Math. Slovaca, **48** (1998), No. 4, 411–421

PARAMETRIC HOMOTOPY PRINCIPLE OF SOME PARTIAL DIFFERENTIAL RELATIONS

MAHUYA DATTA* — AMIYA MUKHERJEE**

(*Communicated by Július Korbaš*)

ABSTRACT. An rth order partial differential relation is a subspace in the space of r-jets of C^r sections of a fibre bundle $p\colon E \to X$. In this paper we consider an open G-invariant relation for equivariant sections of a G-fibre bundle, where G is a compact Lie group, and consider the homotopy classification of equivariant solutions of the differential relation. We also obtain an equivariant analogue of the Smale-Hirsch immersion theorem.

1. Introduction

The open extension theorem of Gromov [4] provides a unifying principle for the work of Smale [9], Hirsch [5], Phillips [8], Feit [3], and others on immersion and submersion problems. The main purpose of the present paper is to study the theorem in an equivariant context, and obtain as applications a generalization of the transversality theorem of Gromov, and an equivariant version of the Smale-Hirsch immersion theorem.

Let G be a compact Lie group, X a differentiable G-manifold with a G-invariant Riemannian metric, and $p\colon E \to X$ a G-locally trivial differentiable G-fibre bundle. Recall that a G-fibre bundle $p\colon E \to X$ is a locally trivial G-map, and that this is G-locally trivial if for every x in X there exists a G_x-invariant open neighbourhood U_x of x such that $p^{-1}(U_x)$ is differentiably G_x-equivalent to the trivial G_x-fibre bundle $U_x \times p^{-1}(x)$. As has been shown in Bierstone [1; Theorem 4.1], a differentiable G-fibre bundle is G-locally trivial if and only if it has the equivariant covering homotopy property.

Let $p^{(r)}\colon E^{(r)} \to X$ be the bundle of r-jets of local sections of p. Then $p^{(r)}$ inherits a natural differentiable G-fibre bundle structure, where the action of G on $E^{(r)}$ is given by $g \cdot j^r_x f = j^r_{gx}(gfg^{-1})$, for a local section f of p at $x \in X$ and

AMS Subject Classification (1991): Primary 58A30, 58D10, 57S25.
Key words: equivariant parametric homotopy principle, transversality, immersion, submersion, partial differential relation.

$g \in G$. Then a *partial differential relation*, or simply a *relation*, is a G-invariant subspace $\mathcal{R}$ of $E^{(r)}$.

Let $E_G^{(r)} \subset E^{(r)}$ be the subspace of $E^{(r)}$ consisting of r-jets of equivariant local sections of p defined on G-invariant open sets of X. Then $E_G^{(r)}$ is a G-invariant subspace of $E^{(r)}$. We shall denote the subset $\mathcal{R} \cap E_G^{(r)}$ by $\mathcal{R}_G$.

A section $f \colon X \to E$ of p is called a *solution* of the partial differential relation $\mathcal{R}$, if the corresponding r-jet map $j^r f$ has its image in $\mathcal{R}$.

We shall denote the space of equivariant C^∞ solutions of $\mathcal{R}$ by $\operatorname{Sol}\mathcal{R}$, and the space of equivariant C^0 sections of $p^{(r)}$ with images in $\mathcal{R}_G$ by $\Gamma(\mathcal{R})$. The former space has the C^∞ compact-open topology, whereas the latter one has the C^0 compact-open topology. The r-jet map j^r maps $\operatorname{Sol}\mathcal{R}$ into $\Gamma(\mathcal{R})$, and is continuous with respect to the above topologies.

A relation $\mathcal{R} \subset E^{(r)}$ is said to satisfy *equivariant parametric h-principle* (h for homotopy), if the r-jet map $j^r \colon \operatorname{Sol}\mathcal{R} \to \Gamma(\mathcal{R})$ is a weak homotopy equivalence.

The manifolds $X \times \mathbb{R}$ and $E \times \mathbb{R}$ are G-manifolds under the diagonal G-action on them, the G-action on $\mathbb{R}$ being the trivial one. Moreover, $p \times \mathrm{id} \colon E \times \mathbb{R} \to X \times \mathbb{R}$ is a G-locally trivial G-fibre bundle. Let $\pi \colon X \times \mathbb{R} \to X$ be the canonical projection on the first factor. There is a natural bundle map $\pi^{(r)} \colon (E \times \mathbb{R})^{(r)} \to E^{(r)}$ covering the projection π which sends the r-jet $j^r_{(x,t)} f$ onto the r-jet $j^r_x(\pi \circ f \circ i_t)$, where $i_t \colon X \to X \times \mathbb{R}$ is given by $i_t(y) = (y, t)$ for $y \in X$. We call a relation $\widetilde{\mathcal{R}} \subset (E \times \mathbb{R})^{(r)}$ an *extension* of $\mathcal{R}$, if $\pi^{(r)}$ sends $\widetilde{\mathcal{R}}_G$ onto $\mathcal{R}_G$.

