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OPTIMUM SAMPLING PLAN FOR DETECTION OF INDEPENDENT DEFECT CATEGORIES IN CASE OF DEFECT INTEFERENCE

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ABSTRACT : Defect interference is a situation where an item stamped as a defective of one category may have defects of other categories also. This is particularly noticed in the inspected data of rail inspection where an inspector stops inspecting items for ascertaining defects of other categories as soon as one defect category is first noticed. In this paper we provide an optimal sampling procedure for the estimation of defect categories in case of Interference when defects occur independently.

1. INTRODUCTION

In [3] Mukherjee demonstrated the defect interference problem in rail inspection were an inspector stops inspecting defects of other categories as soon as one defect category is first noticed. He provided the statistical process control procedure of incidence of defect categories in case of defect interference when an inspector inspects two defect categories in the order A, B in a random sample of n items. He assumed the defect categories A and B to be probabilistically independent and provided the estimators for defect categories A and B from the inspected data of such items. In [1] Chandra and Sinha found the optimal sampling strategies for defect detection in case of defect interference when two defect categories are not probabilistically independent. They found the estimates of parameter of interest in an optimal manner with and without taking into consideration the budgetary constraint.

We propose here an optimum sampling plan for two defect categories A and B where they occur independently by assuming their respective true proportions of defectives as π_A and π_B in a particular reference population. In order to estimate π_A and π_B based on a random sample we will follow the following sampling inspection scheme :

First draw a sample of size n and after inspecting for category A first, suppose n_A of them are rejected due to A and of the remaining $(n-n_A)$ items n_B are rejected due to B. Next draw a second sample, independent of the first one, of size m after inspecting for category B first, suppose m_B of them are rejected due to B and of the remaining $(m-m_B)$ items m_A are rejected due to A. Our purpose is to find the optimal sampling plan for the estimation of π_A and π_B. After finding the maximum likelihood estimators (m.l.e.s) of π_A and π_B, we find the optimal values of n and m by minimising any one of the following three viz.

i) tr(D)

ii) generalised variance or |D|

iii) $v(p_A \cup {}_B)$

subject to m+n = N, where D is the dispersion matrix

$$\begin{bmatrix} \sigma_{AA} & \sigma_{AB} \\ & \sigma_{BB} \end{bmatrix}$$

of $(\pi_A, \pi_B)'$. Aditionally, we also propose to include the cost aspect of the plan by maximising the effeciency per unit cost which is equivalent to minimising $\overline{vc}$, where $\bar{v}$ is the average variance and $\bar{c}$ is the average cost.

2. RESULTS

The m.l.e.s of π_A and π_B are :

$\hat{\pi}_A = (n_A + m_A) / (n + m - m_B)$ and

$\hat{\pi}_B = (m_B + n_B) / (m + n - n_A)$.

It is easy to show that the $\hat{\pi}_A$ and $\hat{\pi}_B$ are unbiased with the dispersion matrix given below :

$$D \approx \begin{bmatrix} \dfrac{\pi_A(1-\pi_A)}{n+m(1-\pi_B)} & 0 \\ 0 & \dfrac{\pi_B(1-\pi_B)}{n+m(1-\pi_A)} \end{bmatrix}$$

Theorem 1 : Minimization of tr(D) subject to $n + m = N$ and $m,n>0$ provides the optimal values of m as

$$\frac{m_{opt}}{N} = \frac{N-1}{N} \qquad \text{if } (1-\pi_A) \leq (1-\pi_B)^3$$

$$= \frac{1-\delta(1-\pi_A)}{\pi_B + \delta\pi_A} \qquad \text{if } (1-_B)^3 < (1-\pi_A) < (1-\pi_B)^{1/3}$$

$$= \frac{1}{N} \qquad \text{if } (1-\pi_B)^{1/3} \leq (1-\pi_A)$$

where $\delta = \sqrt{(1-\pi_A)/(1-\pi_B)}.$

Proof : $\text{tr(D)} = \dfrac{\pi_A(1-\pi_A)}{n+m(1-\pi_B)} + \dfrac{\pi_B(1-\pi_B)}{m+n(1-\pi_A)}$

Since n+m = M, tr(D) can be written as

$$\text{tr(D)} = \frac{\pi_A Q_A}{N-m\pi_B} + \frac{\pi_B Q_B}{NQ_A+m\pi_A} = f \text{ say, where}$$

$Q_A = (1 - \pi_A)$ and $Q_B = (1 - \pi_B)$.

