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A note on the κ deformed Landau problem

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Abstract

We formulate and solve the κ deformed Landau problem. It is shown that, unlike in the Dirac–Coulomb problem, in this case the energy levels depend on the deformation parameter.

During the last two years the quantum deformation of Poincaré algebra has been well studied [1–6]. One way to deform the Poincaré algebra is to apply the contraction process to the $SO_q(3, 2)$ algebra. Taking the limit of de Sitter curvature $R \to \alpha$ with a suitable limit of the (real) deformation parameter q such that $\kappa^{-1} = \lim(R \ln q)$, one obtains that κ Poincaré algebra. Subsequently one can obtain the Dirac equation [4,7]. Recently it has been shown that in the case of the κ Dirac–Coulomb problem [8] the first order perturbation vanishes identically, resulting in no change in the energy levels. The same result has also been obtained in the nonrelativistic approximation [9]. Here we shall study the deformed relativistic Landau problem and our aim is to find a deformed system whose energy levels depend on the deformation parameter.

For the sake of completeness we first present some results [7,8] relating to the deformed Poincaré algebra. The deformed algebra structure is given by (we take $\kappa^{-1} = \epsilon$)

$$[P_i, P_j] = 0\,,$$

$$[P_i, P_0] = 0\,,$$

$$[M_i, P_j] = \mathrm{i}\epsilon_{ijk} P_k\,,$$

$$[M_i, P_0] = 0\,,$$

$$[L_i, P_0] = \mathrm{i}P_i\,,$$

$$[L_i, P_j] = \mathrm{i}\delta_{ij}\epsilon^{-1} \sinh(\epsilon P_0)\,,$$

$$[M_i, M_j] = \mathrm{i}\epsilon_{ijk} M_k\,,$$

$$[M_i, L_j] = \mathrm{i}\epsilon_{ijk} L_k\,,$$

$$[L_i, L_j] = -\mathrm{i}\epsilon_{ijk}[M_k \cosh(\epsilon P_0) - \tfrac{1}{4}\epsilon^2 P_k P_i M_i]\,, \quad (1)$$

where $P_\mu \equiv (P_0, P_i)$ are the deformed energy and momenta, M_i and L_i are spatial rotation and deformed boost generators, respectively. The coalgebra and antipode are given by

$$\Delta M_i = M_i \otimes I + I \otimes M_i\,,$$

$$\Delta P_0 = P_0 \otimes I + I \otimes P_0\,,$$

$$\Delta P_i = P_i \otimes \exp(\epsilon P_0) + \exp(-\epsilon P_0) \otimes P_i\,,$$

$$\Delta L_i = L_i \otimes \exp(\epsilon P_0) + \exp(-\epsilon P_0) \otimes L_i + \tfrac{1}{2}\epsilon\epsilon_{ijk}[P_i \otimes M_k \exp(\epsilon P_0) + \exp(-\epsilon P_0) M_j \otimes P_k]\,,$$

$$S(P_\mu) = -P_\mu\,, \quad S(M_i) = -M_i\,,$$

$$S(L_i) = -L_i + \epsilon\tfrac{3}{2}\mathrm{i}P_i\,. \quad (2)$$

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The deformed Casimir operators for the κ Poincaré algebra are given by

$$C_1 = [2(1/\epsilon)\sinh(\epsilon P_0)]^2 - P_i P_i ,$$
$$C_2 = [\cosh(\epsilon P_0) - \tfrac{1}{4}\epsilon^2 P_i P_i] W_0^2 - W_i W_i , \quad (3)$$

where $W_0 = P_i M_i$ and

$$W_i = (1/\epsilon)\sinh(\epsilon P_0) M_i + \epsilon_{ijk} P_j L_k .$$

We now write

$$\mathscr{P}_\mu = P_\mu ,$$
$$\mathscr{M}_i = M_i + m_i ,$$
$$\mathscr{L}_i = L_i + \exp(-\tfrac{1}{2}\epsilon P_0) l_i - \tfrac{1}{2}\epsilon\epsilon_{ijk} m_j P_k ,$$
$$m_i = \tfrac{1}{4}\mathrm{i}\epsilon_{ijk}\gamma_i\gamma_k , \quad l_i = -\tfrac{1}{2}\mathrm{i}\gamma_4\gamma_i . \quad (4)$$

