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Bayesian Implementation: The Necessity of Infinite Mechanisms*

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We demonstrate the existence of a social choice function in an environment where there are two outcomes and two players each of whom can be of two types, which can only be implemented in Bayesian–Nash equilibrium by a mechanism where both players have an infinite number of messages. This stands in dramatic contrast to the case of Nash implementation in complete information, finite environments. *Journal of Economic Literature* Classification Numbers: 025, 026.

1. INTRODUCTION

The theory of implementation is concerned with the decentralisation of decision making when agents have private information. The heart of the implementation problem is the construction of a mechanism or decision procedure which will induce agents to reveal their private information. This need to give agents the right incentives acts as a constraint both on the kind of decentralised procedures which can be used as well as on the class of social objectives which can be *implemented.*

Of course, the choice of mechanisms as well as the nature of implementable social goals will depend on the environment, in particular on the structure of information. In a classic paper, Maskin [6] considered the case of *complete formation*, that is, a framework in which the state of the world is known to all agents. Various issues in the implementation problem with complete information have been analysed subsequently.[1]

One aspect of the literature which has come in for a lot of criticism is that many of the positive results are obtained with the help of mechanisms which possess undesirable features. For instance, Moore [7] remarks that the general theorem on implementation in subgame perfect equilibrium

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[1] See Moore [7] for a recent survey of this literature.

involves "... the construction of an enormously elaborate mechanism, with several stages of simultaneous moves. Worse, the mechanism appeals to clevel but unpalatable devices which exploit the finite details of what constitutes—or rather does not constitute—an equilibrium." Moore goes on to list some of these "unpalatable" devices, one of them being the use of mechanisms whose message sets are infinite even when the environment is finite.

In this paper, we restrict attention to environments with incomplete information, in particular to those in which agents possess *exclusive* information.[2] We show that in *noneconomic environments*[3] with (exclusive) incomplete information, there are social choice correspondences which can only be implemented in Bayesian–Nash equilibrium by mechanisms with message sets which are not finite *even* in the simplest finite environment. Thus, our result shows that the very nature of Bayesian implementation involves the use of mechanisms with "unpalatable" devices.

This result also underlines the difference between implementation in Nash equilibrium and in Bayesian–Nash equilibrium. In the former case, unwanted equilibria can be destroyed by making individuals cycle endlessly over a finite set of strategies. This is accomplished by means of the "modulo game" construction. Indeed, an upper bound on the size (that is the number of messages) of individual message sets can be obtained as a function of the environment.[4] Our result demonstrates that this idea cannot be used in the incomplete information context. Attempts to eliminate equilibria with a finite number of strategies may result in the creation of new unwanted equilibria. Consequently, the general principles underlying mechanism design in incomplete information environments are far more subtle than in the complete information setting. It is also important to realise that the usual "integer games" will not suffice either. Although such games have an infinite number of strategies, it is always possible to replicate them perfectly by using appropriate modulo games.[5]

[2] An alternative framework is one where information is *nonexclusive*; that is, each agent's information is redundant if the other agents pool their information. See Postlewaite and Schmeidler [11] and Palfrey and Srivastava [9] for analyses of implementation in this setting.

[3] We are following the terminology of Jackson [5]. Jackson defined an *economic environment* as one in which at least two agents are never satiated.

[4] See Danilov [1], Dutta and Sen [2, 3], and Moore and Repullo [8] for some recent applications of modulo type constructions in the context of implementation in the perfect information setting.

[5] The referee has drawn our attention to the need for caution in the use of the term "integer game." He points out that the infinite mechanism we employ in our example can be described as an integer game of the following kind: Both players announce integers and the outcome is a_2 if agent one has a higher integer and a_1 otherwise. By "integer game," we refer to games where the player with the higher integer dictates the outcome.

The most general result on Bayesian implementation is that of Jackson [5]. Jackson formulates a condition on social choice functions called monotonicity-no-veto. He shows that if there are three individuals, then the condition is sufficient for implementation. Moreover, implementation is achieved by means of a mechanism which is finite in finite environments. Our example is not covered by Jackson's sufficiency theorem for two reasons. We have two individuals and the social choice function does not satisfy monotonicity-no-veto. However, we show that it is possible to extend our two person example to a three person example where two players require infinite message sets. Therefore, the lack of finiteness is a consequence of the failure of monotonicity-no-veto.