Setting $\dfrac{df}{dm} = 0$ we have $\dfrac{N-m\pi_B}{NQ_A+m\pi_A} = \pm\sqrt{Q_A/Q_B} = \pm\,\delta$, say

i.e., if m_1 and m_2 be two roots then

$$\frac{N-m_1\pi_B}{NQ_A+m_1\pi_A} = \delta \quad \text{and} \quad \frac{N-m_2\pi_B}{NQ_A+m_2 p_A} = -\delta$$

$\Rightarrow m_1 = N(1 - SQ_A)/(\pi_B + \delta\pi_A)$ and $m_2 = N(1 + \delta Q_A) / (\pi_B - \delta\pi_A)$.

Note that i) $-(NQ_A/\pi_A) < m_1 < (N/\pi_B)$, and

ii) either $m_2 > (N/\pi_B)$ or $m_2 < -(NQ_A/\pi_A)$.

To find the minimum, let us note that

$$\left.\frac{d^2f}{dm^2}\right|_{m=m_1} > 0 \text{ and } \left.\frac{d^2f}{dm^2}\right|_{m=m_2} < 0$$

The function tr(D) is discontinuous at two points, N/π_B and $-(NQ_A/\pi_A)$, and is continuous for all m such that $-(NQ_A/\pi_A) < m < (N/\pi_B)$ with a unique minimum at m_1. But for feasibility of the solution we must have $0 < m < N$.

So the following three cases are worth noting -

case (i) : if $0 < (1 - SQ_A) < (\pi_B + \delta\pi_A)$

i.e. $\delta^2 Q_A^2 < 1$ and $Q_B < \delta$

i.e. $3Q_B < Q_A < Q_B/3$, then $m_{opt} = m_1$ and $n_{opt} = N - m_{opt}$,

where $m_1 = N(1 - SQ_A) / (\pi_B + \delta\pi_A)$.

case (ii) : if $(1 - SQ_A) \leq 0$, i.e. $1/Q_A \leq \delta$, then obviously $m_{opt} = 1$ and $n_{opt} = N-1$.

case (iii) : if $(\pi_B + \delta\pi_A) \leq (1 - SQ_A)$

i.e. $Q_A \leq 3Q_B$, then $m_{opt} = N-1$ and $n_{opt} = 1$.

Therefore, the theorem follows from the above three observations.

Remark 1.1 : If $\pi_A \gtrless \pi_B$ and $N \geq 3$ then $(n_{opt}/n) \lessgtr (1/2)$.

Remark 1.2 : Let $\frac{n_{opt}}{N}(p,q)$ and $\frac{m_{opt}}{N}(q,p)$ denote the optimum ratios of $\frac{n}{N}$ and $\frac{m}{N}$ respectively when $\pi_A = p$ and $\pi_B = q$, then

$$\frac{n_{opt}}{N}(p,q) = \frac{m_{opt}}{N}(q,p) .$$

Remark 1.3 : Let $\beta = \pi_A/\pi_B$, then for small values of π_A and π_B we have $\frac{n_{opt}}{N} \approx \frac{3-\beta}{2(1+\beta)}$ if $\frac{1}{3} < \beta < 3$.

In Table 1 we display the values of n_{opt}/N for different values of β so as to highlight the change in the values of n_{opt}/N for various choices of β.

TABLE 1 : *n_{opt}/N for various* β

β	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.80	0.90	1.00
$\frac{n_{opt}}{N}$	0.98	0.93	0.88	0.83	0.79	0.75	0.71	0.68	0.61	0.55	0.50

Remark 1.4 : n_{opt}/N need not be studied seperately for $1 < \beta < 3$ since $\frac{n_{opt}}{N}(\beta) = 1 - \frac{n_{opt}}{N}(1/\beta)$.

Theorem 2 : Minimisation of generalised variance $|D|$ subject to $m+n = N$ provides the optimum values of m as

$$\frac{m_{opt}}{N} = 1/2\ (1 - \alpha) \text{ if } |\alpha| < 1$$

$$= (N-1) / N \text{ if } \alpha \leq -1$$

$$= 1 / N \qquad \text{if } \alpha \geq 1,$$

where $\alpha = 1/\pi_A - 1/\pi_B$.

