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$C(K, X)$ as an M-ideal in $WC(K, X)$

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Abstract. In this paper we study the classes of Banach spaces X for which the space of continuous X-valued functions forms an M-ideal in the space of weakly continuous functions. We also study a lifting problem for weakly continuous functions.

Keywords. M-ideals; weakly continuous functions; Schur property.

1. Introduction

For an infinite dimensional Banach space X and an infinite compact Hausdroff space K let $C(K, X)$ denote the Banach space of X-valued continuous functions on K equipped with the supremum norm and let $WC(K, X)$ denote the space of functions that are continuous when X has the weak topology, equipped with the supremum norm. In a recent work Diestel *et al* [4] show that any element of $WC(K, X)$ is Bochner μ-integrable w.r.t. each regular Borel probability measure μ on K and thus identify $C(K, X)^*$ as a subspace of $WC(K, X)^*$ and obtain the decomposition

$$WC(K, X)^* = C(K, X)^* \oplus C(K, X)^{\perp}$$

via the restriction map
The question raised by them is "when is the sum above an l_1 sum?"

Let us recall that a subspace $J \subset X$ is said to be an M-deal if $J^{\perp} \oplus_1 N = X^*(l_1 - \text{sum})$ for some closed subspace N. Also if $J \subset X$ is such that $J^{\perp}$ is the kernel of a norm one projection P in X^* and if J is an M-ideal then $N = \text{Range } P$. Hence, in terms of M-structure theory (see [1] for all relevant definitions) an equivalent formulation is that "When is $C(K, X)$ an M-ideal in $WC(K, X)$?"

In this paper we look at this question and obtain some positive and negative results. Our first theorem disposes off the trivial situation and more.

Theorem 1. *The following statements are equivalent.*

1. *X has the Schur property*
2. *$C(K, X) = WC(K, X)$ for any K*
3. *For any K, every element of $WC(K, X)$ attains its norm on K*
4. *For some K, every element of $WC(K, X)$ attains its norm*

Proof. $1 \Rightarrow 2$: Let $f \in WC(K, X)$. Since X has the Schur property, $f(K)$ is a norm compact subset of X and hence on $f(K)$ weak and norm topologies coincide. Therefore f is norm continuous.

$2 \Rightarrow 3 \Rightarrow 4$ are clear.
$4 \Rightarrow 1$: Suppose X fails the Schur property. Assume w.l.o.g $\exists$ a $y_n \in X$, $\|y_n\| = 1$ and $y_n \to 0$ weakly.
Fix any $\alpha \in l^\infty$ with

$$\sup_n |\alpha(n)| = 1 > |\alpha(n)| \forall n$$

Let $x_n = \alpha(n) y_n$ then $x_n \to 0$ weakly. Fix a distinct sequence $k_n \in K$ and a pairwise disjoint sequence of open sets U_n with $k_n \in U_n$. Choose $f_n \in C(K)$, $0 \leqslant f_n \leqslant 1$ and $f_n(k_n) = 1$, $f_n = 0$ on $K \setminus U_n$.

Define $g: K \to X$ by $g(k) = \Sigma f_n(k) x_n$. Clearly g is well defined and $\|g\| = 1$. To see that g is weakly continuous, note that for any $x^* \in X^*$, $x^* \circ g = \Sigma x^*(x_n) f_n$ and since $x^*(x_n) \to 0$ the RHS is a continuous function. To obtain the required contradiction we now show that g fails to attain its norm on K. Suppose for some k_0, $\|g(k_0)\| = 1$. Let n_0 be such that $k_0 \in U_{n_0}$, then $g(k_0) = f_{n_0}(k_0) x_{n_0} = f_{n_0}(k_0) \alpha_{n0} y_{n0}$. Since $\|y_{n0}\| = 1$ and $f_{n0}(k_0) \leqslant 1$, $|\alpha_{n0}| < 1$, we get a contradiction. Therefore X has the Schur property.

In spite of the decomposition of $WC(K,X)^*$ the precise nature of its elements is far from being clear, hence the following corollary is of some interest. Let $\partial e X_1^*$ denote the extreme points of the dual unit ball and for any $k \in K$, let $\delta(k)$ denote the Dirac measure at k. It is well known that

$$\partial e C(K,X)_1^* = \{\delta(k) \oplus x^* : k \in K,\ x^* \in \partial e X_1^*\}.$$

Note that for any function $f: K \to X$,

$$(\delta(k) \oplus x^*)(f) = x^*(f(k)).$$

COROLLARY.