2. Notation and Definitions

In this section, we describe the general framework of Bayesian implementation. For any collection of sets $\{B^i\}$, $i \in I$, B and B^{-i} will denote the Cartesian products $\prod_{i \in I} B^i$ and $\prod_{j \in I \setminus \{i\}} B^j$, respectively. The vector $(\tilde{b}^i, b^{-i}) \in B$ denotes the vector $(b^1, ..., b^{i-1}, \tilde{b}^i, b^{i+1}, ..., b^N)$. In general, lower case letters will denote elements of sets which are represented by corresponding capital letters.

The set of individuals is a finite set $I = \{1, ..., N\}$. Following the formulation of Harsanyi [4], the set of *types* of individual $i \in I$ will be denoted by S^i. Throughout the paper, we assume that S^i is finite. An element $s \in S$ will be referred to as a *state of the world*, or simply as a state. A complete description of individual preferences is associated with each state.

The set of feasible outcomes will be denoted by A. Elements of A may be interpreted as allocations of commodities across individuals, candidates in an election, and so on. It is assumed that A is fixed and independent of the state.

An *allocation* x is a mapping $x: S \to A$. For all $x \in S$, $x[s] \in A$ is the outcome specified by x. Let X denote the set of all allocations. Note that when A is finite, the set of allocations will also be finite. We will refer to this case as the *finite environment.*

Every individual $i \in I$ has a *prior probability distribution* q^i defined on the set S. We assume that $\{s \in S \mid q^i(s) > 0\} = S$ for all $i \in I$.

For all $i \in I$, $s^i \in S^i$ and $s^{-i} \in S^{-i}$, $q^i(s^{-i} \mid s^i)$ is the conditional probability of s^{-i}, given that s^i has occurred.

Each individual i has a *state-dependent utility function* $u^i: A \times S \to \mathbb{R}$. Note that the utility function depends on the entire state s and not just on individual i's type s^i in that state.

For all $i \in I$ and $s^i \in S^i$, the binary relation $R^i(s^i)$ is defined on the elements of X as follows:

$$\text{for all } x, y \in X, \quad xR^i(s^i)\, y \leftrightarrow \sum (u^i(x[s], s) - u^i(y[s], s))\, q^i(s^{-i} \mid s^i) \geqslant 0.$$

One feature of the binary relation $R^i(s^i)$ deserves special mention. Pick an arbitrary $i \in I$ and $\bar{s}^i \in S^i$. Let $x, y, \tilde{x}, \tilde{y} \in X$ be such that $x[\bar{s}^i, s^{-i}] = \tilde{x}[\bar{s}^i, s^{-i}]$ and $y[\bar{s}^i, s^{-i}] = \tilde{y}[\bar{s}^i, s^{-i}]$ for all $s^{-i} \in S^{-i}$. Then $\tilde{x}R^i(\bar{s}^i)\,\tilde{y}$ if and only if $xR^i(\bar{s}^i)\,y$. Thus, for the purpose of ranking any two allocations x and y under $R^i(\bar{s}^i)$, only the values of x and y in states whose ith component is $\bar{s}^i$ matter. This fact follows immediately from the definition of $R^i(\bar{s}^i)$.

A *social choice function* (*SCF*) F is an element of X.

A *mechanism* G is an $(N+1)$ tuple $(M^1, ..., M^N, g)$, where M^i is the message set for individual $i \in I$ and g is the outcome function $g: M \to A$.

Let G be a mechanism. The collection $(I, S, \{q^i\}_{i \in I}, \{u^i\}_{i \in I}, G)$ constitutes a game of incomplete information. We shall refer to this game as the game associated with G, or the G-game. A strategy for individual i, σ^i, is a mapping $\sigma^i: S^i \to M^i$. The set Σ^i is the set of all strategies of i. For all $\sigma \in \Sigma$, $\sigma(s)$ represents the vector $(\sigma^1(s^1), ..., \sigma^N(s^N))$ and $g(\sigma)$ the allocation which results when σ is played.

A *Bayesian–Nash equilibrium* of the G-game is a vector of strategies $\sigma_* \in \Sigma$ such that $g(\sigma_*)\, R^i(s^i)\, g(\sigma^i, \sigma_*^{-i})$ for all $\sigma^i \in \Sigma^i$, $s^i \in S^i$, $i \in I$. Let $\Sigma_*(G)$ denote the set of all Bayesian–Nash equilibria of the G-game.

DEFINITION 2.1. A SCF F is *implementable* if there exists a mechanism G such that $\{g(\sigma_*) \mid \sigma_* \in \Sigma_*(G)\} = F$.