Proof : The problem is equivalent to maximising $1/|D|$ subject to $n+m=N$.

$$\underset{m}{\text{Max}} \frac{1}{|D|} = \underset{m}{\text{Max}} \frac{(N - m\pi_B)(NQ_A + m\pi_A)}{P_A\, Q_A\, \pi_B\, Q_B}.$$ Taking derivative

with respect to m and equating it to 0 yields $N\pi_A - m\pi_A\pi_B - NQ_A\pi_B - m\pi_A\pi_B = 0$. This implies that maximum of f_1 is attained when $m = N(1 - a)/2 = m_1$, say, since the second derivative of f_1 with respect to m taken at m_1 is negative. It is clear that $-(NQ_A/\pi_A) < m_1 < (N/\pi_B)$ where $-(NQ_A/\pi_A)$ and (N/π_B) are two discontinuity points of |D|. However for the solution to be feasible we must have $0 < m < N$. So let us examine the following three cases—

Case (i) : If $0 < (\pi_A - \pi_B + \pi_A\pi_B) < 2\pi_A\pi_B$, i.e. $|a| < 1$ then $m_{opt} = m_1$.

Case (ii) : If $(\pi_A - \pi_B + \pi_A\pi_B) \leq \alpha$ then $m_1 \leq 0$. Since $m > 0$ always, therefore $m_{opt} = 1$ and $n_{opt} = N - 1$ as the function is decreasing on the interval $[m_1,N]$.

Case (iii) : If $2\pi_A\pi_B < (\pi_A - \pi_B + \pi_A\pi_B)$, i.e. $\alpha \leq -1$, then $m_1 \geq N$. Since $m < N$ always, therefore $m_{opt} = N-1$ and $n_{opt} = 1$ as the function is increasing on the interval $[0,m_1]$.

From the previous three cases the theorem easily follows.

Remark 2.1 : If $\pi_A \overset{\geq}{>} \pi_B$ and $N \geq 3$ then $(n_{opt}/n) \overset{\leq}{>} (1/2)$.

Remark 2.2 : $\frac{n_{opt}}{N}(\alpha) = \frac{m_{opt}}{N}(-\alpha)$.

Lemma 1 : $$v(\hat{\pi}_{A\cup B}) = \pi_{A\cup B}(1-\pi_{A\cup B})\frac{1}{N + \frac{nm\,\pi_A\,\pi_B}{N - n\pi_A - m\pi_B}},$$

where $N = n + m$.

Proof : As the defect categories A and B are assumed to be independent,

$$\hat{\pi}_{A\cup B} = \frac{n_A + m_A}{N - m_B} + \frac{m_B + n_B}{N - n_A} - \frac{(n_A + m_A)(m_B + n_B)}{(N - n_A)(N - m_B)}$$

$$= f(n_A, n_B, m_B, m_A), \text{ say}.$$

Let $\underset{\sim}{X} = (n_A, n_B, m_B, m_A)$. Then

$$E(X) = \underset{\sim}{\mu} = (n\pi_A, n\pi_B(1-\pi_A), m\pi_B, m\pi_A(1-\pi_B))'$$

Following Rao [4], we can write

$$V(f) = \underset{\sim}{l}' \underset{\sim}{l} \text{ where}$$

$$\underset{\sim}{l} = (l_1, l_2, l_3, l_4)' \text{ with } l_i = \left.\frac{\delta f}{\delta x_i}\right|_{\underset{\sim}{\mu}}$$

and $\Sigma = E(\underset{\sim}{X} - E\underset{\sim}{X})(\underset{\sim}{X} - E\underset{\sim}{X})'$.

It can be easily shown that

$$\Sigma \approx \begin{bmatrix} n\pi_A Q_A & -n\pi_A Q_A \pi_B & 0 & 0 \\ & nQ_A\pi_B(1-Q_A\pi_B) & 0 & 0 \\ & & m\pi_B Q_B & -m\pi_A\pi_B Q_B \\ & & & m\pi_A Q_B(1-\pi_A Q_B) \end{bmatrix}$$

where $Q_A = 1 - \pi_A$ and $Q_B = 1 - \pi_B$.

Now $l_1 = \dfrac{1}{N - m\pi_B}\left[1 - \dfrac{m\pi_B \pi_{A\cup B}}{N - n\pi_A}\right]$

$$l_2 = (1 - \pi_A) / (N - n\pi_A)$$

$$l_3 = \frac{1}{N - n\pi_A}\left[1 - \frac{n\pi_A \pi_{A\cup B}}{N - m\pi_B}\right]$$

$$l_4 = (1 - \pi_B) / (N - m\pi_B).$$

Hence $V(f) = \underset{\sim}{l}' \Sigma \underset{\sim}{l} = \pi_{A\cup B}(1-\pi_{A\cup B})\left(N + \dfrac{nm\pi_A\pi_B}{N - n\pi_A - m\pi_B}\right)$

Hence, the result.

