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Bounds for the Equivalence of Bayes and Maximum Likelihood Estimators for a Class of Diffusion Processes

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Summary. Let θ be the unknown parameter in the drift coefficient of a diffusion process described by a linear homogeneous stochastic differential equation. Bounds for the equivalence of Bayes and Maximum likelihood estimators of the parameter θ have been obtained in this paper.

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Key words: Linear homogeneous stochastic differential equation, Bayes estimator, Maximum likelihood estimator, drift coefficient of diffusion processes, prior probability measure, posterior density, Ornstein-Uhlenbeck process.

1. Introduction

Prakasa Rao (1979) obtained bounds on $|\tilde{\theta}_T - \hat{\theta}_T|$ for discrete time stationary Markov processes where $\hat{\theta}_T$ is a maximum likelihood estimator (MLE) and $\tilde{\theta}_T$ is a Bayes estimator of the parameter θ corresponding to some smooth loss function and some prior, extending the earlier work of Strasser (1977) in the case of independent and identically distributed observations. It was also proved in Prakasa Rao (1980) that $Q(T)\,(\tilde{\theta}_T - \hat{\theta}_T) \to 0$ a.s. (P_{θ_0}), as $T \to \infty$, where θ_0 is the true value of the parameter and $Q(T)$ is a suitable continuous function monotonically increasing to infinity as $T \to \infty$, for a class of diffusion processes satisfying a linear stochastic differential equation. In particular it was shown that the asymptotic behaviour of the Bayes estimator $\tilde{\theta}_T$ and the maximum likelihood estimator $\hat{\theta}_T$ is the same, as T approaches infinity. In this paper, bounds on $|\tilde{\theta}_T - \hat{\theta}_T|$ are obtained generalizing the earlier work of Prakasa Rao (1979) for discrete time stationary Markov processes and extending results in Prakasa Rao (1980) for diffusion processes satisfying the linear stochastic differential equation

$$\begin{aligned} \mathrm{d}X_t &= \theta a(X_t)\,\mathrm{d}t + b(X_t)\,\mathrm{d}W_t, \quad 0 < t \leqq T, \\ X_0 &= x \in \mathrm{R}, \end{aligned} \tag{1.1}$$

where $\{W_t,\ t \geqq 0\}$ is the standard Wiener process. Interalia, we obtain a Berry-Esseen type bound for the Bayes estimator $\tilde{\theta}_T$ under some regularity conditions. Mishra and Prakasa Rao (1985) derived a Berry-Esseen type bound for the maximum likelihood estimator of the parameter θ for processes defined by (1.1).

2. Assumptions and preliminaries

Let $(\Omega, \mathfrak{J}, \mathsf{P})$ be a probability space. Consider the one-dimensional stochastic differential equation (1.1) defined on $(\Omega, \mathfrak{J}, \mathsf{P})$ where $\{W_t, t \geqq 0\}$ is the standard WIENER process.

Assume that there exists a unique solution $X_0^T = \{X_s, 0 \leqq s \leqq T\}$ to the stochastic differential equation (1.1) for every $\theta \in \Theta$ open in R. Denote by P_θ^T the measure generated by the process $\{X_s, 0 \leqq s \leqq T\}$ on the space (C_T, B_T) of continuous functions on $[0, T]$ with the associated sigma-algebra B_T of BOREL subsets of C_T generated under the supremum norm. Let E_θ be expectation with respect to the measure P_θ^T and P_W^T the measure induced by the standard WIENER process on (C_T, B_T). Let θ_0 denote the true value of the paramelei.

Throughout the paper we shall use C_1, C_2 etc. for positive constants. We assume that the following assumptions hold.

(A_1) Suppose that $\mathsf{P}_{\theta_0}^T$ and $\mathsf{P}_{\theta_1}^T$ are absolutely continuous with respect to each other for all $\theta_1 \in \Theta$. It is known that the RADON-NIKODYM derivative of $\mathsf{P}_{\theta_1}^T$ with respect to $\mathsf{P}_{\theta_0}^T$ is given by

$$\frac{\mathrm{d}\mathsf{P}_{\theta_1}^T}{\mathrm{d}\mathsf{P}_{\theta_0}^T}(X_0^T) = \exp\left[(\theta_1 - \theta_0)\int_0^T \frac{a(X_t)}{b(X_t)}\,\mathrm{d}W_t - \frac{1}{2}(\theta_1 - \theta_0)^2 \int_0^T \frac{a^2(X_t)}{b^2(X_t)}\,\mathrm{d}t\right] \tag{2.1}$$

where $X_0^T = \{X_s, 0 \leqq s \leqq T\}$. This can be seen from LIPTSER and SHIRYAYEV (1977, p. 248). Here $b(X_t)\,\mathrm{d}W_t = \mathrm{d}X_t - \theta_0\, a(X_t)\,\mathrm{d}t$ and the stochastic integral is with respect to the probability measure $\mathsf{P}_{\theta_0}^T$.

(A_2) Suppose that Λ is a prior probability measure on $(\Theta, \mathfrak{B})$ where $\mathfrak{B}$ is the sigma-algebra of BOREL subsets of Θ. Suppose Λ has a density $p(\cdot)$ with respect to the LEBESGUE measure on R. Suppose the density $p(\cdot)$ is continuous, strictly positive on Θ and there exists $C_1 > 0$ such that

$$|p(\theta_1) - p(\theta_2)| \leqq C_1 |\theta_1 - \theta_2| \tag{2.2}$$

for all θ_1 and $\theta_2 \in \Theta$.

