PRODUCT OF TRANSITION SEMI-GROUPS AND THE HYPOTHESIS (L)

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SUMMARY. An improved version of the theorem of Cairoli on the permanence of hypothesis (L) is given.

1. Introduction and notation

We assume familiarity of Cairoli (1967) especially of Theorems 6 and 7 therein.

Let E be a locally compact second countable topological space and $\mathcal{B}_E$ the σ-algebra of its Borel subsets. A sub-Markov transition semi-group (to be referred to as semi-group, in the sequel) $(P_t)_{t>0}$ on $(E, \mathcal{B}_E)$ is said to be weakly continuous, if one has $\lim_{t \to 0} P_t f(x) = f(x)$ for every continuous function f with compact support on E. In what follows, we take all semi-groups to be weakly continuous.

Definition : (P_t) is said to satisfy the hypothesis (L) of Meyer if there exists a finite measure θ on $(E, \mathcal{B}_E)$ such that the θ-negligible sets are precisely the potential null sets. Such a measure is said to be a fundamental measure for (P_t). To simplify notation we will write just '(P_t) satisfies (L)'.

Let $(Q_t)_{t>0}$ be another semi-group on $(F, \mathcal{B}_F)$ where F, again, is a locally compact second countable space and $\mathcal{B}_F$ the σ-algebra of its Borel subsets.

Definition : A semi-group $(R_t)_{t>0}$ on $(E \times F, \mathcal{B}_{E\times F})$ is said to be the product of (P_t) and (Q_t) if

$$R_t((x, y), \Gamma) = P_t^{\epsilon_x} \otimes Q_t^{\epsilon_y}(\Gamma)$$

for every $x \epsilon E$, $y \epsilon F$ and $\Gamma \epsilon \mathcal{B}_{E\times}$ (where $P_t^{\epsilon_x}(.) = P_t(x, .)$ etc.)

In symbols,

$$(R_t) = (P_t \otimes Q_t).$$

Cairoli showed that if both the semi-groups satisfy (L) and at least one of them is a strong Feller semi-group then the product semi-group also satisfies (L) and the product of the fundamental measures of the coordinates is a fundamental measure for the product.

We improve upon this theorem by giving a weaker sufficient condition.

2. Main theorems

Let $(P_t)_{t>0}$, $(Q_t)_{t>0}$ be semi-groups on $(E, \mathcal{B}_E)$ and $(F, \mathcal{B}_F)$ respectively. Let

$$(R_t) = (P_t \otimes Q_t).$$

Let (U_p), (V_p) and (W_p) be the resolvents associated with (P_t), (Q_t) and (R_t) respectively.

It is easy to check that (P_t) satisfies (L) and (Q_t) satisfies (L) does not, in general, imply that (R_t) satisfies (L) (counter-example : product of two semi-groups of uniform translation). Regarding the implication in the other direction the following is true.

Theorem 2.1 : *(R_t) satisfies (L) implies that (P_t) and (Q_t) satisfy (L).*

Proof : Recall that any family of finite measures on a measurable space, if absolutely continuous with respect to a finite measure, admits a countable equivalent subfamily; and hence admits a finite measure equivalent to that family. (See, for instance, Theorem 2, page 354 of Lehmann (1959)).

So, enough to show that

$$\{U_p(x, .),\ x\epsilon E \text{ and } p \text{ fixed}\}$$

and

$$\{V_p(y, .),\ y\epsilon F \text{ and } p \text{ fixed}\}$$

are families absolutely continuous with respect to finite measures on $(E, \mathcal{B}_E)$ and $(F, \mathcal{B}_F)$ respectively.

Let μ be a fundamental measure for (R_t).

Let

$$\theta(A) = \mu(A \times F) \ \forall\ A \epsilon \mathcal{B}_E$$

and

$$\nu(B) = \mu(E \times B) \ \forall\ B \epsilon \mathcal{B}_F$$

We shall show that θ and ν would do the job. Let $\theta(A) = 0$. This implies that

$$\mu(A \times F) = 0 \text{ and consequently that}$$

$$W_p I_{A \times F}(x, y) = 0 \ \forall\ x \text{ and } y \text{ for fixed } p.$$

From the equation connecting $U_p \otimes V_p$ and W_{2p}, namely,

$$U_p \otimes V_p = ((U_p \otimes I_F) + (I_E \otimes V_p)) W_{2p}$$

(see, for instance page 32 of Cairoli (1967))

we get

$$U_p \otimes V_p I_{A \times F}(x, y) = 0 \ \forall\ x \text{ and } y$$

i.e.

$$U_p(x, A) V_p(y, F) = 0 \ \forall\ x \text{ and } y$$

Now, weak continuity of (Q_t) ensures that $V_p(y, F) > 0 \ \forall\ y$. So, $U_p(x, A) = 0 \ \forall\ x$. i.e., $\{U_p(x, .),\ x\epsilon E$ and p fixed$\}$ is dominated by θ.

A similar reasoning would show that

$$\{V_p(y, .),\ y\epsilon F \text{ and } p \text{ fixed}\} \text{ is dominated by } \nu.$$

Hence the theorem.

In what follows we abbreviate 'lower semi-continuous' by l.s.c.

Lemma 2.2 : *For every universally measurable subset Δ of $E \times F$ and every y and p fixed, let the mapping $x \to W_p I_\Delta(x, y)$ be l.s.c. in x. Then for every fixed p and non-negative universally measurable function f on E the mapping $x \to U_p f(x)$ is l.s.c. in x.*

Proof : Let p be fixed. Let g be a p-excessive function for the semi-group (P_t). Define a function h on $E \times F$ by setting $h(x, y) = g(x)$, $x\epsilon E$, $y\epsilon F$.

