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Differential Topology and Supersymmetric Quantum Mechanics

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Abstract This article is based on a one-hour talk of mine in a conference on Algebraic and Geometric Topology at the Department of Mathematics of Delhi University held in January 1–4, 2002. The lecture was meant to give a general audience some idea of the research work of Edward Witten on sypersymmetry and Morse theory in an informal way. The work is highly influential towards the development of a subject called Topological Quantum Field Theory. The ideas in it have become of central importance in the study of differential geometry. Atiyah had written that this work of Witten "is obligatory reading for geometers interested in understanding modern quantum field theory." The aim of this article is to prepare the ground for supersymmetric quantum field theory as the Hodge-de Rham theory of infinite dimensional manifolds. Of course there are many important aspects that are not discussed here.

1 Introduction

We begin with the supersymmetry needed in the present article. Modern theories of particle physics deal with subatomic particles known as fermions (particles with half-integer values of spin) and bosons (particles with integer values of spin). The spin is the intrinsic angular momentum, that is, the total angular momentum which is not due to the motion of the particle. All the particles have either integer spin or half-integer spin, in unit of the reduced Plank constant $\hbar$. Fermions are basic constituents of matter, and their examples are quarks and leptons (which include electrons, muons, neutrinos). Bosons are particles that convey the fundamental forces. For example, electromagnetic forces are carried by bosons called photons, and strong forces, which bind nuclei of atoms together, are mediated by bosons called gluons. Two types of statistics are used to describe these elementary particles. Fermions obey Fermi-Dirac statistics that apply to particles restricted by the Pauli Exclusion Principle (two or more fermions are not allowed to occupy the same quantum state). Fermions contrast with bosons which are particles not covered by the exclusion principle (several bosons can occupy the same quantum state) and they obey Bose-Einstein statistics.

An entity is said to exhibit symmetry when it appears unchanged after undergoing a transformation operation. A square, for example, has a four-fold symmetry by which it appears the same when rotated about its centre through 90, 180, 270, and 360 degrees; four 90 degree rotations bring the square back to its original position. With supersymmetry, fermions can be transformed into bosons and conversely without changing the structure of the underlying theory of particles and their interactions.

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However, when a fermion is transformed into a boson, and then back into a fermion, it turns out that the particle has moved in space (this is an effect that is related to special relativity). Thus supersymmetry relates transformations of an internal property of particles (spin) to transformations in space-time. Supersymmetric quantum theories are theories which possess symmetric properties under the exchange of fermions and bosons. It may be mentioned that supersymmetry is not observed in the world today; bosons and fermions are not equally paired in the universe we see today. As the universe expanded from the Plank era (an incredibly small interval of time after big bang) supersymmetry broke down.

Traditional symmetries in physics are generated by objects that transform under the tensor representation. Supersymmetries, on the other hand are generated by objects that transform under the spinor representation. According to the spin-statistics theorem, bosonic fields commute while fermionic fields anti-commute. Combining two kinds of fields into a single algebra requires the introduction of a $\mathbb{Z}_2$-grading under which the bosons are even elements and the fermions are odd elements. Thus the Hilbert space $\mathbf{H}$ of a quantum field theory has a decomposition into bosonic and fermionic parts

$$\mathbf{H} = \mathbf{H}_B \oplus \mathbf{H}_F.$$

Then supersymmetries are defined by Hermitian operators $Q_i, i = 1, 2, \ldots, N$, on $\mathbf{H}$, which map $\mathbf{H}_B$ to $\mathbf{H}_F$ and vice versa. These are the generators of the supersymmetric algebra satisfying certain conditions, which include anti-commutation

$$\{Q_i, Q_j\} = Q_iQ_j + Q_jQ_i = 0 \text{ for } i \neq j,$$

and commutation with the Hamiltonian operator $\mathcal{H}$

$$[\mathcal{H}, Q_i] = \mathcal{H}Q_i - Q_i\mathcal{H} = 0 \text{ for each } i.$$

In this context, it is important to discuss solutions of the following equation

$$Q_i|S>= 0,$$

and also non-existence of solutions. The problem is that whether there exists a physical state S in $\mathbf{H}$ which is annihilated by the supersymmetric operators Q_i. The existence would mean spontaneous breaking of supersymmetry. This can be seen as an index problem for operators on a differential manifold. By this approach, supersymmetry is related to differential topology. In this article we shall discuss one such relation, namely, the simplest supersymmetric algebra that yields a new insight into Morse theory.

