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ON A MARTINGALE REPRESENTATION THEOREM OF M. H. A. DAVIS—A MARKOVIAN APPROACH

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In this paper, we give a Markovian proof of a well-known result of M. H. A. Davis on representation of square-integrable martingales associated with arbitrary pure-jump processes. The main tool used to replace the original (non-Markovian) set up by a Markovian one is an interesting idea originally due to F. B. Knight. The main analysis then follows the classical line of investigating structures of martingale additive functionals of a Markov process, wherein a Lévy system of the Markov process (in the classical sense—due to S. Watanabe) plays a vital role.

KEY WORDS: Jump-processes, Markov processes, Levy system, martingale representation.

1. INTRODUCTION

Stochastic processes $\{X_t, t \geqq 0\}$ that evolve purely through jumps, but have otherwise arbitrary laws, have been the subject of study in a number of papers. The list includes [2], [3], [7] and [4] to name a few. One of the key issues had been to understand the structure of square-integrable martingales "associated with" (that is, adapted to the natural filtration of) such processes. In 1976, M. H. A. Davis ([4]) proved a beautiful theorem that gave a complete description of such martingales. On the other side of the story, investigation of the structure of square integrable martingales associated with Markov processes has been a classical problem. Methods were developed by Motoo and Watanabe ([11]), Watanabe ([12]) to describe square integrable martingales associated with a Hunt process. The aim of the present paper is to place the problem of analysing martingales of a jump process in the latter setting. Using an interesting idea proposed by F. B. Knight ([8]), the original pure-jump process $\{X_t, t \geqq 0\}$ (non-Markov!) is replaced by a strong Markov process $\{Z_t, t \geqq 0\}$ with r.c.l.l. paths (indeed, a Borel right process), whose natural filtration is the same (after completion) as that of the original process. This puts the problem of studying square-integrable martingales adapted to this filtration on the same footing as the problems considered by Motoo and Watanabe. This constitutes the first and the key step in our approach. In this part, we omit proofs of the results stated, for the sake of brevity of this paper. The interested reader is referred to [6]. After this, we simply follow Watanabe's method ([12]) to get a Markovian proof of Davis' martingale representation theorem. The fact that the filtration in question does not permit non-trivial continuous martingales reduces the problem to investigating only martingales (indeed, only certain martingale additive functionals) that are compensated sum of jumps. Watanabe ([12]) resalised the potential usefulness, in such context, of what came to be known as a "Levy System" for the underlying Markov process. We exhibit such a system for our Markov process, and, Davis' theorem comes out as a natural consequence.

Two cases, namely that of a single jump and that of infinitely many jumps are presented separately in this paper. This is primarily because, even though the ideas are exactly same in both the cases, the case of a single jump is notationally somewhat simpler, and, thus makes access to the principal idea easier. The understanding gained from the single-jump case would hopefully prevent one from getting lost in the (naturally) somewhat more complicated notations for the multi-jump case.

2. SINGLE-JUMP CASE

The basic process considered by Davis ([4]) can be described in its canonical setting as follows:

Let $\Omega=[0,\infty]\times\mathbb{R}_0$ where $\mathbb{R}_0$, here as well as throughout this paper, denotes the set $\mathbb{R}\backslash\{0\}$, and, let $\mathfrak{F}^0=\mathfrak{B}([0,\infty)\times\mathbb{R}_0)\vee(\{\infty\}\times\mathbb{R}_0)$. Let T and X be defined on Ω by $T(w)=t, X(w)=x$ for $w=(t,x)\in\Omega$. Clearly, T and $X1_{\{t<\infty\}}$ are random variables on $(\Omega,\mathfrak{F}^0)$. The process $\{X_t, t\geqq 0\}$ is now defined on $(\Omega,\mathfrak{F}^0)$ by

$$X_t(w)=X(w)1_{\{T(w)\leqq t\}} \quad \text{for } w\in\Omega,\ t\geqq 0. \tag{1}$$

As usual, let $\{\mathfrak{F}_t^0\}_{t\geqq 0}$ denote the natural filtration of the process $\{X_t, t\geqq 0\}$, and for $t>0$, let $\mathfrak{F}_{t-}^0=\bigvee_{s<t}\mathfrak{F}_s^0$. Clearly, $\bigvee_{t>0}\mathfrak{F}_t^0=\bigvee_{t>0}\mathfrak{F}_{t-}^0=\mathfrak{F}^0$. The law of the process $\{X_t, t\geqq 0\}$ in this setting is determined by a probability on $(\Omega,\mathfrak{F}^0)$.

Let us introduce a few notations to be used in the sequel. For a probability P on $(\Omega,\mathfrak{F}^0)$, and, for $t\in[0,\infty)$ with $P(T>t)>0$, P_t will denote the P-conditional distribution of $(T-t, X1_{\{T<\infty\}})$ given $T>t$, and, if further $P(T\geqq t)>0, P_{t-}$ will denote the P-conditional distribution of $(T-t, X1_{\{T<\infty\}})$ given $T\geqq t$. It is clear that P_t as well as P_{t-}, whenever defined, are once again probabilities on $(\Omega,\mathfrak{F}^0)$. Also for $x\in\mathbb{R}_0$, we denote by δ_x the degenerate probability on $(\Omega,\mathfrak{F}^0)$ concentrated at the point $(0,x)\in\Omega$.

Let Q be any given probability on $(\Omega,\mathfrak{F}^0)$. Our entire analysis from this point on will be starting from this given Q. We consider now a collection of probabilities on $(\Omega,\mathfrak{F}^0)$, namely,

$$\mathbb{P}=\{Q_t: t\in[0,\infty), Q(T>t)>0\}\cup\{Q_{t-}: t\in[0,\infty), Q(T\geqq t)>0\}\cup\{\delta_x: x\in\mathbb{R}_0\} \tag{2}$$

This set $\mathbb{P}$ is of central importance in our approach, because this will be the state space for the Markov process we are going to construct. Note that $Q\in\mathbb{P}$, since $Q_0=Q$. We give a topology on $\mathbb{P}$ as follows.

Consider the smallest topology on Ω making all the functions $w\mapsto\int_0^t f(X_s(w))\,ds$, $t\in[0,\infty)$, $f\in C_{b,u}(\mathbb{R})$ continuous. It is easy to see that this topology makes Ω into a metrizable Lusin space with $\mathfrak{F}^0$ as the Borel σ-field on it. $\mathbb{P}$ is now endowed with the topology of weak convergence. Clearly, $\mathbb{P}$ then becomes a metrizable

Lusin space. Further, the Borel σ-field on $\mathbb{P}$ coincides with the σ-field $\underline{\mathbb{P}}$ generated by the class of functions $\{e_S:S\in\mathfrak{F}^0\}$ where $e_S:\mathbb{P}\to[0,1]$, for $S\in\mathfrak{F}^0$, is defined as $e_S(P)=P(S)$. This topology on $\mathbb{P}$ ensures that our Markov process will have quite regular paths (see Theorem 2.1 (ii) below).

For each $P\in\mathbb{P}$, we define two processes $\{Z_t^P,t\geqq 0\},\{Z_{t-}^P,t\geqq 0\}$ (note their dependence on P) on $(\Omega,\mathfrak{F}^0,P)$ and taking probabilities on $(\Omega,\mathfrak{F}^0)$ as their values, by

$$Z_t^P(w)=\begin{cases}P_t & \text{if } t<T(w) \text{ and } P(T>t)>0\\ \delta_{X(w)} & \text{otherwise.}\end{cases}$$

$$Z_{t-}^P(w)=\begin{cases}P_{t-} & \text{if } t\leqq T(w) \text{ and } P(T\geqq t)>0\\ \delta_{X(w)} & \text{otherwise.}\end{cases}\qquad \text{for}\quad t\geqq 0, w\in\Omega. \tag{3}$$

We then have the following theorem.

