Int. J. Non-Linear Mechanics, Vol. 28, No. 5, pp. 489 501, 1993

FLOW OF A THIN LIQUID FILM OVER A COLD/HOT ROTATING DISK

B. S. DANDAPAT and P. C. RAY

Physical and Earth Sciences Division, Indian Statistical Institute, 203 B. T. Road, Calcutta 700 035, India

Abstract—The unsteady flow of a liquid film on a cold/hot rotating disk is analysed by means of matched asymptotic expansion under the assumptions of radially uniform film thickness that varies with time. The velocity, temperature and rate of heat transfer are determined for both the cases. Two new non-dimensional parameters α and β are defined and the variation of the uniform film thickness with these two parameters are shown graphically. Depending on α, a novel feature of flow reversal on the free surface of the film is obtained when the disk is heated from below axisymmetrically. A physical explanation of this flow reversal is also provided. It is further shown that heat flows from the disk to the film for $R < R_c$ and from the film to the disk for $R > R_c$ when the disk is cooled from below. These heat flow directions are reversed when the disk is heated from below. Moreover, in this case, heat also flows from the free surface to the curve $T_\xi = 0$.

1. INTRODUCTION

Flow of thin liquid film on a smooth solid surface by the action of gravity either on stationary vertical walls or on inclined planes has attracted scientists due to its enormous applications in the modern technological world. Thin film can also be produced by the action of centrifugal force on a smooth rotating disk. It is surprising that most of the theoretical and experimental work on the production of thin film is confined to the former type of analysis; not much attention has been paid in the literature to the latter one, although the application of the formation of thin film on a rotating disk is increasing with the advancement of science and technology. For example, the study of heat transfer from a rotating disk to a thin film is used to promote the absorption of vapour into the liquid on the disk. Specifically the absorber unit of a space-based vapour-absorption refrigeration system will use a liquid film thinned by the centrifugal force on a rotating disk to enhance the absorption of the refrigerant vapour into the absorbent. Since a falling film cannot be produced in a micro-gravity environment, the vapour-absorption cycle is more appropriate than a vapour-compression cycle for a micro-gravity application.

Production of thin film on a rotating disk can also be used in the micro-electronics industry to coat the photoresist on silicon wafers for integrated circuits, for magnetic storage disks, for magnetic paint coating on the substrate, etc. This technique of coating is known as spin coating in the literature. In this process, a thick layer of fluid is distributed initially over a horizontal disk, often containing concentric grooves, and the layer is subsequently thinned by spinning the disk at high angular velocities. At the start of the spinning, most of the liquid is ejected from the wafer leaving a thin film which flows slowly outward from the centre of the disk under the action of centrifugal force. As the film thins the liquid evaporates causing the increase of fluid viscosity, which reduces the radial flow. Eventually, the viscosity increases to such an extent that the radial motion virtually ceases. After a while the spinner is stopped and the process is completed by evaporating the residual in an oven.

It is well known that the evaporation starts from the surface layer of the film and during the process of evaporation the latent heat is extracted from the film as a result of which a solid skin is formed on the surface layer which puts greater resistance to the remaining liquid for evaporation. During baking, the inner liquid starts evaporating, as a result, there may appear cracks, waves, etc. on the surface layer of the film. To suppress these unwanted cracks, waves, etc., the film may be cooled by cooling the disk from the underside. This paper is addressed to the study of heat transfer from disk to thin film as well as the development of a velocity field on the rotating disk when it is either cooled or heated from below.

Earlier works on the production of thin film on a rotating disk and the study of heat transfer may be classified into two groups depending on the deposition of liquid on the rotating disk. In the first group the liquid is deposited either in the form of an impinging jet or as a continuous stream at the centre of the rotating disk. In this group Watson [1] analysed a free-falling jet which impinges on a horizontal plane. Espig and Hoyle [2], Miyasaka [3] and many others measured the film thickness while Butuzov and Rifert [4] and Ishigai *et al.* [5] examined the flow pattern, heat transfer and film thickness. In the study of the other group, a thick uniform layer of liquid is distributed over the disk which is then spun. Emslie *et al.* [6], first initiated the study on the development of thin film. In this group of investigation, later on, Meyerhofer [7], Tu [8], Jenekhe [9], Higgins [10] and Hwang and Ma [11] studied the film thickness and its dependence on various parameters like rotational speed, concentration of the liquid, surface tension, non-Newtonian effects, surface roughness, etc. Dandapat and Ray [12] examined the heat transfer from the film to the disk as well as the film thickness when the disk is cooled axisymmetrically from the underside. To solve the non-linear coupled differential equations, they have neglected the effect of thermal stress on the free surface as well as the variation of surface tension with temperature in their investigation. But it is well known that the thermal stress and the surface tension plays a vital role on the thin film development on a rotating disk. So it may be worthy to reconsider the problem of Dandapat and Ray [12] (hereafter referred to as DR) by taking into account the effect of the variation of surface tension with temperature and thermal stress on the free surface. In this investigation we shall study the film thickness, velocity field and heat transfer when the disk is either cooled or heated axisymmetrically from below. Following ref. [12] we also assume:

(i) The disk radius is much larger than the film thickness, so that the edge effect can be neglected.

(ii) The film flows under planar interface under the conditions justified by earlier researchers, starting from Emslie *et al.* [6].

(iii) The Reynolds number $Re = U_0 h_0/\nu$ is small and the dimensionless parameters $\beta = g'\alpha'\Delta T/h_0\Omega^2$ and $\alpha = \gamma\,\Delta T/\rho_0 h_0^3\Omega^2$ are less than 10, where U_0, h_0, Ω, ν, g', α', ρ_0 and γ are the characteristic velocity, initial film thickness, angular velocity, kinematic viscosity, gravitational acceleration, thermal expansion coefficient, density at room temperature T_0 and the variation of surface tension with temperature, respectively. ΔT stands for the difference in temperature between the centre and a point at radial distance $\sqrt{2}h_0$ from the centre of the disk.

(iv) Adjacent to the liquid film at the free surface is a gas or liquid vapour, and therefore the viscosity ratio, μ_g/μ_l (where μ_l and μ_g are the viscosities of the liquid and gas phases, respectively) is much less than unity and any motion of the gas is neglected.

(v) All physical properties except density and surface tension are constant and independent of temperature.

2. MATHEMATICAL FORMULATION

Consider a uniform film of viscous heat-conducting liquid on a disk whose radius is large compared with the thickness of the film. Initially, the system is at the room temperature T_0. Simultaneously the system starts rotating with a uniform angular velocity about an axis normal to the plane of the disk and an axially symmetric temperature distribution which decreases/increases radially outward from the axis of rotation is imposed on the disk. The origin is fixed at the centre of the disk and the z-axis pointing vertically upwards is the axis of rotation.