Theorem 3 : Minimisation of $V(\hat{\pi}_{A\cup B})$ subject to n+m=N provides the optimum value of n as $n_{opt}/N = 1/(1+\delta)$, where $\delta = \sqrt{(1-\pi_A)/(1-\pi_B)}$.

Proof :
$$\min_{\substack{m+n=N \\ 0<n,\ m<N}} V(\hat{\pi}_{A\cup B}) \iff \min_{\substack{n+m=N \\ 0<n<N}} \left[\frac{1-\pi_A}{N-n}+\frac{1-\pi_B}{n}\right]$$

Let $f(n) = (1-\pi_A)/(N-n) + (1-\pi_B)/n$.

Setting $f'(n) = 0$ we get $\frac{N-n}{n} = \pm\sqrt{(1-\pi_A)/(1-\pi_B)} = \pm\,\delta$.

clearly $f''(n) > 0$ for all $n \in (0, N)$.
For n to be feasible $(N - n) / n > 0$ and hence $(N - n) / n = \delta$ for optimum n.
Therefore, $n_{opt}/N = 1/(1 + \delta)$.

Remark 3.1 : If $\pi_A \lesseqgtr \pi_B$ and $N \geq 3$ then $(n_{opt}/N) \lesseqgtr (1/2)$.

Remark 3.2 : Here also it can be shown that

$$\frac{n_{opt}}{N}(\pi_A, \pi_B) = \frac{m_{opt}}{N}(\pi_B, \pi_A).$$

Note : Let $\tau = \pi_A - \pi_B$; clearly $|\tau| < 1$ and $\delta \approx 1 - \tau/2$, assuming $\pi_A \cdot \pi_B$ to be small. Hence $n_{opt}/N = 1/2(1 + \tau/4)$ and $m_{opt}/N = (1 + \tau/4)/2$.

So $\frac{n_{opt}}{N}(t) = \frac{m_{opt}}{N}(-t)$. Therefore it sufficient to study n_{opt}/N for $0 < \tau < 1$.

3. OPTIMUM COST BASED PLANS

If C_A and C_B denote the costs of inspecting one unit each of defect categories A and B respectively, then the expected cost under the plan is $\bar{c} = C_B\,[n(1 + \Gamma - \pi_A) + m(1 + \Gamma - \Gamma\pi_B)]$ and $\Gamma = C_A/C_B$.

For finding the optimum values of n and m, we maximise the efficiency measured as inverse of variance per unit cost. Therefore it is equivalent to maximising $\bar{V}\,\bar{c}$ subject to a budgetary constraint C", where $\bar{V} = tr(D)$.

Theorem 4 : Mimnimisation of $\bar{V}\,\bar{c}$ subject to a budgetary constraint of C", where $\bar{V}$ is tr(D), yields the optimum $(m/n)_{opt}$ as

$$(m/n)_{opt} = \frac{C*-C_B(1+\Gamma-\pi_A)}{C_B(1+\Gamma-\Gamma\pi_B)} \text{ if } \frac{\pi_A(1-\pi_A)}{\pi_B(1-\pi_B)^3} \le \Gamma$$

$$= \frac{1-\phi(1-\pi_A)}{\phi-(1-\pi_B)} \text{ if } \frac{\pi_A(1-\pi_A)^3}{\pi_A(1-\pi_B)} < \Gamma < \frac{\pi_A(1-\pi_A)}{\pi_B(1-\pi_B)^3}$$

$$= \frac{C_B(1+\Gamma-\Gamma\pi_B)}{C^{*}-C_B(1+\Gamma-\pi_A)} \text{ if } \Gamma \le \frac{\pi_A(1-\pi_A)^3}{\pi_B(1-\pi_B)},$$

where $\phi^2 = \frac{\pi_A(1-\pi_A)}{\pi_B(1-\pi_B)} \cdot \frac{C_B}{C_A}$.

Proof : Let $\theta = m/n$. Then the problem is

$$\min_{\substack{\theta>0 \\ C*=\bar{c}}} \left[k_1\left(M + \frac{N}{1+\theta(1-\pi_B)}\right) + k_2\left(P + \frac{R}{\theta+(1-\pi_A)}\right)\right] = F(\theta), \text{ say,}$$

where $k_1 = C_B\pi_A(1-\pi_A)/(1-\pi_B)$,
$k_2 = C_B\pi_B(1-\pi_B)$,
$M = (1+\Gamma-\pi_B) = P$,
$N = -\pi_{A\cup B}$,
$R = \Gamma\,\pi_{A\cup B}$.