In addition to (A_1) and (A_2) assume that

(A_3) the density function $p(\cdot)$ satisfies the inequality

$$\int_{-\infty}^{\infty} |u|\, p(u)\,\mathrm{d}u < \infty$$

and

(A_4) $\quad 0 < \mathsf{E}_\theta \int_0^T \frac{a^2(X_t)}{b^2(X_t)}\,\mathrm{d}t < \infty$ for all $T > 0$, $\theta \in \Theta$.

Define

$$\alpha_T = Q^{-1}(T) \int_0^T \frac{a(X_s)}{b(X_s)} \mathrm{d}W_s,$$

and

$$\beta_T = Q^{-2}(T) \int_0^T \frac{a^2(X_s)}{b^2(X_s)} \mathrm{d}s,$$

where $Q(T)$ is as given by (A_5) given below. Here too $b(X_t)\mathrm{d}W_t = \mathrm{d}X_t - -\theta_0 a(X_t)\mathrm{d}t$.

(A_5) Assume that there exists a sequence of positive numbers $Q(T)$ possibly depending on θ_0 and tending to infinity as $T \to \infty$ such that $\beta_T \to 1$ as $T \to \infty$ in $\mathsf{P}_{\theta_0}^T$-probability.

Observe that

$$\log \frac{\mathrm{d}\mathsf{P}_\theta^T}{\mathrm{d}\mathsf{P}_{\theta_0}^T}\bigg|_{\theta=\hat{\theta}_T} = (\hat{\theta}_T - \theta_0)\, Q(T)\,\alpha_T - \frac{1}{2}(\hat{\theta}_T - \theta_0)^2 \beta_T Q^2(T)$$

and, using the above relations, it is easy to see that the M.L.E. $\hat{\theta}_T$ satisfies the relation $\alpha_T = (\hat{\theta}_T - \theta_0)\,\beta_T\, Q(T)$. For simplicity we write

$$\frac{\mathrm{d}\mathsf{P}_{\hat{\theta}_T}^T}{\mathrm{d}\mathsf{P}_{\theta_0}^T} \text{ for } \frac{\mathrm{d}\mathsf{P}_\theta^T}{\mathrm{d}\mathsf{P}_{\theta_0}^T} \text{ evaluated at } \theta = \hat{\theta}_T.$$

(A_6) Suppose that $\hat{\theta}_T \xrightarrow{a.s} \theta_0$ under $\mathsf{P}_{\theta_0}^T$ probability as $T \to \infty$.

Let us denote the posterior density of θ given X_0^T as $p^*(\theta | X_0^T)$. Then we obtain

$$p^*(\theta | X_0^T) = \frac{\mathrm{d}\mathsf{P}_\theta^T}{\mathrm{d}\mathsf{P}_{\theta_0}^T}(X_0^T)\, p(\theta) \bigg/ \int_\Theta \frac{\mathrm{d}\mathsf{P}_\theta^T}{\mathrm{d}\mathsf{P}_{\theta_0}^T}(X_0^T)\, p(\theta)\,\mathrm{d}\theta$$

and write

$$t = Q(T)\,(\theta - \hat{\theta}_T).$$

Then the posterior density of $Q(T)\,(\theta - \hat{\theta}_T)$ is given by

$$p^{**}(t | X_0^T) = Q^{-1}(T)\, p^*(\hat{\theta}_T + t\,Q^{-1}(T) | X_0^T).$$

Let

$$\zeta_T(t)=\frac{\mathrm{d}\,\mathrm{P}^T_{\hat\theta_T+tQ^{-1}(T)}}{\mathrm{d}\,\mathrm{P}^T_{\hat\theta_T}}=\frac{\mathrm{d}\,\mathrm{P}^T_{\hat\theta_T+tQ^{-1}(T)}/\mathrm{d}\,\mathrm{P}^T_{\theta_0}}{\mathrm{d}\,\mathrm{P}^T_{\hat\theta_T}/\mathrm{d}\,\mathrm{P}^T_{\theta_0}}$$

and

$$C_T=\int_{-\infty}^{\infty}\zeta_T(T)\,p(\hat\theta_T+tQ^{-1}(T))\,\mathrm{d}t.$$

Then

$$p^{**}(t\,|\,X_0^T)=C_T^{-1}\zeta_T(t)\,p(\hat\theta_T+tQ^{-1}(T))$$

and

$$\log\zeta_T(t)=-\frac{1}{2}\beta_T t^2 \text{ (cf. PRAKASA RAO (1979)).}$$

Let $\tilde\theta_T$ be the BAYES estimator of the parameter θ occuring in the stochastic differential equation (1.1) corresponding to prior density $p(\cdot)$. It is easy to see that

$$Q(T)\,(\tilde\theta_T-\hat\theta_T)=\frac{\int_{-\infty}^{\infty} s\zeta_T(s)\,p\left(\hat\theta_T+\frac{s}{Q(T)}\right)\mathrm{d}s}{\int_{-\infty}^{\infty}\zeta_T(t)\,p\left(\hat\theta_T+\frac{t}{Q(T)}\right)\mathrm{d}t} \tag{2.3}$$

(cf. BASAWA and PRAKASA RAO (1980, p. 242)). This follows from the relation for the generalized BAYESian estimator given in equation (126) in the reference cited. The generalized BAYESian estimator reduces to usual BAYES estimator in our discussion, for quadratic loss function, since $p(\cdot)$ is a density.