X has the Schur property iff

$$\partial e WC(K,X)_1^* = \{\delta(k) \oplus x^* : k \in K,\ x^* \in \partial e X_1^*\}.$$

Proof. Suppose X fails the Schur property and

$$\partial e WC(K,X)_1^* = \{\delta(k) \oplus x^* : k \in K,\ x^* \in \partial e X_1^*\}.$$

Let g be the function constructed during the proof of $4 \Rightarrow 1$ above. By the Hahn-Banach theorem,

$$1 = \|g\| = \Lambda(g) \text{ for some } \Lambda \in \partial e WC(K,X)_1^*.$$

By our assumption, $\Lambda = \delta(k) \oplus x^*$ for some $k \in K$ and $x^* \in \partial e X_1^*$.
Now $1 = \Lambda(g) = x^*(g(k)) \leqslant \|g(k)\| \leqslant 1$. Hence $\|g(k)\| = 1$ contradicting the fact that g fails to attain its norm.

From now on we assume that $C(K,X)$ is a proper subspace of $WC(K,X)$. For Banach spaces X, Y let us denote by $\mathscr{K}(X,Y)$ = space of compact operators, $\mathscr{F}(X,Y)$ = space of weakly compact operators and $\mathscr{L}(X,Y)$ = space of bounded operators.

For any index set Γ, let $\oplus_0^\Gamma X$ denote X-valued functions defined on Γ and vanishing at ∞ and let $\oplus_\infty^\Gamma X$ denote the space of X-valued bounded functions defined on Γ. Both these spaces are equipped with the supremum norm.

It is well-known that

$$\oplus_0^\Gamma X \text{ is an M-ideal in } \oplus_\infty^\Gamma X$$

for any Banach space X and index set Γ. Our first result is based on the following easy observation about M-ideals.

Observation. For Banach spaces X, Y, Z with $Z \subset Y \subset X$, if Z is an M-ideal in X then Z is an M-ideal in Y.

PROPOSITION 1.

For any discrete set Γ, and for any compact K,

$$C(K, c_0(\Gamma)) \text{ is an M-ideal in } WC(K, c_0(\Gamma)).$$

Proof. Let us note the canonical identification

$$C(K, c_0(\Gamma)) = \oplus_0^\Gamma C(K)$$

and

$$WC(K, c_0(\Gamma)) \subset \oplus_\infty^\Gamma C(K)$$

via evaluation at elements of Γ. Since $\oplus_0^\Gamma C(K)$ is an M-ideal in $\oplus_\infty^\Gamma C(K)$, using the observations mentioned above we get that $C(K, c_0(\Gamma))$ is an M-ideal in $WC(K, c_0(\Gamma))$.

In [10] the authors study a class of Banach spaces Y (the so called M_∞-spaces) with the property, $\mathscr{K}(X, Y)$ is an M-ideal in $\mathscr{L}(X, Y)$ for any Banach spaces X, our next result involves subspaces of such a space Y.

Theorem 2. *Let K be any compact extremally disconnected space and let X be a closed subspace of an M_∞-space then $C(K,X)$ is an M-ideal in $WC(K,X)$.*

Proof. Case i: Suppose K is the Stone-Čech compactification $\beta(\Gamma)$ of some discrete space Γ. It is easy to identify $C(\beta(\Gamma), X)$ as $\mathscr{K}(l^1(\Gamma), X)$ by restricting the functions to Γ and the same mapping allows one to identify $WC(\beta(\Gamma), X)$ as a closed subspace of $\mathscr{L}(l^1(\Gamma), X)$. In view of our observation, the result is proved once we note that $\mathscr{K}(l^1(\Gamma), X)$ is an M-ideal in $\mathscr{L}(l^1(\Gamma), X)$. That this is indeed the case can be proved by using arguments identical to the ones given in the proof of Proposition 2.9 in [8].

Case ii: Let K be any compact extremally disconnected space. By well known results in topology (see [6]), there exist a discrete set Γ, an into homeomorphism $\psi: K \to \beta(\Gamma)$ and a continuous onto map $\phi: \beta(\Gamma) \to K$ such that $\phi \circ \psi =$ identity on K. Let Φ denote the canonical isometry $f \to f \circ \phi$ taking function spaces on K isometrically into function spaces on $\beta(\Gamma)$ and let P denote the norm one projection $g \to \Phi(g \circ \phi)$ on the appropriate spaces.

