maps $f_0(G/G_x)$ and $f_1(G/G_x)$ are homotopic, $f_0(G/G_x)^* = f_1(G/G_x)^*$. Consequently $\Gamma f_0 = \Gamma f_1$.

Conversely, suppose that $f_0, f_1: \Phi K \to K(\lambda, n)$ are such that Γf_0 and Γf_1 are cohomologous, that is, $\Gamma f_0 = \Gamma f_1 + dh$, where $h \in C_G^{m-1}(RK; \lambda)$. It suffices to find a $\gamma \in Z_G^n(RK \otimes RI; \lambda)$ such that $i_0^*(\gamma) = \Gamma f_0$ and $i_1^*(\gamma) = \Gamma f_1$ where $i_0, i_1: RK \to RK \otimes RI$ are the inclusions as in Lemma 3.2. Then the natural transformation

$$\Gamma'(\gamma): \Phi K \otimes \underline{RI} \to K(\lambda, n)$$

will be a homotopy from f_0 to f_1. To get such a γ, write $\gamma_0 = p^*(\Gamma f_0) \in Z_G^n(RK \otimes RI; \lambda)$, where p is projection $RK \otimes RI \to RK$. Then

$$i_0^*(\gamma_0) = i_1^*(\gamma_0) = \Gamma f_0.$$

Further, regarding $h \in C_G^{n-1}(RK; \lambda)$ as a cochain defined on $i_1(RK)$, we may choose a cochain $\beta \in C_G^{n-1}(RK \otimes RI; \lambda)$ which extends h and vanishes on $i_0(RK)$. Thus $i_0^*(\beta) = 0$ and $i_1^* \beta = h$. Now take $\gamma = \gamma_0 - d\beta$. This completes the proof. ■

We have in effect proved the following theorem.

Theorem 4.5. (*Classification*) *For any G-simplicial set K, there is a bijection*

$$[\Phi K, K(\lambda, n)] \leftrightarrow H_G^n(K; \lambda).$$

5. Proofs of Theorems 1.4 and 1.5

We begin by proving a lemma. Recall from § 1 that given a G-cohomology theory $A: \mathbf{O}_G \to \mathscr{C}_R$, each $A(G/H)$ is a cohomology theory in the sense of Cartan, and we have a $\lambda_A \in \mathbf{RO}_G$**-mod** defined by $\lambda_A(G/H) = (Z^0(A(G/H)))_0$, where $Z^0(A(G/H))$ is the kernel of the homomorphism $\delta_H: A^0(G/H) \to A^1(G/H)$. We also have contravariant functors

$$A^n, Z^n A: \mathbf{O}_G \to \mathbf{sR\text{-}mod},$$

where $A^n(G/H)$ and $Z^nA(G/H)$ are simplicial R-modules with the set of p-simplexes as $A(G/H)^n_p$ and $\mathrm{Ker}\{\delta: A(G/H)^n_p \to A(G/H)^{n+1}_p\}$ respectively.

Lemma 5.1. *If* $A: \mathbf{O}_G \to \mathscr{C}_R$ *is a G-cohomology theory over R, then each* $A^n: \mathbf{O}_G \to$ **sR-mod** *is contractible as an object of* $\mathbf{sRO}_G$**-mod**.

Proof. For every $H \subseteq G$, and $n \geq 0$, the exact sequence $A^0(G/H) \xrightarrow{\delta} A^1(G/H) \to \dots$ gives rise to a short exact sequence

$$0 \to Z^nA(G/H) \to A^n(G/H) \to Z^{n+1}A(G/H) \to 0.$$

This amounts to saying that $A^n(G/H) \to Z^{n+1}A(G/H)$ is a principal fibration with fibre $Z^nA(G/H)$ in the category of simplicial sets, and hence a principal twisted cartesian product (PTCP) of type (W) with group complex $Z^nA(G/H)$ in the sense of May [7], where $A^n(G/H)$ is contractible. This PTCP of type (W) is naturally isomorphic to the universal PTCP of type (W), $W(Z^nA(G/H)) \to \bar{W}(Z^nA(G/H))$, constructed by means of $\bar{W}$- and W-constructions on $Z^nA(G/H)$. It can be checked easily that $W(Z^nA(G/H))$ is contractible and the contraction can be chosen so as to be natural with respect to the morphisms in $\mathbf{O}_G$. Consequently, the contraction of $A^n(G/H)$ is also natural. The resulting contractions are all the necessary equipments one needs for the construction of a contraction of $A^n: \mathbf{O}_G \to$ **sR-mod.** ■

