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A CYLINDRICAL WAVE-MAKER IN LIQUID OF FINITE DEPTH WITH AN INERTIAL SURFACE

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This paper is concerned with the generation of waves in a liquid of uniform finite depth with an inertial surface composed of uniformly distributed non-interacting floating particles, due to forced symmetric motion prescribed on the surface of an immersed circular vertical wave-maker. The techniques of Laplace transform in time and a suitable Weber transform in the radial co-ordinate are used to solve the problem. It is shown that if the inertial surface is too heavy, time-harmonic disturbance due to the wave-maker has only local influence on the liquid.

1. INTRODUCTION

The classical wave-maker problem for the case of deep water with a free surface was solved long back by Havelock[2] wherein the wave-maker is either in the form of a vertical plane or a cylinder with circular cross-section. Later Rhodes-Robinson[7] extended them to include the effect of surface tension at the free surface. Recently there has been a considerable interest in different problems concerning generation of waves in a liquid with an inertial surface composed of a thin but uniform distribution of disconnected particles (e.g. broken ice, floating mat on water). Rhodes-Robinson[8], Mandal and Kundu[4,5], Mandal[3] considered problems involving generation of waves at an inertial surface due to different types of sources with time dependent strengths submerged in a fluid of both infinite and uniform finite depths. Rhodes-Robinson[8] also pointed out briefly the method of solving the plane-vertical wave-maker problem in a liquid with an inertial surface by a suitable use of Green's integral theorem in the liquid region after taking Laplace transform in time. However the circular cylindrical wave-maker problem needs attention as it can, but not easily be solved by this method. Here we use a suitable Weber transform[1] in the radial coordinate after employing Laplace transform in time to solve the problem. The important time-harmonic case is considered and the inertial surface depression is calculated. It is observed that the time-harmonic wave-maker affects the inertial surface only locally if it is too heavy.

2. STATEMENT AND FORMULATION OF THE PROBLEM

We consider the motion under gravity in an ideal liquid of volume density ρ covered by an inertial surface composed of uniformly distributed floating particles of area density $\rho\,\epsilon$. $\epsilon = 0$ corresponds to a liquid with a free surface. On an immersed vertical circular cylindrical wave-maker, the normal fluid velocity is supposed to be prescribed which is both time and depth dependent. We choose a cylindrical coordinate system (r, θ, y) in which the y-axis is taken as the axis of the cylinder with radius a so that $r = a$ is the wave-maker $0 < y < h$ $r > a$ is the fluid region and $y = 0$, $r > a$ is the position of the inertial surface at rest. The wave-maker starts operating from time $t = 0$ with outward normal velocity $U(y, t)$ on its boundary $r = a$. We consider only the axisymetric case in which the resulting motion in the liquid is independent of θ. Since the motion starts from rest it is irrotational and can be described by a velocity potential $\varphi(r, y, t)$ satisfying the Laplace's equation

$$\varphi_{rr} + \frac{1}{r}\varphi_r + \varphi_{yy} = 0,\ r > a,\ 0 < y < h. \qquad ...(2.1)$$

The condition at the inertial surface $y = 0$ relating the potential function and the inertial surface depression ζ, within the frame-work of linearised theory, consists of the kinematic condition

$$\varphi_y = \zeta_t \text{ on } y = 0 \qquad ...(2.2)$$

and the dynamic condition

$$\varphi_t = g\,\zeta + \epsilon\,\zeta_{tt} \qquad ...(2.3)$$

where g is the gravity. Elimination of ζ produces the inertial surface condition

$$\Phi_{tt} - g\,\varphi_y = 0 \text{ on } y = 0 \qquad ...(2.4)$$

where

$$\Phi = \varphi - \epsilon\,\varphi_y. \qquad ...(2.5)$$

The condition at the wave-maker is

$$\varphi_r = U(y, t) \text{ on } r = a \qquad ...(2.6)$$

and the condition at the bottom is

$$\varphi_y = 0 \text{ on } y = h. \qquad ...(2.7)$$

There are also initial conditions at the inertial surface given by

$$\frac{\partial\Phi}{\partial t} = \Phi = 0 \text{ on } y = 0 \text{ at } t = 0. \qquad ...(2.8)$$