Let $\mathcal{D}_G(X \times \mathbb{R})$ denote the pseudogroup of equivariant local diffeomorphisms on $X \times \mathbb{R}$. We shall be interested in the subpseudogroup $\mathcal{D}_G(X \times \mathbb{R}, \pi)$ of $\mathcal{D}_G(X \times \mathbb{R})$ consisting of fibre-preserving diffeomorphisms, which are local diffeomorphisms λ such that $\pi \circ \lambda = \pi$. The pseudogroup $\mathcal{D}_G(X \times \mathbb{R}, \pi)$ has a natural action on $E \times \mathbb{R}$. This action may be described by a map $\varrho \colon \mathcal{D}_G(X \times \mathbb{R}, \pi) \to \mathcal{D}_G(E \times \mathbb{R})$ in the following way. If $\lambda \colon U \times J \to U \times J'$ is in $\mathcal{D}_G(X \times \mathbb{R}, \pi)$, where J and J' are open intervals of the real line $\mathbb{R}$, then we define $\varrho(\lambda) \colon p^{-1}(U) \times J \to p^{-1}(U) \times J'$ by $\varrho(\lambda)(e,t) = \big(e, \lambda'(p(e), t)\big)$, where $\lambda' \colon U \times J \to J'$ is the G-equivariant map satisfying $\lambda(x,t) = \big(x, \lambda'(x,t)\big)$ so that $\pi_2 \circ \lambda = \lambda'$ (π_2 denotes the projection on the second factor). The map ϱ is continuous with respect to C^∞ compact-open topologies, and it induces an action of $\mathcal{D}_G(X \times \mathbb{R}, \pi)$ on the space of local sections of $p \times \mathrm{id}$, and hence an action on the jet space $(E \times \mathbb{R})^{(r)}$. The actions are given by $(\lambda, f) \mapsto \lambda^* f = \varrho(\lambda)^{-1} \circ f \circ \lambda$ and $\big(\lambda, j^r_{\lambda(x,t)} f\big) \mapsto j^r_{(x,t)}(\lambda^* f)$, where $\lambda \in \mathcal{D}_G(X \times \mathbb{R}, \pi)$ is a local G-diffeomorphism at x and f is a local G-section of $p \times \mathrm{id}$ at $\lambda(x,t)$.

A relation $\mathcal{R}$ is said to be $\mathcal{D}_G(X \times \mathbb{R}, \pi)$-*invariant*, if $\lambda^*(\mathcal{R}) \subset \mathcal{R}$ for every $\lambda \in \mathcal{D}_G(X \times \mathbb{R}, \pi)$.

The main theorem of the paper is:

THEOREM 1.1. *If $\mathcal{R} \subset E^{(r)}$ is an open relation which admits a G- and $\mathcal{D}_G(X \times \mathbb{R}, \pi)$-invariant open extension $\widetilde{\mathcal{R}} \subset (E \times \mathbb{R})^{(r)}$, then $\mathcal{R}$ abides by the equivariant parametric h-principle.*

The main theorem reduces to the "open extension theorem" of Gromov [4; p. 86] when G is trivial, and generalizes the theorems of Bierstone [2], and Izumiya [6] in the sense that we do not require the invariance of the basic partial differential relation $\mathcal{R}$ under the action of the pseudogroup of local diffeomorphisms.

Our next theorem is an application of Theorem 1.1, and is a generalization of the transversality theorem of Gromov [4; p. 87]. First recall that if X is a G-manifold, then its tangent bundle TX is a G-vector bundle over X under the differential action of G. In fact, since TX has a Lie structure group, TX is actually G-locally trivial (see Bierstone [1; Theorem 4.2]). Also, if H is a closed subgroup of G, then the H-fixed point set X^H is a submanifold of X, and $T(X^H) = (TX)^H$ ([7; §11.13]). Thus $(TX)^H$ is a vector bundle over X^H. Now consider a G-locally trivial G-fibre bundle $p\colon E \to X$, and let ξ and η be G-subbundles of TE and TX respectively. Let $\mathcal{R} \subset E^{(1)}$ be the relation consisting of 1-jets of germs of local sections, $j^1_x\sigma$ for $x \in X$, such that $j^1_x\sigma(\eta_x) \cap \xi_{\sigma(x)} = \{0\}$. Thus the solutions of $\mathcal{R}$ are sections of p which are transversal to ξ on η.

THEOREM 1.2. *If for each isotropy subgroup H of the action of G on X we have locally*

$$\dim X^H + \dim \xi^H < \dim E^H ,$$

where $\dim \xi^H$ means the fibre dimension of ξ^H, then $\mathcal{R}$ satisfies equivariant parametric h-principle.

Explicitly, the condition means that for each $x \in X^H$ and each $e \in p^{-1}(x) \cap E^H$, $\dim X^H$ at x is strictly less than $\dim E^H - \dim \xi^H$ at e.