3. Main results

We now state the main results.

Theorem 3.1. *Let the assumptions* (A_1) *to* (A_4) *hold. Let* $Z(T)\uparrow\infty$ *and* $r_1(T)\downarrow 0$ *such that* $Z(T)/Q(T)\downarrow 0$ *as* $T\to\infty$. *Let us denote* $\frac{1}{Z(T)}=r_2(T)$, $Q(T)\,\mathrm{e}^{-Z^2(T)/2}=r_3(T)$ *and* $Q^2(T)\,\mathrm{e}^{-Z^2(T)/2}=r_4(T)$. *Define*

$$d_T=\frac{[\sqrt{2\pi}\,C_1 r_2(T)+2p(\hat\theta_T)+C_2 r_4(T)]\,\mathrm{e}^{r_1^2(T)}}{p(\hat\theta_T)\sqrt{2\pi}\,(1-R(T))}$$

where $R(T)=R_1(T)+R_2(T)$ *with*

$$R_1(T)=\frac{\sqrt{2\pi}\,p(\hat{\theta}_T)\,(\mathrm{e}^{r_1^2(T)}-1)+2C_1 r_2(T)\,\mathrm{e}^{r_1^2(T)}}{\sqrt{2\pi}\,p(\hat{\theta}_T)}$$

and

$$R_2(T)=\left\{r_3(T)\,\mathrm{e}^{r_1^2(T)}+\frac{2p(\hat{\theta}_T)}{\lambda}\sqrt{\frac{2\pi}{1-\lambda}}\,r_2^2(T)\right\}\Big/p(\hat{\theta}_T)\sqrt{2\pi},$$

for some $0<\lambda<1$. *Then, for T large,*

$$\mathsf{P}_{\theta_0}^T\{|\tilde{\theta}_T-\hat{\theta}_T|>d_T r_2(T)\}\leqq 6\,\mathsf{P}_{\theta_0}^T\{|\beta_T-1|>2r_1^2(T)\,r_2^2(T)\}+2\,\mathsf{P}_{\theta_0}^T\{|\beta_T-1|>1\}.$$

Remarks 3.0. If $a(\cdot)=b(\cdot)$ and $Q(T)=T^{1/2}$, then $\beta_T=1$ and Theorem 3.1 implies that for T large

$$\mathsf{P}_{\theta_0}^T(|\tilde{\theta}_T-\hat{\theta}_T|>d_T\,r_2(T))=0.$$

Hence $|\tilde{\theta}_T-\hat{\theta}_T|\leqq d_T\,r_2(T)$ with $\mathsf{P}_{\theta_0}^T$-probability one for T large. Note that $d_T\xrightarrow{a\cdot s}\sqrt{\frac{2}{\pi}}$ and $r_2(T)\to 0$ as $T\to\infty$.

Theorem 3.2. *Let the assumptions* (A_1) *to* (A_6) *hold. Furthermore suppose* d_T *is as defined in Theorem* 3.1. *Let* $r_6(T)$ *be a positive sequence decreasing to zero such that* $r_5(T)=(Q(T)\,r_6(T))^{-1}\downarrow 0$ *as* $T\to\infty$. *Then there exists a constant* $C_3>0$ *such that, for T large,*

$$\begin{aligned}\mathsf{P}_{\theta_0}^T\{|\tilde{\theta}_T-\theta_0|>d_T\,r_2(T)+r_5(T)\}\\ \leqq 6\,\mathsf{P}_{\theta_0}^T\{|\beta_T-1|>2r_1^2(T)\,r_2^2(T)\}+2\,\mathsf{P}_{\theta_0}^T\{|\beta_T-1|>r_6(T)\}\\ +2\,\mathsf{P}_{\theta_0}^T\{|\beta_T-1|>1\}+C_3(r_6(T))^{1/2}.\end{aligned}$$

We shall use the following lemmas for the proofs of theorems. Proofs of the lemmas are given in the appendix.

Lemma 3.1. *Let* $(\Omega,\mathfrak{J},\mathsf{P})$ *be a probability space and* $f:\Omega\to\mathbb{R}$ $g:\Omega\to\mathbb{R}$ *be* $\mathfrak{J}$*-measurable functions. Let* $d>0$ *and* $0\leqq r\leqq 1$. *Then*

$$\mathsf{P}\left\{w:\frac{f(w)}{g(w)}\geqq d\right\}\leqq\mathsf{P}\{w:f(w)\geqq d(1-r)\}+\mathsf{P}\{w:|g(w)-1|>r\}.$$

[For the proof, see Lemma 2.1 of MISHRA and PRAKASA RAO (1987)]

In the following lemmas, let $r_1(T) \downarrow 0$ as $T \to \infty$ and $r_2(T)$ be as defined earlier.

Lemma 3.2. $\mathsf{P}_{\theta_0}^T\{|\int_{|t| \leqq Z(T)} p(\hat{\theta}_T)(\mathrm{e}^{-\frac{1}{2}\beta_T t^2} - \mathrm{e}^{-\frac{1}{2}t^2})\,\mathrm{d}t| > \sqrt{2\pi}\, p(\hat{\theta}_T)(\mathrm{e}^{r_1^2(T)} - 1)\}$
$\leqq \mathsf{P}_{\theta_0}^T\{|\beta_T - 1| > 2r_1^2(T)\, r_2^2(T)\}$.