Jackson [5] contains a review of other definitions of implementation in the incomplete information context. The reader is also referred to Palfrey [10].

3. The Necessity of Infinite Mechanisms

We present an example to demonstrate the striking fact that implementation *even* in finite environments may involve the use of infinite mechanisms. This has serious implications. The most general result on Bayesian implementation so far (Jackson [5]) uses the Bayesian equivalent of the "modulo game" widely used in mechanisms for implementation in complete information settings. These mechanisms are finite when the environment is finite. In our example, the unique implementing mechanism is infinite even though the environment is finite. This suggests immediately that it would be impossible to use "integer" or "modulo game" constructions to obtain a necessary and sufficient condition.

The necessity of infinite mechanisms also stands in sharp contrast to canonical mechanisms in the complete information case. Here, it is well known that if a SCF is implementable, we need not look beyond the class of modulo games in our search for the mechanism that implements it. The example shows that this result does not carry over to the incomplete information case. We now proceed to the example.

Let $I=\{1,2\}$, $S^1=\{s^1,s^2\}$, $S^2=\{t^1,t^2\}$, and $A=\{a_1,a_2\}$. An allocation will be represented by a 4 tuple whose first, second, third, and fourth components refer to the outcomes in states s^1t^1, s^1t^2, s^2t^1, and s^2t^2, respectively. Thus, (a_1,a_2,a_1,a_2) is the allocation which specifies a_1,a_2,a_1, and a_2 in states s^1t^1, s^1t^2, s^2t^1, and s^2t^2, respectively. There are 16 allocations altogether and they are numbered as follows:

$$(a_1,a_1,a_1,a_1)=x^1,\quad (a_1,a_1,a_1,a_2)=x^2,$$

$$(a_1,a_1,a_2,a_2)=x^3,\quad (a_1,a_1,a_2,a_1)=x^4,$$

$$(a_1,a_2,a_1,a_1)=x^5,\quad (a_1,a_2,a_1,a_2)=x^6,$$

$$(a_1,a_2,a_2,a_1)=x^7,\quad (a_1,a_2,a_2,a_2)=x^8,$$

$$(a_2,a_1,a_1,a_1)=x^9,\quad (a_2,a_1,a_1,a_2)=x^{10},$$

$$(a_2,a_1,a_2,a_1)=x^{11},\quad (a_2,a_1,a_2,a_2)=x^{12},$$

$$(a_2,a_2,a_1,a_1)=x^{13},\quad (a_2,a_2,a_1,a_2)=x^{14},$$

$$(a_2,a_2,a_2,a_1)=x^{15},\quad (a_2,a_2,a_2,a_2)=x^{16},$$

The utility functions of the individuals are as follows:

$$u^1(a_1,(s^1t^1))=2;\quad u^1(a_2,(s^1t^1))=1$$

$$u^1(a_1,(s^1t^2))=1;\quad u^1(a_2,(s^1t^2))=2$$

$$u^1(a_1,(s^2t^1))=1;\quad u^1(a_2,(s^2t^1))=2$$

$$u^1(a_1,(s^2t^2))=2;\quad u^1(a_2,(s^2t^2))=2$$

$$u^2(a_1,(s^1t^1))=2;\quad u^2(a_2,(s^1t^1))=0$$

$$u^2(a_1,(s^2t^1))=2;\quad u^2(a_2,(s^2t^1))=2$$

$$u^2(a_1,(s^1t^2))=1.5;\quad u^2(a_2,(s^1t^2))=2$$

$$u^2(a_1,(s^2t^2))=2;\quad u^2(a_2,(s^2t^2))=0.$$

The prior probability distributions for the players are given by

$$q^1(t^1 \mid s^1) = q^1(t^2 \mid s^1) = q^1(t^1 \mid s^2) = q^1(t^2 \mid s^2) = 0.5$$

and

$$q^2(s^1 \mid t^1) = q^2(s^2 \mid t^1) = q^2(s^1 \mid t^2) = q^2(s^2 \mid t^2) = 0.5.$$