By Case (i) $C(\beta(\Gamma), X)$ is an M-ideal in $WC(\beta(\Gamma), X)$. We shall verify the restricted 3 ball property for $C(K, X) \subset WC(K, X)$ to conclude that it is an M-ideal (see [1]). Let $f_i \in C(K, X)_1$, $1 \leqslant i \leqslant 3$, $g \in WC(K, X)_1$ and $\varepsilon > 0$. Since $C(\beta(\Gamma), X)$ is an M-ideal in $WC(\beta(\Gamma), X)$, applying the restricted 3 ball property to $\Phi(f_i)$, $\Phi(g)$ we get a

$h' \in C(\beta(\Gamma), X)$ such that $\| \Phi(g) + \Phi(f_i) - h' \| \leq 1 + \varepsilon \forall i$. Since P is a projection of norm one

$$\| \Phi(g) + \Phi(f_i) - P(h') \| \leq 1 + \varepsilon$$

$$\text{i.e. } \| \Phi(g) + \Phi(f_i) - \Phi(h' \circ \psi) \| \leq 1 + \varepsilon \text{ or}$$

$$\| g + f_i - h' \circ \psi \| \leq 1 + \varepsilon \forall i$$

Now $h' \circ \psi \in C(K, X)$. Hence $C(K, X)$ is an M-ideal in $WC(K, X)$.

Remark. Whether the above theorem is valid for any compact space K is not known. The properties of M_∞-spaces and their subspaces seem to indicate that this should be so.

Related to the above ideas is a question of lifting weakly compact sets. We are interested in the following two situations.

(a) X is a Banach space, $Y \subset X$ is a closed subspace and $\pi: X \to X/Y$ is the quotient map. Given a weakly compact set K in X/Y and $\varepsilon > 0$ there is a weakly compact set $\tilde{K}$ in X such that $\pi(\tilde{K}) = K$ and $\mathrm{Sup}_{\tilde{K}} \| \, \| \leq (1 + \varepsilon) \sup_K \| \, \|$.

(b) For a compact K and $f \in WC(K, X/Y)$, $\varepsilon > 0$ there is a $g \in WC(K, X)$ such that $\pi \circ g = f$ and $\| g \| \leq (1 + \varepsilon) \| f \|$.

Let us note that this is trivial when X/Y has the Schur property and a quotient map from a $l^1(\Gamma)$ onto a Banach space X does the lifting in (a) only when X has the Schur property (ie in general there is no weakly continuous cross-section map for π).

Examples

1) The authours of [13] show that if Y is a reflexive subspace of a Banach space X, π has lifting as in (a).
2) Let T denote the unit circle and H_0^1 the Hardy space in $L^1(T)$, then a classical theorem in analysis (see [11]) says that π has lifting as in (a). Note that the norm-restrictions are valid since $\tilde{K}$ is the weak closure of image of K under the nearest point cross-section map in this case.
3) X a Banach space, $Y \subset X$ be an L^1-predual. Consider $\pi: X^* \to X^*/Y^\perp$.

Let K be any compact set and let $f \in WC(K, Y^*)$. Define $T: Y \to C(K)$ by $T(y)(k) = f(k)(y)$. It is well known that T is a weakly compact operator and $\| T \| = \| f \|$. Since $Y \subset X$ and Y is an L^1-predual by Theorem 6.1 of [9], $\exists$ a weakly compact operator $\tilde{T}: X \to C(K)$, extending T and such that $\| \tilde{T} \| = \| T \| = \| f \|$. Now $g = (\tilde{T})^* \circ \delta$ (where $\delta: K \to C(K)^*$ the Dirac map) is the necessary weakly continuous lifting.

PROPOSITION 2.

Let X be a Banach space and let $Y \subset X$ be a closed subspace. $B \Rightarrow A$, and $A \Rightarrow B$ for compact extremally disconnected spaces. When B holds and if $C(K,X)$ is an M-ideal in $WC(K,X)$ then the same is true of $C(K, X/Y)$ in $WC(K, X/Y)$.

Proof. $B \Rightarrow A$ is clear.

Let K be compact, extremally disconnected. As in the proof of case [ii] of Theorem 2, get a discrete set Γ and mappings $\psi: K \to \beta(\Gamma)$, $\phi: \beta(\Gamma) \to K$ with $\phi \circ \psi = \text{identity}$.