2 Morse theory

The object of the Morse theory is to study the critical points of a suitable smooth function $f : M \longrightarrow \mathbb{R}$, where M is a compact smooth manifold without boundary. For a nice class of such functions f there exists a relationship between the number of critical points of f, and certain topological invariants of the manifold M, namely, the Betti numbers of M.

Critical points of f are the solutions of the equation

$$df = 0.$$

Thus at a critical point p of f all the partial derivatives $\partial f/\partial x_i$ vanish. A critical point p is called non-degenerate if the Hessian matrix of f at p

$$H_p f = [\partial^2 f/\partial x_i \partial x_j(p)]$$

is non-singular. Clearly, the non-degenerate critical points are discrete, and, since M is compact, there number must be finite. The index of f at p, denoted by λ_p, is the number of negative eigenvalues of $H_p f$. It is a standard fact that in a neighbourhood of a non-degenerate critical point p of index λ_p we can represent f as

$$f(x) = f(p) - x_1^2 - x_2^2 - \cdots - x_{\lambda_p}^2 + x_{\lambda_{p+1}}^2 + \cdots + x_n^2,$$

where $(x_1, x_2, \ldots, x_n)$ is a local coordinate system about $p, n = \dim M$.

The function f is called a Morse function if all its critical points are non-degenerate. The Morse series $M_t(f)$ is defined by

$$M_t(f) = \sum_p t^{\lambda_p} = \sum_k m_k t^k,$$

where m_k is the number of critical points of f of index k. On the other hand, the Poincaré series of M relative to any field K is

$$P_t(M; K) = \sum_{k=0}^{n} \dim H^k(M; K) \cdot t^k = \sum_{k=0}^{n} b_k t^k,$$

where $b_k = \dim H^k(M; K)$ is the k-th Betti number of M. The fundamental result of the Morse theory is the following relation, called the strong Morse inequalities,

$$M_t(f) \geq P_t(M), \quad t \in \mathbb{R}.$$

The weak Morse inequalities follow from this result, and they are

$$m_k \geq b_k \text{ for } k = 0, 1, 2, \ldots, n.$$

Precisely formulated, the expression for the strong Morse inequalities takes the following form

$$M_t(f) - P_t(M) = (1 + t) \cdot Q_t(f; K),$$

where $Q_t(f; K) = \sum_k a_k t^k$, $a_i \geq 0$, $k = 0, \ldots, n$ (it is called the K-error of f). The Morse function f is called K-perfect, if its K-error is zero. It can be shown that if $M_t(f) = P_t(M; K)$ for all K, then the cohomology ring $H^*(M, \mathbb{Z})$ is torsion free.

If $t = -1$, we have

$$M_{-1}(f) = P_{-1}(M) = \sum_{k=0}^{n} (-1)^k b_k = \chi(M).$$

Thus the Euler-Poincaré characteristic $\chi(M)$ of M is completely independent of f. This result reveals the power of the Morse inequalities.

The proof of the Morse inequalities comes from an investigation of the level surfaces $f^{-1}(a) = \{x \in M | f(x) = a\}$ and the associated half spaces

$$M_a = \{x \in M | f(x) \leq a\}.$$

The idea is that as a varies, the topology of M_a will not change, unless a passes through a critical value of f. Explicitly

Theorem 2.1. *If there is no critical value of f in an interval $[a, b]$, then M_a is diffeomorphic to M_b. If the interval (a, b) contains just one non-degenerate critical value of f of index λ, then the homotopy type of M_b is obtained from M_a by attaching a cell E_λ of dimension λ*

$$M_b \simeq M_a \cup_\alpha E_\lambda,$$

where $\alpha : \partial E_\lambda \longrightarrow M_a$ is an attaching map.

Using these results in the exact cohomology sequences associated to attaching cells, one gets the Morse inequalities for any cohomology theory satisfying Eilenberg-Steenrod axioms. Thus, up to homotopy, M admits a cellular decomposition

$$M = \cup_\lambda E_\lambda,$$

where the number of cells is equal to the number of critical points, and the dimension of a cell is given by the index of the critical point.