It is easy to verify that with these data, the binary relations $R^1(s^1)$, $R^1(s^2)$, $R^2(t^1)$, and $R^2(t^2)$ are as follows:

$$R^1(s^1)$$

$$x^5 \sim x^6 \sim x^7 \sim x^8$$

$$x^{13} \sim x^{14} \sim x^{15} \sim x^{16} \sim x^1 \sim x^2 \sim x^3 \sim x^4$$

$$x^9 \sim x^{10} \sim x^{11} \sim x^{12}$$

$$R^1(s^2)$$

$$x^4 \sim x^7 \sim x^{11} \sim x^{15} \sim x^3 \sim x^8 \sim x^{12} \sim x^{16}$$

$$x^2 \sim x^6 \sim x^{10} \sim x^{14} \sim x^1 \sim x^5 \sim x^9 \sim x^{13}$$

$$R^2(t^1)$$

$$x^1 \sim x^2 \sim x^3 \sim x^4 \sim x^5 \sim x^6 \sim x^7 \sim x^8$$

$$x^9 \sim x^{10} \sim x^{11} \sim x^{12} \sim x^{13} \sim x^{14} \sim x^{15} \sim x^{16}$$

$$R^2(t^2)$$

$$x^5 \sim x^7 \sim x^{13} \sim x^{15}$$

$$x^1 \sim x^4 \sim x^9 \sim x^{11}$$

$$x^6 \sim x^8 \sim x^{14} \sim x^{16}$$

$$x^2 \sim x^3 \sim x^{10} \sim x^{12}.$$

In all the four binary relations, any two allocations on the same row are indifferent to each other. If one allocation is above the other, then the first is strictly preferred to the second.

Consider the SCF, $F = \{x^4\}$.[6] Let G denote the mechanism described below:

[6] Observe that x^7 is (weakly) maximal for players of all types but is not F. Therefore, F does not satisfy Jackson's monotonicity-no-veto condition.

	m_1^2	m_2^2	m_3^2	m_4^2	m_5^2	$\cdots$
m_1^1	a_1	a_1	a_1	a_1	a_1	$\cdots$
m_2^1	a_2	a_1	a_1	a_1	a_1	$\cdots$
m_3^1	a_2	a_2	a_1	a_1	a_1	$\cdots$
m_4^1	a_2	a_2	a_2	a_1	a_1	$\cdots$
m_5^1	a_2	a_2	a_2	a_2	a_1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

(Here, player 1 chooses rows, while player 2 chooses columns).

PROPOSITION 3.1. *The mechanism G implements F. Moreover, it is the unique (upto a relabelling of rows and columns) mechanism which implements it.*

Proof. We first establish that G implements F. Let $\sigma_*^1(s^1)=m_1^1$, $\sigma_*^1(s^2)=m_2^1$, $\sigma_*^2(t^1)=m_1^2$, $\sigma_*^2(t^2)=m_2^2$. Clearly, $g(\sigma_*)=x^4$. We first show that $\sigma_*\in\Sigma_*(G)$, where $\Sigma_*(G)$ is the set of equilibrium strategies in G. By deviating from σ_*, player 1 can get any of the allocations x^1, x^9, x^{11}, x^{13}, and x^{16}. However, x^4 is weakly preferred to all these allocations according to both $R^1(s^1)$ and $R^1(s^2)$. Similarly, deviation from σ_* allows player 2 to get allocations x^1, x^2, and x^3. However, none of these are better than x^4 under either $R^2(t^1)$ or $R^2(t^2)$. Therefore, $\sigma_*\in\Sigma_*(G)$.

We now show that if $\sigma\in\Sigma_*$, then $g(\sigma)=x^4$. Pick an arbitrary $\sigma\in\Sigma$. We first claim that it cannot be the case that $g(\sigma)=x^7$ or x^{10}. To see this, suppose that $g(\sigma)=y=x^{10}$. Assume without loss of generality that $\sigma^1(s^1)=m_r^1$. Let $\sigma^2(t^1)=m_k^2$ and $\sigma^2(t^2)=m_l^2$. It must be the case that $k<l$. Otherwise, one of the following must hold: (i) $y[s^1t^1]=y[s^1t^2]=a_1$, (ii) $y[s^1t^1]=y[s^1t^2]=a_2$, or (iii) $y[s^1t^1]=a_1$ and $y[s^1t^2]=a_2$. In each case we contradict the hypothesis that $y=x^{10}$. Suppose that $\sigma^2(s^2)=m_v^1$ and $v\leqslant r$. Then $y[s^2t^2]=a_1$ or $y=x^{16}$. Therfore, $v>r$. But now $y[s^1t^1]=a_2$ implies that $y[s^2t^1]=a_2$. Therefore $y\neq x^{10}$. Suppose that $g(\sigma)=x^7$. Let $\tilde\sigma\in\Sigma$ be such that $\tilde\sigma^1=\sigma^1$, $\tilde\sigma^2(t^1)=\sigma^2(t^1)=\sigma^2(t^2)$ and $\tilde\sigma^2(t^2)=\sigma^2(t^1)$. Then, $g(\tilde\sigma)=x^{10}$. However, this is impossible. Therefore, $g(\sigma)\neq x^7$.