For axisymmetric motion, the governing equations, after using a Boussinesq approximation in the cylindrical coordinate (r, θ, z) system, become

$$u_r + (u/r) + w_z = 0, \tag{1}$$

$$u_t + uu_r - (v^2/r) + wu_z = -p_r + \nu[u_{rr} + (u/r)_r + u_{zz}], \tag{2}$$

$$v_t + uv_r + (uv/r) + wv_z = \nu[v_{rr} + (v/r)_r + v_{zz}], \tag{3}$$

$$w_t + uw_r + ww_z = -p_z + \nu[w_{rr} + (1/r)w_r + w_{zz}] + g'\alpha'(T - T_0), \tag{4}$$

where u, v, w denote the velocity components along the radial, circumferential and axial directions, respectively, and p is the pressure. Here the subscripts denote derivatives with respect to the indicated variables.

The energy equation after neglecting viscous dissipation becomes

$$T_t + uT_r + wT_z = K[T_{rr} + (1/r)T_r + T_{zz}], \tag{5}$$

where K is the thermal diffusivity. It should be noted here that in deriving equation (4) we have used the variation of density

$$\rho = \rho_0[1 - \alpha'(T - T_0)],$$

in the momentum equation.

For $t > 0$, the boundary conditions are:

(1) no-slip conditions at the disk

$$u(r, 0, t) = 0, \qquad v(r, 0, t) = \Omega r, \qquad w(r, 0, t) = 0 \tag{6}$$

(2) the imposed temperature distribution

$$T(r, 0, t) = T_0 - \lambda(r^2/2)T_1, \tag{7}$$

where T_0 and T_1 are positive constants and λ takes values either 1 or -1 depending on whether the disk is cooled or heated from below.

At the free surface $z = h(t)$, the jump in the normal stress across the interface is balanced by surface tension times curvature, and the shear stress equals the surface tension gradient along the interface. Under the assumption of a planar interface these are given by

$$-p + 2\mu w_z = 0, \tag{8}$$

$$\mu(u_z + w_r) = -(\bar{\sigma}_T)T_r, \tag{9}$$

$$\mu v_z = -(\bar{\sigma}_T)T_z, \tag{10}$$

where $\bar{\sigma}$ denotes surface tension. Further, the thermal boundary condition at the free surface $z = h(t)$ is given by Newton's law of cooling

$$T_z + L(T - T_g) = 0, \tag{11}$$

when L and T_g denote the heat transfer coefficient at the free surface and the temperature in the gas phase, respectively. The kinematic condition at the free surface is

$$h_t = w(r, h, t). \tag{12}$$

The initial conditions for the velocity components and the temperature are as follows:

$$u(r, z, 0) = v(r, z, 0) = w(r, z, 0) = 0, \tag{13a}$$

$$T(r, z, 0) = T_0. \tag{13b}$$

Further, the film thickness satisfies the initial conditions

$$h = h_0, \qquad h_t = 0 \quad \text{at } t = 0. \tag{13c}$$

Following von Kármán [13], the solution of the above system may be assumed in the well-known similarity form (for details see ref. [12])

$$\begin{aligned} &u = rf(z,t), \qquad v = rg(z,t), \qquad w = w(z,t), \\ &p = -(r^2/2)A(z,t) + B(z,t), \\ &T = T_0 - \lambda(r^2/2)M(z,t) - \lambda N(z,t). \end{aligned} \tag{14}$$

Equations (1)–(5) after using (14), reduce to the form

$$2f + w_z = 0, \tag{15a}$$

$$f_t + f^2 - g^2 + wf_z - \nu f_{zz} = A(z,t), \tag{15b}$$

$$g_t - gw_z + wg_z = \nu g_{zz}, \tag{15c}$$

$$A_z = \lambda g'\alpha' M, \tag{15d}$$

$$M_t - w_z M + wM_z = KM_{zz}, \tag{15e}$$

$$N_t + wN_z = 2KM + KN_{zz}. \tag{15f}$$

Here equations (15a), (15b) and (15c) are the consequence of equations (1), (2) and (3). Again equations (15d), (15e) and (15f) are obtained from equations (4) and (5) after using equation (14) and equating the terms of order r^2 and r^0. It is clear that the similarity solution for temperature as assumed in equation (14) is compatible with the temperature boundary condition (7). It should be emphasized here that the last equation of (14) holds for large but finite values of r (since the radius of the disk is large compared with the film thickness) so that T can never tend to $-\infty$. Further, the r-independent term of equation (4) after using (14) becomes

$$w_t + \tfrac{1}{2}(w^2)_z - \nu w_{zz} + B_z + \lambda g'\alpha' N = 0. \tag{16}$$

Now, $B(z,t)$ can be evaluated by integrating (16) with respect to z, from $z = 0$ to $z = h(t)$, and thus we can evaluate the pressure from equation (14) after finding $A(z,t)$ from (15d) (for details see ref. [12]).

The boundary conditions (8)–(11) on the free surface $z = h(t)$ can also be simplified by using equation (14) and, equating the terms of the different powers of r, we get

$$A(h,t) = 0, \tag{17}$$

$$f_z = (\gamma\lambda/\mu)M, \tag{18}$$

$$g_z = 0, \qquad M_z = 0, \qquad N_z = 0 \tag{19}$$

$$M_z + LM = 0, \qquad \lambda N_z + L(\lambda N + T_g - T_0) = 0. \tag{20}$$

To study the effect of the variation of surface tension on film development we assume $L = 0$ in this analysis and obtain

$$A(h,t) = 0,$$

$$f_z = (\gamma\lambda/\mu)M,$$

$$g_z = M_z = N_z = 0 \quad \text{on } z = h(t). \tag{21a}$$

In the above system of equations we have assumed the variations of surface tension in the form

$$\bar\sigma = \bar\sigma_0 - \gamma(T - T_0),$$

where $\gamma = -\bar\sigma_T$, which is positive for most of the liquids, and $\bar\sigma_0$ is the surface tension at temperature T_0.

Using equation (14) in the boundary conditions (6) and (7) on the disk, the kinematic boundary condition (12) at the free surface can be simplified. The set of simplified boundary conditions are given by

$$f(0,t) = w(0,t) = 0, \qquad g(0,t) = \Omega, \tag{21b}$$

$$M(0,t) = T_1, \qquad N(0,t) = 0, \tag{21c}$$

$$h_t = w(h,t) \quad \text{at } z = h(t). \tag{21d}$$

Similarly the initial conditions (13) reduce to the form:

$$\left.\begin{array}{l} f(z,0) = g(z,0) = w(z,0) = 0 \\ h = h_0, \qquad h_t = 0 \quad \text{and} \quad T(r,z,0) = T_0 \end{array}\right\} \text{ at } t = 0. \tag{21e}$$

3. SOLUTION FOR SMALL REYNOLDS NUMBER

During the course of spinning we may arrive at the situation when the centrifugal force and the viscous shear across the film are of comparable magnitude. At this stage the

Reynolds number $Re(\equiv U_0 h_0/\nu)$ will be much less than unity and this balance of forces defines a characteristic time (cf. refs [6, 10, 12]) t_b such that

$$t_b = \nu/(h_0^2\Omega^2), \tag{22}$$

where the characteristic velocity scale U_0 is defined as h_0/t_b. We introduce the dimensionless variables as

$$\tau = t/t_b, \quad \xi = z/h_0, \quad H = h/h_0, \quad R = r/h_0, \quad F = h_0 f/U_0,$$
$$G = g/\Omega, \quad M' = (h_0^2/\Delta T)M, \quad N' = N/\Delta T, \quad A' = A/\Omega^2, \quad W = w/U_0, \tag{23}$$

where $\Delta T = |\lambda(h_0^2)T_1|$.