This is the theory for finite dimensional manifolds. Originally, Marston Morse used his theory for infinite dimensional manifolds in proving that there exist infinitely many geodesics joining a pair of points on an n-dimensional sphere S^n with any Riemannian metric.

Let $L(M)$ be the space of parameterized paths $\sigma : [0, 1] \longrightarrow M$. This can be considered as an infinite dimensional manifold whose tangent space at a path $\sigma \in L(M)$ is the space of vector fields X along σ such that $X(0) = X(1) = 0$. Morse used the energy functional or free particle Lagrangian $\mathcal{E} : L(M) \longrightarrow \mathbb{R}$, given by

$$\mathcal{E}(\sigma) = \int_0^1 \left\| \frac{d\sigma(t)}{dt} \right\|^2 dt = \int_0^1 g_{ij} \frac{d\sigma_i(t)}{dt} \frac{d\sigma_j(t)}{dt} dt,$$

where (g_{ij}) is the Riemannian metric on M. For a pair of points $p, q \in M$, the subspace $L(M)_{(p,q)}$ of $L(M)$ consisting of paths which start at p and end at q, is an infinite dimensional submanifold. Morse proved that the restriction of $\mathcal{E}$ to $L(M)_{(p,q)}$ is a non-degenerate $\mathbb{Z}_2$-perfect Morse function, whose critical points are geodesics joining p and q. Here non-degeneracy means that the critical points of $\mathcal{E}$ occur along manifolds on which the Hessian $H_\sigma \mathcal{E}$ is non-degenerate in the normal direction. In the special case when $M = S^n$, Morse proved that the Poinceré series of the manifold $L(S^n)_{(p,q)}$ is given by

$$P_t(L(S^n)_{(p,q)}) = \frac{1}{1 - t^{(n-1)}}$$

for any field K. Since the critical points are geodesics joining p and q, it follows that there are an infinite number of geodesics in S^n from p to q.

The idea of Morse initiated a variety of research which are quite impressive and widespread. A few notable examples are Bott's proof of his celebrated periodicity theorems on the homotopy of Lie groups, work of Bott and Samelson on symmetric spaces, Milnor's construction of the first exotic spheres, Smale's refinement of Theorem 2.1 in the form of his handlebody theory, Solution of the Poincaré conjecture in dimensions ≥ 5, and the h-cobordism theorem.

3 Work of Smale

Stephen Smale fitted Morse theory into the framework of a dynamical system by creating a situation under which Theorem 2.1 holds. By this approach Morse theory found a variety of applications in both classical and quantum physics. It was Raoul Bott who potentiated our understanding of the work of Smale.

The starting point of this development is the gradient vector field ∇f of the function f with respect to a Riemannian structure on M. In the non-degenerate case, ∇f vanishes at the finite number of critical points of f. Then through every non-critical point r of f there passes a unique integral curve X_r of ∇f which starts at the critical point p and ends at some other critical point q. These integral curves are paths of quickest or steepest descent on M, and they are solutions $\sigma(t)$ of the differential equation

$$\frac{d\sigma(t)}{dt} = -\Delta f(\sigma(t)), \qquad \sigma(0) = r$$

such that $\lim_{t\to-\infty}\sigma(t) = p$ (the initial point), and $\lim_{t\to\infty}\sigma(t) = q$ (the terminal point). Physicists call such an integral curve instanton, because it stays near the initial point p for most of the time $t < t_0$ and near the terminal point q for most of the time $t > t_1$ and then moves from the vicinity of p to that of q in an instant. It may also be called a descending gradient flow of f on M.

According to René Thom, the set W_p of instantons of ∇f which start at the critical point p of index λ_p is a cell of dimension λ_p. This is called a descending cell. These cells $\{W_p\}$ as p runs over critical points of f describes M as a disjoint union of them. But this is not a cellular decomposition of M in the true sense of the word, the boundary of the cells may be very wild. To derive the Morse inequalities from this construction, Smale proposed the use of transversality into the Thom's cell decomposition $M = \cup_p W_p$. If f is changed to $-f$, a critical point p of f of index λ_p becomes a critical point of $-f$ of index $n - \lambda_p$. One then gets a new descending cell W'_p of dimension $n - \lambda_p$, which is called an ascending cell. The cell W'_p is generated by instantons of $-\nabla f$ starting at p. The corresponding decomposition M becomes $M = \cup_p W'_p$. Two types of cells W_p and W'_r are said to intersect transversally if for each $x \in W_p \cap W'_r$ the tangent spaces $T_x(W_p)$ and $T_x(W'_r)$ span the tangent space $T_x(M)$, that is,