Therefore $\frac{dF}{d\theta} = 0 \Rightarrow \theta = -(1-\pi_A) + \frac{\pi_{A\cup B}}{-(1-\pi_B)\pm\phi}$ if $\phi \ne (1-\pi_B)$.

$$\Rightarrow \theta_1 = \frac{1-\phi(1-\pi_A)}{\phi-(1-\pi_B)} \text{ and } \theta_2 = \frac{1-\phi(1-\pi_A)}{\phi-(1-\pi_B)}$$

If $\phi = (1-\pi_B)$ then $\frac{dF}{d\theta}=0$ has only one root

$$\theta_3 = [(1-\pi_A)+\frac{\pi_{A\cup B}}{2(1-\pi_B)}].$$

It can be easily shown that

$$\left.\frac{d^2F}{d\theta^2}\right|_{\theta 1} > 0, \quad \left.\frac{d^2F}{d\theta^2}\right|_{\theta 2} < 0 \quad \text{and} \quad \left.\frac{d^2F}{d\theta^2}\right|_{\theta 3} < 0 .$$

Note : (1) If $\phi < (1-\pi_B)$ then $-(1/(1-\pi_B)) < (\theta_1 < \theta_2 < -(1-\pi_A) < 0$ and $\left.\frac{d^2F}{d\theta^2}\right|_{\theta 2} < 0$ implying that the maximum is attained at θ_2 and $-(1-\pi_A)$ is a discontinuity point, consequently we study $\frac{F}{d\theta}$ for $\theta > -(1-\pi_A)$.

$$\frac{dF}{d\theta} = k_2R \left[\frac{\phi^2}{[1+\theta(1-\pi_B)]^2} - \frac{1}{[\theta+(1-\pi_A)]^2}\right]$$

Now $-1/(1-\pi_B) < -(1-\pi_A) < \theta \Rightarrow (1-\pi_B)\,[\theta+(1-\pi_A)] < 1+\theta(1-\pi_B)$.

Therefore $\frac{dF}{d\theta} < k_2R \left[\frac{\phi^2}{(1-\pi_B)^2\{\theta+(1-\pi_A)\}^2} - \frac{1}{\{\theta+(1-\pi_A)\}^2}\right]$

i.e. $\frac{dF}{d\theta} < \frac{k_2R\{\phi+(1-\pi_B)\}\ \{\phi-(1-\pi_B)\}}{[\theta+(1-\pi_A)]^2\ (1-\pi_B)^2} < 0.$

$\Rightarrow$ the function F is decreasing for all $\theta > -(1-\pi_A)$. Therefore $\min_{\theta>0} F$ with budget constraint C* is attained when n = 1 and

$$m = [\frac{C^*}{C_B} - (1+\Gamma-\pi_A)] \,/\, (1+\Gamma-\Gamma\pi_B)$$

$$\text{i.e. } \theta_{opt} = \frac{C_B(1+\Gamma-\Gamma\pi_B)}{C^* - C_B(1+\Gamma-\pi_A)} \qquad \text{... (3.1)}$$

(2) If $\phi = (1-\pi_B)$ then $-(1/(1-\pi_B)) < \theta_3 < -(1-\pi_A) < 0$ and second derivative of F with respect to θ at θ_3 is negative. This implies that the maximum is attained at θ_3 with $-(1-\pi_A)$ as a point of discontinuity. Following similar steps as in (1) it can be easily shown that $\frac{dF}{d\theta} < 0.$

$$\text{Therefore } \theta_{opt} = \frac{C_B(1+\Gamma-\Gamma\pi_B)}{C^* - C_B(1+\Gamma-\pi_A)} \qquad \text{...(3.2)}$$

(3) If $(1-\pi_B) < \phi < 1/(1-\pi_A)$ then $\left.\frac{d^2F}{d\theta^2}\right|_{\theta_1} > 0$ and $\theta_1 > 0$.

$$\text{Thus } \theta_{opt} = \theta_1 = \frac{\phi-(1-\pi_B)}{1-\phi(1-\pi_A)} \qquad \text{...(3,3)}$$

(4) If $(1-\pi_B) < 1/(1-\pi_A) \le \phi$ then $\theta_2 < \theta_1 \le 0$. Also note that θ_1 is greater than both the discontinuity points and the second derivative at θ_1 is positive. This implies that miximum is attained at θ_1 and $\frac{dF}{d\theta}$ does not change sign. Therefore F is at least nondecreasing for $\theta \ge \theta_1$. Hence $\min_{\theta>0} F$ with a budgetary restriction C* is attained for m = 1 and

$$n = [\,\frac{C^*}{C_B} - (1 + \Gamma - \Gamma\pi_B)] \,/\, (1+ \Gamma - \Gamma\pi_A)$$

$$\text{i.e., } \theta_{opt} = \frac{C^* - C_B(1+\Gamma-\Gamma\pi_B)}{C_B(1+\Gamma-\pi_A)} \qquad \text{... (3,4)}$$