Given $f \in WC(K,X/Y)$, $\varepsilon > 0$ since $f \circ \phi \in WC(\beta(\Gamma), X/Y)$ by property (A) $(f \circ \phi)(\beta(\Gamma))$ can be lifted and hence we can define a $g' \in WC(\beta(\Gamma), X) \ni \|g'\| \leqslant (1+\varepsilon)\|f\|$ and $\pi \circ g' = f \circ \phi$.
Put $g = g' \circ \psi$, $g \in WC(K,X)$,

$$\|g\| \leqslant (1+\varepsilon)\|f\|$$

and for $k \in K$,

$$\begin{aligned}\pi(g(k)) &= \pi(g'(\psi(k))) \\ &= f(\phi(\psi(k))) = f(k)\end{aligned}$$

so that $\pi \circ g = f$.
Proof of the rest of the proposition can be completed as in Theorem 2 using the "restricted 3-ball" characterization of M-ideals.

Problem. Does $A \Rightarrow B$?

Even though we do not have a complete description of situations when $C(K,X)$ is an M-ideal in $WC(K,X)$, our last proposition shows that they exhibit properties similar to c_0-spaces.

PROPOSITION 3.

If $C(K,X)$ is an M-*ideal in $WC(K,X)$ then $0 \in \overline{\partial e X_1^*}$ (Closure taken in the w^*-topology).*

Proof. Let us observe that

$$WC(K,X)_1^* = \overline{CO}(\delta(k) \oplus x^* : k \in K,\ x^* \in \partial e X_1^*)$$

(w^*-closed convex hull), since the functionals on the RHS determine the norm. If $WC(K,X)^* = C(K,X)^* \oplus_1 C(K,X)^\perp$ then choose

$$\Lambda \in C(K,X)^\perp \cap \partial e WC(K,X)_1^*.$$

By Milman's converse to the Krein-Milman theorem ([3]) $\Lambda \in \{\delta(k) \oplus x^* : k \in K, x^* \in \partial e X_1^*\}^{-w^*}$.
Let

$$\Lambda = \lim_\alpha \delta(k_\alpha) \oplus x_\alpha^*,\ k_\alpha \in K,\ x_\alpha^* \in \partial e X_1^*.$$

For any $x \in X$ considered as constant function in $C(K,X)$

$$0 = \Lambda(x) = \lim_\alpha x_\alpha^*(x)$$

Therefore $0 \in \overline{\partial e X_1^*}$

Negative results.
As before these observations are based on the corresponding facts known for operator spaces. An observation due to Saatkamp [12] in operator theory says that when $\mathscr{K}(X,Y) \neq \mathscr{L}(X,Y)$, $\mathscr{K}(X,Y)$ is not an M-summand in $\mathscr{L}(X,Y)$. Similar argument

works to show that $C(K,X)$ is not an M-summand in $WC(K,X)$. Hence, a standard procedure now to show that $C(K,X)$ is not an M-ideal is to notice when $C(K,X)$ has the intersection property (I.P, See [2], [5]) and then appeal to Theorem 4.3 of [2] to conclude that $C(K,X)$ is not an M-ideal in $WC(K,X)$.

It has been observed in [5] that when X has the I.P, $C(K,X)$ has the I.P and examples of Banach spaces X with I.P include $C(K)$ spaces, reflexive Banach spaces and more generally spaces with the Radon-Nikodým property, spaces with a non-trivial l^p-summand for $p<\infty$ (see [2]). In all these situations $C(K,X)$ is not an M-ideal in $WC(K,X)$.

For a dual space X^*, identifying

$$C(K,X^*)=\mathscr{K}(X,C(K)),\ WC(K,X^*)=\mathscr{F}(X,C(K))$$

when X^* fails the I.P, since X^* has a copy of c_0 (see [2])., arguments given during the Proof of Proposition 2.2 [8] work to show that if $\mathscr{K}(X,C(K))$ is an M-ideal in $\mathscr{F}(X,C(K))$ then $\mathscr{K}(l^1,C(K))$ is an M-ideal in $\mathscr{F}(l^1,C(K))$. But as we have noted before $\mathscr{K}(l^1,C(K))=C(\beta(N),C(K))$ and $\mathscr{F}(l^1,C(K))=WC(\beta(N),C(K))$ and since $C(K)$ has the I.P this cannot happen. So for no dual space X^*, $C(K,X^*)$ can be an M-ideal in $WC(K,X^*)$.