Non-dimensional forms of equations (15) and (21) (after dropping primes on M, N and A) are

$$2F + W_\xi = 0,$$
$$Re(F_\tau + F^2 + WF_\xi) = F_{\xi\xi} + G^2 + A,$$
$$Re(G_\tau - GW_\xi + WG_\xi) = G_{\xi\xi},$$
$$\sigma Re(M_\tau + WM_\xi - W_\xi M) = M_{\xi\xi}$$
$$\sigma Re(N_\tau + WN_\xi) = N_{\xi\xi} + 2M,$$
$$A_\xi = \beta\lambda M(\xi,\tau), \tag{24}$$

where $\sigma = \nu/K$ is the Prandtl number.

The dimensionless boundary conditions are, for $\tau > 0$,

$$\left.\begin{aligned} F(0,\tau) = W(0,\tau) = 0, \quad & G(0,\tau) = 1, \\ M(0,\tau) = 1, \quad & N(0,\tau) = 0 \end{aligned}\right\} \text{ at } \xi = 0, \tag{25}$$

$$\left.\begin{aligned} F_\xi(H,\tau) = \alpha\lambda M(H,\tau), \quad & G_\xi(H,\tau) = 0, \\ M_\xi(H,\tau) = N_\xi(H,\tau) = A(H,\tau) = 0, & \\ H_\tau(\tau) = W(H,\tau) & \end{aligned}\right\} \text{ at } \xi = H \tag{26}$$

and the corresponding initial conditions are

$$F(\xi,0) = G(\xi,0) = W(\xi,0) = M(\xi,0) = N(\xi,0) = A(\xi,0) = 0,$$
$$H(0) = 1, \quad H_\tau(0) = 0. \tag{27}$$

The above coupled, non-linear system of equations (24)–(27) can be solved by expanding the dependent variables in terms of the powers of Re in the form:

$$\phi(\xi,\tau) = \sum_{j=0}^{\infty} Re^j \phi_j(\xi,\tau) \tag{28}$$

and

$$H(\tau) = H_0(\tau) + ReH_1(\tau) + \cdots.$$

Substituting equation (28) in the system of equations (24)–(27) and equating the different powers of Re we can get different sets of equations involving the dependent variables F, G, W, . . . Solutions to the zero-order set are:

$$F_0(\xi,\tau) = -\frac{\lambda\beta\xi^3}{6} + \frac{1}{2}(\lambda\beta H - 1)\xi^2 + (\alpha\lambda + H - \frac{1}{2}\lambda\beta H^2)\xi,$$

$$G_0(\xi,\tau) = 1,$$

$$W_0(\xi,\tau) = \frac{\lambda\beta\xi^4}{12} - \frac{1}{3}(\lambda\beta H - 1)\xi^3 - (\alpha\lambda + H - \frac{1}{2}\lambda\beta H^2)\xi^2,$$

$$A_0(\xi,\tau) = \beta\lambda(\xi - H),$$

$$M_0(\xi,\tau) = 1,$$

$$N_0(\xi,\tau) = -\xi^2 + 2H\xi. \tag{29}$$

It is to be noted here that the effect of a temperature boundary condition as well as the effect of the variation of surface tension with temperature (viz. parameter α) can be seen in the zero-order solutions.

To obtain the solutions for higher-order terms straightforward but lengthy calculations need to be performed. So we avoid the details of these repetitions (please see ref. [12]) here. To determine the film thickness we use the kinematic condition given in equation (26) as follows:

$$H_{0\tau} + ReH_{1\tau} + O(Re^2) = W_0(H,\tau) + ReW_1(H,\tau) + O(Re^2).$$

Comparing the orders, we have

$$H_{0\tau} = \frac{\lambda\beta}{4}H_0^4 - \frac{2}{3}H_0^3 - \alpha\lambda H_0^2, \tag{30}$$

$$H_{1\tau} = (\lambda\beta H_0^3 - 2H_0^2 - 2\lambda\alpha H_0)H_1 + C\lambda^2\beta^2H_0^9 + D\beta\lambda H_0^8 + EH_0^7 + F\lambda\alpha H_0^6 + \tfrac{2}{3}\lambda^2\alpha^2\sigma H_0^5, \tag{31}$$

where

$$C = \frac{299}{9072}\sigma - \frac{127}{2268}, \qquad D = \frac{107 - 131\sigma}{1440},$$

$$E = \frac{68}{315} + \lambda^2\alpha\beta\frac{149}{630} - \lambda^2\alpha\beta\sigma\frac{37}{126},$$

$$F = \frac{61}{180} + \frac{5}{12}\sigma.$$

The solution we have obtained so far by finding F, G, W, M and N does not satisfy the initial conditions (27) due to the large-time-scale assumptions as explained earlier. For a uniformly valid solution, an *inner* expansion for short time is needed and this solution must be matched with the long-time-scale solution by using Van Dyke's technique [14] of composite matched asymptotic expansion.

3.1. *Short-time-scale analysis*

In this analysis, we shall define a new time scale in such a way that the local inertial term (e.g. the term f_t in equation (15b)) is of the same order of magnitude as the viscous and the centrifugal terms in the governing equations. The dimensionless variables in this part are used as

$$\bar{\tau} = \frac{t\Omega}{\sqrt{Re}} = \frac{\tau}{Re}, \quad \bar{F} = F,\ \bar{G} = G,\ \bar{W} = W,\ \bar{H} = H,\ \bar{M} = M,\ \bar{N} = N,\ \bar{A} = A \text{ and } \eta = \xi.$$

In this stretching of the temporal coordinate system, the corresponding equations become

$$\begin{aligned} 2\bar{F} + \bar{W}_\eta &= 0, \\ \bar{F}_{\bar{\tau}} + Re(\bar{F}^2 + \bar{W}\bar{F}_\eta) &= \bar{F}_{\eta\eta} + \bar{G}^2 + \bar{A}, \\ \bar{G}_{\bar{\tau}} + Re(-\bar{G}\bar{W}_\eta + \bar{W}\bar{G}_\eta) &= \bar{G}_{\eta\eta}, \\ \sigma\bar{M}_{\bar{\tau}} + Re\sigma(\bar{W}\bar{M}_\eta - \bar{W}_\eta\bar{M}) &= \bar{M}_{\eta\eta}, \\ \sigma\bar{N}_{\bar{\tau}} + Re\sigma\bar{W}\bar{N}_\eta &= \bar{N}_{\eta\eta} + 2\bar{M}, \\ \bar{A}_\eta &= \lambda\beta\bar{M}. \end{aligned} \tag{32}$$