This theorem leads to an equivariant version of the Smale-Hirsch immersion theorem. Let X and Y be smooth G-manifolds with $\dim X < \dim Y$. Let $\mathrm{Imm}_G(X, Y)$ denote the space of equivariant smooth immersions of X in Y, and $\mathrm{R}_G(TX, TY)$ denote the space of equivariant continuous monomorphisms $F\colon TX \to TY$ such that $F_x|_{T_x(Gx)}$ is given by the differential of the map $gx \mapsto gf(x)$ of the orbit Gx onto the orbit $Gf(x)$, where $f\colon X \to Y$ is the map covered by $F\colon TX \to TY$.

THEOREM 1.3. *The differential map* $\mathrm{d}\colon \mathrm{Imm}_G(X, Y) \to \mathrm{R}_G(TX, TY)$ *is a weak homotopy equivalence, provided* $\dim X^H < \dim Y^H$ *locally for every isotropy subgroup H of the G-action on X.*

This theorem may be compared with earlier work on equivariant immersions by Bierstone [2] and Izumiya [6]. Bierstone used a dimension condition which may be described as follows. Recall that an invariant component of a G-manifold X is the inverse image under the orbit map $X \to X/G$ of a component of X/G, and that the saturation of a fixed point set X^H is the closed G-subspace $X^{(H)} = G \cdot X^H$ of X. Let $\{X_i^j\}$ be the set of invariant components of the saturations $X^{(H_j)}$ partially ordered by inclusion, where H_j runs over the isotropy subgroups of G over X. Then the equivariant immersion theorem of Bierstone demands that $\dim(X_i^j)^{H_j}$ for each minimal component X_i^j should be strictly less than the dimension of each component of Y^{H_j}. On the other hand, if $n = \max\{\dim X^H\}$ where H runs over isotropy subgroups of G over X, and if $m = \min\{\dim Y^K\}$ where K runs over isotropy subgroups of G over Y, then the equivariant immersion theorem of Izumiya assumes that $n < m$. It follows then that Izumiya's theorem is weaker than Theorem 1.3, and Theorem 1.3 is weaker than Bierstone's theorem.

2. Proof of Theorem 1.1

We shall resort to the sheaf theoretic treatment of Gromov [4]. Let Φ be the sheaf on X with $\Phi(U)$, where U is an open set in X (not necessarily G-invariant), as the space of equivariant C^∞ solutions of $\mathcal{R}$ over GU, and with obvious restriction maps which are continuous with respect to the C^∞ compact-open topologies. If C is a subset of X, we let $\Phi(C)$ to be the direct limit of the spaces $\Phi(U)$ over all open sets U containing C. Thus $\Phi(C)$ consists of germs of equivariant C^∞ solutions of $\mathcal{R}$ near C, and $\Phi(C) = \Phi(GC)$. We endow $\Phi(C)$ with the following quasi-topological structure, in order to avoid certain awkward situations (see Gromov [4; p. 35]). If P is any topological space, then the space $C^0(P, \Phi(C))$ of quasi-continuous maps (which will also be referred to as continuous maps) from P to $\Phi(C)$ is the direct limit of the spaces $C^0(P, \Phi(U))$ of continuous maps $P \to \Phi(U)$ over all open sets U containing C. Thus the restriction maps $r\colon \Phi(C) \to \Phi(C')$, $C' \subset C \subset X$, are continuous in the sense that for any $f \in C^0(P, \Phi(C))$ the composition $r \circ f \in C^0(P, \Phi(C'))$.

Similarly, we define the sheaf Ψ of equivariant C^0 sections of $p^{(r)}$ whose images lie in $\mathcal{R}_G$.

It is easy to see that $\Phi(X)$ and $\Psi(X)$ are respectively the spaces $\operatorname{Sol}\mathcal{R}$ and $\Gamma(\mathcal{R})$, and the r-jet map induces a continuous sheaf homomorphism $j^r\colon \Phi \to \Psi$.

In view of the sheaf homomorphism theorem of Gromov [4; p. 77], the proof of our theorem consists in showing that the sheaf Φ is flexible, which means that for every pair of compact sets (C, C') in X the restriction map $r\colon \Phi(C) \to \Phi(C')$ is a Serre fibration. The other prerequisites, namely, flexibility of Ψ, and local weak homotopy equivalence of $j^r\colon \Phi \to \Psi$ can be worked out easily following respectively the arguments of the Flexibility sublemma of Gromov [4; p. 40], and Lemma 5.4 of Bierstone [2] ($\mathcal{R}$ being open).

To prove the flexibility of Φ, we need to consider the solution sheaf $\widetilde{\Phi}$ of the relation $\widetilde{\mathcal{R}}$. Using the fact that $\widetilde{\mathcal{R}}$ is an open extension of $\mathcal{R}$, it is not difficult to show that the canonical restriction $\alpha\colon \widetilde{\Phi}|_X \to \Phi$ is a microextension. Therefore, our objective is to show that the sheaf $\widetilde{\Phi}|_X$ is flexible, because once this is done, the flexibility of Φ will follow directly from the Microextension theorem of Gromov [4; p. 85].