Denote, for $P\in\mathbb{P}$, the P-completion of $\mathfrak{F}^0$ by $\mathfrak{F}^P$, and, for $t\geqq 0$ (resp., $t>0$), the augmentation of $\mathfrak{F}_t^0$ (resp., $\mathfrak{F}_{t-}^0$) with P-null sets of $\mathfrak{F}^P$ by $\mathfrak{F}_t^P$ (resp., $\mathfrak{F}_{t-}^P$). Also, let $\mathbb{P}^+=\{Q_t:t\in[0,\infty),\ Q(T>t)>0\}\cup\{\delta_x:x\in\mathbb{R}_0\}$. Clearly, $\mathbb{P}^+\in\underline{\mathbb{P}}$. Let $\underline{\mathbb{P}}^+$ denote the restriction of $\underline{\mathbb{P}}$ to $\mathbb{P}^+$.

THEOREM 2.1 *(i) For every $p\in\mathbb{P}$, the process $\{Z_t^P,t\geqq 0\}$ is a strong Markov process on $(\Omega,\mathfrak{F}^P,\mathfrak{F}_t^P,P)$ with state space $(\mathbb{P}^+,\underline{\mathbb{P}}^+)$. Moreover, all these processes have the same borel transition function q defined by $q(t,P,A)=P[Z_t^P\in A]$, $t\geqq 0$, $P\in\mathbb{P}$, $A\in\underline{\mathbb{P}}^+$.*

(ii) For every $P\in\mathbb{P}$, the process $\{Z_t^P,t\geqq 0\}$ has all *paths right continuous on* $[0,\infty)$, *and, has, P-almost surely, $\{Z_{t-},t>0\}$ as left limits in $\mathbb{P}$ on $(0,\infty)$.*

(iii) $(\Omega,\mathfrak{F}^P,\mathfrak{F}_t^P,Z_t^P,P\in\mathbb{P}^+)$ is a borel right process.

For a complete proof of the theorem, we refer to [6]. However, just as an illustration, we present below the proof of (i). This rests on the following well-known result (see [6]).

LEMMA *Let τ be any $(\mathfrak{F}_t^P)$-stopping time. If $P(\tau<T)>0$, then there exists a unique $t_0<\infty$ such that $P[(T\leqq t_0\wedge\tau)\cup(T>t_0=\tau)]=1$, and, in this case $\mathfrak{F}_\tau^P=\mathfrak{F}_{t_0}^P$. Otherwise, $\mathfrak{F}_\tau^P=\mathfrak{F}^P$.*

Proof of Theorem 2.1(i) Let $P\in\mathbb{P}$. We simply have to show that for any $(\mathfrak{F}_t^P)$-stopping time τ, any $t\geqq 0$ and any $A\in\underline{\mathbb{P}}$.

$$P[Z_{\tau+t}^P\in A\,|\,\mathfrak{F}_\tau^P]=Z_\tau^P[Z_t^{Z_\tau^P}\in A]\quad P\text{-a.s. on } \{\tau<\infty\} \tag{4}$$

First of all, if $P(\tau<T)=0$, then $Z_\tau^P=Z_{\tau+t}^P=\delta_X$ P-a.s. on $\{\tau<\infty\}$, so that, by the above lemma,

$$P[Z_{\tau+t}^P\in A\,|\,\mathfrak{F}_\tau^P]=1_{\{\delta_X\in A\}}\ P\text{-a.s. on } \{\tau<\infty\},$$

while

$$Z_\tau^P[Z_t^{Z_\tau^P}\in A]=\delta_X[Z_t^{\delta_X}\in A]=1_{\{\delta_X\in A\}}\ P\text{-a.s.}$$

On the other hand, if $P[\tau<T]>0$, then, by the lemma again, there is a unique $t_0<\infty$ such that, P-a.s., $\tau=t_0$ on $\{T>t_0\}$ and $\tau\geqq T$ on $\{t_0\geqq T\}$, and, also $\mathfrak{F}_\tau^P=\mathfrak{F}_{t_0}^P$. Therefore, one has P-a.s. on $\{\tau<\infty\}$,

$$P[Z_{\tau+t}^P\in A\,|\,\mathfrak{F}_\tau^P]=1_{\{\delta_X\in A,\,T\leqq t_0\}}+P[Z_{t_0+t}^P\in A\,|\,T>t_0]1_{\{T>t_0\}}$$

while

$$Z_\tau^P[Z_t^{Z_\tau^P}\in A]=1_{\{\delta_X\in A,\,T\leqq t_0\}}+Z_{t_0}^P[Z_t^{Z_{t_0}^P}\in A]1_{\{T<t_0\}}.$$

Thus, in view of (3), we have only to verify that

$$P[Z_{t_0+t}^P\in A\,|\,T>t_0]=P_{t_0}[Z_t^{P_{t_0}}\in A].$$

But this is immediate from the definition of P_{t_0}. □

As already pointed out, the process (Z_t^Q,Q) rather than (X_t,Q) is to be regarded as the central process in our approach. But the probability space $(\Omega,\mathfrak{F}^0)$—a natural one for the process $\{X_t,t\geqq 0\}$—does not, however, provide a convenient setting for the processes $\{Z_t^P,t\geqq 0\}$, $P\in\mathbb{P}$. So we transport everything to a natural setting for the latter processes, as follows.

Let $\bar{\Omega}$ be the space of all $\mathbb{P}^+$-valued right-continuous functions on $[0,\infty)$ with left limits in $\mathbb{P}$ on $(0,\infty)$, and, let $\{Z_t,t\geqq 0\}$ denote the coordinate process on $\bar{\Omega}$. Let $\bar{\mathfrak{F}}_t=\sigma(Z_s,s\leqq t)$ for $t\geqq 0$, $\bar{\mathfrak{F}}=\bigvee_{t\geqq 0}\bar{\mathfrak{F}}_t$, and $\bar{\mathfrak{F}}_{t-}=\bigvee_{s>t}\bar{\mathfrak{F}}_s$ for $t>0$, as usual. Then, clearly for each $P\in\mathbb{P}$, there is a unique probability, to be called P again, on $(\bar{\Omega},\bar{\mathfrak{F}})$ such that $\{Z_t,t\geqq 0\}$ is a strong Markov process under P with transition function q. Our using the same notation P for a probability on $(\Omega,\mathfrak{F}^0)$ as well as the associated probability on $(\bar{\Omega},\bar{\mathfrak{F}})$ should not lead to any major confusion, since the interpretation will always be clear from the use. We shall write E^P for expectation under P, once again with dual interpretation. The following theorem, which was proved in [6], ensures that from a probabilistic point of view nothing is lost in shifting from $(\Omega,\mathfrak{F}^0,\mathfrak{F}_t^0,P)$ to $(\bar{\Omega},\bar{\mathfrak{F}},\bar{\mathfrak{F}}_t,P)$.

THEOREM 2.2 *Let $\phi:\mathbb{P}\to\mathbb{R}$ be defined by $\phi(P)=E^P(X_0)$.*

(i) ϕ is a borel function, and, for all $P\in\mathbb{P}$, the P-law of $\{X_t,Z_t^P;t\geqq 0\}$ on $(\Omega,\mathfrak{F}^0)$ is the same as that of $\{\phi(Z_t),Z_t;t\leqq 0\}$ on $(\bar{\Omega},\bar{\mathfrak{F}})$.