The boundary conditions are

$$\bar{F}(0,\bar{\tau}) = \bar{W}(0,\bar{\tau}) = 0, \quad \bar{G}(0,\bar{\tau}) = 1, \quad \bar{M}(0,\bar{\tau}) = 1, \quad \bar{N}(0,\bar{\tau}) = 0, \tag{33}$$

$$\bar{F}_\eta(\bar{H},\bar{\tau}) = \lambda\alpha\bar{M}(\bar{H},\bar{\tau}), \quad \bar{G}_\eta(\bar{H},\tau) = \bar{M}_\eta(\bar{H},\bar{\tau}) = 0,$$

$$\bar{N}_\eta(\bar{H},\bar{\tau}) = \bar{A}(\bar{H},\bar{\tau}) = 0, \quad \bar{H}_{\bar{\tau}} = Re\,\bar{W}(\bar{H},\bar{\tau}) \tag{34}$$

and the initial conditions are

$$\bar{F} = \bar{W} = \bar{G} = \bar{M} = \bar{N} = \bar{A} = 0, \text{ at } \bar{\tau} = 0, \quad \bar{H} = 1, \qquad \bar{H}_{\bar{\tau}} = 0. \tag{35}$$

Expanding the dependent variables according to the perturbation scheme (28), and solving the zero-order set we get

$$\bar{H}_0(\bar{\tau}) = 1,$$

$$\bar{G}_0 = 1 - 2\sum_{n=1(2)}^{\infty} \frac{e^{-p_n^2\bar{\tau}}\sin(p_n\eta)}{p_n}, \qquad p_n = \frac{n\pi}{2} \quad \text{and} \quad \bar{\tau} > 0,$$

$$\bar{M}_0 = 1 - 2\sum_{n=1(2)}^{\infty} \frac{e^{-(p_n^2/\sigma)\bar{\tau}}\sin(p_n\eta)}{p_n},$$

$$\bar{A}_0 = \lambda\beta(\eta - 1) + 2\beta\lambda\sum_{n=1(2)}^{\infty} \frac{e^{-(p_n^2/\sigma)\bar{\tau}}}{p_n^2}\cos(p_n\eta), \quad 0 \le \eta \le 1. \tag{36}$$

Expressions for $\bar{F}_0$ and $\bar{W}_0$ are quite lengthy and straightforward so we dropped them and they will be supplied on request. First-order correction to the film thickness for a short-time-scale can be obtained from the kinematic condition as

$$\bar{H}_{1\bar{\tau}} = \bar{W}_0(1,\bar{\tau}). \tag{37}$$

The solution of equation (37) satisfying equation (35) gives the $O(Re)$ correction for film thickness as follows:

$$\begin{aligned}
\bar{H}_1(\bar{\tau}) = & -(\lambda\alpha + \tfrac{2}{3})\bar{\tau} + 4\lambda\beta\sum_{m=1(2)}^{\infty}\frac{1}{p_m^4}\left[1 - \frac{(-1)^{(m-1)/2}}{p_m}\right]\bar{\tau} \\
& - 4\sum_{m=1(2)}^{\infty}\frac{1}{p_m^5}\left[(-1)^{(m-1)/2}\lambda\alpha + (1-\lambda\beta)\frac{1}{p_m} + \lambda\beta\frac{(-1)^{(m-1)/2}}{p_m^2}\right](e^{-p_m^2\bar{\tau}} - 1) \\
& - 2\lambda\alpha\sigma\sum_{m=1(2)}^{\infty}\frac{(-1)^{(m-1)/2}}{p_m^3}(e^{-p_m^2\bar{\tau}/\sigma} - 1) - 8\sum_{m=1(2)}^{\infty}\frac{1}{p_m^4}\left[e^{-p_m^2\bar{\tau}}\left(\bar{\tau} + \frac{1}{p_m^2}\right) - \frac{1}{p_m^2}\right] \\
& + 8\lambda\alpha\sigma\sum_{m=1(2)}^{\infty}\sum_{n=1(2)}^{\infty}\frac{(-1)^{(m+n-2)/2}}{p_m^3 p_n^3}(e^{-p_n^2\bar{\tau}/\sigma} - 1) \\
& - 8\lambda\alpha\sum_{m=1(2)}^{\infty}\sum_{n=1(2)}^{\infty}\frac{(-1)}{p_m p_n(p_m^2 - p_n^2/\sigma)}\left[\frac{(e^{-p_n^2\bar{\tau}/\sigma} - 1)}{p_n^2/\sigma} - \frac{(e^{-p_m^2\bar{\tau}} - 1)}{p_m^2}\right] \\
& + 4\lambda\beta\sum_{m=1(2)}^{\infty}\sum_{n=1(2)}^{\infty}\left\{\frac{1-(-1)^{(m+n)/2}}{p_m + p_n} + \left(\frac{1-(-1)^{(m-n)/2}}{p_m - p_n}\right)_{m\neq n}\right\} \\
& \times\frac{1}{p_m p_n^2(p_m^2 - p_n^2/\sigma)}\left[\frac{(e^{-p_n^2\bar{\tau}/\sigma} - 1)}{p_n^2/\sigma} - \frac{(e^{-p_m^2\bar{\tau}} - 1)}{p_m^2}\right] \\
& - 32\sum_{m=1(2)}^{\infty}\sum_{n=1(2)}^{\infty}\frac{A_{nm}^0}{p_m(p_m^2 - 2p_n^2)}\left[\frac{(e^{-2p_n^2\bar{\tau}} - 1)}{2p_n^2} - \frac{(e^{-p_m^2\bar{\tau}} - 1)}{p_m^2}\right] \\
& - 64\sum_{m=1(2)}^{\infty}\sum_{n=1(2)}^{\infty}\sum_{l>n}\frac{A_{nm}^l}{p_m(p_m^2 - p_n^2 - p_l^2)}\left[\frac{(e^{-(p_n^2+p_l^2)\bar{\tau}} - 1)}{(p_n^2 + p_l^2)} - \frac{(e^{-p_m^2\bar{\tau}} - 1)}{p_m^2}\right],
\end{aligned} \tag{38}$$

where

$$A_{nm}^0 = 1/[p_m(p_m^2 - 4p_n^2)],$$

$$A_{nm}^l = p_m/[(p_n^2 - p_m^2)^2 + p_l^4 - 2p_l^2(p_n^2 + p_m^2)].$$

By using the technique of composite matched asymptotic expansions due to Van Dyke [14], we have matched the short- and long-time-scale solutions and obtained the expression for the film thickness which is uniformly valid for all times as

$$\begin{aligned}
H^c(\tau) = & H_0(\tau) + (\tfrac{3}{2} + \lambda\alpha)\tau - 0.24998\beta\lambda\tau \\
& + Re[H_1(\tau) + \bar{H}_1(\tau/Re) - 0.66087 - 3.91666\alpha\lambda \\
& + 0.87500\beta\lambda - 0.00000056\alpha\sigma\lambda].
\end{aligned}$$

It is to be noted here that for large $\bar{\tau}$, $\bar{H}_1$ has been calculated and then used as the initial condition for equations (30) and (31) along with $\bar{H}_0 = 1$. These equations are then numerically solved using Gill's modified method [15].

4. RESULTS AND DISCUSSION

Setting $\lambda = 1$ or -1 we shall obtain the case of cooling or heating. Figures 1 and 2 depict the variation of the composite film thickness with time for various values of β and α when the disk is cooled or heated from below, respectively. It is clear from Fig. 1 that for fixed α, the film thickness increases with β. Here β acts as a heat sucking parameter.