Before taking on the relation $\widetilde{\mathcal{R}}$, we observe the following simple but extremely important fact. Let S be a compact G-invariant hypersurface lying in a G-invariant open set $U \subset X$ and δ be a positive real. Let $\mathcal{E}_G(U, U \times (-\delta, \delta))$ be the space of equivariant C^∞ embeddings of $\operatorname{Op} U$ in $U \times (-\delta, \delta)$ with C^∞ compact-open quasi-topology, where $\operatorname{Op} U$ denotes an arbitrary open invariant neighbourhood of U in $U \times (-\delta, \delta)$ which may be different for different embeddings. Suppose that for some $\tau > 0$, the τ-neighbourhood U_τ of S is contained in U. Then we have:

LEMMA 2.1. *For every real number a, $0 < a < \delta$, there exists an isotopy $\sigma\colon I \to \mathcal{E}_G(U, U \times (-\delta, \delta))$ such that*

(i) *for each $t \in I$, σ_t is a fibre-preserving diffeomorphism; in particular σ_0 is the inclusion map,*
(ii) *for each t, $\sigma_t(x, s) = (x, s)$ whenever x lies outside U_τ,*
(iii) *for each x lying in a fixed neighbourhood of S, $d(\sigma_1(x, s), X) > a$, where d denotes the distance with respect to the G-invariant metric on $X \times \mathbb{R}$.*

Note that following Gromov, the diffeotopy σ_t may be said to sharply move X locally in $X \times \mathbb{R}$ at the hypersurface S.

Proof. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function defined by

$$f(u) = \begin{cases} \exp 1/(u^2 - 1) & \text{if } |u| < 1, \\ 0 & \text{if } |u| \geq 1. \end{cases}$$

Next, define a 1-parameter family of maps

$$\sigma_t\colon \operatorname{Op} U \to U \times (-\delta, \delta), \qquad 0 \leq t \leq 1,$$

by

$$\sigma_t(x, s) = \Big(x, tcf\big(d(x, S)/\tau\big) + s\Big),$$

where c is a constant (the value of which will be determined later according to our requirements).

It is clear from the definition that σ_0 is the inclusion map, σ_t is fibre-preserving, and $\sigma_t(x, s) = (x, s)$ if x lies outside the τ-neighbourhood of S in X. Also, each σ_t is an equivariant map, because $d(gx, S) = d(gx, gS) = d(x, S)$.

To prove that σ_t is an embedding, it is enough to observe that σ_t is fibre-preserving, and that, for a fixed x, the map $s \mapsto tcf\big(d(x,S)/\tau\big)+s$ is a one-one immersion.

Now observe that $\max\limits_{t,x} tcf\big(d(x,S)/\tau\big) = cf(0)$, and choose c so that $a/f(0) < c < \delta/f(0)$. Then, it is easy to verify that there exists an $\varepsilon > 0$ such that $\sigma\colon I \to \mathcal{E}_G\big(U \times (-\varepsilon,\varepsilon), U \times (-\delta,\delta)\big)$ has all the required properties. □

We now turn to the proof of flexibility of the sheaf $\widetilde{\Phi}|_X$. Since compressibility of deformations over compact sets is equivalent to flexibility of the sheaf [4; p. 80–81], it is sufficient to prove that an arbitrary deformation $\psi\colon Q\times I \to \widetilde{\Phi}(A)$, where $A \subset X$ is a compact set, is compressible. To see this, let us take a G-invariant open neighbourhood $\bar{U}$ of A in $U \cap X$, where $U \subset X \times \mathbb{R}$ is a common domain for the family of maps $\psi(q,t)$ parametrized by $Q \times I$ (such an U exists by the quasi-continuity of ψ). Since A is compact, we get a G-invariant open neighbourhood U_1 of A in X (with closure $\mathrm{cl}\, U_1$ compact) and an $a > 0$ such that

$$U_1 \subset \bar{U} \qquad \text{and} \qquad \mathrm{cl}\, U_1 \times [-2a, 2a] \subset U\,.$$

Choose G-invariant open sets V_0 and V such that $\mathrm{cl}\, V_0$ and $\mathrm{cl}\, V$ are compact and

$$A \subset V_0 \subset \mathrm{cl}\, V_0 \subset V \subset \mathrm{cl}\, V \subset U_1\,.$$

Set

$$X_0 = \mathrm{cl}\, V_0 \times [-a/2, a/2]\,,$$
$$Y_0 = \mathrm{cl}\, U_1 \times [-2a, 2a] \setminus V \times (-a, a)\,.$$

The sets X_0 and Y_0 are compact, G-invariant and disjoint from each other. Let Δ denote the diagonal subset of $I \times I$. Define a map $\varphi_1\colon Q \times \Delta \to \widetilde{\Phi}\big(\mathrm{cl}\, U_1 \times [-2a, 2a]\big)$ by

$$\varphi_1(q,t,t) = \psi(q,t) \qquad \text{for} \quad (q,t) \in Q \times I\,.$$

Define another map $\varphi_2\colon Q \times I \times I \to \widetilde{\Phi}(X_0 \cup Y_0)$ by

$$\varphi_2(q,t,s)(x) = \begin{cases} \psi(q,s)(x) & \text{if } x \in X_0\,, \\ \psi(q,t)(x) & \text{if } x \in Y_0\,. \end{cases}$$