$$T_x W_p + T_x W'_r = T_x M.$$

The vector field ∇f is called transversal if the two types of cells always intersect transversally. Then, because the tangent space to the integral manifold X_x of ∇f through x is contained in both $T_x(W_p)$ and $T_x(W'_r)$ with opposite orientations, we have an exact sequence

$$0 \longrightarrow T_x(X_x) \longrightarrow T_x(W_p) \oplus T_x(W'_r) \longrightarrow T_x(M) \longrightarrow 0.$$

The intersection $W_p \cap W'_r$ is generated by instantons of ∇f which start at p and end at r, or instantons of $-\nabla f$ which start at r and end at p. The transversality condition implies that

$$\dim(W_p \cap W'_r) = \lambda_p - \lambda_r + 1.$$

This means that if $\lambda_p - \lambda_r = -1$, then the number of instantons of ∇f which start at p and end at r is finite. Bott called these proper instantons.

Now given a non-degenerate function f with transversal gradient ∇f, orient the descending cells arbitrarily, and form the chain complex $C_f(M)$ whose λ_p dimensional chain group is freely generated over $\mathbb{Z}$ by cells W_p of dimension λ_p. Then define the boundary operator $\partial : C_k \longrightarrow C_{k-1}$, where $k = \lambda_p$, by setting $\partial[W_p] = \sum \epsilon(\sigma)[W_r]$, where σ is a proper instanton of ∇f from the critical point p of index k to the critical point r of index $k-1$, and $\epsilon(\sigma)$ is $+1$ or -1 according to whether for some $x \in \sigma$ the above exact sequence is orientation preserving or not. (Here M is oriented, and an orientation of W_p induces an orientation of W_r', while $T_x(X_x)$ is oriented by $-\nabla f_x$, so $\epsilon(\sigma)$ is well-defined.) Smale proved that $\partial^2 = 0$, and as a consequence he obtained the following theorem.

Theorem 3.1. *The cohomology groups $H^*(C_f(M))$ are isomorphic to the singular cohomology groups $H^*(M, \mathbb{Z})$.*

Remark 3.2.

(1) One can get the Morse inequalities from this set up from the purely algebraic fact that for a finite dimensional chain complex $C = \oplus_k C_k$ the series $C_t = \sum_k \dim C_k t^k$ (called the counting series of the complex C) satisfies the inequalities

$$C_t \geq P_t, \quad t \in \mathbb{R},$$

where $P_t = \sum_k (\dim H^k) \cdot t^k$ is the Poincaré series of the cohomology of the complex C.

(2) The existence of a boundary operator ∂ is equivalent to the strong Morse inequalities. However, the Morse inequalities give no canonical form for ∂.

(3) The dual of the chain complex $C_f(M)$ may be identified with the cochain complex $\oplus_k C^k$, where C^k is the group freely generated over $\mathbb{Z}$ by the set of critical points of index k, and the coboundary $\delta : C^k \longrightarrow C^{k+1}$ is given by $\delta(p) = \sum \epsilon(\sigma) \cdot q$, the summation is over all proper instantons σ from the critical point p of index k to other critical points q of index $k+1$.

4 Work of Witten

We first recall the Hodge-de Rham theory. Consider a compact oriented manifold M without boundary equipped with a Riemannian structure g, and a smooth map $f : M \longrightarrow \mathbb{R}$. Let $\Omega^*(M)$ be the de Rham complex of M with exterior derivative $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$, which maps k-forms into $(k+1)$-forms, $k = 0, 1, \ldots, n-1$, satisfying $d^2 = 0$

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \longrightarrow \cdots \longrightarrow \Omega^n(M).$$

The metric g defines the adjoint $d^* : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ of d, which satisfies $(d^*)^2 = 0$. Then the Hodge Laplacian $\Delta = dd^* + d^*d$, which maps k-forms into k-forms, gives a decomposition of $\Omega^k(M)$ into a direct sum of finite dimensional eigenspaces of Δ

$$\Omega^k(M) = \oplus\, \Omega^k_\lambda(M),$$

where $\Omega^k_\lambda(M) = \{\phi \in \Omega^k(M) | \Delta\phi = \lambda\phi\}$. The forms in the null space $\ker \Delta = \Omega^k_0(M)$ are called harmonic forms.

