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Mathematical Analysis in India

V. S. Sunder

Abstract As part of the Platinum Jubilee celebrations of the Indian Academy of Sciences, many mathematicians have been asked to write something about the areas in which there is work done by the fellowship of the academy and about the contributions to these areas by Indians. Mathematics being too vast a canvas, this writing has been broken down further into sub-disciplines. More specifically, regarding work in mathematical analysis, three people – namely Alladi Sitaram, V. S. Sunder and M. Vanninathan – were entrusted with the task of coming up with something representative, subject to some constraints of length. This article comprises the inputs of Sitaram and Sunder, in (the comparatively close disciplines of) harmonic analysis and operator theory/algebras; whereas the input of Vanninathan, on PDEs, will appear in a separate article. Sunder gratefully acknowledges the inputs and help provided by many people, such as Gadadhar Misra, Rajarama Bhat and Kalyan Sinha.

Keywords Functional analysis · Operator theory · Operator algebras · Quantum probability · Quantum dynamical systems · Subfactors · Harmonic analysis.

1 Functional analysis

Research in functional analysis in India is pursued in varied forms and at various levels of seriousness. Thus you will find some Banach space theorists, some numerical functional analysts, operator theorists and operator algebraists of varying hues, ... The 'opinions' stated in this article are naturally influenced by the author's own tastes and limitations, and if some areas and/or people are not mentioned, that is partly due to limitations of space, partly to the author's own inadequacies, and any perceived slight is unintentional, and the author craves indulgence from the injured parties.

1.1 Operator theory

The most outstanding open problem in operator theory is *the invariant subspace problem*, which is easy to state: for every bounded operator T on a separable complex Hilbert space $\mathcal{H}$, does there exist a closed non-trivial subspace $\mathcal{M} \subseteq \mathcal{H}$ which is invariant under T, that is, $T\mathcal{M} \subseteq \mathcal{M}$. (However, there does exist an operator on the Banach space $\ell^1(\mathbb{N})$ which does not admit an invariant subspace [30].) It was shown, as early as 1954, that any non-zero compact operator admits an invariant subspace

V. S. Sunder
The Institute for Mathematical Sciences, CIT Campus, Taramani, Chennai 600 113, India
E-mail: sunder@imsc.res.in

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[5]. More recently, a striking result of Lomonosov [24] shows that if a bounded linear operator T on a complex separable Hilbert space $\mathcal{H}$ commutes with a non-zero compact operator on $\mathcal{H}$ then T admits a non-trivial closed invariant subspace. Since there exist [20] bounded linear operators that do not commute with any non-zero compact operators, the invariant subspace problem remains open! There is one other class, namely subnormal operators, for which the invariant subspace theorem has been proved (by Brown [11]). Later, a simple proof was given by [31]. The notes [19] provide an account of more recent developments in this area with an emphasis on the invariant subspace problem for operators on a Banach space.

The spectral theorem for normal operators provides both a complete set of unitary invariants (=invariants for unitary equivalence) and a canonical model. Beyond normal operators, there are only few instances where complete unitary invariants have been found, and canonical models constructed.

The first such results are due to B. Sz Nagy and Foias. These involve the class of completely non-unitary (cnu) contractions [26]. They call their unitary invariant the characteristic function of the operator. The generalization of this theory to the multi-variate context has begun over the last decade [3,8,9,28,29].

Secondly, in the work of Carey and Pincus [13], they associate with a hyponormal operator T an operator-valued function called the mosaic of T. The mosaic of a pure hyponormal operator T with trace class self commutator is then shown to be a complete unitary invariant for T. They also prove the existence of a hyponormal operator with a prescribed mosaic.

Finally, the work of Cowen and Douglas [14] associates (to a certain class of operators, and more generally) to a commuting tuple $\mathbf{T}$ of operators, a holomorphic Hermitian vector bundle $E_{\mathbf{T}}$. They show that the equivalence class of the vector bundle $E_{\mathbf{T}}$ and the unitary equivalence class of the operator $\mathbf{T}$ determine each other. Thus a correspondence is set up between complex geometry and operator theory. More recently, some of this work has been recast in the language of Hilbert modules over a function algebra making it possible to ask natural questions involving submodules and quotient modules [17,16]. An identification of those homogeneous holomorphic Hermitian vector bundles over the unit disc which correspond to a Hilbert space operator has been completed [25]. For bounded symmetric domains, some partial answers are to be found in [4,10]. The question of when two operators in the Cowen Douglas class are similar remained open until very recently. Now, it is answered completely, using the ordered K-group of the commutant algebra as an invariant [23].

