Linear Algebra Midterm Examination

M. Math First Year Program
Indian Statistical Institute
September 04, 2012

General Instructions: Write your answers legibly. Begin each problem on a separate page. If you are unable to solve some part of a problem, you may however use the result of that part to solve subsequent parts if necessary. Alloted time for this test is 2 h 30 m. Maximum marks = 35.

1. Short answer type problems. Justify your answers to true/false statements only.

(i) Fill in the following blanks using statements/expressions that involve 'trace(A)' only.

Let A be a matrix with rank$(A) = 1$. Then

(a) A is nilpotent if and only if ————————. **(1)**

(b) A is diagonalizable if and only if ————————. **(1)**

(c) $\det(A + I) =$ ————————. **(1)**

(ii) True/False: Let A be a $n \times n$ real matrix and 0 is the only eigenvalue of A over $\mathbb{R}$. Then the characteristic polynomial for A over $\mathbb{R}$ is x^n. **(1)**

(iii) True/False: A (3×3) matrix A satisfying $A^3 = A$ is diagonalizable. **(1)**

(iv) Fill in the blanks: Let A and B be two invertible matrices over a field $\mathbb{F}$, and satisfying $B^{-1}AB = 2A$. Then the characteristic of $\mathbb{F}$ is ———————— and trace$(A) =$ ————————. **(2)**

2. Let V and W be finite dimensional vector spaces over a field $\mathbb{F}$. Let $\mathscr{B}$ be an ordered basis for V with dual basis $\mathscr{B}^*$, and let $\mathscr{B}'$ be an ordered basis for W with dual basis $\mathscr{B}'^*$. Let T be a linear transformation from V into W, and let T^t denotes the corresponding transpose transformation from W^* into V^*. State neatly and prove the Theorem that relates the entries of the matrices $[T]_{\mathscr{B},\mathscr{B}'}$ and $[T^t]_{\mathscr{B}'^*,\mathscr{B}^*}$. (6).

3. For a matrix A, let $r(A)$ denote its rank. Using the fact that $r(AB) \leq \min\{r(A), r(B)\}$, show that similar matrices have the same rank. **(3)**

4. If two complex matrices A and B satisfy $AB - BA = A$, then show that $\det A = 0$. **(3)**

5. Matrices considered in this problem are over arbitrary field of characteristic $\neq 2$.

(a) Given a projection E, and its matrix $[E]$ with respect to some basis, find a matrix P such that $P^2 = I$. Such a matrix P is called an *involution.* **(2)**

(b) Show that if a matrix A can be expressed as a product of two involutions, say $A = PQ$, where P and Q are involutions, then $A \sim A^{-1}$. **(2)**

6. Let T be a nilpotent linear operator on some finite dimensional vector space V over a field $\mathbb{F}$ and $c_0, c_1, c_2, \cdots c_k$ be a scalars, where $c_0 \neq 0$. Prove that the operator given by $c_0 + c_1T + c_2T^2 + \cdots + c_kT^k$ is invertible. **(3)**

7. Let V be a finite dimensional vector space over a field $\mathbb{F}$, and S and T be linear operator on V.

   (a) Show that $\dim(\text{Im}(T) \cap \ker(S)) = \text{rank}(T) - \text{rank}(ST)$. (Hint: Try working with S', the restriction of S to $\text{Im}(T)$) **(3)**

   (b) Deduce from part (a) that $\text{rank}(ST) \geq \text{rank}(S) + \text{rank}(T) - n$. **(2)**

   (c) Let $A_1, A_2, \cdots A_k$ be $n \times n$ matrices, where $k < n$. Suppose it is also given that $\text{rank}(A_i) = n-1$ for $i = 1, 2 \cdots k$. Use part (b) and an induction argument to concluded that the product $A_1 \cdot A_2 \cdots A_k \neq 0$. **(4)**

Indian Statistical Institute
Semester 1 (2012-2013)
M. Math. 1st Year
Mid-semester Examination
Measure Theoretic Probability

Date and Time: 10.9.12, 2:15 - 4:45 Total Points: $6 \times 5 = 30$

Answers must be justified with clear and precise arguments. If you use any theorem/proposition proved in class state it explicitly. All the functions are assumed measurable and the integrals are Lebesgue integrals wrt the appropriate measure.

1. Give the definition of completeness of a measure and prove that Lebesgue measure on the real line is complete.

2. Let f_n be a sequence of nonnegative measurable functions on $(-\infty, \infty)$ such that $f_n \to f$ a.e., and suppose that $\int f_n \to \int f < \infty$. Then show that for each measurable set E we have $\int_E f_n \to \int_E f$. (The measure is Lebesgue measure.)

3. Show that $\lim \int_1^n [1 - (t/n)]^n \log t \, dt = \int_1^\infty e^{-t} \log t$.

4. Suppose μ is a finite measure on the real line, i.e. $\mu(-\infty, \infty) < \infty$ and $f \in L^p(\mu)$, i.e. $\int_{\mathbb{R}} |f|^p d\mu < \infty$ for some fixed $p > 1$. Then show that $f \in L^1(\mu)$.