Lemma 3.3. $\mathsf{P}_{\theta_0}^T\{|\int_{|t| \leqq Z(T)} \mathrm{e}^{-\frac{1}{2}\beta_T t^2}\left(p\left(\hat{\theta}_T + \frac{t}{Q(T)}\right) - p(\hat{\theta}_T)\right)\mathrm{d}t| > 2C_1 r_2(T)\,\mathrm{e}^{r_1^2(T)}\}$
$\leqq \mathsf{P}_{\theta_0}^T\{|\beta_T - 1| > 2r_1^2(T)\, r_2^2(T)\}$.

Remark 3.1. From Lemma 3.2 and Lemma 3.3 we get,

$$\begin{aligned}\mathsf{P}_{\theta_0}^T\{|\int_{|t| \leqq Z(T)} \mathrm{e}^{-\frac{1}{2}\beta_T t^2} p\left(\hat{\theta}_T + \frac{t}{Q(T)}\right)\mathrm{d}t - \int_{|t| \leqq Z(T)} p(\hat{\theta}_T)\,\mathrm{e}^{-\frac{1}{2}t^2}\,\mathrm{d}t| >&\\ > (\sqrt{2\pi}\, p(\hat{\theta}_T)(\mathrm{e}^{r_1^2(T)} - 1) + 2C_1 r_2(T)\,\mathrm{e}^{r_1^2(T)})\}&\\ \leqq 2\,\mathsf{P}_{\theta_0}^T\{|\beta_T - 1| > 2r_1^2(T)\, r_2^2(T)\}.&\end{aligned}$$

Hence,

$$\begin{aligned}\mathsf{P}_{\theta_0}^T\{|\int_{|t| \leqq Z(T)} (\mathrm{e}^{-\frac{1}{2}\beta_T t^2} p\left(\hat{\theta}_T + \frac{t}{Q(T)}\right) - p(\hat{\theta}_T)\,\mathrm{e}^{-\frac{1}{2}t^2})\,\mathrm{d}t| > p(\hat{\theta}_T)\sqrt{2\pi}\, R_1(T)\}&\\ \leqq 2\,\mathsf{P}_{\theta_0}^T\{|\beta_T - 1| > 2r_1^2(T)\, r_2^2(T)\},& \qquad (3.1)\end{aligned}$$

where

$$R_1(T) = \frac{\sqrt{2\pi}\, p(\hat{\theta}_T)(\mathrm{e}^{r_1^2(T)} - 1) + 2C_1 r_2(T)\,\mathrm{e}^{r_1^2(T)}}{\sqrt{2\pi}\, p(\hat{\theta}_T)}.$$

Lemma 3.4. *For T large,*

$$\begin{aligned}&\mathsf{P}_{\theta_0}^T\{\int_{|t| > Z(T)} \mathrm{e}^{-\frac{1}{2}\beta_T t^2} p\left(\hat{\theta}_T + \frac{t}{Q(T)}\right)\mathrm{d}t > C_2 r_3(T)\,\mathrm{e}^{r_1^2(T)}\}\\ &\leqq \mathsf{P}_{\theta_0}^T\{|\beta_T - 1| > 2r_1^2(T)\, r_2^2(T)\} + \mathsf{P}_{\theta_0}^T\{|\beta_T - 1| > 1\}.\end{aligned}$$

Remark 3.2. $p(\hat{\theta}_T)\int_{|t| > Z(T)} \mathrm{e}^{-\frac{1}{2}t^2}\,\mathrm{d}t$

$$\begin{aligned}&\leqq p(\hat{\theta}_T)\,\mathrm{e}^{-(\lambda/2)Z^2(T)} \int_{|t| > Z(T)} \mathrm{e}^{-(t^2/2)(1-\lambda)}\,\mathrm{d}t, \quad (0 \leqq \lambda < 1)\\ &\leqq \frac{p(\hat{\theta}_T)}{\sqrt{(1-\lambda)}}\sqrt{2\pi}\,\frac{2}{\lambda}\, r_2^2(T). \qquad (3.2)\end{aligned}$$

Lemma 3.5. *For T large,*

$$\mathsf{P}^T_{\theta_0}\{|\int_{|t|\leqq Z(T)} t\zeta_T(t)|\, p\left(\hat{\theta}_T+\frac{t}{Q(T)}\right)-p(\hat{\theta}_T)\,|\mathrm{d}t|>\sqrt{2\pi}\; C_1 r_2(T)\mathrm{e}^{r_1^2(T)}\}$$
$$\leqq \mathsf{P}^T_{\theta_0}\{|\beta_T-1|>2r_1^2(T)\, r_2^2(T)\}.$$

Lemma 3.6. $\mathsf{P}^T_{\theta_0}\{|\int_{|t|\leqq Z(T)} p(\hat{\theta}_T)\, t\,\zeta_T(t)\,\mathrm{d}t|>2p(\hat{\theta}_T)\mathrm{e}^{r_1^2(T)}\}$
$$\leqq \mathsf{P}^T_{\theta_0}\{|\beta_T-1|>2r_1^2(T)\, r_2^2(T)\}.$$

Lemma 3.7. *For T large,*

$$\mathsf{P}^T_{\theta_0}\{|\int_{|t|>Z(T)} t\zeta_T(t)\, p\left(\hat{\theta}_T+\frac{t}{Q(T)}\right)\mathrm{d}t|>C_2 r_4(T)\mathrm{e}^{r_1^2(T)}\}$$
$$\leqq \mathsf{P}^T_{\theta_0}\{|\beta_T-1|>2r_1^2(T)\, r_2^2(T)\}+\mathsf{P}^T_{\theta_0}\{|\beta_T-1|>1\}.$$

We now discuss proofs of Theorems 3.1 and 3.2.