(ii) If $\bar{\mathfrak{F}}^P$ and $\bar{\mathfrak{G}}^P$ denote the P-completions of $\bar{\mathfrak{F}}$ and $\sigma(\phi(Z_t),t\geqq 0)$ respectively, then $\bar{\mathfrak{F}}^P=\bar{\mathfrak{G}}^P$; and, for each $t\geqq 0$, if $\bar{\mathfrak{F}}_t^P$ and *$\bar{\mathfrak{G}}_t^P$ denote the augmentations of $\bar{\mathfrak{F}}_t$ and $\sigma(\phi(Z_s),s\leqq t)$ respectively by P-null sets of $\bar{\mathfrak{F}}^P$, then $\bar{\mathfrak{F}}_t^P=\bar{\mathfrak{G}}_t^P$.*

Having thus laid down the Markovian set-up, we now proceed to exhibit a Levy system for our Markov process $\{Z_t,t\geqq 0\}$. Following [1], this would consist of a pair (N,H), where $N(P,A)$, $P\in\mathbb{P}$, $A\in\underline{\mathbb{P}}$, is a kernel on $(\mathbb{P},\underline{\mathbb{P}})$ with $N(P,\{P\})=0$ for all $P\in\mathbb{P}$, and, H is an increasing previsible additive functional such that

$$\text{for all } 0\leqq f\in\underline{\mathbb{P}}\otimes\underline{\mathbb{P}}\quad\text{with}\quad f(P,P)=0\quad\text{for all}\quad P\in\mathbb{P},$$

$$E^P\left[\sum_{0<s\leqq t}f(Z_{s-},Z_s)\right]=E^P\left[\int_{(0,t]}dH_s\int_{\mathbb{P}}N(Z_{s-},dP')f(Z_{s-},P')\right]\quad\forall p\in\mathbb{P},\,t>0\qquad(5)$$

In a heuristic way, what it means is that the increasing process under the brackets on the right is, for each $P \in \mathbb{P}$, the dual previsible projection of the increasing process under the brackets on the left. Watanabe ([12]) proved the existence of such a system for a Hunt process satisfying a certain hypothesis called "Hypothesis (L)". Later Benveniste and Jacod ([1]) constructed a Levy system for any Ray process. But even that does not directly apply to our process which is only a right process. Nevertheless, we are able to directly construct such a system for our process. First, let us define counterparts of T and X on our new probability space $(\bar{\Omega}, \bar{\mathfrak{F}})$ by

$$\bar{T}(\bar{w}) = \inf\{t \geqq 0 : Z_t(\bar{w}) \in \{\delta_x : x \in \mathbb{R}_0\}\}$$

$$\bar{X}(\bar{w}) = \phi(Z_{\bar{T}}(\bar{w})) \qquad , \qquad \text{for } \bar{w} \in \bar{\Omega}. \tag{6}$$

Also, for any $P \in \mathbb{P}$, let F^P denote the distribution function of T (equivalently, $\bar{T}$) under P, and, λ^P on $[0, \infty) \times \mathfrak{B}(\mathbb{R}_0)$ denote a version of the regular conditional distribution, under P, of X given T on $\{T < \infty\}$. We can and do choose it so as to be jointly $\underline{\mathbb{P}} \otimes \mathfrak{B}([0, \infty))$-measurable. Clearly, $\lambda^{P_t}(.,.) = \lambda^P(t+.,.)$.

Now, let us define $\{H_t, t \geqq 0\}$ on $(\bar{\Omega}, \bar{\mathfrak{F}})$ by

$$H_0 \equiv 0, \quad \text{and, for} \quad t > 0,$$

$$H_t = \int_{(0, t \wedge \bar{T}]} (1 - F^P(s-))^{-1} F^P(ds) \qquad \text{on } \{Z_0 = P\} \tag{7}$$

and N on $\mathbb{P} \times \underline{\mathbb{P}}$ by

$$N(P, dP') \begin{cases} = 0 \quad \text{if} \quad P = \delta_x \quad \text{for some} \quad x \in \mathbb{R}_0 \\ = \dfrac{1}{F^P(0)} q(0, P, dP') \quad \text{if} \quad P \neq \delta_x \quad \text{and} \quad P = Q_{t-} \neq Q_t \\ = \lambda^P(0, B) = \lambda^Q(t, B) \quad \text{if} \quad P = Q_t \quad \text{and where } B = \{x \in \mathbb{R}_0 : \delta_x \in dP'\} \end{cases} \tag{8}$$

THEOREM 2.3 *(N, H) is a Levy system for the process $(\bar{\Omega}, \bar{\mathfrak{F}}^P, \bar{\mathfrak{F}}_t^P, Z_t, P \in \mathbb{P})$.*

Proof Other things being clear from definitions, we really need only to show that (5) holds. For f as in (5), the l.h.s. equals

$$E^P\left[\sum_{0 < s \leqq t} f(Z_{s-}, Z_s) 1_{\{t < \bar{T}\}}\right] + E^P\left[\sum_{0 < s \leqq \bar{T}} f(Z_{s-}, Z_s) 1_{\{t \geqq \bar{T} > 0\}}\right]$$

$$= (1 - F^P(t))\left(\sum_{0 < s \leqq t} f(P_{s-}, P_s)\right) + \int_{(0, t] \times \mathbb{R}_0} \left(\sum_{0 < s < u} f(P_{s-}, P_s) + f(P_{u-}, \delta_x)\right) P(du, dx)$$

which after a little algebra involving a change of order of summation and integration and a proper grouping of terms yields

$$\sum_{0<s\leqq t}(1-F^P(s))f(P_{s-},P_s)+\int_{(0,t]\times\mathbb{R}_0} f(P_{s-},\delta_x)P(ds,dx) \qquad (*)$$

On the other hand, similar calculations makes the r.h.s. equal

$$(1-F^P(t))\left[\int_{(0,t]}(1-F^{P_0}(s-))^{-1}F^{P_0}(ds)\int_{\mathbb{P}}N(P_{s-},dP')f(P_{s-},P')\right]$$

$$+\int_{(0,t]}F^P(du)\left[\int_{(0,u]}(1-F^{P_0}(s-))^{-1}F^{P_0}(ds)\int_{\mathbb{P}}N(P_{s-},dP')f(P_{s-},P')\right]$$

which upon noting that $(1-F^{P_0}(s-))^{-1}F^{P_0}(ds)=(1-F^P(s-))^{-1}F^P(ds)$ and then interchanging the order of integrations w.r.t. u and s in the seond term yields simply

$$\int_{(0,t]}F^P(ds)\int_{\mathbb{P}}N(P_{s-},dP')f(P_{s-},P')$$

$$=\sum_{\substack{0<s\leqq t\\ P_{s-}\neq P_s}}\Delta F^P(s)\cdot\frac{1}{F^{P_s}_{(0)}}\int_{\mathbb{P}}q(0,P_{s-},dP')f(P_{s-},P')$$

$$+\int_{\substack{s\in(0,t]\\ P_{s-}=P_s}}F^P(ds)\int_{\mathbb{R}_0}\lambda^P(s,dx)f(P_{s-},\delta_x),$$

which upon puting the expression for $q(0,P_{s-},dP')$ and simplifying gives (*). □

Remark Davis introduced in [4] what he called the "Local description of the process $\{X_t,t\geqq 0\}$" and hinted at a connection of it with the "Levy System" as introduced in [12]. The relationship of his "Local description" (Λ,n) with our "Levy System" is clear, namely $H_t=\Lambda(t\wedge\bar{T})$, and, $N(Q_t,\{\delta_x:x\in B\})=n(t,B)$. A little reflection will reveal that times t for which $Z_{t-}=Q_t$ are all that matter for the purpose of compensation. Indeed, the totally inaccessible times of discontinuity of the process $\{Z_t,t\geqq 0\}$ lie in $\{t\in[0,\infty):Q_{t-}=Q_t\}$, and, it can be shown easily that the totally inaccessible times of discontinuity of any square integrable $\{\mathfrak{F}_t\}$-adapted martingale are almost surely contained in those of $\{Z_t,t\geqq 0\}$—and, of course, the only jumps of a martingale that need to be compensated are the totally inaccessible ones. Thus, in a way, our approach provides a connection between Davis' notion of "Levy System for the jump process $\{X_t,t\geqq 0\}$" and the classical notion of Levy System for a Markov process.