5. State the Caratheodory Extension Theorem.

6. Show that $e^{-xt} \sin x$ is integrable over $(t, x) \in (0, \infty) \times (0, A)$ for any finite $A > 0$. Also evaluate the following limit rigorously:

$$\lim_{A \to \infty} \left| \int_0^\infty \frac{e^{-At} \cos A + te^{-At} \sin A}{1 + t^2} dt \right|.$$

INDIAN STATISTICAL INSTITUTE

MID-SEMESTER EXAMINATION : (2012-2013)
M. MATH I
ALGEBRA I
SEPTEMBER 12, 2012 MAXIMUM MARKS : 35 DURATION : $2\frac{1}{2}$ HOURS

---

(1) (a) Let R be a commutative ring with 1 and $R[X]$ denote the polynomial ring. Let I be an ideal of $R[X]$ such that $I \cap R = \{0\}$. Prove that there is a ring S and an element $u \in S$ such that R can be identified with a subring of S and that $R[u] \simeq R[X]/I$.
(b) Give an example of a ring R and $I \subset R$ such that I is a left ideal but not a right ideal.
(c) Give an example of a commutative ring R and a map $\phi : R \longrightarrow R$ such that ϕ is a ring morphism but not an R-module morphism.
(d) Give an example of a commutative ring R and a map $\phi : R \longrightarrow R$ such that ϕ is an R-module morphism but not a ring morphism. [4+2+2+2]

(2) Let $R = \mathbb{Z}[\sqrt{-5}] := \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$. Prove that
(a) R is not a PID;
(b) $(1 + \sqrt{-5})$ is an irreducible element of R which is not prime;
(c) $\alpha = 2(1 + \sqrt{-5})$ and $\beta = 6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ do not have a gcd in R. [3+3+4]

(3) Let R be an integral domain. Let $a, b \in R$ be non-zero. A *least common multiple* of a, b is an element e such that : (i) $a|e$ and $b|e$, (ii) if $a|e'$ and $b|e'$ then $e|e'$. Now assume that for any two non-zero elements x, y of R, a gcd of x, y exists in R and answer the following questions.
(a) For any non-zero $t \in R$, prove that $\gcd(ta, tb) = t\gcd(a, b)$.
(b) Prove that any two non-zero elements $a, b \in R$ have a least common multiple $\text{lcm}(a, b)$ which is unique up to a unit factor. Also, prove that $ab = \gcd(a, b)\text{lcm}(a, b)$.
(c) Prove that $\text{lcm}(a, b)$ is a generator for the unique largest principal ideal contained in $< a > \cap < b >$. [2+5+3]

(4) Let R be a commutative ring with 1 and $0 \longrightarrow M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \longrightarrow 0$ be an exact sequence of R-modules. Assume that there is an R-module morphism $u : M'' \longrightarrow M$ such that $\psi u = id_{M''}$. Prove that
(a) there is an R-module morphism $v : M \longrightarrow M'$ such that $v\phi = id_{M'}$
(b) $M = \phi(M') + u(M'')$ and $\phi(M') \cap u(M'') = \{0\}$. [5+5]

INDIAN STATISTICAL INSTITUTE

SEMESTRAL EXAMINATION : (2012-2013)
M. MATH II
ALGEBRA I
NOVEMBER 16, 2012   MAXIMUM MARKS : 65   DURATION : $3\frac{1}{2}$ HOURS

R will denote a commutative ring with unity

(1) Indicate True/False for the following statements with justification. [$3 \times 10 = 30$]

(a) Let R be a ring and M, N be R-modules. If $x \otimes y \in M \otimes N$ is zero then $x = 0$ or $y = 0$.

(b) An abelian simple group is cyclic of prime order.

(c) A subring of a UFD is also a UFD.

(d) Let R be a PID. Then any non-zero prime ideal of R is maximal.

(e) Given any positive integer n, there is an irreducible polynomial of degree n in $\mathbb{Q}[X]$.

(f) A local ring does not contain an idempotent other than 0 and 1.

(g) Let R be an integral domain and $a, b \in R$ be two non-zero non-unit elements. Let $\gcd(a, b) = d$. Then there exist $r, s \in R$ such that $d = ra + sb$.

(h) There is a group G such that $G/Z(G)$ is cyclic of order 3.

(i) The ring $\frac{\mathbb{Q}[X,Y]}{<Y^3 - X>}$ is a Noetherian integral domain.

(j) $\mathbb{Z}/3\mathbb{Z}$ is a projective $\mathbb{Z}$-module.

(2) Let R be a UFD and F be its quotient field. Let $f, g \in F[X]$ be such that $fg \in R[X]$. Then prove that the product of any coefficient of f with any coefficient of g belongs to R. [6]

(3) Let R be a ring and $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of R-modules. If both M', M'' are Noetherian R-modules then prove that M is also Noetherian. [6]

(4) Let R be a ring and M be a Noetherian R-module. Let $f : M \longrightarrow M$ be a surjective R-module morphism. Then prove that f is an isomorphism. [6] [P.T.O.]

(5) Let R be a ring and M, N be R-modules. Define the tensor product of M with N. Prove that the tensor product is unique up to isomorphism (no need to prove the existence of tensor product). [6]

(6) Let R be a ring and I, J be ideals of R such that $I + J = R$. Let M be an R-module. Prove that

$$\frac{M}{(IJ)M} \simeq \frac{M}{IM} \times \frac{M}{JM}.$$

[6]

(7) Let G be a group of order 105. Prove that G has a normal Sylow 5-subgroup and a normal Sylow 7-subgroup. [6]

(8) (a) Let G be a finite group, acting on the set of all its subgroups by conjugation. Let H be a subgroup of G. Then prove that the number of elements in the orbit C_H of H is equal to the index of the normalizer $N(H)$ in G. [2]

(b) Let $X = \{z = x + iy \in \mathbb{C} \mid y > 0\}$. Let $G = SL_2(\mathbb{R})$. For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z \in X$ let $\alpha z = \frac{az+b}{cz+d}$. Prove that $\alpha z \in X$. This defines an action of G on X (no need to prove this). Now prove that the isotropy subgroup of i is the group of matrices $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, with $\theta \in \mathbb{R}$. [4]

INDIAN STATISTICAL INSTITUTE
Semester Examination: 2012-2013, First Semester
M-Stat II and M-Math I
Subject: Set Theory and Topology/General Topology

Date: 26.11.12 Max. Marks 60 Duration: 3 Hours

Note: Answer all questions.
All the spaces considered are Hausdorff
unless otherwise stated.

1. a) Let X be locally compact and Y a subspace of X. If Y is open in X, show that Y is locally compact.

   b) Let $X := \mathbf{R}^{\mathbf{N}}$ with product topology, where $\mathbf{N}$ is the set of natural numbers. Show that X is not locally compact.