Proof of Theorem 3.1. Observe that, for T large,

$$\mathsf{P}^T_{\theta_0}\{|\int_{|t|>Z(T)} \mathrm{e}^{-\frac{1}{2}\beta_T t^2} p\left(\hat{\theta}_T+\frac{t}{Q(T)}\right)\mathrm{d}t-\int_{|t|>Z(T)} p(\hat{\theta}_T)\mathrm{e}^{-\frac{1}{2}t^2}\,\mathrm{d}t|>$$
$$>r_3(T)\mathrm{e}^{r_1^2(T)}+\frac{2p(\hat{\theta}_T)}{\lambda}\sqrt{\left(\frac{2\pi}{1-\lambda}\right)}\, r_2^2(T)\}$$
$$\leqq \mathsf{P}^T_{\theta_0}\{\int_{|t|>Z(T)} \mathrm{e}^{-\frac{1}{2}\beta_T t^2} p\left(\hat{\theta}_T+\frac{t}{Q(T)}\right)\mathrm{d}t+\int_{|t|>Z(T)} p(\hat{\theta}_T)\mathrm{e}^{-\frac{1}{2}t^2}\,\mathrm{d}t>$$
$$>r_3(T)\mathrm{e}^{r_1^2(T)}+\frac{2p(\hat{\theta}_T)}{\lambda}\sqrt{\left(\frac{2\pi}{1-\lambda}\right)}\, r_2^2(T)\}$$
$$\leqq \mathsf{P}^T_{\theta_0}\{\int_{|t|>Z(T)} \mathrm{e}^{-\frac{1}{2}\beta_T t^2} p\left(\hat{\theta}_T+\frac{t}{Q(T)}\right)\mathrm{d}t>r_3(T)\mathrm{e}^{r_1^2(T)}\}$$

(By inequality (3.2))

$$\leqq \mathsf{P}^T_{\theta_0}\{|\beta_T-1|>2r_1^2(T)\, r_2^2(T)\}+\mathsf{P}^T_{\theta_0}\{|\beta_T-1|>1\}. \tag{3.3}$$

(By Lemma (3.4))

Now, using the inequalities (3.1) and (3.3), we get that, for T large,

$$\mathsf{P}^T_{\theta_0}\{|\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}\beta_T t^2} p\left(\hat{\theta}_T+\frac{t}{Q(T)}\right)\mathrm{d}t-\int_{-\infty}^{\infty} p(\hat{\theta}_T)\mathrm{e}^{-\frac{1}{2}t^2}\,\mathrm{d}t|>$$
$$>p(\hat{\theta}_T)\sqrt{2\pi}\,(R_1(T)+R_2(T))\}$$
$$\leqq 3\,\mathsf{P}^T_{\theta_0}\{|\beta_T-1|>2r_1^2(T)\, r_2^2(T)\}+\mathsf{P}^T_{\theta_0}\{|\beta_T-1|>1\}.$$

The above statement can be rewritten as,

$$\mathsf{P}_{\theta_0}^T \left| \frac{\int_{-\infty}^{\infty} e^{-\frac{1}{2}\beta_T t^2} p\left(\hat{\theta}_T + \frac{t}{Q(T)}\right) dt}{\int_{-\infty}^{\infty} p(\hat{\theta}_T) e^{-\frac{1}{2}t^2} dt} - 1 \right| > R(T) \tag{3.4}$$

$$\leqq 3\,\mathsf{P}_{\theta_0}^T \{|\beta_T - 1| > 2 r_1^2(T)\, r_2^2(T)\} + \mathsf{P}_{\theta_0}^T \{|\beta_T - 1| > 1\}.$$

Observe that $R(T) \geqq 0$ and $R(T) \xrightarrow{a.s} 0$ as $T \to \infty$ by assumptions (A_2) and (A_6).

Again, following the above procedure and using Lemma 3.5 to 3.7, we obtain, that for T large,

$$\mathsf{P}_{\theta_0}^T \{| \int_{-\infty}^{\infty} t \zeta_T(t)\, p\left(\hat{\theta}_T + \frac{t}{Q(T)}\right) dt| > (\sqrt{2\pi}\; C_1 r_2(T) e^{r_1^2(T)}$$

$$+ 2 p(\hat{\theta}_T) e^{r_1^2(T)} + C_2 r_4(T) e^{r_1^2(T)})\}$$

$$\leqq 3\,\mathsf{P}_{\theta_0}^T \{|\beta_T - 1| > 2 r_1^2(T)\, r_2^2(T)\} + \mathsf{P}_{\theta_0}^T \{|\beta_T - 1| > 1\}. \tag{3.5}$$

