A NOTE ON THE MULTIVARIATE EXTENSION OF SOME THEOREMS RELATED TO THE UNIVARIATE NORMAL DISTRIBUTION

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1. Introduction

This note is of an expository nature. It is hoped that the approach will be found useful to the students. It is shown how certain theorems connected with the univariate normal (N_1) distribution may be immediately generalized to the multivariate case if we define the multivariate normal distribution as follows (Frechet, 1951).

Definition: The random p-vector $\mathbf{x}$ is said to have the p-variate normal (N_p) distribution if for every constant p-vector $\mathbf{t}$ the distribution of $\mathbf{t}\,\mathbf{x}'$ is univariate normal (N_1).

That the above definition is equivalent to the usual definition is proved as follows. Since every linear function (functional) of $\mathbf{x}$ is N_1, the dispersion matrix Λ of $\mathbf{x}$ must exist. Let μ be the mean vector of $\mathbf{x}$. Then, for every p-vector $\mathbf{t}$, the distribution of $\mathbf{t}\,\mathbf{x}'$ is N_1 with mean $\mathbf{t}\,\mu'$ and variance $\mathbf{t}\,\Lambda\,\mathbf{t}'$

Hence $$E\, e^{i\mathbf{t}\mathbf{x}'} = e^{i\mathbf{t}\mu' - \frac{1}{2}\mathbf{t}\Lambda\mathbf{t}'}$$

and the rest follows (Cramer 1946).

2. Some properties of N_p

Theorem 1: *If $\mathbf{x}$ is N_p and A is any constant $p \times q$ matrix then $\mathbf{x}\,A$ is N_q.*

Proof: For any q-vector $\mathbf{t}$

$$\mathbf{t}(\mathbf{x}A)' = (\mathbf{t}A')\mathbf{x}'$$

But $(\mathbf{t}A')\mathbf{x}'$ is N_1 because $\mathbf{x}$ is N_p.

Theorem 2: *If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are mutually independent N_p's then for any set of constants $c_1, c_2, \ldots, c_n$ the p-vector $\mathbf{y} = c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n$ is N_p.*

Proof: For any p-vector $\mathbf{t}$, the set of random variables $\mathbf{t}\mathbf{x}_1', \mathbf{t}\mathbf{x}_2', \ldots, \mathbf{t}\mathbf{x}_n'$ are mutually independent N_1's and hence

$$\mathbf{t}\mathbf{y}' = \Sigma\, c_j\,\mathbf{t}\mathbf{x}_j' \text{ is } N_1$$

Theorem 3: *If $\mathbf{x}_1$ and $\mathbf{x}_2$ are independent random p-vectors and $\mathbf{x}_1 + \mathbf{x}_2$ is N_p then both $\mathbf{x}_1$ and $\mathbf{x}_2$ are N_p's (a constant p-vector is a degenerate case of N_p.)*

Proof: Since $\mathbf{x}_1+\mathbf{x}_2$ is N_p we have, for every $\mathbf{t}$,

$$\mathbf{t}\mathbf{x}_1'+\mathbf{t}\mathbf{x}_2' = \mathbf{t}(\mathbf{x}_1+\mathbf{x}_2)' \text{ is } N_1$$

And since $\mathbf{t}\mathbf{x}_1'$ and $\mathbf{t}\mathbf{x}_2'$ are independent it follows (Cramer, 1937) that they are both N_1's.

Theorem 4: *If $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ are mutually independent N_p's with common dispersion matrix Λ and*

$$\mathbf{y}_i = \sum_j a_{ij}\,\mathbf{x}_j \qquad i, j = 1, 2, ..., n$$

where $A = (a_{ij})$ is a unitary orthogonal matrix then $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n$ are mutually independent N_p's with common dispersion matrix Λ.

Proof: Let $\mathbf{t}_1, \mathbf{t}_2, ..., \mathbf{t}_n$ be arbitrary p-vectors and let $\mathbf{z} = (z_1, z_2, ..., z_n)$ where

$$z_i = \mathbf{t}_i\mathbf{y}_i' = \Sigma\, a_{ij}\,(\mathbf{t}_i\,\mathbf{x}_j') \qquad i = 1, 2, ..., n.$$

Now, every linear functional of $\mathbf{z}$ is a linear combination of linear functionals of $\mathbf{x}_i$'s and hence is N_1. That is $\mathbf{z}$ is N_p.