Therefore, as β increases, the temperature of the film decreases and hence the density of the fluid increases which results in higher resistance for film thinning. On the other hand, for fixed β, as α increases, the film thickness decreases. Here α is the measure of thermocapillary force which is induced owing to the variation of surface tension with temperature. Since the disk is cooled axisymmetrically, the surface tension is low at the centre, and hence a thermocapillary flow is induced at the free surface in the favourable flow direction. Thus, α enhances the film thinning when the disk is cooled from below. Figure 2 shows the reverse action for β and α when the disk is heated from below. This is due to the fact that for fixed α, β, the heating parameter, enhances the film thinning with its increment, whereas for fixed β, α introduces an adverse thinning effect. Thus, the thermocapillary flow which is induced in this case has opposite flow direction, i.e. towards the centre. This fact can be seen in Fig. 3, in which we have plotted F against ξ for different values of α, β and τ. One can find from Fig.

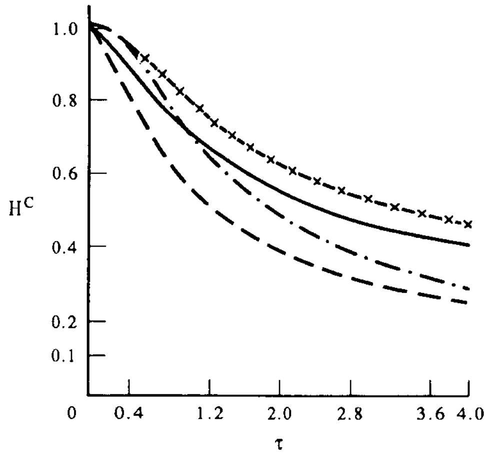

Fig. 1. Variation of composite height H^c with respect to time τ for $Re = 0.25$, $\sigma = 0.5$ and for different values of (α, β). For *cooling* (—) for (0.001, 0.001); (- - -) for (0.5, 0.001); (-·-·-·-) for (0.5, 2.5) and (- × - × - × -) for (0.001, 1.0).

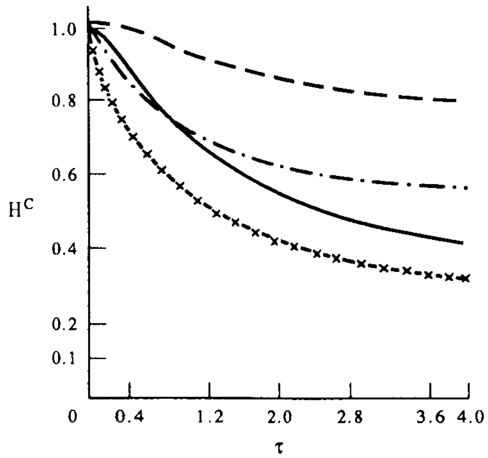

Fig. 2. Variation of composite height H^c with respect to time τ for $Re = 0.25$, $\sigma = 0.5$ and for different values of (α, β). For *heating* ( ) for (0.001, 0.001); (- - -) for (0.5, 0.001); (-·-·-·-) for (0.5, 2.5) and (- × - × - × -) for (0.001, 4.0).

3 that for $\alpha = 0.5$, $\beta = 0.001$, $\sigma = 0.5$, $\tau = 3.0$, F becomes negative for $\xi > \xi_c$ such that $F = 0$ at $\xi = \xi_c$. This shows that α introduces a reverse flow (towards the centre of the disk) at the free surface. For cooling, Fig. 4 shows that F increases as α increases for fixed β, whereas F decreases with the increase of β for fixed α. Thus, for cooling, thermocapillary effect enhances the film thinning. Figures 5 and 7 (Figs 6 and 8) show the changes of G and W with α and β when the disk is cooled (heated) from below.

Non-dimensional temperature distribution in the film which is developed due to cooling or heating the disk may be expressed as

$$\frac{T_0 \sim T}{\Delta T} = \pm\lambda\left(\frac{R^2}{2}\right)M(\xi,\tau) \pm \lambda N(\xi,\tau), \tag{39}$$

where $R = (r/h_0)$. In equation (39), the positive sign corresponds to cooling and the negative sign to heating the disk from below. This gives

$$T_\xi = -\lambda\left(\frac{R^2}{2}\right)M_\xi(\xi,\tau) - \lambda N_\xi(\xi,\tau). \tag{40}$$

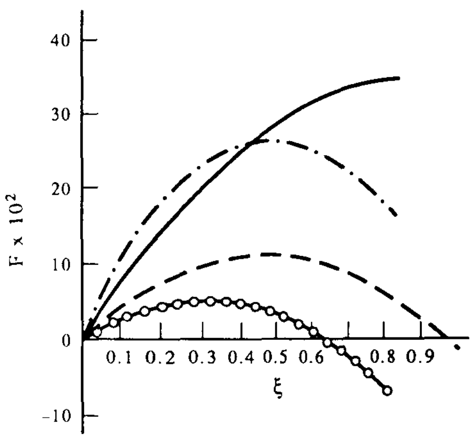

Fig. 3. Variation of radial velocity with ξ for different values of α, β and τ. For *heating* (—) with $\alpha = 0.001$, $\beta = 0.001$ and $\tau = 0.5$; (- - - -) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 0.5$; (-·-·-·-) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 0.5$; (-○-○-○-○-) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 3.0$.

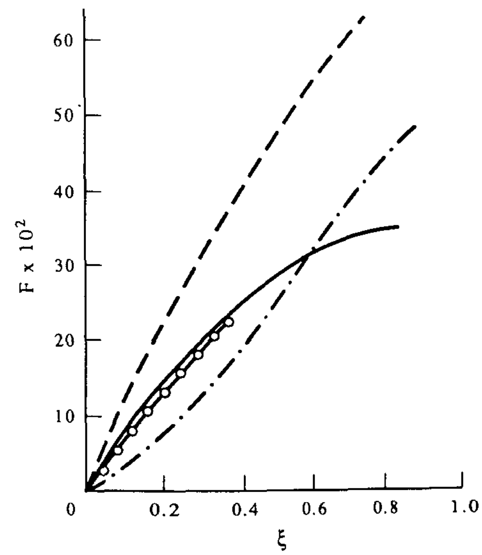

Fig. 4. Variation of radial velocity with ξ for different values of α, β and τ. For *cooling* (—) with $\alpha = 0.001$, $\beta = 0.001$ and $\tau = 0.5$; (- - - -) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 0.5$; (-·-·-·-) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 0.5$; (-○-○-○-○-) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 3.0$.

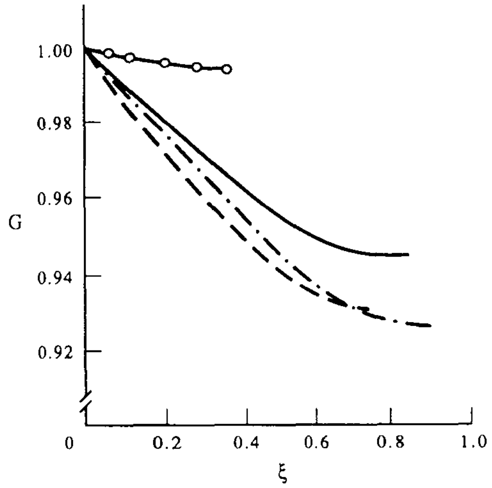

Fig. 5. Variation of cross-radial velocity with ξ for different values of α, β and τ. For *cooling* (—) with $\alpha = 0.001$, $\beta = 0.001$ and $\tau = 0.5$; (- - - -) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 0.5$; (-·-·-·-) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 0.5$; (–○–○–○–) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 3.0$.