Now,

$$\mathsf{P}_{\theta_0}^T \{|\tilde{\theta}_T - \hat{\theta}_T| > d_T r_2(T)\}$$

$$= \mathsf{P}_{\theta_0}^T \left\{ \left| \frac{\int_{-\infty}^{\infty} r_2(T)\, t\, \zeta_T(T)\, p\left(\hat{\theta}_T + \frac{t}{Q(T)}\right) dt}{\int_{-\infty}^{\infty} \zeta_T(t)\, p\left(\hat{\theta}_T + \frac{t}{Q(T)}\right) dt} \right| > d_T r_2(T) \right\}$$

(by using equation (2.3))

$$\leqq \mathsf{P}_{\theta_0}^T \left\{ \left| \frac{\int_{-\infty}^{\infty} r_2(T)\, t\, \zeta_T(t)\, p\left(\hat{\theta}_T + \frac{t}{Q(T)}\right) dt}{p(\theta_0)\sqrt{2\pi}} \right| > d_T r_2(T)\,(1 - R(T)) \right\}$$

$$+ \mathsf{P}_{\theta_0}^T \left\{ \left| \frac{\int_{-\infty}^{\infty} \zeta_T(t)\, p\left(\hat{\theta}_T + \frac{t}{Q(T)}\right) dt}{p(\hat{\theta}_T)\sqrt{2\pi}} - 1 \right| > R(T) \right\}$$

(by Lemma 3.1)

$$= J_1 + J_2 \text{ (say)}.$$

The first expression

$$J_1 = \mathrm{P}^T_{\theta_0}\{|r_2(T) \int_{-\infty}^{\infty} t\zeta_T(t)\, p\left(\hat{\theta}_T + \frac{t}{Q(T)}\right) \mathrm{d}t| >$$

$$> d_T r_2(T)\, p(\hat{\theta}_T) \sqrt{2\pi}(1 - R(T))\}$$

$$= \mathrm{P}^T_{\theta_0}\{| \int_{-\infty}^{\infty} t\zeta_T(t)\, p\left(\hat{\theta}_T + \frac{t}{Q(T)}\right) \mathrm{d}t| > (\sqrt{2\pi}\; C_1 r_2(T)\,\mathrm{e}^{r_1^2(T)} +$$

$$+ 2p(\hat{\theta}_T)\,\mathrm{e}^{r_1^2(T)} + C_2 r_4(T)\,\mathrm{e}^{r_1^2(T)})\}$$

$$\leqq 3\,\mathrm{P}^T_{\theta_0}\{|\beta_T - 1| > 2r_1^2(T)\, r_2^2(T)\} + \mathrm{P}^T_{\theta_0}\{|\beta_T - 1| > 1\},$$

by using the inequality (3.5).

Similar type of bound for the second expression J_2 has been mentioned in the inequality (3.4). Using these two results, we obtain that, for T large,

$$\mathrm{P}^T_{\theta_0}\{|\tilde{\theta}_T - \hat{\theta}_T| > d_T r_2(T)\} \leqq 6\,\mathrm{P}^T_{\theta_0}\{|\beta_T - 1| > 2r_1^2(T)\, r_2^2(T)\} + 2\,\mathrm{P}^T_{\theta_0}\{|\beta_T - 1| > 1\}. \tag{3.6}$$

This completes the proof of Theorem 3.1.

Proof of the Theorem 3.2. Taking into consideration the assumption (A_4), it can be seen from Theorem 3.2 of MISHRA and PRAKASA RAO (1985) that,

$$\mathrm{P}^T_{\theta_0}\{|\hat{\theta}_T - \theta_0| > r_5(T)\} \leqq C_3 \sqrt{r_6(T)} + 2\,\mathrm{P}^T_{\theta_0}\{|\beta_T - 1| > r_6(T)\}. \tag{3.7}$$

Thus, from the inequalities (3.6) and (3.7), we prove the Theorem 3.2.

Remark 3.3. The bound is uniform over compact subsets K of Θ provided $p(\cdot)$ is bounded above and bounded away from zero for $\theta \in K$. The expression $R_1(T) + R_2(T)$ in Theorem 3.1 can be explicitly computed in terms of $r_1(T)$ to $r_6(T)$ and $p(\theta)$. Observe that $d_T \xrightarrow{a.s} \sqrt{2\pi}$ as $T \to \infty$ and hence bounded in T.

4. Example

We now illustrate the above result by considering the linear stochastic differential equation

$$\mathrm{d}X_t = -\theta X_t\,\mathrm{d}t + \mathrm{d}W_t, \quad t > 0, \tag{4.1}$$

$$X_0 = 0$$

where $\theta \in [\alpha, \gamma]$, $\alpha > 0$, $\gamma > 0$. Let us choose $Z^2(T) = T^{1/5}$, $r_1^2(T) = T^{-1/5}$. MISHRA and PRAKASA RAO (1985) have shown that for equation (4.1), $Q^2(T) = T/2\theta_0$ and there exists a constant $C_4 > 0$ such that, for $\varepsilon(T) = T^{-2/5}$,

$$\mathsf{P}_{\theta_0}^T\{|\beta_T - 1| \geqq \varepsilon(T)\} \leqq C_4(T\varepsilon^2(T))^{-1} = C_4 T^{-1/5}.$$