Let a bar above a function denote its Laplace transform in time. Then $\bar{\varphi}(r, y; p)$ satisfies the boundary value problem

$$\bar{\varphi}_{rr} + \frac{1}{r}\bar{\varphi}_r + \bar{\varphi}_{yy} = 0,\ r > a,\ 0 < y < h$$

$$p^2\bar{\varphi} - (g + \epsilon\,p^2)\,\bar{\varphi}_y = 0 \text{ on } y = 0,\ r > a$$

$$\bar{\varphi}_r = \bar{U}(y, p) \quad \text{on} \quad r = a$$

$$\bar{\varphi}_y = 0 \quad \text{on} \quad y = h. \qquad \ldots(2.9)$$

3. Solution by Weber-transform Method

We use the following form of Weber-transform of a function $f(r)$ defined in (a, ∞) by

$$g(t) = \int_a^\infty r A(r, \xi) f(r)\, dr \quad (\xi > 0) \qquad \ldots(3.1)$$

where

$$A(r, \xi) = J_1(a\xi) Y_0(r\xi) - J_0(r\xi) Y_1(a\xi) \qquad \ldots(3.2)$$

J_n, Y_n $(n = 0, 1)$ are the Bessel functions of the first and second kinds respectively. (cf. Davies[1], p. 252).

The inverse transform formula is

$$f(r) = \int_0^\infty \frac{\xi A(r, \xi)}{J_1^2(a\xi) + Y_1^2(a\xi)} g(\xi)\, d\xi. \qquad \ldots(3.3)$$

It may be noted that

$$\int_a^\infty \left(f_{rr} + \frac{1}{r} f_r \right) r A(r, \xi)\, dr = -\frac{2}{\pi\xi} f_r(a) - \xi^2 g(\xi). \qquad \ldots(3.4)$$

Let $\psi(\xi, y; p)$ denote the Weber transform of $\bar{\varphi}(r, y; p)$ as defined by (3.1). Then in view of (3.4), $\psi(r, \xi)$ satisfies

$$\left.\begin{aligned} &\psi_{yy} - \xi^2 \psi = \frac{2}{\pi\xi} \bar{U}(y; p), \; 0 < y < h \\ &p^2 \psi - (g + \epsilon p^2) \psi_y = 0 \quad \text{at } y = 0 \\ &\psi_y = 0 \quad \text{at} \quad y = h. \end{aligned}\right\} \qquad \ldots(3.5)$$

The solution of (3.5) is

$$\psi(\xi, y; p) = -\frac{2}{\pi\xi} \int_0^h G(y, \alpha) \bar{U}(\alpha, p)\, d\alpha \qquad \ldots(3.6)$$

where $G(y, \alpha)$ is the associated Green's function given by (cf. Mikhlin[6])

$$G(y, \alpha) = \frac{\{p^2 \sinh \xi y + (g + \epsilon p^2) \xi \cosh \xi y\} \cosh \xi (h - \alpha)}{\xi \{p^2 \cosh \xi h + (g + \epsilon p^2) \xi \sinh \xi h\}} \qquad \ldots(3.7)$$

for $0 < y < \alpha$. For $\alpha < y < h$, y and α are to be inter-changed in the expression (3.7). Using the inverse Weber transform formula (3.3) we obtain

$$\bar{\varphi}(r, y, p) = -\frac{2}{\pi}\int_0^\infty \frac{A(r,\xi)}{J_1^2(a\xi) + Y_1^2(a\xi)} \int_0^h G(y,\alpha)\,\bar{U}(\alpha, p)\,d\alpha\,d\xi.$$