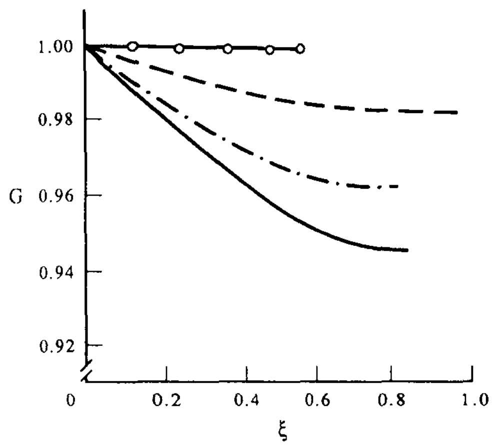

Fig. 6. Variation of cross-radial velocity with ξ for different values of α, β and τ. For *heating* (—) with $\alpha = 0.001$, $\beta = 0.001$ and $\tau = 0.5$; (- - - - -) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 0.5$; (-·-·-·-) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 0.5$; (–○–○–○–○–) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 3.0$.

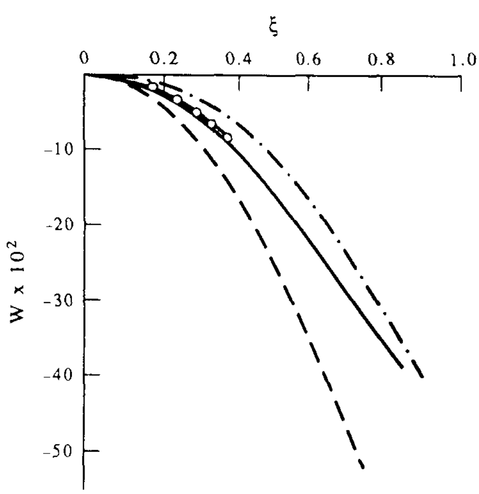

Fig. 7. Variation of axial velocity with ξ for different values of α, β and τ. For *cooling* (—) with $\alpha = 0.001$, $\beta = 0.001$ and $\tau = 0.5$; (- - - -) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 0.5$; (-·-·-·-) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 0.5$; (–○–○–○–) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 3.0$.

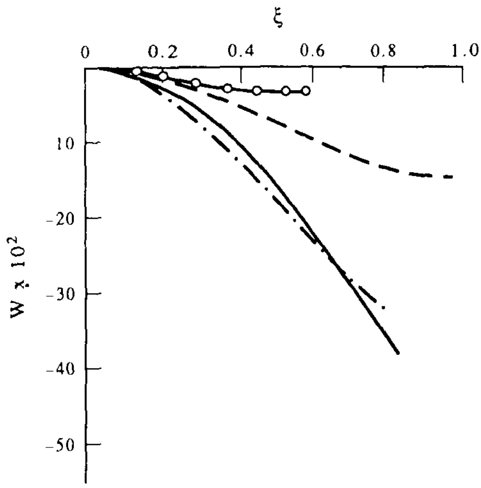

Fig. 8. Variation of axial velocity with ξ for different values of α, β and τ. For *heating* (—) with $\alpha = 0.001$, $\beta = 0.001$ and $\tau = 0.5$; (- - - -) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 0.5$; (- · · · · · -) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 0.5$; (-○-○-○-○-) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 3.0$.

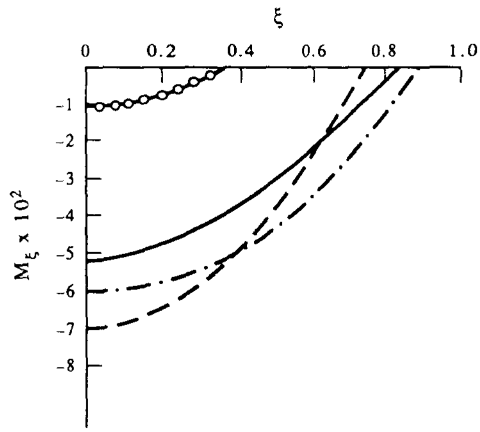

Fig. 9. Variation of M_ξ with respect to ξ for different values of α, β and τ. For *cooling* (--) with $\alpha = 0.001$, $\beta = 0.001$ and $\tau = 0.5$; (- - - -) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 0.5$; (- · · · · · -) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 0.5$; (-○-○-○-) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 3.0$.

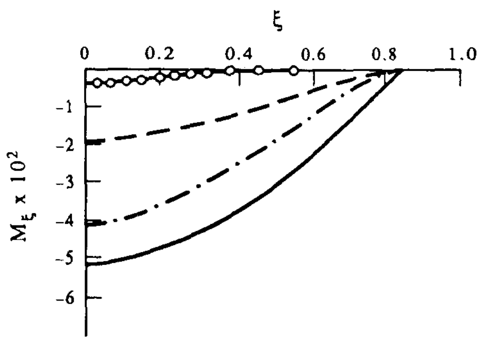

Fig. 10. Variation of M_ξ with respect to ξ for different values of α, β and τ. For *heating* ( ) with $\alpha = 0.001$, $\beta = 0.001$ and $\tau = 0.5$; (- - - -) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 0.5$; (- · · · · · -) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 0.5$; (-○-○-○-) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 3.0$.

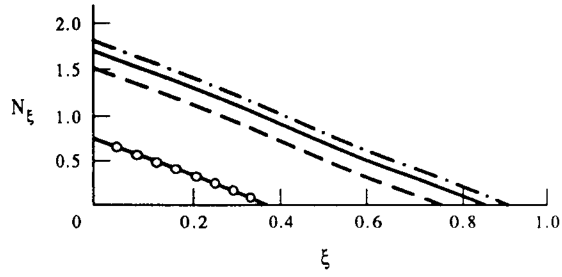

Fig. 11. Variation of N_ξ with respect to ξ for different values of α, β and τ. For *cooling* (—) with $\alpha = 0.001$, $\beta = 0.001$ and $\tau = 0.5$; (- - - -) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 0.5$; (-·-·-·-) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 0.5$; (–O–O–O–) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 3.0$.

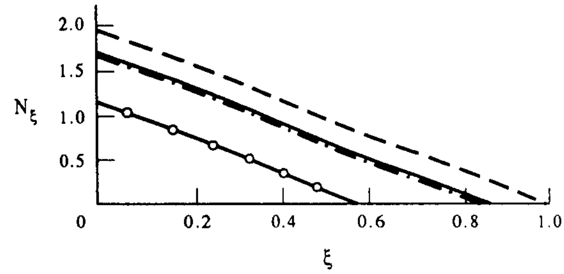

Fig. 12. Variation of N_ξ with respect to ξ for different values of α, β and τ. For *heating* (—) with $\alpha = 0.001$, $\beta = 0.001$ and $\tau = 0.5$; (- - - -) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 0.5$; (-·-·-·-) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 0.5$; (–O–O–O–O–) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 3.0$.