   [7+8]

2. Let $X := \mathcal{C}[0,1]$ with uniform metric ρ, where

$$\mathcal{C}[0,1] = \{f : [0,1] \to \mathbf{R},\ f \text{ continuous}\}$$

$$\text{and } \rho(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

   Define $f_n \in X$ by

$$\begin{aligned} f_n(x) &= nx,\ 0 \le x \le 1/n \\ &= 1,\ \ 1/n \le x \le 1 \end{aligned}$$

   a) Show that $S := \{f_n : n \ge 1\}$ is closed in X.

   b) Show that S is not compact. [6+6]

3. Let X be a second countable regular space. Let U be an open set in X.

   a) Show that U is an F_σ set.

b) Show that there exists a continuous function $f : X \to [0, 1]$ such that $f(x) > 0$ for all $x \in U$ and $f(x) = 0$ for all $x \notin U$.

[6+7]

4. Show that a connected normal space with more than one point is uncountable. [10]

5. Let $(G, \cdot)$ be a topological group.

a) Let $f : G \times G \to G$ be a function defined by $f(x, y) = x.y^{-1}$. Show that f is continuous.

b) Let $A \subset G$ be closed and $B \subset G$ be compact. Show that $A.B$ is closed.

c) Give an example to show that if B in (b) is closed but not compact, then (b) is not necessarily true.

[5+10+5]

Linear Algebra Final Examination

M. Math First Year Program
Indian Statistical Institute
November 30, 2012

General Instructions: Begin each problem on a *separate* page. You must show your *complete* work in order to obtain full credit. Alloted time for this test is **3** hours. Maximum marks = **55**.

1. Determine upto isomorphism the number of abelian groups of order $648 = 2^3 3^4$. [4]

2. Let T be a linear operator on a vector space V over a field $\mathbb{F}$, and let $Z(v\,; T)$ be the T-cyclic subspace generated by v. Show that $\dim Z(v\,; T) = 1$ if and only if v is a eigenvector for T. [4]

3. Let V be an inner product space over $\mathbb{C}$ or $\mathbb{R}$, and let T be a linear operator on V. Show that T preserves the inner product (that is, T is unitary) if and only if $||T(v)|| = ||v||$ for every $v \in V$. Here $||\cdot||$ denotes the norm defined by the inner product. [5]

4. Exhibit a 2×2 matrix B such that $B^2 = A$, where A is the following matrix. [6]

$$A = \begin{pmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix}.$$

5. Find all possible Jordan forms of a matrix A whose characteristic polynomial is $(x-2)^2(x-5)^3$. [5]

6. Let A be a self-adjoint $n \times n$ matrix. Prove that there is a real number c such that $cI + A$ is a positive matrix. [5]

7. Let V be an n-dimensional vector space, and let T be a *diagonalizable* operator on V.

(a) Show that, if T has a cyclic vector, then T has n distinct eigenvalues. [5]

(b) If T has n distinct eigenvalues, and if $\{v_1, \cdots, v_n\}$ is a basis of eigenvectors for T, show that $v = v_1 + \cdots + v_n$ is a cyclic vector for T. [5]

8. Suppose A is a *real symmetric* $n \times n$ matrix such that $X^t A X > 0$ for all nonzero $X \in \mathbb{R}^{n\times 1}$, where X^t is the transpose of X. Show that $X^* A X > 0$ for all nonzero $X \in \mathbb{C}^n$, where X^* denotes the adjoint of X. [5]

9. Let V be a vector space over $\mathbb{R}$, and let f be a symmetric bilinear form on V. Suppose W_1 is a subspace of V on which f is positive definite, and W_2 is a subspace of V on which f is negative definite. Show that W_1 and W_2 are independent subspaces. $\boxed{5}$

10. Let M be a finitely generated module over a principal ideal domain R, and let M_{tor} denote the set of all *torsion* elements of M. Recall that we had shown M_{tor} is a submodule of M.

(a) Use the conclusion of the structure theorem for finitely generated modules over principal ideal domains, to show that, if M is *torsion* free, then M is free. $\boxed{2}$

(b) Show that there exists a submodule N of M such that $M = M_{tor} \oplus N$. $\boxed{4}$

Indian Statistical Institute

Mid-semsetral examination : (2012-2013)

M. Math. I year

Differential Geometry I

Date : 22/2/13 Maximum marks : 40 Duration : 2 hours.

Answer ALL questions. Marks are indicated in bracket.

(1) Let S be an n-dimensional regular level hypersurface in $\mathbb{R}^{n+1}$ and $\gamma : I \to S$ be a smooth curve (where I is an open subinterval of $\mathbb{R}$). Prove that there exist n smooth vector fields $X_1, \ldots X_n$ along γ satisfying the following two conditions:

(a)for every $t \in I$, $\{X_1(t), \ldots, X_n(t)\}$ forms an orthonormal basis of the tangent space $T_{\gamma(t)}S$;

(b) given any smooth vector field $V(t)$ of unit norm along γ, it is parallel if and only if for each i, the angle between $V(t)$ and $X_i(t)$ is independent of t.

[10]

(2) Let S be as in Question (1). Given two smooth vector fields X and Y in S, defined on some open set U of $\mathbb{R}^{n+1}$ containing S. Define a new vector field $[X, Y]$ by setting

$$[X, Y](p) := (\nabla_{X(p)} Y)(p) - (\nabla_{Y(p)} X)(p),$$

for $p \in U$. Prove that for $[X, Y](p) \in T_pS$ for all $p \in S$, i.e. the restriction of $[X, Y]$ to S gives a smooth vector field in S. [15]

(3) Compute the signed curvature at every point of the oreinted planar curve $\{(x, y) \in \mathbb{R}^2 : f(x, y) \equiv x^2 - y^2 = 1,\ x > 0\}$, with the usual orientation given by $\frac{\nabla f}{\|\nabla f\|}$. [8]

(4) Prove that a smooth regular curve γ in $\mathbb{R}^3$ is a planar curve (i.e. the image of γ lies in a plane) if and only if the torsion function τ is identically zero. [7]

INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination: 2012-13 (Second Semester)

M. MATH. I YEAR
Algebra II

Date: 25.2.2013 Maximum Marks: 60 Duration: $3\frac{1}{2}$ Hours

Answer FIVE questions from Group A, TWO from Group B and ONE from Group C. Throughout the paper, k will denote a field.