The Hodge theory says that

(1) $\Omega_0^k(M) = H^k(M)$
(2) for each $\lambda > 0$ the following sequence is exact

$$0 \longrightarrow \Omega_\lambda^k \xrightarrow{d_\lambda} \Omega_\lambda^{k+1} \longrightarrow \cdots \longrightarrow \Omega_\lambda^n \longrightarrow 0,$$

where d_λ is the restriction of d on $\Omega^k = \oplus\Omega_\lambda^k$ to Ω_λ^k.

These two conditions imply that all finite dimensional complexes

$$\Omega_a^* = \oplus_{\lambda \leq a}\Omega_\lambda^*, \quad a > 0$$

have $H^*(M)$ as cohomology. Also, all the counting series $\Omega_{a,t}^* = \sum_k \dim(\Omega_a^k) \cdot t^k$ satisfy Morse inequalities relative to $P_t(M)$. Since all the complexes Ω_a^* have the same cohomology $H^*(M)$, an alteration of the metric g on M will not affect these inequalities.

In his generalization of the above theory, Witten considered the decomposition of the Hilbert space $\mathbf{H}$ of one particle state, as we have described in §1,

$$\mathbf{H} = \mathbf{H}_B \oplus \mathbf{H}_f,$$

where the algebra of operators on $\mathbf{H}$ has only two sypersymmetric generators Q_1 and Q_2. These satisfy the anti-commutative law

$$\{Q_1, Q_2\} = Q_1Q_2 + Q_2Q_1 = 0.$$

This model corresponds to the two-dimensional theory where both space and time are one-dimensional. To introduce squares of the generators, they are expressed in terms of the momentum $\mathcal{P}$ and the Hamiltonian $\mathcal{H}$ by the equations

$$Q_1^2 = \mathcal{H} + \mathcal{P}, \quad Q_2^2 = \mathcal{H} - \mathcal{P}.$$

These relations together with the anti-commutation give

$$[Q_i, \mathcal{H}] = 0, \quad \text{and} \quad \mathcal{H} = \frac{1}{2}(Q_1^2 + Q_2^2).$$

To distinguish bosons (B) and fermions (F), the counting operator $(-1)^F$ is introduced

$$\begin{aligned}(-1)^F|\, S\, \rangle &= -S|\ \rangle \text{ if } S = F\\ &= \ \ S|\ \rangle \text{ if } S = B.\end{aligned}$$

The operator $(-1)^F$ counts the number fermions modulo two. With respect to a basis of $\mathbf{H}_B$ and $\mathbf{H}_F$, $(-1)^F$ has a representation

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

The sypersymmetric generators Q_1 and Q_2 anti-commute with $(-1)^F$. We have

$$Q_i = \begin{pmatrix} 0 & Q_i^{BF} \\ Q_i^{FB} & 0 \end{pmatrix}, \quad i = 1, 2.$$

where the operators Q_i^{BF} and Q_i^{FB} transform one type of particle into the other

$$Q_i^{BF}: \mathbf{H}_F \longrightarrow \mathbf{H}_B \quad \text{and} \quad Q_i^{FB}: \mathbf{H}_B \longrightarrow \mathbf{H}_F, \quad i = 1, 2.$$

Witten constructed the above supersymmetric algebra by the following definitions

$$\mathbf{H} = \oplus_{k\geq 0}\Omega^k(M), \quad \mathbf{H}_B = \oplus_{k\geq 0}\Omega^{2k}(M), \quad \mathbf{H}_F = \oplus_{k\geq 0}\Omega^{2k+1}(M),$$
$$Q_1 = (d + d^*), \qquad Q_2 = i(d - d^*).$$

Then Q_1 and Q_2 anti-commute with each other, because

$$Q_1 Q_2 = i(d + d^*)(d - d^*) = i(d^*d - dd^*) = -Q_2 Q_1.$$

One can also verify that $Q_1^2 = Q_2^2$, so in this model the momentum $\mathcal{P} = 0$. Moreover,

$$\mathcal{H} = \frac{Q_1^2 + Q_2^2}{2} = dd^* + d^*d = \oplus_{k\geq 0}\Delta_k.$$

Witten brought in a Morse function f into the model without disturbing the supersymmetric algebra by introducing the operators $d_s = e^{-sf} \circ d \circ e^{sf}$, $s \in \mathbb{R}$, which is the conjugation of d by e^{sf}. Clearly $d_s^2 = 0$, and the cohomology $H_s^*(M) = \ker d_s / \operatorname{Im} d_s$ is the same as the de Rham cohomology $H^*(M)$.