We finally go on to show how Davis' representation theorem follows as a natural consequence.

THEOREM 2.4 *Any square-integrable $(\mathfrak{F}^Q_t)$-adapted martingale $\{M_t\}$ with $M_0\equiv 0$ admits a representation given by*

$$M_t = h(\bar{T}, \bar{X})1_{\{t \geq \bar{T}\}} - \int_{(0, t \wedge \bar{T}] \times \mathbb{R}_0} (1 - F^Q(s-))^{-1} h(s, x) Q(ds, dx) \tag{9}$$

for some function h on $(0, \infty) \times \mathbb{R}_0$.

Proof First of all, it is well known (see [6] for a complete proof) that there are no non-trivial $(\mathfrak{F}_t^Q)$-adapted martingales with continuous paths. By Meyer's decomposition theorem, we have, therefore, that any square-integrable $(\mathfrak{F}_t^Q)$-adapted martingale $\{M_t\}$ with $M_0 \equiv 0$ is necessarily of the form

$$M_t = \sum_{0 < s \leq t} \Delta M_s - A_t, \tag{10}$$

where A_t is a previsible process. Next, let $(\bar{R}_\lambda)_{\lambda > 0}$ denote the resolvent of the Ray–Knight compactification of $\mathbb{P}^+$. Consider martingales of the form

$$M_t = \bar{R}_\lambda g(Z_t) - \bar{R}_\lambda g(Z_0) - \int_0^t (\lambda \bar{R}_\lambda g(Z_s) - g(Z_s))\, ds, \tag{11}$$

where $\lambda > 0$ and g is continuous on the Ray–Knight compactification of $\mathbb{P}^+$. Since $\{\bar{R}_\lambda g(Z_{t-}), t > 0\}$ is the left limit process of $\{\bar{R}_\lambda g(Z_t), t > 0\}$ (see [9] for a simple argument), it follows that $\Delta M_s = f(Z_{s-}, Z_s)$, for $s > 0$, where $f \in \underline{\mathbb{P}} \otimes \underline{\mathbb{P}}$ is defined by $f(P', P'') = \bar{R}_\lambda g(P'') - \bar{R}_\lambda g(P')$. But then by Theorem 2.3, and, the uniqueness of the previsible process $\{A_t\}$ in (10), we get

$$M_t = \sum_{0 < s \leq t} f(Z_{s-}, Z_s) - \int_{(0, t]} dH_s \int_{\mathbb{P}} N(Z_{s-}, dP') f(Z_{s-}, P'). \tag{12}$$

It is now a matter of simple algebra to see that Eq. (12) can be transformed into the form (9) where h is the function on $(0, \infty) \times \mathbb{R}_0$ defined by

$$h(t, x) = f(Q_{t-}, \delta_x) - f(Q_{t-}, Q_t).$$

Thus, any martingale of the form (11) can be represented as in (9). Next, recalling the well-known fact (see [6]) that for any $(\mathfrak{F}_t^Q)$-previsible process $\{Y_t\}$, there is a *fixed* function $h'(t)$ such that

$$Y_t(\bar{w}) 1_{\{t \leq \bar{T}(\bar{w})\}} = h'(t) \quad \textit{for all} \quad t,\ Q\text{-a.s.},$$

it is easy to see that if $\{M_t\}$ is a square-integrable martingale of the form (9) and if $\{Y_t\}$ is a previsible process such that $\{\int_0^t Y_s dM_s\}$ is also a square-integrable martingale, then this latter martingale is also of the form (9). The proof is now complete in view of the fact that martingales of the form (11) generate all square-integrable $(\mathfrak{F}_t^Q)$-adapted martingales (a basic result of Kunita and Watanabe, which extends without alteration to any realisation of a right process; for example, one can simply repeat the proof in [10]). □

3. CASE OF INFINITELY MANY JUMPS

As in the previous section, we start with a canonical space for the jump process.

Let $\Omega_0 = \mathbb{R}$, $\mathfrak{A}_0 = \mathfrak{B}(\mathbb{R})$, and, for $i \geqq 1$, $\Omega_i = (0, \infty) \times \mathbb{R}_0$, $\mathfrak{A}_i = \mathfrak{B}((0, \infty) \times \mathbb{R}_0)$. Let $(\Omega', \mathfrak{F}')$ denote the product space $(\prod_{i=0}^{\infty} \Omega_i, \bigotimes_{i=0}^{\infty} \mathfrak{A}_i)$, and, $(X^0, S_1, J_1, S_2, J_2, \ldots)$ denote the sequence of random variables defined on $(\Omega', \mathfrak{F}')$ by the coordinate mappings, namely, for $w' = (x^0, (s_1, j_1), (s_2, j_2), \ldots) \in \Omega'$, $X^0(w') = x^0$, and, for $i \geqq 1$, $S_i(w') = s_i$, $J_i(w') = j_i$. With this set-up, we can now define, in a natural way, a jump process on $(\Omega', \mathfrak{F}')$ for which X^0 will be the initial state, and, for $i \geqq 1, S_i$ the time between the ith and the $(i-1)$st jump, and, J_i the size of the jump. However, we want to impose one simplifying assumption, namely, that there is no finite time accumulation of jumps (we like to point out that this assumption is, by no means, crucial for the main idea). We therefore restrict ourselves to

$$\Omega = \left\{ w' \in \Omega' : \sum_{i=1}^{\infty} S_i(w') = \infty \right\}, \quad \text{and} \quad \mathfrak{F}^0 = \mathfrak{F}'|_{\Omega}.$$

Denoting $T_n = \sum_{i=1}^{n} S_i$ on Ω, we have the jump process on $(\Omega, \mathfrak{F}^0)$ defined by

$$X_t(w) = X^0(w) 1_{\{0 \leqq t < T_1(w)\}} + \sum_{n=1}^{\infty} (X^0(w) + J_1(w) + \cdots + J_n(w)) 1_{\{T_n(w) \leqq t < T_{n+1}(w)\}} \quad (13)$$

Let $\mathbb{P}$ denote the set of all probabilities on $(\Omega, \mathfrak{F}^0)$. We endow $\mathbb{P}$ with a topology similar to that in Section 2. $\mathbb{P}$ then becomes a metrizable Lusin space, and, as in Section 2, the Borel σ-field $\underline{\mathbb{P}}$ in this topology coincides with the smallest σ-field on $\mathbb{P}$ making all the functions $P \mapsto P(S)$, $s \in \mathfrak{F}^0$, measurable.