Using these, we obtain from (3.6),

$$\mathsf{P}_{\theta_0}^T\{|\tilde{\theta}_T - \hat{\theta}_T| > d_T T^{-1/10}\} \leqq C_5 T^{-1/5} \tag{4.2}$$

and from (3.7)

$$\mathsf{P}_{\theta_0}^T\{|\hat{\theta}_T - \theta_0| > T^{-1/5}\} \leqq C_6 T^{-3/2\theta_0} \tag{4.3}$$

by choosing

$$r_5(T) = \frac{\sqrt{2\theta_0}}{T^{1/5}}, \quad r_6(T) = T^{-3/10} \quad \text{and} \quad Q^2(T) = \frac{T}{2\theta_0}.$$

Suppose conditions (A_2) and (A_3) hold for the density $p(\cdot)$. It can be seen from MISHRA and PRAKASA RAO (1985) and FEIGIN (1976), that other conditions hold for the stochastic differential equation (4.1). Thus, using the results in Theorem 3.2, we obtain from (4.2) and (4.3) that for any constant $C_7 > 0$ there corresponds another constant $C_8 > 0$ such that,

$$\mathsf{P}_{\theta_0}^T\{|\tilde{\theta}_T - \theta_0| > d_T C_7 T^{-1/10}\} \leqq C_8 T^{-3/20}.$$

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Appendix

Proof of Lemma 3.2.

$$\mathsf{P}_{\theta_0}^T\{|\int_{|t| \leqq Z(T)} (e^{-\frac{1}{2}\beta_T t^2} - e^{-\frac{1}{2}t^2})\, p(\hat{\theta}_T)\, dt| > \sqrt{2\pi}\, p(\hat{\theta}_T)\, (e^{r_1^2(T)} - 1)\}$$

$$\leqq \mathsf{P}_{\theta_0}^T\{\int_{|t| \leqq Z(T)} e^{-\frac{1}{2}t^2} |e^{-\frac{1}{2}t^2(\beta_T - 1)} - 1|\, dt > \sqrt{2\pi}\, (e^{r_1^2(T)} - 1)\}$$

$$\leqq \mathsf{P}_{\theta_0}^T\{\int_{|t| \leqq Z(T)} e^{-t^2/2} |(e^{|\beta_T - 1|\frac{t^2}{2}} - 1)|\, dt > \sqrt{2\pi}\, (e^{r_1^2(T)} - 1)\}$$

(since $|1 - e^{-x}| \leqq e^{|x|} - 1$ for all x)

$$\leqq \mathsf{P}_{\theta_0}^T\{\sqrt{2\pi}\, (e^{|\beta_T - 1|\frac{Z^2(T)}{2}} - 1) > \sqrt{2\pi}\, (e^{r_1^2(T)} - 1)\}$$

$$= \mathsf{P}_{\theta_0}^T\{|\beta_T - 1| > 2 r_1^2(T)\, r_2^2(T)\}.$$

Proof of Lemma 3.3.

$$P^T_{\theta_0}\{|\int_{|t|\leqq Z(T)} e^{-\frac{1}{2}\beta_T t^2}\left(p\left(\hat{\theta}_T+\frac{t}{Q(T)}\right)-p(\hat{\theta}_T)\right)dt|>2C_1 r_2(T)e^{r_1^2(T)}\}$$

$$\leqq P^T_{\theta_0}\{\int_{|t|\leqq Z(T)} e^{-\frac{1}{2}\beta_T t^2}|p\left(\hat{\theta}_T+\frac{t}{Q(T)}\right)-p(\hat{\theta}_T)|\,dt>2C_1 r_2(T)e^{r_1^2(T)}\}$$

$$\leqq P^T_{\theta_0}\{C_1 r_2(T)\int_{|t|\leqq Z(T)} |t|e^{-\frac{1}{2}\beta_T t^2}dt>2C_1 r_2(T)e^{r_1^2(T)}\}$$

(by assumption (A_2))

$$\leqq P^T_{\theta_0}\{C_1 r_2(T)\int_{|t|\leqq Z(T)} |t|e^{-\frac{1}{2}t^2(1-|\beta_T-1|)}dt>2C_1 r_2(T)e^{r_1^2(T)}\}$$

$$=P^T_{\theta_0}\{C_1 r_2(T)e^{-\frac{1}{2}Z^2(T)|\beta_T-1|}\int_{|t|\leqq Z(T)} |t|e^{-\frac{1}{2}t^2}dt>2C_1 r_2(T)e^{r_1^2(T)}\}$$

$$\leqq P^T_{\theta_0}\{2C_1 r_2(T)e^{\frac{1}{2}Z^2(T)|\beta_T-1|}>2C_1 r_2(T)e^{r_1^2(T)}\}$$

$$=P^T_{\theta_0}\{|\beta_T-1|>2r_1^2(T)\,r_2^2(T)\}.$$

Proof of Lemma 3.4. For T large,

$$P^T_{\theta_0}\{\int_{|t|>Z(T)} e^{-\frac{1}{2}\beta_T t^2}p\left(\hat{\theta}_T+\frac{t}{Q(T)}\right)dt>r_3(T)e^{r_1^2(T)}\}$$

$$\leqq P^T_{\theta_0}\{\int_{|t|>Z(T)} e^{-(t^2/2)(1-|\beta_T-1|)}p\left(\hat{\theta}_T+\frac{t}{Q(T)}\right)dt>r_3(T)e^{r_1^2(T)}\}$$

$$+P^T_{\theta_0}\{|\beta_T-1|>1\}$$

$$\leqq P^T_{\theta_0}\{\int_{|u|>Z(T)/Q(T)} e^{-(Z^2(T)/2)(1-|\beta_T-1|)}Q(T)\,p(\hat{\theta}_T+u)\,du>r_3(T)e^{r_1^2(T)}\}$$

$$+P^T_{\theta_0}\{|\beta_T-1|>1\}$$

(By assumption (A_6))

$$\leqq P^T_{\theta_0}\{r_3(T)e^{|\beta_T-1|\frac{Z^2(T)}{2}}>r_3(T)e^{r_1^2(T)}\}+P^T_{\theta_0}\{|\beta_T-1|>1\}$$

$$=P^T_{\theta_0}\{|\beta_T-1|>2r_1^2(T)\,r_2^2(T)\}+P^T_{\theta_0}\{|\beta_T-1|>1\}.$$