Observe that $r \circ \varphi_1 = \varphi_2|_{Q \times \Delta}$, where r is the restriction $\widetilde{\Phi}\big(\mathrm{cl}\, U_1 \times [-2a, 2a]\big) \to \widetilde{\Phi}(X_0 \cup Y_0)$. Since $\widetilde{\mathcal{R}}$ is open, there exists a neighbourhood N of $Q \times \Delta$ in $Q \times I \times I$ and a map $\widetilde{\psi}\colon N \to \widetilde{\Phi}\big(\mathrm{cl}\, U_1 \times [-2a, 2a]\big)$ such that $\widetilde{\psi}|_{Q \times \Delta} = \varphi_1$ and $r \circ \widetilde{\psi} = \varphi_2$. Since $Q \times \Delta$ is compact, we can find a positive number $\varepsilon \le 1$

such that $(q, t, s) \in N$ whenever $|t - s| < \varepsilon$. We now partition the interval $[0, 1]$ as follows:

$$0 = t_0 < t_1 < \cdots < t_n = 1 \qquad \text{such that} \quad |t_k - t_{k+1}| < \varepsilon \text{ for all } k,$$

and define, for each k, a map

$$\lambda_k \colon Q \times [t_k, t_{k+1}] \to \widetilde{\Phi}(\mathrm{cl}\, U_1 \times [-2a, 2a])$$

by the rule

$$\lambda_k(q, t)(x) = \widetilde{\psi}(q, t_k, t)(x) .$$

Then λ_k has the following properties:

(i) for all x, $\lambda_k(q, t_k)(x) = \psi(q, t_k)$,
(ii) for $x \in X_0$, $\lambda_k(q, t)(x) = \psi(q, t)(x)$,
(iii) for $x \in Y_0$, $\lambda_k(q, t)(x) = \psi(q, t_k)(x)$, that is, non-fixed points of λ_k lie inside $V \times (-a, a)$.

We are now in a position to define the required deformation $\bar{\psi}$ using the above λ_k's and the sharply moving diffeotopies. Suppose that, for some k, $1 \leq k \leq n - 1$, we have an $\varepsilon_k > 0$ and a map $\psi_k \colon Q \times [0, t_k] \to \widetilde{\Phi}(U_1 \times (-\varepsilon_k, \varepsilon_k))$ such that, for all $q \in Q$ and $t \in [0, t_k]$,

(iv) $\psi_k(q, t) = \psi(q, t)$ on $V_k \times (-\varepsilon_k, \varepsilon_k)$, where V_k is a G-invariant neighbourhood of A in V_0,
(v) $\psi_k(q, 0) = \psi(q, 0)$,
(vi) $\mathrm{supp}\, \psi_k \subset V \times (-\varepsilon_k, \varepsilon_k)$, where $\mathrm{supp}\, \psi_k$ denotes the set of non-fixed points of ψ_k ([4; p. 80]).

We shall construct $\psi_{k+1} \colon Q \times [0, t_{k+1}] \to \widetilde{\Phi}(U_1 \times (-\varepsilon_{k+1}, \varepsilon_{k+1}))$ for some positive number $\varepsilon_{k+1} < \varepsilon_k$. Choose open G-invariant neighbourhoods V_k' and W_k (with compact closures) of A in V_k satisfying

$$A \subset W_k \subset \mathrm{cl}\, W_k \subset V_k' \subset \mathrm{cl}\, V_k' \subset V_k .$$

Now, if τ is such that $0 < \tau < \min\Big(d\big(A, \partial(\mathrm{cl}\, W_k)\big), d\big(W_k, \partial(\mathrm{cl}\, V_k')\big)\Big)$, where d is the G-invariant Riemannian metric on X, then the τ-neighbourhood of $\partial(\mathrm{cl}\, W_k)$ in X is contained in $V_k' \setminus A$.

Let us consider the open subset $U' = U_1 \times (-2a, 2a)$ of $X \times \mathbb{R}$. By Lemma 2.1, there exists a positive number $\varepsilon_{k+1} < \varepsilon_k$, and an isotopy

$$\sigma \colon I \to \mathcal{E}_G(U_1 \times (-\varepsilon_{k+1}, \varepsilon_{k+1}), U')$$

which lies in $\mathcal{D}_G(X \times \mathbb{R}, \pi)$ and sharply moves U_1 at $\partial(\mathrm{cl}\, W_k)$. Then, $\sigma_t^* \lambda_k(q, s) \in \widetilde{\Phi}(U_1 \times (-\varepsilon_{k+1}, \varepsilon_{k+1}))$ for each $t \in I$, $q \in Q$ and $s \in [t_k, t_{k+1}]$, since $\widetilde{\mathcal{R}}$ is invariant under the action of $\mathcal{D}_G(X \times \mathbb{R}, \pi)$.