Again $d_s^* = e^{sf} \circ d \circ e^{-sf}$. The new generators are given by

$$Q_1(s) = (d_s + d_s^*), \qquad Q_2(s) = i(d_s - d_s^*),$$

and the associated Laplacian is

$$\Delta(s) = d_s d_s^* + d_s^* d_s = \sum_{k\geq 0} \Delta_k(s),$$

with $\Delta(0) = \Delta$. We have the decomposition of the corresponding complex of differential forms

$$\Omega^*(s) = \oplus\, \Omega_\lambda^*(s)$$

into eigenspaces of the Hamiltonian $\mathcal{H}_s = \Delta(s)$.

Although the spectrum of $\mathcal{H}_s$ depends on s, the null space of $\mathcal{H}_s$ is independent of s (it is not changed by conjugation by e^{sf}). Therefore, by Hodge theory the Betti numbers of M are given by

$$b_k = \dim \ker \Delta_k(s) \text{ for all } s.$$

As before, we also have for each $a > 0$ the finite dimensional complex of differential forms $\Omega_a^*(s)$ spanned by eigenforms of $\mathcal{H}_s$ with eigenvalues $\lambda \leq a$.

The counting series of any of these complexes $\Omega_a^*(s)$ must be greater than or equal to the Poincaré series of M, for the same reason as given in Remark 3.2 (1). These will become the Morse inequalities if $\dim \Omega_a^k(s) = m_k$, where m_k is the number of critical points of f of index k.

Witten obtained the Morse theoretic results by studying this family of cochain complexes $\{\Omega_a^*(s)\}$. His argument is that if the parameter s very large, the cochain complex $\Omega_a^*(s)$ becomes independent of s, and behaves like the dual of Smale's chain complex.

We describe briefly the ideas behind Witten's approach. The point is that $\Delta = dd^* + d^*d$ is a second order operator, and so its expansion terminates at the t^2 term, giving an explicit formula for $\mathcal{H}_s$

$$\begin{aligned}\mathcal{H}_s &= d_s d_s^* + d_s^* d_s = e^{-sf}(dd^* + d^*d)\, e^{sf} \\ &= (dd^* + d^*d) + s\cdot\sum_{i,j}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\left[dx_i, \frac{\partial}{\partial x_j}\right] + s^2\cdot(df)^2 \\ &= \Delta + V,\end{aligned}$$

where V is the sum of the last two terms.

Here the x_i are the local coordinates, $\partial/\partial x_i$ is considered as the operation of internal multiplication ι on the exterior algebra $(\psi \mapsto \iota(\partial/\partial x_i)\psi)$, and dx_i as the operation of exterior multiplication by dx_i (in quantum field theory, the operators dx_i and $\partial/\partial x_i$ are called fermion creation and annihilation operators respectively), the $\partial^2 f/\partial x_i \partial x_j$ are the components of the second derivative of f in the basis (dx_i), and lastly $(df)^2 = g^{ij}(\partial f/\partial x_i)(\partial f/\partial x_j)$ is the square of the gradient of f with respect to the Riemannian metric g on M.

The above formula for $\mathcal{H}_s$ suggests that $\mathcal{H}_s$ may be viewed as a Schrödinger type operator with potential term V. Therefore in order to obtain Morse theoretic results, one should investigate $\mathcal{H}_s$ for large s. For large s the potential is dominated by its s^2 term whose coefficient is $(\nabla f)^2$. Also the enormous growth of the potential energy will force the eigenfunctions of $\mathcal{H}_s$ to be zero. Therefore for very large s the potential energy becomes very larger, except in the vicinities of the critical points of f where $\nabla f = 0$, and the supports of the eigenfunctions of $\mathcal{H}_s$ are concentrated at the critical points of f. Therefore in order to make contact with the Morse theory one should approximate quantities by taking s large and expanding them about critical points of f.