(5) Aditionally the restrictions on ϕ can be rewritten as :

1. $0 < \phi \leq (1-\pi_B) => [\pi_A(1-\pi_A) / \pi_B(1-\pi_B)^3] \leq \Gamma$

2. $(1 - \pi_B) < \phi < 1/(1-\pi_A) => \dfrac{\pi_A(1-\pi_A)^3}{\pi_B(1-\pi_B)} < \Gamma < \dfrac{\pi_A(1-\pi_A)}{\pi_B(1-\pi_B)^3}$

3. $1/(1-\pi_A) \leq \phi => \Gamma \leq \dfrac{\pi_A(1-\pi_A)^3}{\pi_B(1-\pi_B)}$

In view of the above notes (1) - (5), the theorem follows.

Theorem 5 : Minimisation of $\overline{VC}$, where $\overline{V}$ is the generalised variance or |D|, subject to a budgetary constraint C*, provides the optimum ratio of m/n as :

$$(m/n)_{opt} = \frac{C_B(1+\Gamma = \pi_A)}{C^* - C_B(1+\Gamma - \Gamma\pi_B)} \quad \text{if } \Gamma \leq (1-\pi_A)$$

$$= \frac{\Gamma-(1-\pi_A)}{1-\Gamma(1-\pi_B)} \quad \text{if } (1-\pi_A) < \Gamma < 1/(1-\pi_B)$$

$$= \frac{C^* - C_B(1+\Gamma-\pi_A)}{C_B(1+\Gamma-\Gamma\pi_B)} \quad \text{if } 1/(1-\pi_B) \leq \Gamma.$$

Proof : Let q = m/n, $Q_A = 1 - \pi_A$ and $Q_B = 1 - \pi_B$, then the problem is to minimize $[\dfrac{\pi_A(1-\pi_A)}{n+m(1-\pi_B)} \times \dfrac{\pi_B(1-\pi_B)}{m+n(1-\pi_A)}]$ $[C_B\,(n(1 + \Gamma - \pi_A)$ $+ m(1 + \Gamma - \Gamma\pi_B)]^2 = F(\theta)$ subject to $\theta > 0$ and $C^* = \bar{c}$

Let $k_1 = C_B\pi_A Q_A/Q_B$, $k_2 = C_B\pi_B Q_B$,

$M = (1+\Gamma - \Gamma\pi_B) = P$, $N = -\pi_{A\cup B}$, $R = \Gamma\,\pi_{A\cup B}$.

$\dfrac{dF}{d\theta} = 0 => \theta_1 = - \dfrac{(1+\Gamma-\pi_A)}{(1+\Gamma-\Gamma\pi_B)}$ and $\theta_2 = \dfrac{\Gamma - Q_A}{1-\Gamma Q_B}$ are the two solutions.

Now if $1/\Gamma > Q_B$ then $(\theta_2 + Q_A) > 0$ and if $1/\Gamma < Q_B$ then $\theta_2 + Q_A < 0$.

It is easy to see that

$$\frac{d^2F}{d\theta^2} = Y + 2,$$

$$\text{where } Y = k_1 \left[M + \frac{N}{1+\theta Q_B}\right] \frac{2k_2R}{(\theta+Q_A)^3} + \frac{k_2R}{(\theta+Q_A)^2}\,\frac{k_1Q_BN}{(1+\theta Q_B)^2}$$

$$\text{and } Z = k_2 \left[P + \frac{R}{\theta+Q_A}\right] \frac{2k_1NQ_B}{(1+\theta Q_B)^3} + \frac{k_1NQ_B}{(1+\theta Q_B)^2}\,\frac{k_2R}{(\theta+Q_A)^2}$$

It can be easily shown that

$$\left.\frac{d^2F}{d\theta^2}\right|_{\theta_2} = Y|\theta_2 + Z|\theta_2 = \frac{2k_1k_2Q_B\pi^2_{A\cup B}}{\Gamma(1+\theta_2Q_B)^4} > 0.$$

This shows that the minimum of F is attained at $\theta = \theta_2$. Similarly it can be verified that

$$\left.\frac{d^2F}{d\theta^2}\right|_{\theta_1} = Y'|\theta_1 + Z'|\theta_1 = \frac{-2k_1k_2\Gamma Q_B\pi^2_{A\cup B}}{(\theta_1+Q_A)^2\,(1+\theta_2Q_B)^2} < 0$$