Then, one has,

$$e^{-pt} R_t h(x, y) = e^{-pt} \int P_t(x, du) \int h(u, v) Q_t(y, dv) = e^{-pt} P_t g(x) Q_t(y, F) \leqslant g(x)$$
$$= h(x, y).$$

Also,
$$\lim_{t \downarrow 0} e^{-pt} R_t h(x, y) = (\lim_{t \downarrow 0} e^{-pt} P_t g(x))(\lim_{t \downarrow 0} Q_t(y, F))$$
$$= g(x) = h(x, y)$$

(since (Q_t) is weakly continuous $\lim_{t \downarrow 0} Q_t 1(y) = 1 \;\forall\; y$).

Therefore, h is p-excessive for (R_t). Hence it is the increasing limit of p-potentials of (R_t). From the hypothesis of lemma one can easily conclude that the mapping $x \to W_p k(x, y)$ is l.s.c. in x for every fixed y where k is a non-negative universally measurable function. Therefore, the mapping $x \to h(x, y)$, being the limit of an increasing sequence of l.s.c. functions, is itself l.s.c. That is, $x \to g(x)$ is l.s.c. in x.

In particular, $x \to U_p f(x)$ is l.s.c. in x. Hence the lemma.

Theorem 2.3 : *For every universally measurable subset Δ of $E \times F$ and every p and y fixed, let the mapping $x \to W_p I_\Delta(x, y)$ be l.s.c. in x. Then (P_t) and (Q_t) satisfy (L) implies that (R_t) satisfies (L).*

In such a case, denoting the fundamental measures of (P_t) and (Q_t) by θ and ν respectively, we have that $\theta \otimes \nu$ is a fundamental measure for (R_t).

Proof : From Lemma 2.2, for every fixed p and non-negative universally measurable function f, on E, the mapping $x \to U_p f(x)$ is l.s.c. in x.

So, Theorem 6 of Cairoli (1967) holds.

Note that, though in that theorem, Cairoli assumes, besides the existence of the fundamental measures for (P_t) and (Q_t), strong Feller nature for (U_p), only the l.s.c. property of $x \to U_p f(x)$ is used in the proof.

Again, though the assumption of strong Feller nature for (P_t) is made in Theorem 7 of Cairoli (1967) what is used is only the l.s.c. property of $x \to W_p I_\Delta(x, y)$ and Theorem 6 of Cairoli (1967).

Hence, under our hypothesis Theorem 7 of Cairoli (1967) also holds.

Therefore we have the conclusion of Theorem 7 of Cairoli (1967) namely that (R_t) satisfies (L) and $\theta \otimes \nu$ is a fundamental measure for (R_t).

(*Remark :* We have not used the same symbols, for fundamental measures as in Cairoli (1967). Also, clearly the theorem is true if for every fixed x and p, the mapping $y \to W_p I_\Delta(x, y)$ is l.s.c.).

3. An example

Here we show that our sufficient condition is weaker than that of either (P_t) or (Q_t) being strong Feller semi-groups.

Evidently, the strong Feller property of either (P_t) or (Q_t) would imply l.s.c. of $W_p I_\Delta(.,.)$ in one of the variables. Hence our condition is apparently weaker.

That it is strictly so follows from the example given below :

Let E be the real line and $(P_t)_{t>0}$ the semi-group of uniform translation with unit speed. Let F be any countable set which has a non-discrete locally compact Hausdorff second countable topology; for instance we could take F to be the one point compactification of the discrete set of natural numbers. On $(F, \mathcal{B}_F)$ define

$$Q_t(x, .) = \epsilon_x \;\forall\; x \epsilon F \text{ and } t > 0.$$

(P_t) is not a strong Feller semi-group is known. Since F is not a discrete space, there exists a sequence $\{x_n\}$ of points in F converging to a point x^0 in F and satisfying the condition that $x_n \neq x^0$ for any n. Hence, (Q_t) is not a strong Feller semi-group; for instance, the mapping $x \to Q_t I_{\{x^0\}}(x)$ is not continuous.

(P_t) satisfies (L) is known. Since no non-empty subset of F is of potential zero and F is countable (Q_t) also satisfies (L) and any fundamental measure for (Q_t) assigns positive mass to singletons in F.

Let $$R_t = P_t \otimes Q_t$$

For any non-negative measurable function f on $(E \times F, \mathcal{B}_{E\times F})$

$$R_t f(x, y) = f(x+t, y).$$

Using the same symbols for resolvents as in 2,

$$W_p f(x, y) = \int_0^\infty e^{-pt} f(x+t, y) dt$$
$$= U_p f_y(x),$$

where $f_y(.)$ stands for the function on E defined by

$$f_y(x) = f(x, y) \;\forall\; x \epsilon E.$$

Therefore, the mapping $x \to W_p f(x, y)$ is continuous.

Let λ be a fundamental measure for (P_t) and μ that for (Q_t). For any $A \epsilon \mathcal{B}_{E\times F}$

$$\lambda \otimes \mu(A) = 0$$

implies that $$\int \lambda(A^y)\mu(dy) = 0.$$

That is, $\lambda(A^y) = 0 \;\forall\; y$, since $\mu(\{y\}) > 0$ for all $y \epsilon F$. Hence $U_p I_{A^y}(x) = 0 \;\forall\; x$ and for every y in F.

So, $W_p I_A(x, y) = 0$ for all $(x, y) \epsilon E \times F$. Thus, we get that for any fixed p, the family $\{W_p((x, y), .), \; (x, y) \epsilon E \times F\}$ is dominated by the finite measure $\lambda \otimes \mu$. Hence (R_t) satisfies (L).

ACKNOWLEDGEMENT

The author is indebted to Professor R. Cairoli for some improvements over an earlier version of the note.

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Paper received : March, 1972.