Let us indicate roughly how Witten obtained the weak Morse inequalities $m_k \geq b_k$ by this principle. As stated in §2, the function f can be expressed in a neighbourhood of a critical point of index λ in the form

$$f(x) = f(0) + \frac{1}{2}\sum_i \epsilon_i x_i^2,$$

where $\epsilon_1 = \cdots = \epsilon_\lambda = -2$, and $\epsilon_{\lambda+1} + \cdots + \epsilon_n = 2$. In this local coordinate system, Witten approximated the Hamiltonian $\mathcal{H}_s$ as

$$\tilde{\mathcal{H}}_s = \left(\sum_i -\frac{\partial^2}{\partial x_i^2} + s^2\epsilon_i^2 x_i^2 + s\,\epsilon_i[dx_i, \partial/\partial x_i]\right),$$

by neglecting the quantities in x_i of order > 2 (this is permissible, because for large s the eigenfunctions are concentrated very near the critical point). Write $\tilde{\mathcal{H}}_s = \sum_i (\mathcal{H}_i + s\,\epsilon_i K_i)$. The term $\mathcal{H}_i = -\frac{\partial^2}{\partial x_i^2} + s^2\epsilon_i^2 x_i^2$ represents the Hamiltonian of the simple harmonic oscillation. It operates on functions, and its spectrum is $s|\epsilon_i|(1+2N_i)$ for $N_i = 0, 1, 2, \ldots$, each with multiplicity one. On the other hand, the operator $K_i = [dx_i, \partial/\partial x_i]$ operates on forms, and it has eigenvalues ± 1 (note that $K_i(dx_i) = dx_i$ and $K_i(dx_j) = -dx_j$ for $i \neq j$). The operators $\mathcal{H}_i$ and K_i commute and can be diagonalized simultaneously. Therefore the spectrum of $\tilde{\mathcal{H}}_s$ consists of the eigenvalues

$$s\sum_i |\epsilon_i|(1+2N_i) + \epsilon_i n_i, \quad n_i = \pm 1.$$

and it acts on the whole of the exterior algebra. If we restrict the action of $\tilde{\mathcal{H}}_s$ on k-forms, then we must have that the number of n_i, which take the value $+1$, is equal to k. To make the eigenvalue of $\tilde{\mathcal{H}}_s$ vanish, we must take $N_i = 0$, and must choose n_i to be $+1$ if and only if ϵ_i is negative, that is, choose $n_i = 1$ for $i \leq \lambda$ and $n_i = -1$ for $i > \lambda$. Therefore we may conclude by approximating $\mathcal{H}_s$ about a critical point that $\tilde{\mathcal{H}}_s$ has precisely one zero eigenvalue corresponding to a k-form where k is the index of the critical point. Taking into account that some of these forms may disappear in the process of approximation, we conclude that $m_k \geq b_k$.

Witten obtained the strong Morse inequalities by investigating the rest of the spectrum of $\mathcal{H}_s$ around a critical point. This necessitates some technicalities of physics. Let us denote the complex $\Omega^*_a(s)$ when s is large by $\Omega^*_a(\infty)$. Its coboundary operator d_∞ is induced by the differential operator d on $\Omega^k_a(\infty)$, where d is the exterior derivative of the de Rham complex $\Omega^*(M)$. The observation of Witten is that dim $\Omega^k_a(\infty)$ is the number of critical points of f of index k. The operator d_∞ corresponds to the dual of the boundary operator ∂ of the Smale's theory, and maps proper instantons from critical points of index k to those of index $k+1$. The verification of the fact that $d^2_\infty = 0$ with this combinatorial realization is difficult. To get around to this difficulty, one has to turn to the quantum mechanical framework of instantons and tunnelling, From the point of view of quantum mechanics d_∞ corresponds to the 'tunnelling effect' between critical points, that is, tunnelling between the minima of the potential $s^2(\nabla f)^2$. The advantage of tunnelling is that, unlike spectral analysis, it is not limited to working in the neighbourhood of only one critical point.

One needs to consider physics to see these facts, and once these are established the strong Morse inequalities can be deduced following the line of arguments of Smale.

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