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Note

After submitting this paper for publication, I have received an expanded version of [4] from Professor J. Diestel. The authors of [14] now have also some answers to the M-ideal question. Their approach is different from the M-structure theoretic approach that I have taken. The purpose of this note is to illustrate this point. The following characterization of the class M_∞ appears in [16]. $X \in M_\infty$ iff there is a net K_α in the unit ball of $\mathscr{K}(X)$ such that

$$K_\alpha x \to x \forall x \in X \text{ and } K_\alpha^* x^* \to x^* \forall x^* \in X^*$$

and for any $\varepsilon > 0$ there is an $\alpha_0 \ni \forall \alpha > \alpha_0$.

$$\| K_\alpha x + (I - K_\alpha) y \| \leqslant (1+\varepsilon) \max \{ \|x\|, \|y\| \} \text{ for all } x, y \in X (\dagger).$$

Note that if a K_α satisfying (†) is a projection then one has

$$\|x\| \leqslant (1+\varepsilon)\max\{\|K_\alpha x\|, \|x - K_\alpha x\|\}$$

and

$$\|K_\alpha\| \leqslant 1 + \varepsilon$$

and

$$\|I - K_\alpha\| \leqslant 1 + \varepsilon.$$

Projections satisfying these conditions are called almost L^∞-projections (see [15]).

Now let us recall from [14] the definition of Schur approximation property. A Banach space X has the Schur approximation property (SAP for shor) if for any compact set $K \subset X$ and $\varepsilon > 0$ there is a projection P with range (P) having the Schur property such that

$$\|x - Px\| < \varepsilon \forall x \in K$$

$$\|P\| \leqslant 1 + \varepsilon, \|I - P\| \leqslant 1 + \varepsilon$$

and

$$\|x\| \leqslant (1+\varepsilon)\max\{\|Px\|, \|x - Px\|\}.$$

PROPOSITION 4.

Let K be a compact Hausdorff space and let $X \in M_\infty$. $C(K,X)$ is an M-*ideal in $WC(K,X)$.*

Proof. As before we shall verify the restricted 3-ball property.

Let $f \in WC(K,X)$, $f_i \in C(K,X)$ be in their respective unit balls and let $\varepsilon > 0$. Put

$K^{\sim} = \cup_{i=1}^{3} f_i(K)$ and use (†) to get a compact operator K_α such that

$$\| K_\alpha x + (I - K_\alpha) y \| \leqslant (1+\varepsilon) \max\{ \|x\| \|y\| \} \forall x, \quad y \in X$$

and

$$\| K_\alpha x - x \| < \varepsilon \forall x \in K^{\sim}.$$

Put $g = K_\alpha \circ f$. Clearly $g \in C(K, X)$.
For any $k \in K$

$$\begin{aligned} \|f_i(k) + f(k) - g(k)\| &\leqslant \|(I - K_\alpha)(f(k)) + K_\alpha(f_i(k)\| + \|f_i(k) - K_\alpha(f_i(k))\| \\ &\leqslant (1+\varepsilon)\max\{\|f(k)\|, \|f_i(k)\|\} + \varepsilon \leqslant 1 + 2\varepsilon \forall i \end{aligned}$$

Remark 1. It follows from the results in Chapter VI of [16] that for any $Y \in M_\infty$ of infinite dimension, every infinite dimensional subspace has an isomorphic copy of c_0. Consequently only finite dimensional subspaces here have the Schur property. So for a $X \subset Y$, $Y \in M_\infty$ the SAP for X already implies the bounded approximation property. However in Theorem 2 above we have made no assumptions about approximation property and such spaces X without the bounded approximation property are known to exist.

Remark 2. An argument similar to the one above gives an M-structure theoretic proof of "$C(K, X)$ is an M-ideal in $WC(K, X)$ when X has the SAP," which is Theorem 7 of [14].

PROPOSITION 5.

If a Banach space X has the Schur property then every M-*ideal in X is an M-summand and there are only finitely many M-summands.*

Proof. Key fact is that X and none of its subspaces have an isomorphic copy of c_0. So if $M \subset X$ is an M-ideal and infinite dimensional then since M has no copy of c_0, M must be an M-summand see [2]. Of course when M is finite dimensional it is already an M-summand, see [1].

An example of a space with the SAP mention in [4] is a c_0 direct sum of spaces with the Schur property. We now show

PROPOSITION 6.

If X has the SAP with M-projections then X is isometric to a c_0 direct sum of spaces with the Schur property.

Proof. Here we consider the maximal function module representation of X ([1]). Then the base spaces have the Schur property and in view of Proposition 5 M-projections correspond to multiplication operator by indicator functions of finite sets, one concludes that X is isometric to a c_0 direct sum of spaces with the Schur property.

Remark. The above formulation and proof are inspired by Proposition 6.5 and its proof in [17].