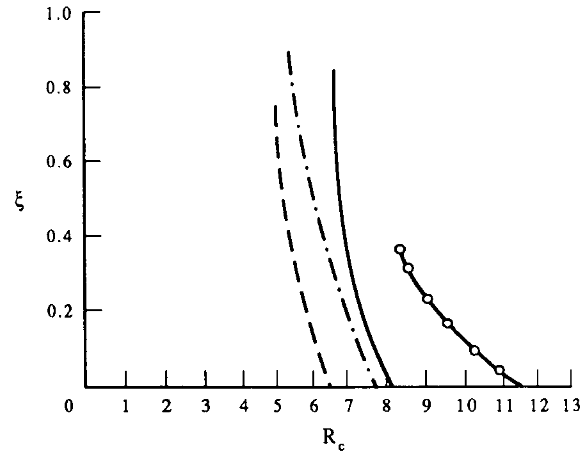

Fig. 13. Variation of R_c with respect to ξ for different values of α, β and τ. For *cooling* (—) with $\alpha = 0.001$, $\beta = 0.001$ and $\tau = 0.5$; (- - - -) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 0.5$; (-·-·-·-) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 0.5$; (–O–O–O–O–) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 3.0$.

Figures 9 and 11 (Figs 10 and 12) show that $M_\xi < 0$ and $N_\xi > 0$ for all $\xi \neq H$ irrespective of cooling (heating). Thus, the sign of the heat flux T_ξ at a point depends on the position in the R–ξ plane. Therefore, it is always possible to draw in this plane the locus of $T_\xi = 0$, which separates the zones of heat flow directions, i.e. from the disk to the film and from the film to the disk. It is to be noted that for cooling (see Fig. 13) heat flows from the disk to the film or from the film to the disk depending, respectively, on whether one is in zone $R < R_c$ or in the zone $R > R_c$, where $R = R_c$ represents the locus of $T_\xi = 0$. Figure 14 depicts the heat flow zones for the case of heating. From equation (40) one can see that the direction of heat flow for heating is exactly the reverse of that in the case of cooling, i.e. heat flows out from the

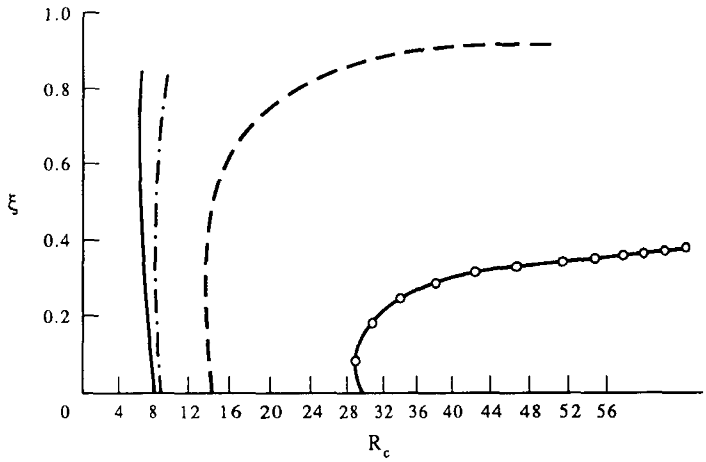

Fig. 14. Variation of R_c with respect to ξ for different values of α, β and τ. For *heating* (—) with $\alpha = 0.001$, $\beta = 0.001$ and $\tau = 0.5$; (- - - - -) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 0.5$; (-······-) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 0.5$; (–O–O–O–O–) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 3.0$.

film to the disk for $R < R_c$ and from the disk to the film for $R > R_c$. Moreover, heat will also flow from the free surface to the curve $T_\xi = 0$. It should be pointed out here that the curve $T_\xi = 0$ touches the free surface in the case of cooling whereas there is always a gap between the free surface and the curve $T_\xi = 0$ depending on α, in the case of heating. This gap is due to the reverse flow which is induced by the thermocapillary force through α.

REFERENCES

1. E. J. Watson, The radial spread of a liquid jet over a horizontal plane. *J. Fluid Mech.* **20**, 481 (1964).
2. H. Espig and R. Hoyle, Waves in a thin liquid layer on a rotating disk. *J. Fluid Mech.* **22**, 671 (1965).
3. Y. Miyasaka, On the flow of a viscous free boundary jet on a rotating disk. *Bull. Japan Soc. Mech. Engng* **17**, 1469 (1974).
4. A. I. Butuzov and V. F. Rifert, An experimental study of heat transfer during condensation of steam at a rotating disk. *Heat Transfer Soviet Res.* **4**, 150 (1972).
5. S. Ishigai, S. Nakanishi, M. Mizuno and T. Imamura, Heat transfer of the impinging round water jet in the interference zone of film flow along the wall. *Bull. Japan Soc. Mech. Engng* **20**, 85 (1977).
6. A. G. Emslie, F.T. Bonner and L.G. Peck, Flow of a viscous liquid on a rotating disk. *J. appl. Phys.* **29**, 858 (1958).
7. D. Meyerhofer, Characteristics of resist films produced by spinning. *J. appl. Phys.* **49**, 3993 (1978).
8. Y. O. Tu, Depletion and retention of fluid on a rotating disk. *J. Lub. Tech.* **105**, 625 (1983).
9. S. A. Jenekhe, Effects of solvent mass transfer on flow of polymer solutions on a flat rotating disk. *Ind. Engng Chem. Fundam.* **23**, 425 (1984).
10. B. G. Higgins, Film flow on a rotating disk. *Phys. Fluids* **29**, 3522 (1986).
11. J. H. Hwang and F. Ma, On the flow of a thin liquid film over a rough rotating disk. *J. appl. Phys.* **66**, 388 (1989).
12. B. S. Dandapat and P. C. Ray, Film cooling on a rotating disk. *Int. J. Non-linear Mech.* **25**, 569 (1990).
13. T. von Kármán, Uber laminare und turbulente reibung. *Z. angew. Math. Mech.* **1**, 233 (1921).
14. M. Van Dyke, *Perturbation Methods in Fluid Mechanics*. Academic Press, New York (1964).
15. A. Ralston and H. S. Wilf, *Mathematical Methods for Digital Computers*. Wiley, New York (1964).

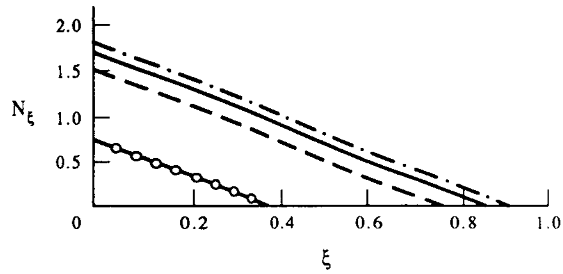

Fig. 11. Variation of N_ξ with respect to ξ for different values of α, β and τ. For *cooling* (—) with $\alpha = 0.001$, $\beta = 0.001$ and $\tau = 0.5$; (----) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 0.5$; (-·-·-·-) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 0.5$; (–O–O–O–) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 3.0$.