It is easily verified that

$$\text{cov}\,(z_i z_s) = \begin{cases} 0 & \text{if } i \neq s \\ \mathbf{t}_i\,\Lambda\,\mathbf{t}_i' & \text{if } i = s \end{cases}$$

Hence $z_i = \mathbf{t}_i\;\mathbf{y}_i'$, $i = 1, 2, ..., n$, are mutually independent normal variables and, since this is true whatever be the $\mathbf{t}_i$'s if follows that $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n$ are mutually independent normal variables (To prove this just consider the joint characteristic function of $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n$). That the dispersion matrix of $\mathbf{y}_i$, $i = 1, 2, ..., n$, is Λ follows easily from the fact that the variance of $\mathbf{t}_i\;\mathbf{y}_i'$ is $\mathbf{t}_i\,\Lambda\;\mathbf{t}_i'$ for all $\mathbf{t}_i$.

Theorem 5: *If $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ are mutually independent random p-vectors such that for two given sets of constants $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ the random p-vectors*

$$\mathbf{y}_1 = \Sigma\, a_i\,\mathbf{x} \text{ and } y_2 = \Sigma\, b_i\,\mathbf{x}_i$$

are independent then every $\mathbf{x}_i$, for which $a_i b_i \neq 0$, must be an N_p $(i = 1, 2, ..., n)$.

This is the multivariate extension of the corresponding well-known result for $p = 1$. (Basu, 1951 ; Darmois, 1953 ; Skitovitch, 1953).

Proof: For any p-vector $\mathbf{t}$, the random variable

$$\mathbf{t}\;\mathbf{y}_1' = \Sigma\, a_i\,(\mathbf{t}\;\mathbf{x}_i')$$

is independent of

$$\mathbf{t}\;\mathbf{y}_2' = \Sigma\, b_i\,(\mathbf{t}\;\mathbf{x}_i')$$

and hence $\mathbf{t}\;\mathbf{x}_i'$ is N_1 if $a_i b_i \neq 0$.

Geary (1936) proved under certain restrictive assumptions that if, for n independent observations $x_1, x_2, \ldots, x_n$ on the random variable x, the sample mean is independent of the sample variance then x must be normal. Extending Geary's result Laha (1953) proved that if x has finite variance σ^2 and if there exists an unbiased (whatever be the distribution of x) quadratic estimator $\mathbf{x} A \mathbf{x}'$ of σ^2 with the property that

$$E[\mathbf{x} A \mathbf{x}' \mid \Sigma x_i] = \sigma^2$$

then x must be normal. The multivariate extension (Lukacs, 1942 ; Laha, 1955) of the above result is what follows.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_p)$ be a random p-vector with finite dispersion matrix Λ and let

$$\mathbf{x}_j = (x_{1j}, x_{2j}, \ldots, x_{pj}) \quad j = 1, 2, \ldots, n$$

be n independent observations on $\mathbf{x}$.

Let $$X = (x_{ij}) \quad i = 1, 2, \ldots, p; \quad j = 1, 2, \ldots, n$$

be the matrix of observations and let XAX' be an unbiased (whatever be the distribution of $\mathbf{x}$) estimator of Λ. Let $\mathbf{s} = (s_1, s_2, \ldots, s_p)$ where $s_i = \sum_j x_{ij}$

Theorem 6 : *If* $E[XAX' | \mathbf{s}] = \Lambda$ *then* $\mathbf{x}$ *must be* N_p.

Proof: Let $\mathbf{t}$ be an arbitrary p-vector.

Note that $\mathbf{t}X = (\mathbf{t}\,\mathbf{x}_1', \mathbf{t}\,\mathbf{x}_2', \ldots, \mathbf{t}\,\mathbf{x}_n')$ is the vector of n independent observations on $\mathbf{t}\,\mathbf{x}'$ and that

$$V(\mathbf{t}\,\mathbf{x}') = \mathbf{t}\,\Lambda\mathbf{t}'. \quad \text{Also } \Sigma\, \mathbf{t}\,\mathbf{x}_j' = \mathbf{t}\,\mathbf{s}'.$$

Since $$E(XAX') = \Lambda$$

we have $$E[(\mathbf{t}X)A(\mathbf{t}X)'] = \mathbf{t}(EXAX')\mathbf{t}' = \mathbf{t}\Lambda\mathbf{t}'.$$

Again since $$E(XAX' \mid \mathbf{s}) = \Lambda \text{ we have}$$

$$E[(\mathbf{t}X)A(\mathbf{t}X)' \mid \mathbf{t}\,\mathbf{s}'] = \mathbf{t}\Lambda\mathbf{t}'.$$

Thus, all the requirements of Laha's extension of Geary's theorem are satisfied. Hence $\mathbf{t}\,\mathbf{x}'$ is N_1. Since $\mathbf{t}$ is arbitrary, $\mathbf{x}$ must be N_p.

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Paper received : May, 1955.