Expressing the Bessel functions of the first and second kinds in terms of the Hankel functions functions and rearranging G we obtain

$$\bar{\varphi}(r, y, p) = -\frac{1}{\pi i}\int_0^\infty \frac{B(r,\xi)}{\xi D(\xi)}\left[E(\xi, y) + \frac{\cosh\xi(h-y)}{\sinh\xi h}\,\frac{\mu^2}{\mu^2+p^2}\right]$$

$$\times \int_0^h \cosh\xi(h-\alpha)\,\bar{U}(\alpha, p)\,d\alpha\,d\xi \qquad \text{...(3.8)}$$

where

$$B(r,\xi) = \frac{H_0^{(1)}(\xi r)}{H_1^{(1)}(\xi a)} - \frac{H_0^{(2)}(\xi r)}{H_1^{(2)}(\xi a)}$$

and

$$\mu^2 = \frac{g\,\xi\sinh\xi h}{D(\xi)} \qquad \text{...(3.9)}$$

$$D(\xi) = \cosh\xi h + \xi\epsilon\sinh\xi h$$

$$E(\xi, y) = \sinh\xi y + \xi\epsilon\cosh\xi y.$$

Taking Laplace's inversion we obtain

$$\varphi(r, y, t) = -\frac{1}{\pi i}\int_0^\infty \frac{B(r,\xi)}{\xi D(\xi)}\int_0^h\left[E\,U(\alpha, t)\right.$$

$$\left.+\ \frac{\mu\cosh\xi(h-y)}{\sinh\xi h}\int_0^t U(\alpha,\tau)\sin\mu(t-\tau)\,d\tau\right]$$

$$\times\ \cosh\xi(h-\alpha)\,d\alpha\,d\xi. \qquad \text{...(3.10)}$$

(3.10) gives the general result for the potential function due to a vertical cylindrical wave-maker with prescribed time-dependent normal fluid velocity $U(y, t)$. The depression of the inertial surface at time t can be obtained from the relation

$$\zeta(r, t) = \frac{1}{g}\,\frac{\partial}{\partial t}(\varphi - \epsilon\,\varphi_y)(r, 0, t). \qquad \text{...(3.11)}$$

4. Time-harmonic Wave-maker and Steady-state Development

For a time-harmonic wave-maker we take

$$U(y, t) = U(y) \sin \sigma t \qquad \ldots(4.1)$$

where σ is the circular frequency. Then (3.10) gives

$$\varphi(r, y, t) = -\frac{1}{\pi i} \int_0^h U(\alpha) \int_0^\infty \frac{B(r, \xi)}{\xi D(\xi)} \left[E \sin \sigma t + \frac{\mu \cosh \xi (h-y)}{\sinh \xi h} \times \frac{\mu \sin \sigma t - \sigma \sin \mu t}{\mu^2 - \sigma^2} \right] \cosh \xi (h - \alpha)\, d\xi\, d\alpha. \qquad \ldots(4.2)$$

To find the steady-state development in (4.2) we follow the method used by Rhodes-Robinson[8]. Two cases are required to be investigated according as the integrand of the inner integral in the second term of (4.2) has a pole in the range of integration $\xi > 0$ or not. Now $\mu^2 - \sigma^2$ or equivalently $\xi \sinh \xi h - K^* \cosh \xi h$ has a zero at $\xi = \xi_0$, say, for $\xi > 0$ if $0 \leqslant \epsilon K < 1$ and none if $\epsilon K \geqslant 1$ where $K = \sigma^2/g$ and $K^* = K(1 - \epsilon K)^{-1}$. The latter case is physically interpreted as the inertial surface to be "too heavy" while the former as "light". The two cases are now dealt with separately.