Let $\bar{\sigma}_t : U_1 \times (-\varepsilon_{k+1}, \varepsilon_{k+1}) \to U'$, $0 \le t \le t_{k+1}$, be the isotopy obtained by shrinking σ_t:

$$\bar{\sigma}_t = \begin{cases} \sigma_{t/t_k} & \text{if } 0 \le t \le t_k\,, \\ \sigma_1 & \text{if } t_k \le t \le t_{k+1}\,. \end{cases}$$

Now define $\psi_{k+1} : Q \times [0, t_{k+1}] \to \widetilde{\Phi}\big(U_1 \times (-\varepsilon_{k+1}, \varepsilon_{k+1})\big)$ in the following way:

$$\psi_{k+1}(q,t)(x,s) = \begin{cases} \psi_k(q,t)(x,s) & \text{if } x \notin V'_k\,, \quad 0 \le t \le t_k\,, \\ \big[\bar{\sigma}_t^* \psi(q,t)\big](x,s) & \text{if } x \in \operatorname{cl} V'_k\,, \quad 0 \le t \le t_k\,, \\ \big[\bar{\sigma}_t^* \lambda_k(q,t)\big](x,s) & \text{if } x \in \operatorname{cl} W_k\,, \quad t_k \le t \le t_{k+1}\,, \\ \big[\bar{\sigma}_t^* \psi(q,t_k)\big](x,s) & \text{if } x \in \operatorname{cl} V'_k \setminus W_k\,, \quad t_k \le t \le t_{k+1}\,, \\ \psi_k(q,t_k)(x,s) & \text{if } x \notin V'_k\,, \quad t_k \le t \le t_{k+1}\,, \end{cases}$$

where $q \in Q$ and $s \in (-\varepsilon_{k+1}, \varepsilon_{k+1})$.

Observe that ψ_n is the required $\bar{\psi}$ with $\varepsilon = \varepsilon_n$, because $\operatorname{supp} \psi_n \subset V \times (-\varepsilon_n, \varepsilon_n)$, and hence we can extend ψ_n to $U \cap \big(X \times (-\varepsilon_n, \varepsilon_n)\big)$ by defining it to be fixed on the complement of $U_1 \times (-\varepsilon_n, \varepsilon_n)$.

To start the induction we must now define $\psi_1 : Q \times [0, t_1] \to \widetilde{\Phi}(\operatorname{Op} U_1)$. For this construction we simply repeat the above arguments for $k = 0$. Note that we must take $t_k = t_0 = 0$, $V_k = V_0$ and $\bar{\sigma}_t = \sigma_1$ for $0 \le t \le t_1$. Then the definition of ψ_1 can be read out from the definition of ψ_{k+1}.

3. Proof of Theorem 1.2 and Theorem 1.3

Proof of Theorem 1.2. It is easy to see that $\mathcal{R}$ is a G-invariant open subset of $E^{(1)}$. If we show that $\mathcal{R}$ has a G-, and $\mathcal{D}_G(X \times \mathbb{R}, \pi)$-invariant open extension, then the theorem will be proved in view of Theorem 1.1.

A section $\tau : X \times \mathbb{R} \to E \times \mathbb{R}$ is of the form $\tau(x,t) = \big(\tau'(x,t), t\big)$ so that $\pi \circ \tau = \tau'$, where $\tau' : X \times \mathbb{R} \to E$ is a map such that, for each $t \in \mathbb{R}$, $\tau'(\cdot\,, t)$ is a section of p. Define a G-subbundle $\widetilde{\eta}$ of $T(X \times \mathbb{R})$ by $\widetilde{\eta}_{(x,t)} = \eta_x \times \mathbb{R}$.

Let $\widetilde{\mathcal{R}} \subset (E \times \mathbb{R})^{(1)}$ consist of 1-jets of local sections $j^1_{(x,t)}\tau$ satisfying the following two conditions:

(a) $j^1_{(x,t)}\tau' |_{\widetilde{\eta}_{(x,t)}}$ is injective,

(b) $j^1_{(x,t)}\tau' \big(\widetilde{\eta}_{(x,t)}\big) \cap \xi_{\tau'(x,t)} = \{0\}$.

Then $\widetilde{\mathcal{R}}$ is a G-, and $\mathcal{D}_G(X \times \mathbb{R}, \pi)$-invariant open relation. Moreover, $\pi^{(1)}$ maps $\widetilde{\mathcal{R}}$ into $\mathcal{R}$. The proof of the theorem will be complete if we show that $\pi^{(1)}$ maps $\widetilde{\mathcal{R}}_G$ onto $\mathcal{R}_G$.