We now introduce certain notations which will prove to be quite handy in our description of the underlying Markov processes. Note, first of all, that any $P \in \mathbb{P}$ is completely determined by the family $\{\mu_n^P, n \geqq 0\}$ when μ_0^P is the P-distribution of X^0, and, for $n > 1$ and

$$(x^0, (s_1, j_1), \ldots, (s_{n-1}, j_{n-1})) \in \prod_{i=0}^{n-1} \Omega_i, \quad \mu_n^P((x^0, s_1, j_1, \ldots, s_{n-1}, j_{n-1}); .)$$

is a version of a regular P-conditional distribution of (S_n, J_n) given

$$X^0 = x^0, S_1 = s_1, J_1 = j_1, \ldots, S_{n-1} = s_{n-1}, J_{n-1} = j_{n-1}.$$

We denote

$$F_n^P((x^0, s_1, j_1, \ldots, s_{n-1}, j_{n-1}); u) = \mu_n^P((x^0, s_1, j_1, \ldots, s_{n-1} j_{n-1}); (0, u] \times \mathbb{R}_0).$$

For $P \in \mathbb{P}, \tilde{x}^0 \in \mathbb{R}$, and, $t \geqq 0$ such that $F_1^P(\tilde{x}^0; t) < 1$, Let $P_t^0(\tilde{x}^0)$ be defined by

$$\mu_0^{P_t^0(\tilde{x}^0)} = \delta_{\tilde{x}^0}$$

$$\mu^{P_t^0(\tilde{x}^0)}(\tilde{x}^0; .) = \mu_1^P(\tilde{x}^0)\text{-conditional distribution of } (S_1 - t, J_1)$$

given $S_1 > t$, and,

$$\mu_k^{P_t^0(\tilde{x}^0)}((\tilde{x}^0, s_1, j_1, \ldots, s_{k-1}, j_{k-1}); .)$$

$$= \mu_k^P((\tilde{x}^0, s_1 + t, j_1, \ldots, s_{k-1}, j_{k-1}); .), \quad \text{for} \quad k \geqq 2. \tag{14}$$

If, moreover, $\mu_1^P(\tilde{x}^0, t-) < 1$, $P_{t-}^0(\tilde{x}^0)$ is defined by

$$\mu_0^{P_{t-}^0(\tilde{x}^0)} = P\text{-conditional distribution of } X_t \text{ given } X^0 = \tilde{x}^0 \text{ and } S_1 \geqq t$$

$$\mu_1^{P_{t-}^0(\tilde{x}^0)}(x^0; ds_1 \times dj_1 = \begin{cases} \dfrac{\mu_1^P(\tilde{x}^0; (ds_1 + t) \times dj_1)}{1 - F_1^P(x^0; t)} & \text{if} \quad x^0 = \tilde{x}^0 \\ \mu_2^P((\tilde{x}^0, t, x^0 - \tilde{x}^0); ds_1 \times dj_1) & \text{if} \quad x^0 \neq \tilde{x}^0 \end{cases}$$

and,

$$\mu_k^{P_{t-}^0(\tilde{x}^0)}((x^0, s_1, j_1, \ldots, s_{k-1}, j_{k-1}); .)$$

$$= \begin{cases} \mu_k^P((\tilde{x}^0, s_1 + t, j_1, \ldots, s_{k-1}, j_{k-1}); .) & \text{if} \quad x^0 = \tilde{x}^0 \\ \mu_{k+1}^P((\tilde{x}^0, t, x^0 - \tilde{x}^0, s_1, j_1, \ldots, s_{k-1}, j_{k-1}); .) & \text{if} \quad x^0 \neq \tilde{x}^0 \end{cases} \tag{14'}$$

Similarly, for $n \geqq 1$, $(\tilde{x}^0, \tilde{s}_1, \tilde{j}_1, \ldots, \tilde{s}_n, \tilde{j}_n) \in \prod_{i=0}^n \Omega_i$, and $t \geqq \tilde{t}_n$ with $F_{n+1}^P((\tilde{x}^0, \tilde{s}_1, \ldots, \tilde{j}_n); t - \tilde{t}_n) < 1$, where $\tilde{t}_n = \sum_{i=1}^n \tilde{s}_i$ and $\tilde{x}^n = \tilde{x}^0 + \sum_{i=1}^n \tilde{j}_i$, $P_t^n(\tilde{x}^0, \tilde{s}_1, \ldots, \tilde{j}_n)$ is defined by

$$\mu_0^{P_t^n} = \delta_{\tilde{x}n}$$

$$\mu_1^{P_t^n}(\tilde{x}^n; ds_1 \times dj_1) = \frac{\mu_{n+1}^P((\tilde{x}_1^0, \tilde{s}_1, \ldots, \tilde{j}_n); ds_1 + (t - \tilde{t}_n) \times dj_1)}{1 - F_{n+1}^P((\tilde{x}^0, \tilde{s}_1, \ldots, \tilde{j}_n); t - \tilde{t}_n)},$$

and,

$$\mu_k^{P_t^n}((\tilde{x}^n, s_1, j_1, \ldots, s_{k-1}, j_{k-1}); .)$$

$$= \mu_{n+k}^P((\tilde{x}^0, \tilde{s}_1, \ldots, \tilde{j}_n, s_1 + t - \tilde{t}_n, j_1, \ldots, j_{k-1}); .) \tag{15}$$

and, if, moreover

$$F_{n+1}^P((\tilde{x}^0, \tilde{s}_1, \ldots, \tilde{j}_n); (t - \tilde{t}_n)-) < 1, \quad P_{t-}^n(\tilde{x}^0, \tilde{s}_1, \ldots, \tilde{j}_n)$$

is defined by

$$\mu_0^{P_{t-}^n} = P\text{-conditional distribution of } X_t \text{ given } X^0 = \tilde{x}^0,$$

$$s_1=\tilde{s}_1,\ldots,J_n=\tilde{j}_n \text{ and } s_{n+1}\geqq t-\tilde{t}_n$$

$$\mu_1^{P_{t-}^n}(x^0;ds_1\times dj_1)=\begin{cases}\dfrac{\mu_{n+1}^P((\tilde{x}^0,\tilde{s}_1,\ldots,\tilde{j}_n);ds_1+(t-\tilde{t}_n)\times dj_1)}{1-F_{n+1}^P((\tilde{x}^0,\tilde{s}_1,\ldots,\tilde{j}_n);t-\tilde{t}_n)} & \text{if } x^0=\tilde{x}^n\\ \mu_{n+2}^P((\tilde{x}^0,\tilde{s}_1,\ldots,\tilde{j}_n,t-\tilde{t}_n,x^0-\tilde{x}^n);ds_1\times dj_1) & \text{if } x^0\neq\tilde{x}^n\end{cases}$$

and,

$$\mu_k^{P_{t-}^n}((x^0,s_1,\ldots,j_{k-1});.)$$

$$=\begin{matrix}\mu_{n+k}^P((\tilde{x}^0,\tilde{s}_1,\ldots,\tilde{j}_n,s_1+t-\tilde{t}_n,j_1,\ldots,j_k);.) & \text{if } x^0=\tilde{x}^n\\ \mu_{n+k+1}^P((\tilde{x}^0,\tilde{s}_1,\ldots,\tilde{j}_n,t-\tilde{t}_n,x^0-\tilde{x}^n,s_1,j_1,\ldots,j_{k-1});.) & \text{if } x^0\neq\tilde{x}^n.\end{matrix} \quad (15)'$$

Clearly, all the $P_t^0, P_{t-}^0, P_t^n, P_{t-}^n$ as defined above, for $P\in\mathbb{P}$, are also elements of $\mathbb{P}$. Let us point out that the seemingly complicated notations introduced above are just rigorous descriptions of a very simple underlying idea, and, do not remain a barrier to understanding once the idea is understood. Thus, for example, $P_t^0(\tilde{x}^0)$ (resp., $P_{t-}^0(\tilde{x}^0)$) described the P-conditional law of the process $\{X_{t+s}, s\geqq 0\}$ given $X^0=\tilde{x}^0$ and $S_1>t$ (resp., $S_1\geqq t$), in exactly the same way as P describes the law of the process $\{X_s, s\geqq 0\}$.