The spectral theorem shows the usefulness of the L^∞ functional calculus for normal operators, or more generally, selfadjoint (closed) operator algebras. In early attempts to find a useful functional calculus, von Neumann introduced the notion of a spectral set for an operator T. A compact subset X of the complex plane is said to be a spectral set for T if X contains the spectrum σ_T of the operator T and $\|r(T)\| \leq \|r\|_{X,\infty}$ for every $r \in \mathrm{Rat}(X)$, where we write $\mathrm{Rat}(X)$ for the algebra of all rational functions with poles off X and $\|r\|_{X,\infty} = \sup\{|R(z)| : z \in X\}$. Now, if X is a spectral set for the operator T, then the homomorphism $r \mapsto r(T)$ extends to the completion of $\mathrm{Rat}(X)$ with respect to $\|\cdot\|_{X,\infty}$. This enlarged functional calculus has several useful properties. For $X = \bar{\mathbb{D}}$, the closed unit disc, von Neumann established the surprising result: $\|p(T)\| \leq \|p\|_{\mathbb{D},\infty}$ for all polynomials p if and only if $\|T\| \leq 1$. Soon afterwards, B. Sz Nagy and Foias showed that if T is a contraction then the homomorphism $\varrho_T : \mathrm{Pol}(\mathbb{D}) \to \mathcal{L}(\mathcal{H})$ defined by $p \mapsto p(T)$ dilates. In other words, there exists a $*$-homomorphism $\tilde{\varrho}_T : C(\mathbb{T}) \to \mathcal{L}(\mathcal{K})$ of the continuous functions $C(\mathbb{T})$ into the algebra

$\mathcal{L}(\mathcal{K})$ of bounded operators on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $P_{\mathcal{H}}\tilde{\varrho}_T(p) = \varrho_T(p)$ for any polynomial p, as long as $\|T\| \leq 1$ (here $P_{\mathcal{H}}$ is the orthogonal projection from $\mathcal{K}$ to $\mathcal{H}$). The von Neumann inequality follows from this dilation theorem for contractive operators T. If X is a spectral set for the operator T, does it necessarily dilate? An affirmative answer was given for 1-connected compact sets X in [1]. Although, it was widely believed that the answer would be negative in general, examples were found only very recently [18]. However, in the early seventies, Arveson showed that if one assumed X to be a 'complete spectral set' for the operator T then the induced homomorphism admits a dilation [7]. However, corresponding questions involving commuting tuples of operators appear to be very mysterious (cf. [2, 17]). In his famous list of 'ten problems' [21], Halmos notes that if an operator T is similar to a contraction ($T = LSL^{-1}$ for some invertible operator L and some contractive operator S), then $\|p(T)\| \leq K\|p\|_{\mathbb{D},\infty}$ for all polynomials p. He then asks if the converse is true. Recently, Pisier has found a counter example. This example along with lot more on similarity questions from several different areas of mathematics are discussed in [27].

There is another way in which one may go beyond the spectral theorem, and that is to work modulo compact operators. The rationale is that the compact operators are limits of finite dimensional operators and therefore don't count in an essential way. Thus the two operators S, T are said to be essentially unitarily equivalent if $USU^* = T + K$ for some unitary operator U and a compact operator K. Similarly, the operator N is said to be essentially normal if $NN^* - N^*N = K$ for some compact operator K. An operator T has an essential spectrum $\sigma_e(T)$, which is the set of complex numbers λ such that $T - \lambda I$ is not invertible modulo the compacts. The index is a map $\text{ind}_T : \mathbb{C} \setminus \sigma_T \to \mathbb{Z}$, which is defined naturally as the difference $\dim\ker(T - \lambda) - \dim\ker(T - \lambda)^*$. In the early seventies, Brown, Douglas and Fillmore (cf. [15]) proved that two essentially normal operators S and T are essentially unitarily equivalent if and only if $\sigma_e(S) = \sigma_e(T)$ and $\text{ind}_S = \text{ind}_T$. The proof of this theorem involves calculating the 'Ext' group for the algebra $C(X)$ of continuous functions on a subset X of the complex plane $\mathbb{C}$. As they have shown, the context for this calculation is algebraic topology. In particular, they show $\text{Ext}(X) = \text{Hom}(\pi^1(X), \mathbb{Z})$ for any subset of X of the complex plane $\mathbb{C}$. This has led to unexpected connections with K-theory and to the Atiyah- Singer index theorem. The excitement is far from over [22].

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