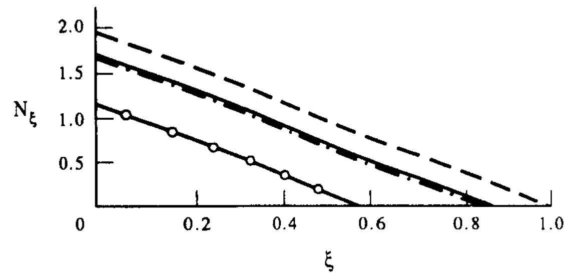

Fig. 12. Variation of N_ξ with respect to ξ for different values of α, β and τ. For *heating* (—) with $\alpha = 0.001$, $\beta = 0.001$ and $\tau = 0.5$; (----) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 0.5$; (-·-·-·-) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 0.5$; (–O–O–O–O–) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 3.0$.

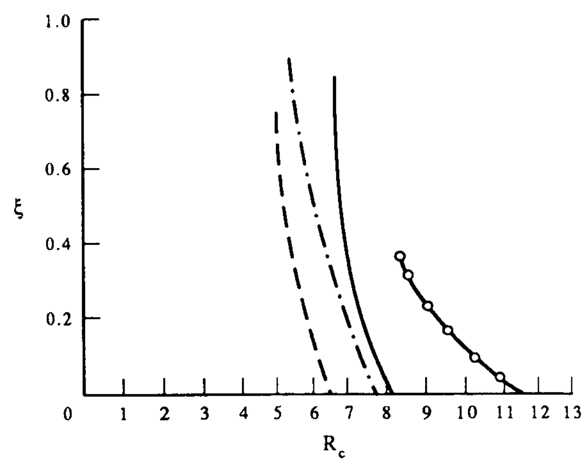

Fig. 13. Variation of R_c with respect to ξ for different values of α, β and τ. For *cooling* (—) with $\alpha = 0.001$, $\beta = 0.001$ and $\tau = 0.5$; (----) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 0.5$; (-·-·-·-) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 0.5$; (–O–O–O–O–) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 3.0$.

Figures 9 and 11 (Figs 10 and 12) show that $M_\xi < 0$ and $N_\xi > 0$ for all $\xi \neq H$ irrespective of cooling (heating). Thus, the sign of the heat flux T_ξ at a point depends on the position in the R–ξ plane. Therefore, it is always possible to draw in this plane the locus of $T_\xi = 0$, which separates the zones of heat flow directions, i.e. from the disk to the film and from the film to the disk. It is to be noted that for cooling (see Fig. 13) heat flows from the disk to the film or from the film to the disk depending, respectively, on whether one is in zone $R < R_c$ or in the zone $R > R_c$, where $R = R_c$ represents the locus of $T_\xi = 0$. Figure 14 depicts the heat flow zones for the case of heating. From equation (40) one can see that the direction of heat flow for heating is exactly the reverse of that in the case of cooling, i.e. heat flows out from the

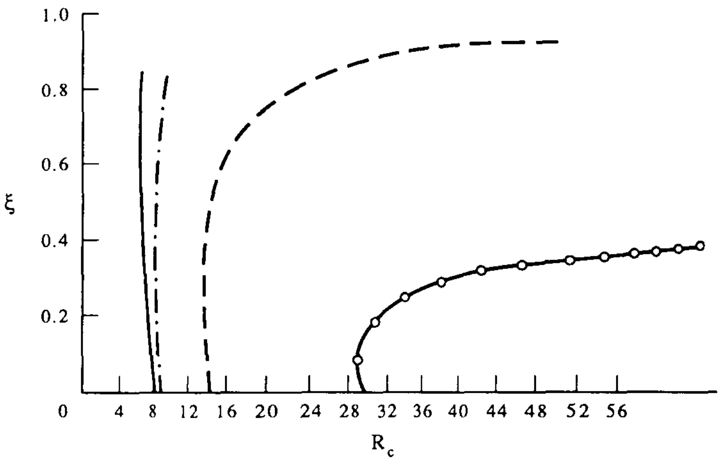

Fig. 14. Variation of R_c with respect to ξ for different values of α, β and τ. For *heating* (—) with $\alpha = 0.001$, $\beta = 0.001$ and $\tau = 0.5$; (- - - - -) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 0.5$; (-·-·-·-) with $\alpha = 0.5$, $\beta = 2.5$ and $\tau = 0.5$; (–○–○–○–○–) with $\alpha = 0.5$, $\beta = 0.001$ and $\tau = 3.0$.

film to the disk for $R < R_c$ and from the disk to the film for $R > R_c$. Moreover, heat will also flow from the free surface to the curve $T_\xi = 0$. It should be pointed out here that the curve $T_\xi = 0$ touches the free surface in the case of cooling whereas there is always a gap between the free surface and the curve $T_\xi = 0$ depending on α, in the case of heating. This gap is due to the reverse flow which is induced by the thermocapillary force through α.

REFERENCES

1. E. J. Watson, The radial spread of a liquid jet over a horizontal plane. *J. Fluid Mech.* **20**, 481 (1964).
2. H. Espig and R. Hoyle, Waves in a thin liquid layer on a rotating disk. *J. Fluid Mech.* **22**, 671 (1965).
3. Y. Miyasaka, On the flow of a viscous free boundary jet on a rotating disk. *Bull. Japan Soc. Mech. Engng* **17**, 1469 (1974).
4. A. I. Butuzov and V. F. Rifert, An experimental study of heat transfer during condensation of steam at a rotating disk. *Heat Transfer Soviet Res.* **4**, 150 (1972).
5. S. Ishigai, S. Nakanishi, M. Mizuno and T. Imamura, Heat transfer of the impinging round water jet in the interference zone of film flow along the wall. *Bull. Japan Soc. Mech. Engng* **20**, 85 (1977).
6. A. G. Emslie, F.T. Bonner and L.G. Peck, Flow of a viscous liquid on a rotating disk. *J. appl. Phys.* **29**, 858 (1958).
7. D. Meyerhofer, Characteristics of resist films produced by spinning. *J. appl. Phys.* **49**, 3993 (1978).
8. Y. O. Tu, Depletion and retention of fluid on a rotating disk. *J. Lub. Tech.* **105**, 625 (1983).
9. S. A. Jenekhe, Effects of solvent mass transfer on flow of polymer solutions on a flat rotating disk. *Ind. Engng Chem. Fundam.* **23**, 425 (1984).
10. B. G. Higgins, Film flow on a rotating disk. *Phys. Fluids* **29**, 3522 (1986).
11. J. H. Hwang and F. Ma, On the flow of a thin liquid film over a rough rotating disk. *J. appl. Phys.* **66**, 388 (1989).
12. B. S. Dandapat and P. C. Ray, Film cooling on a rotating disk. *Int. J. Non-linear Mech.* **25**, 569 (1990).
13. T. von Kármán, Uber laminare und turbulente reibung. *Z. angew. Math. Mech.* **1**, 233 (1921).
14. M. Van Dyke, *Perturbation Methods in Fluid Mechanics*. Academic Press, New York (1964).
15. A. Ralston and H. S. Wilf, *Mathematical Methods for Digital Computers*. Wiley, New York (1964).