For each $P\in\mathbb{P}$, we now consider two $\mathbb{P}$-valued processes on $(\Omega,\mathfrak{F}^0,P)$ defined as:

$$Z_t^P(w)=\begin{cases}P_t^0(X^0(w)) & \text{if } 0\leqq t<T_1(w) \textit{ and } F_1^P(X^0(w);t)<1\\ P_t^n(X^0(w),S_1(w),J_1(w),\ldots,S_n(w),J_n(w)) & \textit{if } T_n(w)\leqq t<T_{n+1}(w) \textit{ and}\end{cases}$$

$$F_{n+1}^P((X^0(w),\ldots,J_n(w));t-T_n(w))<1 \quad \textit{or if} \quad T_{n-1}(w)\leqq t<T_n(w) \quad \textit{and}$$

$$F_n^P((X^0(w),\ldots,J_{n-1}^{(w)});t-T_{n-1}^{(w)})=1. \quad (16)$$

$$Z_{t-}^P(w)=\begin{cases}P_{t-}^0(X^0(w)) & \text{if } 0<t\leqq T_1(w) \textit{ and } F_1^P(X^0(w);t-)<1\\ P_{t-}^n(X^0(w),S_1(w),\ldots,J_n(w) & \text{if } T_n(w)<t\leqq T_{n+1}(w) \textit{ and}\end{cases}$$

$$F_{n+1}^P((X^0(w),S_1(w),\ldots,J_n(w));(t-T_n(w))-)<1$$

$$\textit{or if} \quad T_{n-1}(w)<t\leqq T_n(w) \quad \textit{and}$$

$$F_n^P((X^0(w),S_1(w),\ldots,J_{n-1}(w));(t-T_{n-1}(w))-)=1 \quad (17)$$

With $\mathfrak{F}^P, \mathfrak{F}_t^P, \mathfrak{F}_{t-}^P$, for $P\in\mathbb{P}$, being defined in the same way as in Section 2, we have the following analogue of Theorem 2.1 (see [6]). Let

$\mathbb{P}^+ = \{P \in \mathbb{P}: \mu_0^P \in \{\delta_x: x \in \mathbb{R}\}\}$. Clearly, $\underline{\mathbb{P}}^+ \in \mathbb{P}$. Let $\underline{\mathbb{P}}^+ = \underline{\mathbb{P}}|_{\mathbb{P}^+}$.

THEOREM 3.1 *(i) For every $P \in \mathbb{P}$, the process $\{Z_t^P, t \geqq 0\}$ is a strong Markov process on $(\Omega, \mathfrak{F}^P, \mathfrak{F}_t^P, P)$ with state space $(\mathbb{P}^+, \underline{\mathbb{P}}^+)$; moreover, all these processes have the same borel transition function q defined by $q(t, P, A) = P[Z_t^P \in A]$, $t \geqq 0$, $P \in \underline{\mathbb{P}}$, $A \in \underline{\mathbb{P}}^+$.*

(ii) For every $P \in \mathbb{P}$, the process $\{Z_t^P, t \geqq 0\}$ has, P-a.s., all paths right continuous on $[0, \infty)$ with $\{Z_{t-}^P, t > 0\}$ as left limits in $\mathbb{P}$ on $(0, \infty)$.

(iii) $(\Omega, \mathfrak{F}^P, \mathfrak{F}_t^P, Z_t^P, P \in \mathbb{P}^+)$ is a borel right process.

As in Section 2, we pass on to the natural space for our processes $\{Z_t^P, t \geqq 0\}$, $\{Z_{t-}^P, t > 0\}$ as follows:

Let $\bar{\Omega}$ denote the space of all $\mathbb{P}^+$-valued right continuous functions on $[0, \infty)$ with left limits in $\mathbb{P}$ on $(0, \infty)$, and, let $\{Z_t, t \geqq 0\}$ denote the coordinate process on $\bar{\Omega}$. With $\bar{\mathfrak{F}}, \bar{\mathfrak{F}}_t, \bar{\mathfrak{F}}_{t-}$ defined on $\bar{\Omega}$ as before, we have, for every $P \in \mathbb{P}$, a probability, to be called P again, on $(\bar{\Omega}, \bar{\mathfrak{F}})$ making $\{Z_t, t \geqq 0\}$ a strong Markov process with transition function q. Then the analogue of Theorem 2.2 is (see [6]).

THEOREM 3.2 *Let $\phi: \mathbb{P} \to \mathbb{R}$ be defined by $\phi(P) = E^P(X_0)$.*

(i) ϕ is a borel function, and, for all $P \in \mathbb{P}$, the P-law of $\{X_t, Z_t^P; t \geqq 0\}$ on $(\Omega, \mathfrak{F}^0)$ is the same as that of $\{\phi(Z_t), Z_t; t \geqq 0\}$ on $(\bar{\Omega}, \bar{\mathfrak{F}})$.

(ii) If $\bar{\mathfrak{F}}^P$ and $\mathfrak{G}^P$ denote the P-completions of $\bar{\mathfrak{F}}$ and $\sigma(\phi(Z_t), t \geqq 0)$ respectively, then $\bar{\mathfrak{F}}^P = \mathfrak{G}^P$; and, for each $t \geqq 0$, if $\bar{\mathfrak{F}}_t^P$ and $\mathfrak{G}_t^P$ denote the augmentations of $\bar{\mathfrak{F}}_t$ and $\sigma(\phi(Z_s), s \leqq t)$ respectively by P-null sets of $\bar{\mathfrak{F}}^P$, then $\bar{\mathfrak{F}}_t^P = \mathfrak{G}_t^P$.

On $(\bar{\Omega}, \bar{\mathfrak{F}})$, let us define

$$\bar{T}_1 = \inf\{t \leqq 0: \phi(Z_t) \neq \phi(Z_0)\}$$

and, for $n \geqq 2$,

$$\bar{T}_n = \inf\{t \geqq \bar{T}_{n-1}: \phi(Z_t) \neq \phi(Z_{\bar{T}_{n-1}})\}.$$

Also, let $\bar{X}^0 = \phi(Z_0)$ and, for $n \geqq 1$, $\bar{X}^n = \phi(Z_{\bar{T}_n}) 1_{\{\bar{T}_n < \infty\}}$. It is then clear from (i) of the above theorem that $(\bar{X}^0, \bar{T}_1, \bar{X}^1, \ldots,)$ on $(\bar{\Omega}, \bar{\mathfrak{F}}, P)$ have the same joint distribution as $(X^0, T_1, X^1, \ldots)$ on $(\Omega, \mathfrak{F}^0, P)$, or, equivalently, if we define $\bar{S}_1 = \bar{T}_1$, $\bar{J}_1 = \bar{X}^1 - \bar{X}^0$, and, for $n \geqq 2$, $\bar{S}_n = (\bar{T}_n - \bar{T}_{n-1}) 1_{\{\bar{T}_{n-1} < \infty\}}$, $\bar{J}_n = \bar{X}^n - \bar{X}^{n-1}$, then $(\bar{X}^0, \bar{S}_1, \bar{J}_1, \ldots)$ on $(\bar{\Omega}, \bar{\mathfrak{F}}, P)$ are equivalent in distribution to $(X^0, S_1, J_1, \ldots)$ on $(\Omega, \mathfrak{F}^0, P)$.