Let $g(\sigma)=y$. Player 1, by playing a strategy $\tilde\sigma^1$ such that $\tilde\sigma^1(s^2)=m_r^1$ with r sufficiently large, can ensure that $g(\tilde\sigma^1,\sigma^2)=z$, where $z[s^1t^1]=y[s^1t^1]$, $z[s^1t^2]=y[s^1t^2]$, $z[s^2t^1]=z[s^2t^2]=a_2$. Suppose that $g(\sigma)=x^1$. By deviating, player 1 can get x^3. Since $x^3P^1(s^2)x^1$, σ cannot be an equilibrium. By a similar argument, σ cannot be an equilibrium if

$g(\sigma)=x^5, x^9, x^{13}, x^2, x^6$, or x^{14}. In these cases, player 1 has a deviation which will give him $x^8, x^{12}, x^{16}, x^3, x^8$, and x^{16}, respectively. In each case, the deviation is strictly preferred according to $P^1(s^2)$. For all $\sigma \in \Sigma$, player 2, by playing $\tilde{\sigma}^2$ such that $\tilde{\sigma}^2(t^1)=m_k^2$ and $\tilde{\sigma}^2(t^2)=m_l^2$ with k and l sufficiently large, can ensure that $g(\sigma^1, \tilde{\sigma}^2)=x^1$. This implies that if $g(\sigma) \in \{x^{11}, x^{12}, x^{13}, x^{15}, x^{16}\}$, then σ cannot be an equilibrium. Player 2 will deviate to get x^1 and will be strictly better off according to $R^2(t^1)$. It also implies that if $g(\sigma)=x^3$ or x^8, then σ cannot be an equilibrium. Once again player 2 will deviate to get x^1 and will be strictly better off according to $R^2(t^2)$. This only leaves the case where $g(\sigma)=x^4$. It is acceptable for σ to be an equilibrium since $F=\{x^4\}$. This establishes that G implements F.

We now demonstrate that G is the unique (up to a permutation of rows and columns) mechanism which implements F. Let $\bar{G}=(\bar{M}^1, \bar{M}^2, \bar{g})$ be an arbitrary mechanism which implements F. Let $\bar{\Sigma}^i$ denote the strategy space of player i in the $\bar{G}$-game. There must exist $\sigma_* \in \bar{\Sigma}$ such that $g(\sigma_*)=x^4$. Assume without loss of generality that $\sigma_*^1(s^1)=\bar{m}_1^1$, $\sigma_*^1(s^2)=\bar{m}_2^1$, $\sigma_*^2(t^1)=\bar{m}_1^2$, and $\sigma_*^2(t^2)=\bar{m}_2^2$. The part of $\bar{G}$ we have constructed so far looks as follows:

	$\bar{m}_1^2$	$\bar{m}_2^2$
$\bar{m}_1^1$	a_1	a_1
$\bar{m}_2^1$	a_2	a_1

Consider the strategy $\hat{\sigma} \in \bar{\Sigma}$, where $\hat{\sigma}^1(s^1)=\bar{m}_2^1$, $\hat{\sigma}^1(s^2)=\bar{m}_1^1$, $\hat{\sigma}^2(t^1)=\bar{m}_2^2$, and $\hat{\sigma}^2(t^2)=\bar{m}_1^2$. Then, $\bar{g}(\hat{\sigma})=x^5$. Since $x^5 \notin F$, some player must have a profitable deviation from $\hat{\sigma}$. This player cannot be player 2, since x^5 is both $R^2(t^1)$ and $R^2(t^2)$ maximal. Therefore, player 1 must destroy $\hat{\sigma}$ as a potential equilibrium. Moreover, since x^5 is $R^1(s^1)$ maximal, player 1 must have a profitable deviation according to $R^1(s^2)$. Suppose this deviation yields the allocation y, where $y=x^4, x^7, x^{11}$, or x^{15}. All these allocations share the common feature that in states s^2t^1 and s^2t^2, they specify outcomes a_2 and a_1 respectively. This implies that player 1 must have a message, say $\bar{m}_3^1$, such that $\bar{g}(\bar{m}_3^1, \bar{m}_2^2)=a_2$ and $\bar{g}(\bar{m}_3^1, \bar{m}_1^2)=a_1$. Now consider $\tilde{\sigma} \in \bar{\Sigma}$ such that $\tilde{\sigma}^1(s^1)=\bar{m}_2^1$, $\tilde{\sigma}^1(s^2)=\bar{m}_3^1$, $\tilde{\sigma}^2(t^1)=\bar{m}_2^2$, and $\tilde{\sigma}^2(t^2)=\bar{m}_1^2$. Then $g(\tilde{\sigma})=x^7$. Since x^7 is simultaneously $R^1(s^1)$, $R^1(s^2)$, $R^2(t^1)$, and $R^2(t^2)$ maximal, $\tilde{\sigma}$ must be an equilibrium of $\bar{G}$. However, $x^7 \notin F$ and we have a contradiction. Therefore, the player 1 deviation which knocks out $\hat{\sigma}$ must yield $y \in \{x^3, x^8, x^{12}, x^{16}\}$. All these allocations specify a_2 in states s^2t^1 and s^2t^2; hence, there must be a message for player 1, say $\bar{m}_3^1$, such that $g(\bar{m}_3^1, \bar{m}_1^2)=g(\bar{m}_3^1, \bar{m}_2^2)=a_2$. Now $\bar{G}$ looks as follows:

	$\bar{m}_1^2$	$\bar{m}_2^2$
$\bar{m}_1^1$	a_1	a_1
$\bar{m}_2^1$	a_2	a_1
$\bar{m}_3^1$	a_2	a_2

Consider the strategy $\sigma \in \bar{\Sigma}$, where $\sigma^1(s^1) = \bar{m}_2^1$, $\sigma^1(s^2) = \bar{m}_3^1$, $\sigma^2(t^1) = \bar{m}_2^2$, and $\sigma^2(t^2) = \bar{m}_1^2$. Then $g(\sigma) = x^8$. Since $x^8 \notin F$, some player must deviate from σ. Since x^8 is both $R^1(s^1)$ and $R^1(s^2)$ maximal, it must be player 2 who deviates. Let the allocation which player 2 gets by deviating be y. Since x^8 is $R^2(t^1)$ maximal, it must be the case that $yP^2(t^2)x^8$. There are two cases to consider. In the first, $y \in \{x^5, x^7, x^{13}, x^{15}\}$. In this case, there must exist a message for player 2, say $\bar{m}_3^2$, such that $g(\bar{m}_2^1, \bar{m}_3^2) = a_2$ and $g(\bar{m}_3^1, \bar{m}_3^2) = a_1$. But now, if player 1 plays σ^1 and player 2 plays $\bar{m}_2^2$ when of type t^1 and $\bar{m}_3^2$ when of type t^2, the outcome is x^7. We have argued earlier that such a strategy must be an equilibrium. We are led to a contradiction since $x^7 \notin F$. Therefore $y \in \{x^1, x^4, x^9, x^{11}\}$. This implies that there is a message for player 2, say $\bar{m}_3^2$, such that $\bar{g}(\bar{m}_2^1, \bar{m}_3^2) = \bar{g}(\bar{m}_3^1, \bar{m}_3^2) = a_1$. Suppose that $\bar{g}(\bar{m}_1^1, \bar{m}_3^2) = a_2$. Then the strategy where player 1 plays $\bar{m}_1^1$ and $\bar{m}_2^1$ when of types s^1 and s^2, respectively, and player 2 plays $\bar{m}_1^2$ and $\bar{m}_3^2$ when of types t^1 and t^2, respectively, gives rise to the allocation x^7. We know that this leads to a contradiction. Therefore, $\bar{g}(\bar{m}_1^1, \bar{m}_3^2) = a_1$. The part of $\bar{G}$ constructed so far is shown below.

	$\bar{m}_1^2$	$\bar{m}_2^2$	$\bar{m}_3^2$
$\bar{m}_1^1$	a_1	a_1	a_1
$\bar{m}_2^1$	a_2	a_1	a_1
$\bar{m}_3^1$	a_2	a_2	a_1