GROUP A: Answer ANY FIVE questions.

1. Show that A_6 does not have any subgroup of prime index. [7]

2. Show that any group of order 70 is solvable. [7]

3. Show that $k(X)$ is not a finitely generated k algebra. [7]

4. Let L be a field extension of k. Examine if L can be isomorphic, as a k-algebra, to a proper subfield of L when $L|_k$ is (i) algebraic, (ii) transcendental. [7]

5. (i) Define a normal field extension.
(ii) Let α be a root of $f(X) = X^3 + X^2 - 2X - 1 \in \mathbb{Q}[X]$. Show that $\beta = \alpha^2 - 2$ is also a root of f. Examine whether $\mathbb{Q}(\alpha)$ is a normal extension of $\mathbb{Q}$. [7]

6. Let $f(X) = X^{12} - 1 \in \mathbb{Q}[X]$ and $L = \mathbb{Q}(i, \omega)$, where ω is a non-real cube root of unity. Show that $L = \mathbb{Q}(i\omega)$ and that L is a splitting field of $f(X)$ over $\mathbb{Q}$. Write down all the conjugates of $i\omega$ in L and show that one of them equals $cos(\frac{\pi}{6}) + i\ sin(\frac{\pi}{6})$. [7]

GROUP B: Answer ANY TWO questions.

1. (i) Show that if R is a subring of an algebraic extension L of k and $k \subset R$, then R is a field. Is the result true if $L|_k$ is not algebraic?
(ii) Let $A = k[X_1, \ldots, X_n]$. Suppose that $A/m \cong k$ as k-algebras for every maximal ideal m of A. Show that k is algebraically closed. [5+5=10]

2. (i) Let $f(X)$ and $g(X)$ be coprime polynomials in $k[X]$. Show that $f(X) - Yg(X)$ is irreducible in the polynomial ring $k[X, Y]$ and hence is irreducible in $k(Y)[X]$. Clearly state the result(s) that you use.
(ii) Let $L = k(t)$, where t is transcendental over k, and $F = k(u)$, where $u = f(t)/g(t)$, f and g coprime in $k[t]$. Show that L is algebraic over F and compute $[L : F]$. [5+5=10]

3. Let $f(X)$ be an irreducible polynomial over $\mathbb{Q}[X]$ of the form $X^3 + bX + c$ (where $b, c \in \mathbb{Q}$). Let E be a splitting field for f over $\mathbb{Q}$. Prove that $[E : \mathbb{Q}]$ is either 3 or 6 according as $-4b^3 - 27c^2$ is a square or not. [10]

GROUP C: Answer ANY ONE question.

1. Show that any simple group of order 60 is isomorphic to A_5. (You may assume that A_n is simple for $n \geq 5$.) [10]

2. Define an algebraic closure of a field. Prove that any field has an algebraic closure. [10]

INDIAN STATISTICAL INSTITUTE

2ND MID-SEMESTRAL EXAMINATION (2012–13)

M.MATH I YEAR

COMPLEX ANALYSIS

Date : 28.02.2013 Maximum Marks : 60 Time : $2\frac{1}{2}$ hours

---

(1) Let Ω be an open connected set in $\mathbb{C}$ and u be a harmonic function on Ω (i.e twice continuously differentiable function such that $u_{xx} + u_{yy}$ is identically 0 on Ω). Let (r, θ) be the polar co-ordinates of the point $z = x + iy$. Show that the function

$$r\frac{\partial u}{\partial r} - i\frac{\partial u}{\partial \theta}$$

is holomorphic on $\Omega \setminus \{0\}$. [10]

(2) Find where the series

$$\sum_{n=1}^{\infty} \frac{z^n}{1 + z^{2n}}$$

converges and determine where the sum is holomorphic and justify your answer. [10]

(3) Let f be an entire function and suppose there exist $M, R > 0$ and $n \in \mathbb{N}$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Show that f is a polynomial of degree $\leq n$. [10]

(4) Suppose f is a non-vanishing continuous function on $\bar{\mathbb{D}}$ that is holomorphic on $\mathbb{D}$. Prove that if

$$|f(z)| = 1 \text{ whenever } |z| = 1,$$

then f is constant. [10]

(5) Show that if $f : \mathbb{C} \to \mathbb{C}$ is a continuous function such that f is analytic off $[-1, 1]$, then f is an entire function. [10]

(6) Compute

$$\int_{|z|=1} \frac{1}{z(2\bar{z}^2 + 3)^2(\bar{z}^4 + 2)^3} dz.$$

[10]

(7) Let Ω be an open connected set in $\mathbb{C}$ and f_n be a sequence of holomorphic functions on Ω which are one-one. Suppose f_n converges to some non-constant function f uniformly on Ω. Show that f is an one-one holomorphic function on Ω. [10]

(8) Let f be an entire function and assume it has a nonessential singularity at ∞. Show that f is a polynomial. [10]

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination: 2012-13

M. Math. I Year **Functional Analysis**

Date: 04.03.2013 Maximum Marks: 40 Duration: 3 Hours

---

(1) Let (X, S, μ) be a finite measure space and $p_0 \leq p_1$. Show that $L^{p_0}(X) \subseteq L^{p_1}(X)$ and $\frac{1}{\mu(X)^{1/p_0}}\|f\|_{p_0} \leq \frac{1}{\mu(X)^{1/p_1}}\|f\|_{p_1}$. [5]

(2) Let X be the vector space of all functions $f : [0,1] \to \mathbb{R}$ such that f' is continuous. For $f \in X$, define

$$\|f\|_1 = |f(0)| + \|f'\|_\infty \quad \text{and} \quad \|f\|_2 = \|f\|_\infty + \|f'\|_\infty.$$

Show that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. [5]

(3) Let the operator $T : C[0,1] \to C[0,1]$ be defined by

$$Tf(x) = x^2 f(x) \quad \text{for all } f \in C[0,1] \text{ and all } x \in [0,1].$$

(a) Show that T is a bounded linear operator and find the value of $\|T\|$.