For every $P \in \mathbb{P}$, and $k \geqq 1$, let $\lambda_k^P: (\prod_{i=0}^{k-1} \Omega_i \times (0, \infty)) \times \mathfrak{B}(\mathbb{R}_0) \to [0, 1]$ be a version of regular P-conditional distribution of J_k given $(X^0, S_1, \ldots, J_{k-1}, S_k)$, chosen to be jointly $\underline{\mathbb{P}} \otimes (\bigotimes_{i=0}^{k-1} \mathfrak{A}_i \otimes \mathfrak{B}((0, \infty)))$-measurable. Now let $\{H_t, t \geqq 0\}$ be the increasing process defined on $(\bar{\Omega}, \bar{\mathfrak{F}})$ by

$$H_t=\begin{cases} 0 & \text{if} \quad t=0 \\ H^1(t) & \text{if} \quad 0<t\leqq \bar{T}_1 \\ \sum_{k=1}^{n} H^k(\bar{S}_k)+H^{n+1}(t-\bar{T}_n) & \text{if} \quad \bar{T}_n<t\leqq \bar{T}_{n+1} \end{cases} \tag{18}$$

where

$$H^k(s)=\int_{(0,s]}(1-F_k^{Z_0}((\phi(Z_0),\bar{S}_1,\ldots,\bar{J}_{k-1});u-))^{-1}F_k^{Z_0}((\phi(Z_0),\bar{S}_1,\ldots,\bar{J}_{k-1});du) \tag{19}$$

Next, let N on $\mathbb{P}\times\underline{\mathbb{P}}$ be defined by

$$N(\tilde{P},dP')=\begin{cases} \lambda_1^P(x^0;t,B) & \text{if} \quad \tilde{P}=P_t^0(x^0) \quad \textit{and} \quad \text{where} \\ & B=\{j\in\mathbb{R}_0\colon P_t^1(x^0,t,j)\in dP'\} \\ \dfrac{1-F_1^P(x^0;t-)}{F_1^P(x^0;t)-F_1^P(x^0;t-)}q(0,\tilde{P},dP') & \text{if} \quad \tilde{P}=P_{t-}^0(x^0)\neq P_t^0(x^0) \\ \lambda_{n+1}^P\left((x^0,s_1,\ldots,j_n);t-\sum_1^n s_i,B\right) & \text{if} \quad \tilde{P}=P_t^n(x^0,s_1,\ldots,j_n) \quad \textit{and} \\ \text{where} \quad B=\left\{j\in\mathbb{R}_0\colon P_t^{n+1}\left(x^0,s_1,\ldots,j_n,t-\sum_1^n s_i,j\right)\in dP'\right\} & \\ \dfrac{1-F_{n+1}^P\left((x^0,s_1,\ldots,j_n);\left(t-\sum_1^n s_i\right)-\right)}{F_{n+1}^P\left((x^0,s_1,\ldots,j_n);t-\sum_1^n s_i\right)-F_{n+1}^P\left((x^0,s_1,\ldots,j_n);\left(t-\sum_1^n s_i\right)-\right)} & \\ \times q(0,\tilde{P},dP') & \\ & \text{if} \quad \tilde{P}=P_{t-}^n(x^0,s,\ldots,j_n)\neq P_t^n(x^0,s_1,\ldots,j_n) \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Then we have, analogous to Theorem 2.3, the following

THEOREM 3.3 *(N,H) is a Levy system for the process $\{Z_t,t\geqq 0\}$ on $(\bar{\Omega},\bar{\mathfrak{F}},P)$.*

Proof The previsibility of H is clear. The only thing to prove therefore is the property (5). Let us call the quantity under E^P on the left hand side of (5) as Y_t and that on the right side by $\tilde{Y}_t$. It suffices to show that $E^P(Y_{t\wedge\bar{T}_1})=E^P(\tilde{Y}_{t\wedge\bar{T}_1})$, and that, for all $n\geqq 1$, $E^P(Y_{t\wedge\bar{T}_{n+1}}-Y_{t\wedge\bar{T}_n})=E^P(\tilde{Y}_{t\wedge\bar{T}_{n+1}}-\tilde{Y}_{t\wedge\bar{T}_n})$, since then it will follow, first of all, by induction that $E^P(Y_{t\wedge \bar{T}_n})=E^P(Y_{t\wedge \bar{T}_n})$ for all $n\geqq 1$, and, then, using MCT, that $E^P(Y_t)=E^P(\tilde{Y}_t)$ for all $t>0$ (note that $\bar{T}_n\uparrow\infty, P$-a.s.). We only prove $E^P(Y_{t\wedge \bar{T}_1})=E^P(\tilde{Y}_{t\wedge \bar{T}_1})$ here, since the other identity can be obtained by similar calculations.

Towards this, we note, on one hand, that

$$E^P(Y_{t\wedge \bar{T}_1})=\int_{\mathbb{R}}\mu_0^P(dx^0)\Bigg[(1-F_1^P(x^0;t))\Bigg(\sum_{0<s\leqq t} f(P_{s-}^0(x^0),P_s^0(x^0))\Bigg)$$

$$+\int_{(0,t]}F_1^P(x^0;du)\Bigg(\sum_{0<s<u} f(P_{s-}^0(x^0),P_s^0(x^0))$$

$$+\int_{\mathbb{R}_0}\lambda_1^P(x^0;u,dj_1)f(P_{u-}^0(x^0),P_u^1(x^0,u,j_1))\Bigg)\Bigg]$$

$$=\int_{\mathbb{R}}\mu_0^P(dx^0)\Bigg[\sum_{0<s\leqq t}(1-F_1^P(x^0;s))f(P_{s-}^0(x^0),P_s^0(x^0))$$

$$+\int_{(0,t]\times\mathbb{R}_0}\mu_1^P(x^0;ds,dj_1)f(P_{s-}^0(x^0),P_s^1(x^0,s,j_1))\Bigg], \tag{*}$$

by using Fubini's theorem and then grouping together appropriate terms. On the other hand, similar technique yields

$$E^P(\tilde{Y}_{t\wedge \bar{T}_1})=\int_{\mathbb{R}}\mu_0^P(dx^0)\Bigg[(1-F_1^P(x^0;t))\int_{(0,t]}\frac{F_1^P(x^0;ds)}{1-F_1^P(x^0;s-)}$$

$$\times\int_{\mathbb{P}}N(P_{s-}^0(x^0),dP')f(P_{s-}^0(x^0),P')+\int_{(0,t]}F_1^P(x^0;du)\int_{(0,u]}$$

$$\times\frac{F_1^P(x^0;ds)}{1-F_1^P(x^0;s-)}\int_{\mathbb{P}}N(P_{s-}^0(x^0),dP')f(P_{s-}^0(x^0),P')\Bigg]$$

$$=\int_{\mathbb{R}}\mu_0^P(dx^0)\Bigg[\int_{(0,t]}F_1^P(x^0;ds)\int_{\mathbb{P}}N(P_{s-}^0(x^0),dP')f(P_{s-}^0(x^0),P')\Bigg]$$

$$=\int_{\mathbb{R}}\mu_0^P(dx^0)\Bigg[\sum_{\substack{0<s\leqq t\\ P_{s-}^0(x^0)\neq P_s^0(x^0)}}\Big\{(1-F_1^P(x^0;s))f(P_{s-}^0(x^0),P_s^0(x^0))$$

$$+\int_{\mathbb{R}_0}\lambda_1^P(x^0;s,dj_1)(F_1^P(x^0;s)-F_1^P(x^0;s-))f(P_{s-}^0(x^0),P_s'(x^0,s,j_1))\Big\}$$

$$+\int_{(0,t]}\mu_1^P(x^0;ds)1_{\{P_{s-}^0(x^0)=P_s^0(x^0)\}}(s)\int_{\mathbb{R}_0}\lambda_1^P(x^0;s,dj_1)f(P_{s-}^0(x^0),P_s'(x^0,s,j_1))\Bigg]$$

which is easily seen to be the same as the expression (*). □

Turning towards a proof of Davis' representation theorem, we simply denote by q the random measure on $(0, \infty)\times \mathbb{R}_0$ as introduced by Davis in [4], and, for the sake of brevity of this paper, we refrain from explicitly writing it down here. We then have the following