For $0 \leqslant \epsilon K < 1$, we introduce a Cauchy principal value at $\xi = \xi_0$ (i.e. $\mu = \sigma$) and write the inner involving $\sin \mu t$ in (4.2) as

$$\sigma \int_0^\infty \frac{B(r, \xi) \cosh \xi (h - y) \cosh \xi (h - \alpha)}{\xi D(\xi) \sinh \xi h} \frac{\mu \sin \sigma t\, d\xi}{\mu^2 - \sigma^2}$$

$$= 4\sigma \int_0^{(g/\epsilon)^{1/2}} \left[\frac{\cosh \xi' (h - y) \cosh \xi' (h - \alpha) B(r, \xi')}{P(\xi') (\mu' + \sigma)} \right]_{\xi' = \xi_0}^{\xi} \frac{\sin \mu t}{\mu - \sigma} d\mu$$

$$+ 2 \frac{B(r, \xi_0) \cosh \xi_0 (h - y) \cosh \xi_0 (h - \alpha)}{P(\xi_0)} \int_0^{(g/\epsilon)^{1/2}} \frac{\sin \mu t}{\mu - \sigma} d\mu$$

where

$$P(\xi') = \sinh 2\xi' h + 2\xi' h \text{ and } \mu' = \mu(\xi').$$

By Riemann-Lebesgue lemma the first term is transient and the integral in the second term becomes $\pi \cos \sigma t$ as $t \to \infty$. Thus as $t \to \infty$, we obtain after simplification

$$\varphi(r, y, t) \sim \frac{-\sin \sigma t}{\pi i} \int_0^h \int_0^\infty \frac{B(r, \xi)}{\xi} \frac{\{\xi \cosh \xi y - K^* \sinh \xi y\}}{\Delta(\xi)}$$

(*equation continued on p.* 510)

$$\times \cosh \xi (h - \alpha)\, d\xi\, U(\alpha)\, d\alpha$$

$$+ \frac{2 \cos \sigma t}{i} \frac{B(r, \xi_0) \cosh \xi_0 (h - y) A_0}{P(\xi_0)} \quad \ldots(4.3)$$

where

$$\left.\begin{aligned} \Delta(\xi) &= \xi \sinh \xi h - K^* \cosh \xi h \\ A_0 &= \int_0^h \cosh \xi_0 (h - \alpha)\, U(\alpha)\, d\alpha. \end{aligned}\right\} \quad \ldots(4.4)$$

(4.3) has the alternative representation

$$\varphi(r, y, t) \sim - \frac{4 A_0 \cosh \xi_0 (h - y)}{P(\xi_0)\, H(\xi_0 a)} \{F(\xi_0 r) \cos \sigma t + G(\xi_0 r) \sin \sigma t\}$$

$$- 4 \sin \sigma t \sum_{n=1}^{\infty} A_n \frac{K_0(\xi_n r)}{K_1(\xi_n a)} \frac{\cos \xi_n (h - y)}{\sin 2\xi_n h + 2\xi_n h} \quad \ldots(4.5)$$

where ξ_n' s are the solutions of the transcendental equation

$$\xi_n \sin \xi_n h + K^* \cos \xi_n h = 0,\ n = 1, 2, \ldots \quad \ldots(4.6)$$

and

$$A_n = \int_0^h \cos \xi_n (h - \alpha)\, U(\alpha)\, d\alpha \quad \ldots(4.7)$$

$$\left.\begin{aligned} F(\xi_0 r) &= J_0(\xi_0 r) J_1(\xi_0 a) + Y_0(\xi_0 r) Y_1(\xi_0 a) \\ G(\xi_0 r) &= J_0(\xi_0 r) Y_1(\xi_0 a) - Y_0(\xi_0 r) J_1(\xi_0 a) \\ H(\xi_0 a) &= J_1^2(\xi_0 a) + Y_1^2(\xi_0 a) \end{aligned}\right\} \quad \ldots(4.8)$$

and K_0, K_1 are modified Bessel's function. Hence using (3.11), the depression of the inertial surface as $t \to \infty$ becomes

$$\zeta(r, t) \sim - \frac{4\sigma}{g} (1 + \epsilon K^*) \cos \sigma t \sum_{1}^{\infty} A_n \frac{K_0(\xi_n r)}{K_1(\xi_n a)} \frac{\cos \xi_n h}{2\xi_n h + \sin 2\xi_n h}$$

$$- \frac{4\sigma}{g} (1 - \epsilon K^*) \frac{A_0 \cosh \xi_0 h}{\sinh 2\xi_0 h + 2\xi_0 h} \frac{F \cos \sigma t - G \sin \sigma t}{H(\xi_0 h)} . \quad \ldots(4.9)$$