(b) If I is the identity operator on $C[0,1]$, then show that $\|I + T\| = 1 + \|T\|$. [6]

(4) Let X and Y be non-trivial normed linear spaces such that $B(X, Y)$ is a Banach space. Show that Y is a Banach space.

[Hint. Let $\{y_n\}$ be a Cauchy sequence in Y and $f \in X^*$ with $f \neq 0$. For $n \geq 1$, consider the operators $T_n : X \to Y$ defined by $T_n(x) = f(x)y_n$.] [6]

(5) Let Y be a subspace of the Banach space X such that $\{f(y) : y \in Y\} \subset \mathbb{C}$ is bounded for each $f \in Y^*$. Show that Y is a bounded subset of X, that is, the set $\{\|y\| : y \in Y\}$ is bounded. [6]

(6) Let $X = c_0$ so that $X^* = \ell^1$ and $X^{**} = \ell^\infty$. Define $f : \ell^1 \to \mathbb{C}$ by $f(x) = \sum_{n=1}^{\infty} x_n$ for $x = (x_1, x_2, x_3, \dots) \in \ell^1$.

(a) Show that f is continuous in the weak topology of ℓ^1.

(b) Show that f is not continuous in the weak* topology of ℓ^1.

[Hint. Use the sequence $\{x^{(n)}\}$ in ℓ^1, where $x_n^{(n)} = 1$ and $x_k^{(n)} = 0$ if $k \neq n$.] [7]

(7) Let M be a closed subspace of a reflexive Banach space X. Show that M is reflexive.

[Hint. For $\varphi \in M^{**}$, define $\tilde{\varphi} \in X^{**}$ by $\tilde{\varphi}(f) = \varphi(f|_M)$ for $f \in X^*$. Then, $\tilde{\varphi} = J_X(\tilde{x})$ for some $\tilde{x} \in X$, where J_V is the canonical mapping from a normed linear space V into V^{**}. Show that (i) $\tilde{x} \in M$ and (ii) $\varphi = J_M(\tilde{x})$. For (i), use Hahn-Banach theorem and to show (ii), prove that $\varphi(f) = f(\tilde{x})$ for all $f \in M^*$.] [10]

INDIAN STATISTICAL INSTITUTE
Semestral Examination: 2012-13 (Second Semester)

M. MATH. I YEAR
Algebra II

Date: 29.4.2013 Maximum Marks: 70 Duration: $3\frac{1}{2}$ Hours

Note: (1) Throughout the paper, k denotes a field.
(2) Attempt four questions from Group A, two from Group B and six from Group C.

GROUP A
Attempt ANY FOUR questions.
Each question carries 8 marks.

1. Let $L|_k$ be a cyclic field extension. For each divisor d of $[L:k]$, show that there exists a unique field F such that $k \subset F \subset L$ and $[F:k] = d$.

2. Let $f(X)$ be an irreducible polynomial of degree n over k. Let $g(X) \in k[X]$ and $h(X)$ be an irreducible factor of $f(g(X))$ in $k[X]$. Prove that the degree of $h(X)$ is divisible by n. [Hint: Consider field extensions.]

3. Let $L|_{\mathbb{Q}}$ be a Galois extension of degree 245. Prove that there exist subfields K and F of L such that $K|_{\mathbb{Q}}$ is a Galois extension of degree 49 and $F|_{\mathbb{Q}}$ is a Galois extension of degree 5.

4. Show that if N is an integer ≥ 2 and p is a prime number, then the polynomial $f(X) = X^5 - NpX + p$ cannot be solved by radicals over $\mathbb{Q}$.

5. Let $L = \mathbb{F}_4(t)$, where t is transcendental over $\mathbb{F}_4$. Let $u = t^4 + t$, $K = \mathbb{F}_4(u)$ and $f(X) = X^4 + X + u$. Show that f is irreducible in $K[X]$ and splits completely in $L[X]$. Deduce that L is a Galois extension of K of degree 4.

6. (i) Let $L|_k$ be a field extension and let $L_1(\subset L)$ and $L_2(\subset L)$ be finite Galois extensions of k with Galois groups G_1 and G_2 respectively. Let G be the Galois group of $L_1L_2|_k$. Show that (a) L_1L_2 is a finite Galois extension of k,
(b) G_1, G_2 are quotients of G and G is isomorphic to a subgroup of $G_1 \times G_2$, and
(c) $G \cong G_1 \times G_2$ if $L_1 \cap L_2 = k$.
(ii) Give an example of finite field extensions $F_1|_k$ and $F_2|_k$ such that $F_1 \cap F_2 = k$ but $[F_1F_2 : k] \neq [F_1 : k][F_2 : k]$.

[$8 \times 4 = 32$]

GROUP B
Attempt ANY TWO questions.
Each question carries 12 marks.

1. (i) Let L be a normal extension of k such that $L \setminus k$ does not contain any purely inseparable element over k. Show that L is a separable extension of k.

   (ii) Let $k = \mathbb{F}_2(u, v)$ where u and v are algebraically independent over $\mathbb{F}_2$. Let α be a root of the polynomial $X^2 + X + u \in k[X]$ and $L = k(\sqrt{v\alpha})$. Show that L is an inseparable extension of k but $L \setminus k$ does not contain any purely inseparable element over k.