The κ deformed Dirac operator $\mathscr{D}$ should satisfy the relations

$$[\mathscr{D}, \mathscr{P}_\mu] = [\mathscr{D}, \mathscr{M}_i] = [\mathscr{D}, \mathscr{L}_i] = 0 . \quad (5)$$

Such an operator $\mathscr{D}$ can be found and is given by [7,8]

$$\mathscr{D} = -\exp(-\tfrac{1}{2}\epsilon P_0)\gamma_i P_i + \gamma_4 (1/\epsilon)\sinh(\epsilon P_0) - \tfrac{1}{2}\epsilon\gamma_4 P_i P_i . \quad (6)$$

Also it can be shown that

$$\mathscr{D}^2 = C_1(1 + \tfrac{1}{4}\epsilon^2 C_1) = -\tfrac{4}{3}C_2 . \quad (7)$$

Thus the κ deformed Dirac equation reads

$$[-\exp(-\epsilon P_0)\gamma_i P_i + \gamma_4(1/\epsilon)\sinh(\epsilon P_0) - \tfrac{1}{2}\epsilon\gamma_4 P_i P_i]\psi = m(1 + \tfrac{1}{4}m^2\epsilon^2)^{1/2}\psi . \quad (8)$$

In order to bring this equation to a more tractable form, we operate from the left by $\exp(\epsilon P_0)$, expand and retain terms up to $O(\epsilon)$. The resulting equation is given by

$$[(\gamma_4 P_0 - \gamma_i P_i) + \tfrac{1}{2}\epsilon(\gamma_4(P_0^2 - P_i P_i) - m P_0)]\psi = m\psi . \quad (9)$$

We now introduce a gauge field in the following way:

$$P_0 \to P_0 = H = E , \quad P_i \to \hat{P}_i = P_i - eA_i , \quad (10)$$

and for the vector potential we take

$$A_1 = -\tfrac{1}{2}B_0 y, \quad A_2 = \tfrac{1}{2}B_0 x , \quad A_3 = 0 . \quad (11)$$

Using (10) we can write Eq. (9) as

$$H\psi = [(\gamma_4\gamma_i P_i + \gamma_4 m) - \tfrac{1}{2}\epsilon((H^2 - \hat{P}_i\hat{P}_i) - \gamma_4 m H)]\psi . \quad (12)$$

This equation is highly nonlinear. To bring it to a more reasonable form we follow Ref. [8] and note that the Hamiltonian corresponding to the undeformed part is

$$H_0 = (\gamma_4\gamma_i P_i + \gamma_4 m) . \quad (13)$$

Now we substitute (13) on the right-hand side of (12) and the eigenvalue equation is found to be (to $O(\epsilon)$)

$$[H_0 - \tfrac{1}{2}\epsilon((H_0^2 - \hat{P}_i\hat{P}_i) - \gamma_4 m H_0)]\psi = E\psi . \quad (14)$$

It may be noted that the eigenvalue problem corresponding to H_0 is the relativistic Landau problem and is exactly solvable. We shall solve Eq. (14) exactly without making any more approximation.

In order to solve (14) we choose the following representation of the γ matrices:

$$\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad (15)$$

where the σ_i are the usual Pauli matrices.

Now denoting the left-hand side of Eq. (14) by H_ϵ we can write this equation as

$$H_\epsilon\psi = E\psi , \quad (16)$$

$$H_\epsilon = \begin{bmatrix} (m + \tfrac{1}{2}\epsilon e\sigma\cdot\boldsymbol{B}) & (1 + \tfrac{1}{2}\epsilon m)\sigma\cdot\boldsymbol{P} \\ (1 - \tfrac{1}{2}\epsilon m)\sigma\cdot\boldsymbol{P} & (-m + \tfrac{1}{2}\epsilon e\sigma\cdot\boldsymbol{B}) \end{bmatrix}, \quad (17)$$

where $\boldsymbol{B} = \nabla\times\boldsymbol{A} = B_0\hat{\boldsymbol{Z}}$ is the magnetic field. Then squaring Eq. (16) we find