Let $\sigma\colon U \to E$ be a local G-section of p defined on a G-invariant neighbourhood U of $x \in X$ such that $j^1_x\sigma \in \mathcal{R}_G$. We will produce an equivariant local section τ at $(x,0)$ defined on some G-invariant open neighbourhood $\widetilde{U}$ of $(x,0)$ such that $j^1_{(x,0)}\tau \in \widetilde{\mathcal{R}}_G$ and $\pi^{(1)}j^1_{(x,0)}\tau = j^1_x\sigma$. If there exists such a τ, then

(i) $\tau(y,t)$ can be expressed as $\big(\tau'(y,t),t\big)$ for $(y,t) \in \widetilde{U}$, where τ' is an equivariant map from $\widetilde{U}$ to E, and it satisfies the relation $p\circ\tau'(y,t) = y$. Moreover, $\tau'(x,0) = \sigma(x)$.

(ii) Since τ' is equivariant, it maps $\widetilde{U}^H$ into E^H, where H denotes the isotropy subgroup G_x at x. Let p^H denote the restriction of p to E^H. The relation $p^H\circ\tau'(x,t) = x$ gives $\mathrm{d}p^H_{\sigma(x)}\circ\mathrm{d}\tau'_{(x,0)}(0,w) = 0$ for $(0,w) \in T_xX^H \times T_0\mathbb{R}$. Then $\mathrm{d}\tau'_{(x,0)}(0,w) \in \operatorname{Ker}\mathrm{d}p^H_{\sigma(x)}$. Since $p^H : E^H \to X^H$ is a fibre-bundle with fibre $(E_x)^H$, which is the same as $(E^H)_x$ (E_x being the fibre of p over x), we have $\operatorname{Ker}\mathrm{d}p^H_{\sigma(x)} = T_{\sigma(x)}(E^H_x)$ $(\subset (TE)^H_{\sigma(x)})$. Hence $\mathrm{d}\tau'_{(x,0)}(0,w) \in T_{\sigma(x)}(E^H_x)$.

(iii) Also, by hypothesis, $\mathrm{d}\tau'_{(x,0)}(0,1) \notin \mathrm{d}\sigma_x(\eta_x) \oplus \xi_{\sigma(x)}$.

Therefore, to obtain τ', it is required to find a vector $\boldsymbol{u} \in T_{\sigma(x)}E^H_x$ which does not belong to the intersection $\big(\mathrm{d}\sigma_x(\eta_x)\oplus\xi_{\sigma(x)}\big)\cap T_{\sigma(x)}E^H_x$. Now, since the local condition $\dim X^H + \dim\xi^H < \dim E^H$ is equivalent to $\dim\xi^H < \dim E^H_x$, and since $\mathrm{d}\sigma_x(\eta^H_x)\cap T_{\sigma(x)}(E^H_x) = \{0\}$, $T_{\sigma(x)}(E^H_x)$ is not contained in $\mathrm{d}\sigma_x(\eta^H_x)\oplus \xi^H_{\sigma(x)}$. Also, since η and ξ are G-invariant subbundles, σ is equivariant, and $\mathrm{d}\sigma_x$ is injective, we can prove that

$$\big(\mathrm{d}\sigma_x(\eta^H_x)\oplus\xi^H_{\sigma(x)}\big)\cap T_{\sigma(x)}E^H_x = \big(\mathrm{d}\sigma_x(\eta_x)\oplus\xi_{\sigma(x)}\big)\cap T_{\sigma(x)}E^H_x\,.$$

We may, therefore, choose $\boldsymbol{u}$ as required.

We shall now construct τ' described in (i) above. First identify $E|_U$ with the trivial G_x-bundle $U\times Y$, where Y is G_x-homeomorphic to the fibre E_x. Then σ can be expressed in the following way

$$\sigma(y) = \big(y,\bar\sigma(y)\big) \in U\times Y\,,$$

where $y\in U$, and $\bar\sigma\colon U\to Y$ is a G_x-equivariant map. Therefore, because of (ii), we may assume without loss of generality, that $\boldsymbol{u} \in T_{\bar\sigma(x)}Y^H \subset T_xX \times T_{\bar\sigma(x)}Y$.

Next, note that we can always find a smooth function $\bar f$ (not necessarily equivariant) from a neighbourhood of $(x,0)\in X\times\mathbb{R}$ to Y such that at the point $(x,0)$ it satisfies the following relations

$$\bar f(x,0) = \bar\sigma(x)\,,\qquad \frac{\partial\bar f}{\partial x_i} = \frac{\partial\bar\sigma}{\partial x_i}\,,\qquad \frac{\partial\bar f}{\partial t} = \boldsymbol{u}\,,$$

where x_i's are coordinate functions on a neighbourhood of $x\in X$ and $\boldsymbol{u}$ is as chosen above. These conditions imply that the map $f'\colon\ \mathrm{Op}(x,0)\to E$ defined by the formula $f'(y,t) = \big(y, \bar{f}(y,t)\big)$ has all the properties of τ', except (perhaps) equivariance. Thus $\widetilde{\mathcal{R}}$ is a non-equivariant extension of $\mathcal{R}$. We may also assume that f' agrees with σ on X.