2. Let ρ denote a primitive 7th root of unity and $L = \mathbb{Q}(\rho)$. Let $\alpha = 2\cos\left(\frac{2\pi}{7}\right)$ and $K = \mathbb{Q}(\alpha)$.

(i) Write down the minimal polynomial f of ρ over $\mathbb{Q}$. Determine the Galois group of f, giving an explicit description of the generators (as automorphisms).

(ii) Describe all subfields of L and show that one of them is K.

(iii) Write down the minimial polynomial of ρ over K and the minimal polynomial of α over $\mathbb{Q}$.

(iv) Show that the regular 7-gon is not constructible.

3. Let $f \in k[X]$ be irreducible and let $L|_k$ be a finite normal field extension. Show that if g, h are monic irreducible factors of f in $L[X]$, then there exists a k-automorphism σ of L such that $\sigma(g) = h$. Deduce that, in the prime factorisation of f in $L[X]$, all prime factors have the same degree and exponent. Give an example to show that the normality assumption is required.

4. (i) Let $L = k(\alpha)$, where α is algebraic over k. Show that there are only finitely many intermediary fields between k and L. [Hint: The number of intermediary fields is at most the number of all possible monic factors of the minimal polynomial of α over k.]

(ii) Let $L = \mathbb{F}_2(x, y)$ (where x and y are algebraically independent over $\mathbb{F}_2$) and $k = \mathbb{F}_2(x^2, y^2)$. Show that $[L : k] = 4$ but there are infinitely many intermediary fields between k and L.

(iii) Find an element c in $\mathbb{Q}(x, y)$ such that $\mathbb{Q}(x, y) = \mathbb{Q}(x^2, y^2)(c)$.

[$12 \times 2 = 24$]

GROUP C

State whether the following statements are TRUE or FALSE
with brief justification.
Attempt ANY SIX.
Each statement carries 3 marks.

(i) $\mathbb{C}$ cannot be expressed as the field of fractions of a UFD R ($\neq \mathbb{C}$).

(ii) Every element in a finite field of even order is a square.

(iii) The finite field $\mathbb{F}_{64}$ has a subfield isomorphic to the finite field $\mathbb{F}_{16}$.

(iv) Any finite group is isomorphic to the Galois group of some finite Galois extension over some field of characteristic zero.

(v) If $L|_F$ and $F|_k$ are Galois radical field extensions, then $L|_k$ is a Galois radical extension.

(vi) $\mathbb{Q}(\cos 20°)$ is a Galois extension of $\mathbb{Q}$.

(vii) The polynomial $X^2 + 1 \in \mathbb{F}_{625}[X]$ has a root in $\mathbb{F}_{625}$.

(viii) If ω_1 and ω_2 are primitive n-th roots of unity over a finite field k of characteristic p, where n is a prime $\neq p$, then ω_1 and ω_2 satisfy the same minimal polynomial over k.

[$3 \times 6 = 18$]

INDIAN STATISTICAL INSTITUTE

2ND SEMESTRAL EXAMINATION (2012–13)

M.MATH I YEAR

COMPLEX ANALYSIS

Date : May 02, 2013 Maximum Marks : 100 Time : $3\frac{1}{2}$ hours

---

(1) Let f be analytic in a neighbourhood of the closed unit disc $\bar{\mathbb{D}}$. Suppose that $|f(z)| < 1$ if $|z| = 1$. Show that the equation $f(z) - z = 0$ has unique solution in $\mathbb{D}$. If $|f(z)| \leq 1$ for $|z| = 1$, what can you say? [12]

(2) Use contour integration to evaluate $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$. [15]

(3) Prove that $\{e^{\frac{1}{z}} : 0 < |z| < \varepsilon\} = \mathbb{C} \setminus \{0\}$ for any $\varepsilon > 0$. [10]

(4) Construct a conformal map between the first quadrant $\{z \in \mathbb{C} : \mathrm{Re(z)} > 0 \text{ and } \mathrm{Im(z)} > 0\}$ on to the lower-half of the unit disk $\{z \in \mathbb{D} : \mathrm{Im(z)} < 0\}$. [12]

(5) Let ρ be the Poincaré metric on the unit disc $\mathbb{D}$. Let $f : \mathbb{D} \to \mathbb{D}$ a conformal self map of the disc. Then show that f is an isometry of the pair $(\mathbb{D}, \rho)$ with the pair $(\mathbb{D}, \rho)$. [10]

(6) If $u : \Omega \to \mathbb{R}$ is a harmonic function on a connected open set $\Omega \subset \mathbb{C}$ and if there exists a point $z_0 \in \Omega$ such that $u(z_0) = \inf_{z \in \Omega} u(z)$, then show that u is constant on Ω. [10]

(7) (a) Show that $\frac{1}{|\Gamma(z)|}$ is not $O(e^{c|z|})$ for any $c > 0$ (where the notation $f(z) = O(g(z))$ means that there is a constant $C > 0$ such that $|f(z)| \leq C|g(z)|$).

(b) Show that there is no entire function F with $F(z) = O(e^{c|z|})$ that has simple zeros at $z = 0, -1, \ldots, -n, \ldots$ and that vanishes nowhere else. [8 + 7 = 15]

(8) Prove that every automorphism of the upper half plane $\mathbb{H}$ takes the form of $\frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Conversely every map of this form is an automorphism of $\mathbb{H}$. [15]

(9) Suppose f is entire and never vanishes, and that none of the higher derivatives of f ever vanish. Prove that if f is also of finite order, then $f(z) = e^{az+b}$ for some constants a and b. [12]

(10) If $\{a_n\}$ and $\{b_n\}$ are two disjoint sequences having no finite limit points, then show that there exists a meromorphic function in $\mathbb{C}$ that vanishes at $\{a_n\}$ and has poles exactly at $\{b_n\}$. [12]

Indian Statistical Institute

Semsetral examination : (2012-2013)

M. Math. I year

Differential Geometry I

Date : 08.05.13 Maximum marks : 60 Duration : 3 hours.