As $r \to \infty$, this gives

$$\zeta(r, t) \sim - \frac{4\sigma}{g} \frac{1 - \epsilon K^*}{H(\xi_0 a)} A_0 \left(\frac{2}{\pi \xi_0 r}\right)^{1/2}$$

(*equation continued on p.* 511)

$$\times \left[Y_1 (\xi_0 a) \sin \left(\xi_0 r - \frac{\pi}{4} - \sigma t \right) \right.$$

$$\left. + J_1 (\xi_0 a) \cos \left(\xi_0 r - \frac{\pi}{4} - \sigma t \right) \right]. \qquad ...(4.10)$$

(4.10) represents outgoing waves at large distance from the wave-maker.

For $\epsilon K \geqslant 1$, there is no pole of the integrand in the second term in (4.2) and thus by Riemann-Lebesgue lemma the integral involving sin μt is wholly transient and hence $t \rightarrow \infty$

$$\varphi (r, y, t) \sim - \frac{\sin \sigma t}{\pi t} \int_0^h U(\alpha) \int_0^\infty \frac{B(r, \xi) \{\xi \cosh \xi y + k_0 \sinh \xi y\}}{\xi (\xi \sin \xi h + k_0 \cosh \xi h)}$$

$$\times \cosh \xi (h - \alpha) \, d\xi \, d\alpha \qquad ...(4.11)$$

where

$$k_0 = K (\epsilon K - 1)^{-1}. \qquad ...(4.12)$$

This has the alternative representation

$$\varphi (r, y, t) \sim - 4 \sin \sigma t \sum_{n=1}^{\infty} \frac{K_0 (\zeta_n r)}{K_1 (\zeta_n a)} \frac{\cos \zeta_n (h - y)}{2 \zeta_n h + \sin 2 \zeta_n h}$$

$$\times \int_0^h \cos \zeta_n (h - \alpha) \, U(\alpha) \, d\alpha \qquad ...(4.13)$$

where ζ_n's satisfy

$$\zeta_n \sin \zeta_n h - k_0 h \cos \zeta_n h = 0, \; n = 1, 2, \ldots \qquad ...(4.14)$$

Then the inertial surface depression as $t \rightarrow \infty$ is

$$\zeta (r, t) \sim \frac{4\sigma}{g (\epsilon K - 1)} \cos \sigma t \sum_{1}^{\infty} \frac{K_0 (\xi_n r)}{K_1 (\zeta_n a)} \frac{\cos \zeta_n h}{2 \zeta_n h + \sin 2 \zeta_n h}$$

$$\times \int_0^h \cos \zeta_n (h - \alpha) \, U(\alpha) \, d\alpha. \qquad ...(4.15)$$

As $r \rightarrow \infty$, $\zeta (r, t) \rightarrow 0$. Thus when the inertial surface is too heavy a time-harmanic disturbance on the wave-maker cannot propagate at large distances from the wave-maker.

5. Conclusion

The problem of a vertical circular cylindrical wave-maker immersed in a liquid of finite depth with an inertial surface is solved by the use of Laplace transform in time and a suitable Weber transform in the radial co-ordinate. The steady-state development of the depression of the inertial surface due to a time-harmonic vertical circular cylindrical wave-maker is deduced for a 'light' as well as a 'heavy' inertial surface. In the absence of inertial surface, the results for a time-harmonic wave-maker are recovered which can also be duced from Rhodes-Robinson's[7] results in the absence of surface tension.

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