THEOREM 3.4 *Any square-integrable* $(\mathfrak{F}_t^P)$*-adapted martingale* $\{M_t\}$ *with* $M_0 \equiv 0$ *admits a representation given by*

$$M_t(\bar{w}) = \int_{(0,t]\times \mathbb{R}_0} h(\bar{w}, s, j) q(ds, dj)(\bar{w}) \tag{21}$$

Proof First of all, one argues, in exactly the same way as in the previous section, that if $\{M_t\}$ is a square-integrable $(\mathfrak{F}_t^P)$-adapted martingale of the form

$$M_t = \bar{R}_\lambda g(Z_t) - \bar{R}_\lambda g(Z_0) - \int_0^t (\lambda \bar{R}_\lambda g(Z_s) - g(Z_s))\, ds, \tag{22}$$

where $(\bar{R}_\lambda)_{\lambda>0}$ is the resolvent on the RK compactification $\bar{\mathbb{P}}^+$ of $\mathbb{P}^+$, g continuous on $\bar{\mathbb{P}}^+$ and $\lambda > 0$, then

$$M_t = \sum_{0<s\leq t} f(Z_{s-}, Z_s) - \int_{(0,t]} dH_s \int_{\mathbb{P}} N(Z_{s-}, dP') \Big) f(Z_{s-}, P') \tag{23}$$

where $f \in \underline{\mathbb{P}} \otimes \underline{\mathbb{P}}$ is defined by $f(P', P'') = \bar{R}_\lambda g(P'') - \bar{R}_\lambda g(P')$.

It is now a matter of straight algebra to derive that if, for $k \geq 1$, $h^k: \prod_{i=0}^k \Omega_i \to \mathbb{R}$ is defined by

$$h^k(x^0, s_1, \ldots, j_{k-1}, t, j) = f\left(P_{t-}^{k-1}(x^0, s_1, \ldots, j_{k-1}), P_t^k\left(x^0, s_1, \ldots, j_{k-1}, t - \sum_1^{k-1} s_i, j \right)\right)$$

$$- f(P_{t-}^{k-1}(x^0, s_1, \ldots, j_{k-1}), P_t^{k-1}(x^0, s_1, \ldots, j_{k-1})) \tag{24}$$

and if $h: \bar{\Omega} \times (0, \infty) \times \mathbb{R}_0 \to \mathbb{R}$ is defined by

$$h(\bar{w}, t, j) = \begin{cases} h^1(\phi(Z_0(\bar{w})), t, j) & \text{if } \quad 0 < t \leq \bar{T}_1(\bar{w}) \\ h^k(\phi(Z_0(\bar{w})), \bar{S}_1(\bar{w}), \ldots, \bar{J}_{k-1}(\bar{w}), t, j) & \\ & \text{if } \quad \bar{T}_{k-1}(\bar{w}) < t \leq \bar{T}_k(w), \quad \text{for } \quad k \geq 2 \end{cases} \tag{25}$$

then one has

$$M_t 1_{\{0<t\leq \bar{T}_1\}}(\bar{w}) = h(\bar{w}, \bar{T}_1(\bar{w}), \bar{J}_1(\bar{w})) 1_{\{t=\bar{T}_1(\bar{w})\}}$$

$$- 1_{\{0<t\leq \bar{T}_1(\bar{w})\}} \int_{(0,t]\times \mathbb{R}_0} (1 - F_1^P(\phi(Z_0); s-))^{-1} h(\bar{w}, s, j_1)$$

$$\times \mu_1^P(\phi(Z_0); ds, dj_1) \tag{26}$$

and, for $n \geqq 1$,

$$M_t 1_{\{\bar{T}_n < t \leqq \bar{T}_{n+1}\}}(\bar{w})$$

$$= \sum_{k=1}^{n} h(\bar{w}, \bar{T}_k(\bar{w}), \bar{J}_k(\bar{w})) + h(\bar{w}, \bar{T}_{n+1}(\bar{w}), \bar{J}_{n+1}(\bar{w})) 1_{\{t = \bar{T}_{n+1}(\bar{w})\}}$$

$$- \sum_{k=1}^{n} \int_{(\bar{T}_{k-1}(\bar{w}), \bar{T}_k(\bar{w})] \times \mathbb{R}_0} (1 - F_k^P((\phi(Z_0), \bar{S}_1, \ldots, \bar{J}_{k-1}); (s - \bar{T}_{k-1}) -)^{-1} h(\bar{w}, s, j)$$

$$\times \mu_k^P((\phi(Z_0), \ldots, \bar{J}_{k-1}); d(s - \bar{T}_{k-1}), dj)$$

$$- \int_{(\bar{T}_n(\bar{w}), t] \times \mathbb{R}_0} (1 - F_{n+1}^P((\phi(Z_0), \bar{S}_1, \ldots, \bar{J}_n); (s - \bar{T}_n) -)^{-1} h(\bar{w}, s, j)$$

$$\times \mu_{n+1}^P((\phi(Z_0), \bar{S}_1, \ldots, \bar{J}_n); d(s - \bar{T}_n), dj) \tag{27}$$

But this is the same thing as saying that every square integrable $(\mathfrak{F}_t^P)$-adapted martingale $\{M_t\}$ of the form (22) has a representation given by (21). Let us also note, as in Section 2, that if $\{M_t\}$ is a square-integrable $(\mathfrak{F}_t^P)$-adapted martingale of the form (21), and, if $\{Y_t\}$ *is any* $(\mathfrak{F}_t^P)$-previsible process such that $\{\int_0^t Y_s dM_s\}$ is also a square integrable martingale, then the latter also has the form (21). This uses the fact (see [6]) that $\{Y_t, t \geqq 0\}$ is an $(\mathfrak{F}_t^P)$-previsible process iff it is of the form (P-a.s.).

$$Y_t = g^1(\phi(Z_0), t) 1_{\{0 < t \leqq \bar{T}_1\}} + \sum_{n=1}^{\infty} g^{n+1}(\phi(Z_0), \bar{S}_1, \ldots, \bar{J}_n, t) 1_{\{\bar{T}_n < t \leqq \bar{T}_{n+1}\}} \tag{28}$$

The theorem now follows in view of the fact that martingales of the form (22) generate all square-integrable $(\mathfrak{F}_t^P)$-adapted martingales. □

Remarks 1. That we have chosen our basic jump process $\{X_t, t \geqq 0\}$ to be real-valued instead of taking values in a general metrisable Lusin space (as was done by Davis) is no restriction at all, since the exactly same idea can be carried through in any abstract Lusin space, with possibly minor modifications (due to lack of additive structure).

2. The more general case with possible finite time accumulation of jumps does not pose any significant difficulty in extending our argument.

3. Our work seems to have a potential connection with another work of M.H.A. Davis ([5]), wherein he discusses and studies "piecewise deterministic" Markov processes in terms of their associated jump processes (non-Markov!). It should be noted that the Markov process $\{Z_t, t \geqq 0\}$ introduced by us is a piecewise deterministic process (although we have less regularity structure on the paths than was imposed in [5]), and we have, in a way, used that to study the process $\{X_t, t \geqq 0\}$.

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