Answer ALL questions. Marks are indicated in bracket which add up to 65. However, the maximum you can score is 60.

(1) Compute the Gaussian curvature at an arbitrary point of the ellipsoid S defined as a level surface in $I\!R^3$ by

$$S = \{(x_1, x_2, x_3) : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1\},$$

where a, b, c are positive constants. [12]

(2) Let M be a smooth manifold of dimension n and let TM be the set $\{(x, v) : x \in M, v \in T_x M\}$. Prove the following:

(i) TM is a smooth manifold of dimension $2n$.

(ii) TM is orientable. [5+10=15]

(3) Let $S = \{(x_1, \dots, x_{n+1}) : f(x_1, \dots, x_{n+1}) = 0\}$ be a level surface, where $f : I\!R^{n+1} \to I\!R$ is a smooth function with $\nabla(f) \neq 0$ at all points of S. Prove that S is a smooth manifold of dimension n. [12]

(4) Prove or disprove with justification:

There can be a convex regular 4-surface embedded in $I\!R^5$ such that its Gauss-Kronecker curvature is -1 at every point. [6]

(5) Let S be a 2-surface embedded in $I\!R^3$, U be a subset of S which is open in the subspace topology of S inherited from $I\!R^3$ such that there is a smooth unit vector field X of S defined on U. For any smooth curve $\gamma : [a, b] \to U$ let $\hat{X}$ be the parallel vector field along α with $\hat{X}(a) = X(a)$, and define $\theta(\gamma)$ to be the total angle of rotation of $\hat{X}$ along γ relative to X. Assume furthermore that there is a smooth map ϕ from the unit disc to U with $d\phi$ having rank 2 at every point. For any $\epsilon \in (0, 1)$, denote by γ_ϵ the curve $\gamma_\epsilon(t) = \phi(\epsilon \cos t, \epsilon \sin t)$, $t \in [0, 2\pi]$ and let A_ϵ be the area of the set $\{\phi(r \cos t, r \sin t) : 0 \leq r \leq \epsilon, \ t \in [0, 2\pi]\} \subset U$. Prove that

$$K(p) = \lim_{\epsilon \to 0+} \frac{\theta(\gamma_\epsilon)}{A_\epsilon},$$

where $p = \phi(0, 0)$ and $K(p)$ denotes the Gaussian curvature of S at p. [20]

INDIAN STATISTICAL INSTITUTE
Semesteral Examination : 2012-13
M. Math. - First Year
Topology II

Date : 13. 05. 2013 Maximum Score : 65 Time :3 1/2 Hours

1. The paper carries 80 marks. You are free to answer all the questions. Maximum Score 65

2. You are free to use any theorem proved in the class. However, you must state any theorem that you use at least once in the answer script.

(1) If X is a path connected space, compute $\widetilde{H}_0(X)$. [5]

(2) Let CX denote the cone of a space X and $q : X \hookrightarrow CX$ the canonical inclusion map. Suppose $f : X \to Y$ is a continuous map such that there is a continuous map $g : CX \to Y$ satisfying $g \circ q = f$. For every $p \geq 0$, compute $H_p(f) : H_p(X) \to H_p(Y)$. [8]

(3) For every continuous map $f : S^{2n} \to S^{2n}$, show that there exists a $x \in S^{2n}$ such that $f(x) = x$ or $f(x) = -x$. [8]

(4) Compute all homology groups of $\mathbb{T}\#\mathbb{R}P^2$, the manifold sum of the Torus and the real projective plane $\mathbb{R}P^2$. [12]

(5) If $f, g : X \to Y$ are homotopic maps, show that they induce chain homotopic maps from the singular chain complex of X to that of Y. [15]

(6) If X is a connected n-manifold without boundary, $n > 1$, show that $X \vee S^n$ and X have isomorphic fundamental groups. [15]

(7) Show that for every $p \geq 0$,

$$H_p(X, A) = \widetilde{H}_p(CA \cup_i X),$$

where $A \subset X$ and $i : A \hookrightarrow CA$ is the canonical inclusion map from A into its cone CA. [15]

INDIAN STATISTICAL INSTITUTE

First Semestral Examination: 2012-13 (Backpaper)

M. Math. I Year **Functional Analysis**

Date: 25.02.13 Maximum Marks: 100 Duration: 3 Hours

---

(1) Let X be a normed space and $x, y \in X$ such that $\|x+y\| = \|x\| + \|y\|$. Show that $\|ax+by\| = a\|x\| + b\|y\|$ for all scalars $a, b \geq 0$. [6]

(2) Let X be a normed linear space. Show that X is a Banach space if and only if its unit sphere $B = \{x \in X : \|x\| = 1\|\}$ is a complete metric space under the induced metric $d(x,y) = \|x-y\|$. [8]

(3) Show that a proper closed subspace of a normed linear space is nowhere dense. [6]

(4) Show that a normed linear space is locally compact if and only if it is finite-dimensional. [6]

(5) On the space $C[0,1]$, consider the two norms

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)| \quad \text{and} \quad \|f\|_1 = \int_0^1 |f(x)| dx.$$

Show that the identity operator $I : (C[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_1)$ is continuous, onto, but not open. Why does not this contradict the open mapping theorem? [12]

(6) Let X be the vector space of all real-valued functions on $[0,1]$ that have continuous derivatives with the sup norm. Let $Y = C[0,1]$ with the sup norm. define $D : X \to Y$ by $Df = f'$. Show that D is an unbounded linear operator and the graph of D is closed. Why does not this contradict the closed graph theorem? [12]

(7) Let X be a normed linear space. A subset S of X is called a *fundamental set* if the span of S is dense in X. Show that a sequence $\{x_n\}_{n=1}^\infty$ in X converges weakly to x if and only if $\{\|x_n\|\}_{n=1}^\infty$ is bounded and $f(x_n) \to f(x)$ as $n \to \infty$ for every f of a fundamental set S of X^*. [8]

(8) If X is an inner product space and $x \in X$, show that $\|x\| = \sup_{\|y\|=1} |\langle x, y \rangle|$. [6]

(9) Let E be a nonempty subset of an inner product space X and span E denote the span of E. Show that $E^\perp = (\overline{E})^\perp = (\text{span } E)^\perp = (\overline{\text{span } E})^\perp$. [8]

(10) Let X be a Banach space and P be a projection, that is, $P^2 = P$. Let $Q = I - P$.