Proof of Theorem 1.4. It is required to prove that there is an isomorphism

$$H^*_G(K; \lambda_A) \cong H^*(A_G(K)),$$

where $A_G(K)$ is the differential graded algebra $\mathrm{Hom}(\Phi K, A^n)$ of Definition 1.3. Note that $Z^nA: \mathbf{O}_G \to$ **sR-mod** is an $\mathbf{O}_G$-Eilenberg–MacLane complex of the type (λ_A, n), and that for $n > 0$ we have a short exact sequence

$$0 \to Z^nA \to A^n \to Z^{n+1}A \to 0,$$

in the category $\mathbf{sRO}_G$**-mod.** We may therefore identify $Z^nA_G(K) = \mathrm{Ker}(A^n_G(K) \to A^{n+1}_G(K))$ with $\mathrm{Hom}(\Phi K, Z^nA)$, which is the R-module of morphisms from ΦK to Z^nA. There is an obvious map

$$\mathrm{Hom}(\Phi K, Z^nA) \to [\Phi K, Z^nA] \cong H^n_G(K; \lambda_A),$$

where the isomorphism is as given in Theorem 4.5. We shall show that if $f \in \mathrm{Hom}(\Phi K, Z^nA)$ is homotopic to constant, then it factors through $p: A^{n-1} \to Z^nA$. Consider a commutative diagram:

$$\begin{array}{ccc} \Phi K & \longrightarrow & A^{n-1} \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle p} \\ \Phi K \otimes \underline{RI} & \xrightarrow{F} & Z^nA \end{array}$$

where the horizontal map on the top is the constant map, F is a homotopy between f and the constant map, the vertical map i_0 on the left is a trivial cofibration. Since p is surjective, it is a fibration. Consequently, since **sR-mod** is a closed model category, the above LLP of i_0 with respect to p has a solution

$$\tilde{F}: \Phi K \otimes \underline{RI} \longrightarrow A^{n-1}$$

such that $p \circ \tilde{F} | i_1(\Phi K) = f$. This proves the theorem when $n > 0$.

For $n=0$, it is easy to see that, since $Z^0 A_G(K) = \mathrm{Hom}(\Phi K, Z^0 A)$, two morphisms f, $g \in \mathrm{Hom}(\Phi K, Z^0 A)$ are homotopic if and only if they are equal. This completes the proof of the theorem. ■

Proof of Theorem 1.5. Given $\lambda \in \mathbf{RO}_G$**-mod**, consider the contravariant functor $A:\mathbf{O}_G \to \mathscr{C}_R$ defined by

$$A(G/H) = C^*(\Delta[\ ]; \lambda(G/H)),$$

where $C^p(\Delta[q]; \lambda(G/H))$ denotes the ordinary singular cochain group. Then $A(G/H)$ is a cohomology theory in the sense of Cartan and A is a G-cohomology theory with $\lambda = \lambda_A$. Set $\hat{A}_G(X) = A_G(SX)$. The proof now follows from Theorems 2.2 and 1.4. ■

6. Equivariant commutative cochain problem

We conclude the paper with the observation that for a suitable choice of G-cohomology theory Theorem 1.4 leads to Theorem 4.9 of Triantafillou[12]. Let G be a finite group, $\lambda:\mathbf{O}_G \to \mathbb{Q}$**-mod** a (contravariant) rational coefficient system, and $H_G^*(X;\lambda)$ the Bredon cohomology of a G-complex X. Let $\underline{\mathscr{E}}_X:\mathbf{O}_G \to \mathbf{CDGA}/\mathbb{Q}$ be the covariant functor, where $\underline{\mathscr{E}}_X(G/H) = \mathscr{E}_{X^H}$ which is the de Rham algebra over $\mathbb{Q}$ of rational polynomial forms on X^H, and, for a morphism $\hat{g}:G/H \to G/H'$ in $\mathbf{O}_G$, $g^{-1}Hg \subset H'$, $\underline{\mathscr{E}}_X(\hat{g}) = g^*:\mathscr{E}_{X^{H'}} \to \mathscr{E}_{X^H}$ which is induced by the left translation $g:X^{H'} \to X^H$. Let λ^* and $\underline{\mathscr{E}}_X^*$ denote respectively the functors dual to λ and $\underline{\mathscr{E}}_X$. Then according to Triantafillou[12], Theorem 4.9,

$$H^*(\mathrm{Hom}(\lambda^*, \underline{\mathscr{E}}_x)) \cong H_G^*(X;\lambda).$$

On the other hand, Theorem 1.5 gives

$$H^*(\mathrm{Hom}(\Phi X, \lambda \otimes_{\mathbb{Q}} \underline{\mathscr{E}}_X^*)) \cong H_G^*(X;\lambda).$$

It is not difficult to see that the cochain complexes $\mathrm{Hom}(\lambda^*, \underline{\mathscr{E}}_X)$ and $\mathrm{Hom}(\Phi X, \lambda \otimes_{\mathbb{Q}} \underline{\mathscr{E}}_X^*)$ are isomorphic. Note that we have $R(\lambda \otimes_{\mathbb{Q}} \underline{\mathscr{E}}_X^*) = \lambda$.

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