Thus maximum of F is attained at $\theta = \theta_1$. Therefore we observe the following :

(1) If $\Gamma \le (1-\pi_A)$ then $\theta_1 < -(1-\pi_A) < \theta_2 \le 0$. The second derivative of F with respect to θ at θ_2 is positive. This implies that the minimum of F is achieved at θ_2 and the function is atleast non-decreasing for $\theta > \theta_2$. Hence $\min_{\theta>0} F(\theta)$ subject to a budgetary constraint of C^* is attained when m = 1 and
$n = [C^*/C_B - (1+\Gamma - \Gamma\pi_B)] / (1 + \Gamma - \pi_A)$

$$\text{i.e., } (m/n)_{opt} = \frac{C_B\left(1+\Gamma-\pi_A\right)}{C^* - C_B\left(1+\Gamma-\Gamma\pi_B\right)} \qquad \text{...(3.5)}$$

(2) If $(1-\pi_A) < \Gamma < 1/(1-\pi_B)$ then $\theta_2 > 0$ and the corresponding second derivative is positive. This implies that θ_2 is the solution so that

$$(m/n)_{opt} = \frac{\Gamma-(1-\pi_A)}{1-\Gamma(1-\pi_B)} \qquad ...(3.6)$$

(3) If $\Gamma = 1/(1-\pi_B)$ then there is only one solution θ_1, i.e. $\frac{dF}{d\theta} = 0$ for $\theta = \theta_1 = 1\ [1/(1-\pi_B)+(1-\pi_A)]/2$ and the second derivative of F at θ_1 is negative implying θ_1 to be the maximum point.

(4) If $1/(1-\pi_B) < \Gamma$ then $\theta_2 < -1/(1-\pi_B) < \theta_1 < -(1-\pi_A) < 0$. The second derivative of F at θ_1 is negative. This implies that F attains its minimum at θ_1.

(5) By combining the findings of (3) and (4) we obtain that for

$1/Q_B \le \Gamma$, $\theta_1 < -Q_A < 0$ and $\left.\frac{d^2F}{d\theta^2}\right|_{\theta 1} < 0.$

Since $-Q_A$ is a discontinuity point of $F(\theta)$ we study $\frac{dF}{d\theta}$ for $\theta > -(1-\pi_A)$.

Now $\left.\frac{dF}{d\theta}\right|_{\theta > -Q_A} < 0.$

Thus the function is decreasing for $\theta > -Q_A$ when $1/Q_B \le \Gamma$.

Therefore min $F(\theta)$ for $\theta > 0$ subject to the budget restriction is attained when n = 1 and m = $\{[C^*/C_B - (1 + \Gamma-\pi_A)] / (1+\Gamma-\Gamma\,\pi_B)\}$

$$\text{i.e., } (m/n)_{opt} = \frac{C^* - C_B(1+\Gamma-\pi_A)}{C_B(1+\Gamma-\Gamma\pi_B)} \qquad ...(3.7)$$

Hence the theorem follows.

We next consider the maximisation of efficiency per unit cost when the variance concerned is $V((\hat{\pi}_{A\cup B})$. The optimum plan in this case can be stated through the following theorem.

Theorem 6 : Minimisation of $V((\bar{\pi}_{A\cup B})\ \bar{c}$ subject to a budgetary constraint C* provides the optimum ration (m/n) as

$$(m/n)_{opt} = \frac{C_B(1+\Gamma-\pi_A)}{C^* - C_B(1+\Gamma+\Gamma\pi_B)} \text{ if } \Gamma \le \frac{\pi_A(1-\pi_A)(1-\pi_B)}{\pi_B}$$

$$= \frac{1-\sigma(1-\pi_A)}{\sigma-(1-\pi_B)} \text{ if } \frac{\pi_A(1-\pi_A)(1-\pi_B)}{\pi_B} < \Gamma < \frac{\pi_A}{\pi_B(1-\pi_A)(1-\pi_A)}$$

$$= \frac{C^* - C_B(1+\Gamma-\pi_A)}{C_B(1+\Gamma+\Gamma\pi_B)} \text{ if } \frac{\pi_A}{\pi_B(1-\pi_A)(1-\pi_B)} \le \Gamma$$

where $\sigma^2 = \dfrac{\pi_A(1-\pi_B)}{\Gamma(1-\pi_A)\pi_B}$.