(a) Show that if P is nontrivial (that is, $P \neq 0, I$), then

$$R_\lambda(P) := (P - \lambda I)^{-1} = \tfrac{1}{1-\lambda} P - \tfrac{1}{\lambda} Q, \quad \text{if } \lambda \neq 0, 1.$$

(b) Let $S, T \in B(X)$ such that $ST = I$. Prove that $\sigma(TS) \subseteq \{0, 1\}$. [6]

(11) Let $g \in C[0,1]$ be a fixed function and let T be the bounded linear operator on $C[0,1]$ defined by

$$Tf(x) = g(x)f(x), \quad x \in [0,1].$$

(a) Show that if $\lambda \in \mathbb{C}$ and $\lambda \notin g([0,1])$, then $T - \lambda I$ is invertible.

(b) If $\lambda \in g([0,1])$, then show that ran $(T - \lambda I) \neq C[0,1]$. Conclude that $\sigma(T) = g([0,1])$. [12]

(12) Let $K \subseteq \mathbb{C}$ be an arbitrary nonempty compact set and $\{\lambda_n\}_{n=1}^\infty$ be a dense subset of K. Define the operator $T : \ell^2 \to \ell^2$ as follows.

$$T(x_1, x_2, x_3 \ldots) = (\lambda_1 x_1, \lambda_2 x_2. \lambda_3 x_3 \ldots).$$

Show that $\sigma(T) = K$ by proving the following.

(a) T is a bounded linear operator and λ_k's are its eigenvalues. Conclude that $K \subseteq \sigma(T)$.

(b) If $\lambda \in \mathbb{C} \setminus K$, then $T - \lambda I$ has a bounded inverse and hence $\lambda \notin \sigma(T)$. [12]

INDIAN STATISTICAL INSTITUTE

BACK PAPER EXAMINATION (2012–13)

M.MATH I YEAR

COMPLEX ANALYSIS

Date 26 June, 2013 Maximum Marks : 100 Time : $3\frac{1}{2}$ hours

---

(1) Let f be an entire function and suppose there exist $A, B, R, s > 0$ such that $|f(z)| \leq A|z|^s + B$ for $|z| > R$. Show that f is a polynomial of degree $\leq s$. [15]

(2) Evaluate $\int_0^\infty \frac{\cos 2x}{(1+x^2)^2} dx$ by the method of residues. [15]

(3) Let f be an entire function and assume it has a nonessential singularity at ∞. Show that f is a polynomial. [10]

(4) Let $\mathbb{D}$ be the open unit disc and $f : \mathbb{D} \to \mathbb{D}$ holomorphic, then show that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2} \text{ for all } z \in \mathbb{D}.$$

[10]

(5) Let $\mathbb{H}$ be the upper half plane and $F : \mathbb{H} \to \mathbb{C}$ be a holomorphic function that satisfies $|F(z)| \leq 1$ and $F(i) = 0$. Prove that

$$|F(z)| \leq \left|\frac{z - i}{z + i}\right| \text{ for all } z \in \mathbb{H}.$$

[15]

(6) If ρ is a metric on unit disc $\mathbb{D}$ such that every conformal map of the disc is an isometry of the pair $(\mathbb{D}, \rho)$ with the pair $(\mathbb{D}, \rho)$, then ρ is a constant multiple of the poincare metric. [10]

(7) Suppose f is entire and never vanishes, and that none of the higher derivatives of f ever vanish. Prove that if f is also of finite order, then $f(z) = e^{az+b}$ for some constants a and b. [15]

(8) Prove that every meromorphic function in $\mathbb{C}$ is the quotient of two entire functions. [10]

Indian Statistical Institute

Backpaper examination : (2012-2013)

M. Math. I year

Differential Geometry I

Date : 11.07.13 Maximum marks : 100 Duration : 3 hours.

Answer ALL questions. Each carries 20 marks

(1) Compute the principal curvatures of the surface S given by

$$S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 - x_3^2 = 1\}$$

at an arbitrary point.

(2) Prove that the 2-sphere in $\mathbb{R}^3$ is a smooth manifold of dimension 2.

(3) Let M, N be smooth manifolds and $f : M \to N$ be a smooth map such that $df|_p = 0$ for every $p \in M$. If M is connected, prove that f must be constant.

(4) Let $\phi : (0, \infty) \times \mathbb{R} \to \mathbb{R}^3$ be defined by $\phi(t, \theta) = (x(t), y(t)\cos\theta, y(t)\sin\theta)$, where $x(t) = \int_0^t \sqrt{1 - e^{-2s}}ds$, $y(t) = e^{-t}$. Compute the Gaussian curvature of the parametrized 2-surface given by the image of ϕ in $\mathbb{R}^3$ at an arbitrary point.

(5) Let p be a point of the 2-sphere $S^2 \subset \mathbb{R}^3$ and v, w be two unit vectors of the tangent space of S^2 at p. Prove that we can find a continuous, piecewise smooth curve $\alpha : [a, b] \to S^2$ (for some $a < b \in \mathbb{R}$) such that $\alpha(a) = \alpha(b) = p$ and we get w by parallel translation of v along α.

(Hint: you can assume that great circle arcs are geodesics of the sphere).