Now look at the strategy $\hat{\sigma} \in \bar{\Sigma}$, where $\hat{\sigma}^1(s^1) = \bar{m}_1^3$, $\hat{\sigma}^1(s^2) = \bar{m}_2^1$, $\hat{\sigma}^2(t^1) = \bar{m}_3^2$, and $\hat{\sigma}^2(t^2) = \bar{m}_2^2$. Then $g(\hat{\sigma}) = x^5$. Duplicating earlier arguments, we deduce that there must exist a message for player 1, say $\bar{m}_4^1$, such that $\bar{g}(\bar{m}_4^1, \bar{m}_2^2) = \bar{g}(\bar{m}_4^1, \bar{m}_3^2) = a_2$. Moreover, there must exist a message for player 2, say $\bar{m}_4^2$, such that $\bar{g}(\bar{m}_2^1, \bar{m}_4^2) = \bar{g}(\bar{m}_3^1, \bar{m}_4^2) = a_1$. Also, $\bar{g}(\bar{m}_1^1, \bar{m}_4^1) = a_1$; otherwise it would be possible to construct a strategy whose outcome is x^7. We claim that $\bar{g}(\bar{m}_4^1, \bar{m}_1^2) = a_2$. If $\bar{g}(\bar{m}_4^1, \bar{m}_1^2) = a_1$, then the strategy where player 1 plays $\bar{m}_4^1$ and $\bar{m}_2^1$ when of types s^1 and s^2, respectively, and 2 plays $\bar{m}_2^2$ and $\bar{m}_1^2$ when of types t^1 and t^2, respectively, yields the allocation x^7. After the messages $\bar{m}_4^1$ and $\bar{m}_4^2$ are added, $\bar{G}$ looks as follows:

	$\bar{m}_1^2$	$\bar{m}_2^2$	$\bar{m}_3^2$	$\bar{m}_4^2$
$\bar{m}_1^1$	a_1	a_1	a_1	a_1
$\bar{m}_1^2$	a_2	a_1	a_1	a_1
$\bar{m}_1^3$	a_2	a_2	a_1	a_1
$\bar{m}_1^4$	a_2	a_2	a_2	a_1

Starting with the strategy $\hat{\sigma}$ such that $\hat{\sigma}^1(s^1)=\bar{m}_3^1$, $\hat{\sigma}^1(s^2)=\bar{m}_4^1$, $\hat{\sigma}^2(t^1)=\bar{m}_3^2$, and $\hat{\sigma}^2(t^2)=\bar{m}_4^2$, we can repeat all the earlier arguments to infer that there must exist messages $\bar{m}_5^1$ and $\bar{m}_5^2$ for players 1 and 2, respectively, such that $\bar{g}(\bar{m}_5^1, \bar{m}_l^2)=a_2$ for $l=1, 2, 3, 4$, $\bar{g}(\bar{m}_k^1, \bar{m}_5^2)=a_1$ for $k=1, 2, 3, 4$, and $\bar{g}(\bar{m}_5^1, m_5^2)=a_1$. In fact, it is clear that this argument can be repeated ad infinitum so that $\bar{G}$ must contain an infinite number of messages for both players. Inspection confirms that $\bar{G}=G$. This proves the proposition. ∎

Why does Bayesian implementation require infinite mechanisms in finite environments? More specifically, why is it not sufficient to restrict attention to modulo games? Consider the example presented in this section and the mechanism G which implements $F=\{x^4\}$. We can think of G as a matrix whose entries are elements of A. The only allocation which can be supported by strategies in the G-game are those which form the vertices of a rectangle in the matrix. This is, of course, due to the definition of a player's strategy in an incomplete information game—the message sent by a player depends only on his own type. Let $X^*(G)$ be the set of allocations which can be supported by strategies in G. Observe that $x^7 \notin X^*(G)$. Now let G' be another mechanism which implements F and let $X^*(G')$ be defined in the same manner as $X^*(G)$. Since x^7 is maximal for all individuals of all types, it must be the case that $x^7 \notin X^*(G')$. However, if G' incorporates the modulo game (specifically, we mean that G' is a mechanism of the type used in Jackson [5]), then it is impossible to ensure that $x^7 \notin X^*(G')$. In general, modulo game constructions do offer the players the largest set of deviations from strategies which are not equilibria. However, in doing so, they may enlarge the set of allocations which can be supported by strategies. If any of these newly created allocations are maximal for players of all types (as in the case of x^7), then the mechanism may pick up non-optimal equilibria.

The preceding discussion suggests that the necessity of infinite mechanisms does not depend on a two person assumption.