We shall now modify f' to get the required equivariant map τ'. Define a map τ_1' on the domain of f' by the following rule:

$$\tau_1'(y,t) = \int_H h^{-1} f'\big(h \cdot (y,t)\big)\, \mathrm{d}h\,,$$

where $\mathrm{d}h$ is the normalized Haar measure on $G_x = H$. Then τ_1' is a G_x-equivariant map agreeing with f' (and hence with σ) on $U \times \{0\}$, and is such that, for each $t \in \mathbb{R}$, $\tau_1'(\cdot\,,t)$ is a local section of p, provided the composition $f \circ i_t$ is defined (in fact for a fixed t, $f'\big(h \cdot (y,t)\big) \in E_{hy}$, and therefore $h^{-1} \cdot f'\big(h \cdot (y,t)\big) \in E_y$; consequently, $\tau_1'(y,t) \in E_y$). Moreover, since H fixes both $(x,0)$ and $\boldsymbol{u}$, we have

$$\frac{\partial \tau_1'}{\partial t}(x,0) = \int_H h^{-1} \frac{\partial f'}{\partial t}(x,0)\, \mathrm{d}h = \int_H h^{-1} \cdot \boldsymbol{u}\, \mathrm{d}h = \boldsymbol{u}\,.$$

Therefore, if S_x is a slice at $x \in U$, we may define $\tau' \colon G \times_H S_x \times \mathbb{R} \to E$ by

$$\tau'\big([g,y],t\big) = g\tau_1'(y,t)\,.$$

This completes the proof of Theorem 1.2. □

P r o o f o f T h e o r e m 1.3. Consider first a general situation:

LEMMA 3.1. *Let X, Y be smooth G-manifolds, ξ a G-subbundle of TY, and η a G-subbundle of TX such that $\dim\eta + \dim\xi < \dim Y$. Let $\mathcal{R}$ be the subspace of $J^1(X,Y)$ consisting of 1-jets of germs of local G-maps defined on G-invariant open sets in X, $j_x^1 f$ for $x \in X$, such that*

$$j_x^1 f\big|_{\eta_x} \text{ is injective} \quad \text{and} \quad j_x^1 f(\eta_x) \cap \xi_{f(x)} = \{0\}\,.$$

Then $\mathcal{R}$ satisfies equivariant parametric h-principle (in an obvious sense), if for each isotropy subgroup H of the action of G on X we have locally

$$\dim \eta^H + \dim \xi^H < \dim Y^H\,.$$

P r o o f . Consider the G-locally trivial G-fibre bundle $E = X \times Y \to X$. Then G-sections of E are in one-one correspondence with the G-maps of X in Y and we may write a section σ as $(1_X, \bar{\sigma})$ where $\bar{\sigma} \colon X \to Y$ is a smooth G-map.

Consider the bundle $\bar{\xi}$ on $X \times Y$ defined by $\bar{\xi}_{x,y} = \eta_x \times \xi_y$. Then a section $\sigma\colon X \to E$ satisfies

$$\mathrm{d}\sigma_x(\eta_x) \cap \bar{\xi}_{x,\bar{\sigma}(x)} = \{0\}$$

if and only if $\bar{\sigma}\colon X \to Y$ satisfies the following two conditions:

$$\mathrm{d}\bar{\sigma}_x\big|_{\eta_x} \text{ is injective and } \mathrm{d}\bar{\sigma}_x(\eta_x) \cap \xi_{\bar{\sigma}(x)} = \{0\}.$$

Also, the condition $\dim X^H + \dim \bar{\xi}^H < \dim E^H$ is equivalent to $\dim \xi^H + \dim \eta^H < \dim Y^H$. This completes the proof. □

The proof of the theorem now follows from the above lemma by taking $\eta = TX$ and $\xi = 0$. □

REFERENCES

[1] BIERSTONE, E.: *The equivariant covering homotopy property for differentiable G-fibre bundles*, J. Differential Geom. **8** (1973), 615–622.

[2] BIERSTONE, E.: *Equivariant Gromov theory*, Topology **13** (1974), 327–345.

[3] FEIT, S.: *k-mersions of manifolds*, Acta Math. **122** (1969), 173–195.

[4] GROMOV, M.: *Partial Differential Relations.* Ergeb. Math. Grenzgeb. (3) 9, Springer-Verlag, Berlin, 1986.

[5] HIRSCH, M.: *Immersions of manifolds*, Trans. Amer. Math. Soc. **93** (1959), 242–276.

[6] IZUMIYA, S.: *Homotopy classification of equivariant regular sections*, Manuscripta Math. **28** (1979), 337–360.

[7] PETRIE, T.—RANDALL, J. D.: *Transformation Groups on Manifolds.* Monogr. Textbooks Pure Appl. Math. 82, Marcel Dekker, Inc., New York, 1984.

[8] PHILLIPS, A.: *Submersions of open manifolds*, Topology **6** (1967), 170–206.

[9] SMALE, S.: *The classification of immersions of spheres in Euclidean spaces*, Ann. of Math. (2) **69** (1959), 327–344.