$$H_\epsilon^2\psi = E^2\psi , \quad (18)$$

where H_ϵ^2 is given by

$$H_\epsilon^2 = \begin{bmatrix} D_+ & \epsilon e\boldsymbol{B}\cdot\boldsymbol{P} \\ \epsilon e\boldsymbol{B}\cdot\boldsymbol{P} & D_- \end{bmatrix}, \quad (19)$$

$$D_\pm = (\sigma\cdot\boldsymbol{P})^2 \pm \epsilon e\sigma\cdot\boldsymbol{B} + m^2 , \quad (20)$$

and in deriving (19) we have neglected terms of $O(\epsilon^2)$. Now taking ψ to be of the form

$$\psi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad (21)$$

where φ_1 and φ_2 are two-component spinors, we can write Eq. (19) in the following way:

$$[(\sigma\cdot P)^2+m^2+\epsilon me\sigma\cdot B]\varphi_1+\epsilon eB\cdot P\varphi_2 = E^2\varphi_1\,, \quad (22)$$

$$[(\sigma\cdot P)^2+m^2-\epsilon me\sigma\cdot B]\varphi_2+\epsilon eB\cdot P\varphi_1 = E^2\varphi_2\,. \quad (23)$$

Since the magnetic field is directed along the z axis and $P_z\varphi_i=p_z\varphi_i$, $i=1, 2$ we can write the above equations in the form

$$[D+\epsilon em\sigma\cdot B]\varphi_1+\epsilon ep_z\varphi_2=0\,, \quad (24)$$

$$[D-\epsilon em\sigma\cdot B]\varphi_2+\epsilon ep_z\varphi_1=0\,, \quad (25)$$

where the operator D is defined by

$$D=\hat{P}^2+m^2-E^2-e\sigma\cdot B\,. \quad (26)$$

From (24) we find

$$\varphi_2=-\frac{(D+\epsilon em\sigma\cdot B)\varphi_1}{\epsilon ep_z}\,. \quad (27)$$

Substituting (27) into (25) we get

$$(D-\epsilon e\alpha)(D+\epsilon e\alpha)\varphi_1=0\,, \quad (28)$$

where α is given by

$$\alpha^2=(m^2+p_z^2)B_0^2\,. \quad (29)$$

From (28) it follows that the energy values are given by

$$E_n^2=[m^2+eB_0(2n+1\pm1)eB_0+p_z^2 + \epsilon eB_0(m^2+p_z^2)^{1/2}], \quad n=0, 1, 2, \ldots\,. \quad (30)$$

It is thus seen that the energy levels are deformed. From (30) it also follows that in the limit of $\epsilon\to0$ we recover the standard Landau levels for a Dirac particle.

Conclusion. Here we have solved the κ deformed relativistic Landau problem. It has been shown that while deformation does not destroy the basic symmetry of the problem (i.e. the degeneracy in the angular quantum number), the Landau levels are indeed deformed. We may hope that some future experiment may perhaps detect the effect of deformation.

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References

[1] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. B 264 (1991) 331.
[2] J. Lukierski, A. Nowicki and H. Ruegg, Phys. Lett. B 271 (1991) 321.
[3] S. Giller, J. Kunz, P. Kosinski, M. Majewski and P. Maslanka, Phys. Lett. B 286 (1991) 57.
[4] J. Lukierski, A. Nowicki and H. Ruegg, Phys. Lett. B 293 (1992) 344.
[5] E. Celegnini, R. Giachetti, E. Sorace and M. Tarlini, J. Math. Phys. 31 (1990); 32 (1991) 1155, 1159.
[6] A.A. Kehagias, P. Messen and G. Zoupanos, CERN preprint Th 7101/93.
[7] A. Nowicki, E. Sorace and M. Tarlini, Phys. Lett. B 302 (1993) 419.
[8] L.C. Biedenharn, B. Mueller and M. Tarlini, Phys. Lett. B 318 (1993) 613.
[9] J. Lukierski, H. Ruegg and W. Ruhl, Phys. Lett. B 313 (1993) 357.