We confirm this by means of a simple modification of the example. Suppose there is a third player, player 3, who is either of type r_1 or type r_2. If she is of type r_1, player 1 and 2 have the same utility functions as before, while if she is of type r_2, all outcomes have a utility of zero. Player 3 gets a constant utility (say zero) for all outcomes in all states. Formally, the utility functions $\bar{u}^1$, $\bar{u}^2$, and $\bar{u}^3$ are given as follows: For all $i, j, k, l = 1, 2$,

$$\begin{aligned} \bar{u}^i(a_j, (s^k t^l r^m)) &= u^i(a_j, (s^k t^l)) \quad &\text{if} \quad m = 1 \\ &= 0 \quad &\text{if} \quad m = 2; \end{aligned}$$

For all $j, k, l, m = 1, 2$,

$$\bar{u}^3(a_j, (s^k t^l r^m)) = 0.$$

Assume further that player types are distributed independently and that for all players, each type is equally likely.

We now represented an allocation by a pair such as (x, y), where x and y are the 4 tuples of outcomes in states where player 3 is of types r_1 and r_2, respectively. Let $\bar{R}^1(s^1)$, $\bar{R}^1(s^2)$, $\bar{R}^2(t^1)$, etc., denote the orderings induced on allocations. Observe that for players 1 and 2, differences in utilities associated with outcomes arise only when player 3 is of type r_1. Thus, for all pairs of allocations (x, y) and (w, z), $(x, y)\,\bar{R}^1(s^k)(w, z)$ iff $x\bar{R}^1(s^k)w$, $k = 1, 2$ and $(x, y)\,\bar{R}^2(t^l)(w, z)$ iff $x\bar{R}^2(t^l)w$, $l = 1, 2$. Of course, $\bar{R}^3(r^1) = \bar{R}^3(r^2)$ is the trivial ordering which ranks all pairs of allocations as indifferent.

Let $F = \{(x_4, x_4)\}$. We claim that the mechanism G of Proposition 1 is the unique mechanism which implements F. Thus, player 3's message set contains only one element, while players 1 and 2 have an infinite number of messages. We omit a proof of this claim which hinges on the relationship between the $\bar{R}$ and R orderings and the arguments used in Proposition 3.1. Let us briefly consider the argument to establish the uniqueness of G. Starting from the revelation game, observe that players 1 and 2 have a deception which gives rise to the allocation (x^5, x^5). Since player 3 is always indifferent, either player 1 or 2 must have a deviation to upset this potential equilibrium. The structure of preferences is such that this can occur if and only if one of these players has a successful deviation against x^5 in the original G-game. In addition, these new messages cannot be permitted to allow supportable allocations of the type $(x^7, \cdot)$. Similar arguments can be made to establish an exact correspondence between this construction and the mechanism G. This allows us to deduce the uniqueness of G.

In this three player example, it suffices to give player 3 a single message. However, in spite of her trivial preferences, player 3 is not a "dummy" player because the utility functions of the other players depend on her type. It is possible to construct examples with more than two players when all players have infinite message sets? The logic of our examples leads us to believe that it is possible to do so. We do not attempt such constructions because of the formidable computational difficulties involved.

REFERENCES

1. V. DANILOV, Implementation via Nash implementation, *Econometrica* **61** (1992), 43–56.
2. B. DUTTA AND A. SEN, Implementation under strong equilibrium, *J. Math. Econ.* **21** (1990), 49–68.
3. B. DUTTA AND A. SEN, Necessary and sufficient conditions for 2-Person Nash implementation, *Rev. Econ. Stud.* **58** (1991), 121–128.
4. J. C. HARSANYI, Games with incomplete information played by Bayesian players, *Management Sci.* **14** (1967–1968), 159–189, 320–334, 486–502.
5. M. O. JACKSON, Bayesian implementation, *Econometrica* **59** (1990), 461–477.
6. E. MASKIN, Nash equilibrium and welfare optimality, mimeo.
7. J. MOORE, Implementation, contracts and renegotiation in environments with complete information *in* "Advances in Economic Theory, Sixth World Congress" (J.-J. Laffont, Ed.), Econometric Society Monographs, Cambridge Univ. Press, Cambridge, 1992.
8. J. MOORE AND R. REPULLO, Nash implementation: A full characterization, *Econometrica* **58** (1990), 1083–1100.
9. T. PALFREY AND S. SRIVASTAVA, On Bayesian implementable allocations, *Rev. Econ. Stud.* **54** (1987), 193–208.
10. T. PALFREY, Implementation in Bayesian equilibrium: The multiple equilibrium problem in mechanism design, *in* "Advances in Economic Theory, Sixth World Congress" (J.-J. Laffont, Ed.), Econometric Society Monographs, Cambridge Univ. Press, Cambridge, 1992.
11. A. POSTLEWAITE AND D. SCHMEIDLER, Implementation in differential information economies, *J. Econ. Theory* **39** (1986), 14–33.