Proof : Minimisation of $V((\hat{\pi}_{A\cup B})\ .\bar{c}$ subject to the budgetary constraint is equivalent to

$$\min_{\theta>0}\left[\frac{\pi_{A\cup B}(1-\pi_{A\cup B})}{m+n+\dfrac{nm\pi_A\pi_B}{n(1-\pi_A)+m(1-\pi_B)}}\right] C_B\,[n(1+\Gamma-\pi_A)+m(1+\Gamma-\Gamma\,\pi_B)]$$

subject to the budgetary constraint C*

Thus the problem is equivalent to $\min_{\theta>0} k[(1+\Gamma Q_B) + F\ ]$,

where
$$k = \pi_{A\cup B}(1-\pi_{A\cup B})C_B$$
$$\theta = m/n$$
$$Q_A = 1-\pi_A$$
$$Q_B = 1-\pi_B$$
$$F = \frac{\Gamma Q_A\pi_B}{\theta+Q_A} - \frac{\pi_A}{1+\theta Q_B}.$$

$$\frac{dF}{d\theta} = 0 \Rightarrow \theta_1 = -[Q_A + \frac{\pi_{A\cup B}}{Q_B-\sigma}] \text{ and } \theta_2 = -\ [Q_A + \frac{\pi_{A\cup B}}{Q_B+\sigma}].$$

Now it is easy to see that

$$\left.\frac{d^2F}{d\theta^2}\right|_{\theta_1} = \frac{2\Gamma Q_A \pi_B \pi_{A\cup B}}{\sigma(\theta_1 + Q_A)^4} > 0 \Rightarrow F(\theta) \text{ is minimum at } \theta_1, \text{ and}$$

$$\left.\frac{d^2F}{d\theta^2}\right|_{\theta_2} = \frac{2\Gamma Q_A \pi_B \pi_{A\cup B}}{\sigma(\theta_2 + Q_A)^4} < 0 \Rightarrow F(\theta) \text{ is minimum at } \theta_2.$$

We next consider the following cases —

(1) If $\sigma < Q_B$, then $\theta_1 < \theta_2 < -Q_A < 0$ and the second derivative at θ_2 is negative. This calls for examination of $\frac{dF}{d\theta}$ for $\theta > -Q_A$.

$$\text{Now } \frac{dF}{d\theta} = Q_A \pi_B \left[\frac{\sigma}{(1+\theta Q_B)^2} - \frac{1}{(\theta + Q_A)^2}\right] < 0 \text{ for } \theta > -Q_A.$$

This implies that $F(\theta)$ is decreasing function of θ for $\theta > -Q_A$. Therefore n=1 and $m = [C^* - C_B(1+\Gamma - \pi_A)] / [C_B(1+\Gamma - \Gamma\,\pi_B)]$,

$$\text{i.e., } (m/n)_{opt} = \frac{C^* - C_B(1+G-p_A)}{C_B(1+G-Gp_B)} \qquad ...(3.8)$$

(2) If $\sigma = Q_B$ then $\frac{dF}{d\theta} = 0$ has only one root $\theta' = -\left[Q_A + \frac{p_{A\cup B}}{2Q_B}\right]$

such that $\theta' < -Q_A < 0$ and the second derivative of F at θ' is negative. This indicates $(m/n)_{opt}$ to be the same as that of case (1).

(3) If $Q_B < \sigma < 1/Q_A$ then $\theta_2 < 0 < \theta_1$ and $\left.\frac{d^2F}{d\theta^2}\right|_{\theta_1} > 0.$

Hence $(m/n)_{opt} = \dfrac{1 - sQ_A}{s - Q_B}$. ...(3.9)

(4) If $1/Q_A \le \sigma$ then $\theta_2 < \theta_1 \le 0$ and the second derivative of F w.r.t. θ at θ_1 is positive. Thus $F(\theta)$ is atleast non decreasing for $\theta > 0$. Hence we have $m = 1$ and

$n = [C^* - C_B(1+\Gamma - \pi_B)] / [C_B(1+\Gamma-\Gamma\ \pi_A)]$,

i.e., $(m/n)_{opt} = \dfrac{C_B(1 + G - p_A)}{C^* - C_B(1 + G - Gp_B)}$ ...(3.9)

The theorem follows from the above four cases.

4. CONCLUDING REMARKS

It will be interesting to extend the above problem to the case where defect categories are inspected randomly instead of in an ordered way. For example, p_i may be the probability of noticing the defect category d_i when n_i is the actual number of defectives of category d_i in a sample of size n. In this problem, of course, prior knowledge of p_i should be available from past records of such inspection schemes.

The detailed proofs of all the results may be referred in [2].

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