Proof of Lemma 3.5. For T large,

$$\mathsf{P}_{\theta_0}^T\{|\int_{|t|\leqq Z(T)} t\zeta_T(t)|\, p\left(\hat{\theta}_T+\frac{t}{Q(T)}\right)-p(\hat{\theta}_T)\,|\mathrm{d}t|>\sqrt{2\pi}\; C_1 r_2(T)\mathrm{e}^{r_1^2(T)}\}$$

$$\leqq \mathsf{P}_{\theta_0}^T\{|\int_{|t|\leqq Z(T)} t^2\zeta_T(t)\, C_1 r_2(T)\,\mathrm{d}t|>\sqrt{2\pi}\; C_1 r_2(T)\mathrm{e}^{r_1^2(T)}\}$$

(By assumptions (A_2) and (A_6))

$$\leqq \mathsf{P}_{\theta_0}^T\{\int_{|t|\leqq Z(T)} t^2\mathrm{e}^{-\frac{1}{2}t^2(1-|\beta_T-1|)}\mathrm{d}t>\sqrt{2\pi}\;\mathrm{e}^{r_1^2(T)}\}$$

$$\leqq \mathsf{P}_{\theta_0}^T\{\mathrm{e}^{|\beta_T-1|\frac{Z^2(T)}{2}}\int_{-\infty}^{\infty} t^2\mathrm{e}^{-\frac{1}{2}t^2}\,\mathrm{d}t>\sqrt{2\pi}\;\mathrm{e}^{r_1^2(T)}\}$$

$$=\mathsf{P}_{\theta_0}^T\{|\beta_T-1|>2r_1^2(T)\,r_2^2(T)\}.$$

Proof of Lemma 3.6.

$$\mathsf{P}_{\theta_0}^T\{|\int_{|t|\leqq Z(T)} p(\hat{\theta}_T)\,t\,\zeta_T(t)\,\mathrm{d}t|>2p(\hat{\theta}_T)\mathrm{e}^{r_1^2(T)}\}$$

$$\leqq \mathsf{P}_{\theta_0}^T\{\int_{|t|\leqq Z(T)} |t|\,\mathrm{e}^{-\frac{1}{2}t^2(1-|\beta_T-1|)}\mathrm{d}t>2\mathrm{e}^{r_1^2(T)}\}$$

$$\leqq \mathsf{P}_{\theta_0}^T\{\mathrm{e}^{|\beta_T-1|\frac{Z^2(T)}{2}}\int_{-\infty}^{\infty} |t|\,\mathrm{e}^{-\frac{1}{2}t^2}\,\mathrm{d}t>2\mathrm{e}^{r_1^2(T)}\}$$

$$=\mathsf{P}_{\theta_0}^T\{|\beta_T-1|>2r_1^2(T)\,r_2^2(T)\}.$$

Proof of Lemma 3.7. For T large,

$$\mathsf{P}_{\theta_0}^T\{|\int_{|t|>Z(T)} t\zeta_T(t)\,p\left(\hat{\theta}_T+\frac{t}{Q(T)}\right)\mathrm{d}t|\; C_2 r_4(T)\mathrm{e}^{r_1^2(T)}\}$$

$$\leqq \mathsf{P}_{\theta_0}^T\{\int_{|t|>Z(T)} |t|\,\mathrm{e}^{-(t^2/2)(1-|\beta_T-1|)}\,p\left(\hat{\theta}_T+\frac{t}{Q(T)}\right)\mathrm{d}t>C_3 r_4(T)\mathrm{e}^{r_1^2(T)}\}$$

$$=\mathsf{P}_{\theta_0}^T\{\int_{|u|>Z(T)/Q(T)} |u|\;p(\hat{\theta}_T+u)\,\mathrm{d}u\;Q^2(T)\mathrm{e}^{-Z^2(T)/2(1-|\beta_T-1|)}>$$

$$>C_2 r_4(T)\mathrm{e}^{r_1^2(T)}\}+\mathsf{P}_{\theta_0}^T\{|\beta_T-1|>1\}$$

$$\leqq \mathsf{P}_{\theta_0}^T\{C_2 r_4(T)\mathrm{e}^{|\beta_T-1|\frac{Z^2(T)}{2}}>C_2 r_4(T)\mathrm{e}^{r_1^2(T)}\}+\mathsf{P}_{\theta_0}^T\{|\beta_T-1|>1\}$$

(By assumptions (A_3) and (A_6))

$$=\mathsf{P}_{\theta_0}^T\{|\beta_T-1|>2r_1^2(T)\,r_2^2(T)\}+\mathsf{P}_{\theta_0}^T\{|\beta_T-1|>1\}.$$

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