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On the Numerical Complexity of Short-Circuit Faults in Logic Networks

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Abstract —**The problem of estimating the number of all possible multiple short circuit faults in a network with a given number of lines is settled in this correspondence. A new combinatorial number, namely an associated Bell number $B'(r)$, which enumerates the number of possible partitions of a set $\{1, 2, \cdots, r\}$ with certain constraints, is introduced. This concept immediately resolves the counting problem of short-circuit or bridging faults in an electrical network. A related combinatorial problem is also discussed which shows that under some realistic model of circuit failure, the number of possible ways the network can malfunction is closely connected to the Fibonacci sequence.**

Index Terms —**Bell numbers, bridging faults, Fibonacci numbers, logic networks, short-circuit faults, Stirling numbers.**

I. INTRODUCTION

Apart from the standard stuck-at variety of faults, the occurrence of short-circuit or bridging faults is one of the most common phenomena in digital circuits, particularly in the MOS LSI environment, and therefore their detection plays a significant role in digital logic testing. However, in contrast to the well-formalized methodology of devising a complete test set for detecting stuck-at faults existing nowadays, the bridge fault-detection procedure is still in its infancy, is almost incomplete even from a theoretical viewpoint, and is known only for either a restricted type of bridging faults or for a special class of networks. For instance, methods described by Roth [1], Friedman [2], and Flomenhoft [3] for testing short-circuit faults are based on the detection of an individual bridging fault. Mei [4], on the other hand, considered the detection of a class of short-circuit faults particularly at the input level of the network and some feedback bridging faults. More recently, methods for detecting such faults in two-level logic and unate networks have been reported in [5]. The complexity embedded in the problem of test generation can be attributed to mainly two factors, namely: 1) the astronomical multiplicity of possible bridging faults in a logic circuit, and 2) deep-rooted influence of a short-circuit fault on the functional behavior of the network which is further aggravated by its manifold dependence on the network topology.

The motivation behind this correspondence is to devise an algebraically closed formula that can be used to estimate the number of all possible multiple short-circuit faults in an arbitrary network having k lines. The estimation is of prime importance in appreciating the numerical complexity of possible faulty situations under bridging faults in digital networks for which any precise method of diagnosis is still awaited.

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II. THE NUMERICAL COMPLEXITY OF BRIDGING FAULTS

Clearly, the numbers of possible single and multiple stuck-at faults occurring in a logic circuit having k lines are $2k$ and $(3^k - 1)$, respectively. In contrast, the number of bridging faults involving only two lines at every fault instance (i.e., single bridging faults) is alone $\binom{k}{2}$, and it has been pointed out by Mei [4] and Hayes [12] that if all multiple bridging faults are counted, the number is related to a combinatorial function analogous to the Stirling number of the second kind, which grows much faster than $k^{k/2}$. In the following sections an algebraically closed formula that can be used to estimate the number of all possible multiple short-circuit faults in a circuit with a given number of lines is derived. Moreover, counting of such numbers under a realistic assumption is also considered.

A. Combinatorial Formulation of the Problem

It may be noted that for a bridging fault between two lines h and m in a network, if h happens to be a fan-out stem or a fan-out branch line, then all lines emanating from the parent stem line would also be logically involved in the fault. In the context of the counting problem, therefore, every fan-out stem with its associated branch lines can be thought of as a single line. Considering this fact, we assume that the given network consists of k lines, say $h_1, h_2, \cdots, h_k$. The possible configurations of short-circuit faults can now be phrased as follows.

Case 1—Multiple Bridging Faults of Multiplicity s, $s \geqslant 2$: This is used to denote the situation where s lines are all shorted together and is represented by an unordered set

$$M_s\colon\ (h_{i_1}, h_{i_2}, \cdots, h_{i_s}).$$

In particular for $s = 2$, i.e., when only two lines in the network are shorted together, a single bridging fault is said to occur. Note that the case $s = 1$ does not make any sense since for a short circuit fault to occur in a physical network, at least two lines need to be involved.

Case 2—Multiple Group Bridging Faults (M'_q) of Multiplicity q: It is used to denote the case where q disjoint groups of multiple bridging faults $M_{s_1}, M_{s_2}, \cdots, M_{s_q}$ of multiplicities $s_1, s_2, \cdots, s_q$, respectively, are simultaneously present and is represented by an unordered set

$$M'_q\colon\ (M_{s_1}, M_{s_2}, \cdots, M_{s_q})$$

where

$$\forall\ (s_i, s_j), i \neq j, M_{s_i} \cap M_{s_j} = \emptyset\ \text{(null)}.$$

Since we need at least two lines to make a short-circuit fault meaningful, in a network with k lines one must have: $1 \leqslant q \leqslant \lfloor k/2 \rfloor$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. In particular, a multiple bridging fault is a multiple group bridging fault of multiplicity $q = 1$.

The requirement $M_{s_i} \cap M_{s_j} = \emptyset$ follows from the fact that bridging faults induce an equivalence relation on the set of lines involved in the faulty instance. For example, if we consider a multiple group bridging fault: (h_1, h_2), (h_2, h_4, h_6) of multiplicity 2, then from the transitivity of the short-circuit fault behavior, the line h_1 would be logically shorted to both of h_4 and h_6 and in effect an equivalent multiple bridging fault (h_1, h_2, h_4, h_6) of multiplicity 4 is generated.

Definition 1: We define $N_s(k)$, $N_m(k)$, and $N_{mg}(k)$ to denote the number of all possible single, multiple, and multiple group bridging faults, respectively, in a network with k lines.

The following relation is self-evident: $N_{mg}(k) \geqslant N_m(k) \geqslant N_s(k)$. For example, for $k = 4$, $N_s(4) = 6$, $N_m(4) = 11$, and $N_{mg}(4) = 14$.

Clearly, $N_s(k) = \binom{k}{2}$ and $N_m(k) = 2^k - k - 1$. The difficult problem, however, is to enumerate $N_{mg}(k)$ which can now be combinatorially framed in the following fashion.

Consider a particular multiple group bridging fault M_q' of multiplicity q. We can now think of the multiplicity q as q nondistinct cells, and involved lines as distinct objects to be distributed in q cells, such that the content of each cell corresponds to a multiple bridging fault $M_s \in M_q'$. Since we need at least two lines to make a bridge fault meaningful, we have $1 \leq q \leq \lfloor k/2 \rfloor$ in a network with k lines, and therefore the problem of counting $N_{mg}(k)$ reduces to the enumeration of all possible ways of placing r distinct objects into $\lfloor k/2 \rfloor$ nondistinct cells for all r-element subsets of $(1, 2, \cdots, k)$, with $r \geq 2$, such that some cells may remain empty (which implicitly takes care of different values q can assume), and each nonempty cell contains at least two objects.

B. The Number $S'(r, n)$

To enumerate $N_{mg}(k)$ we would make use of the concept of associated Stirling number of the second kind[1] [6].

Definition 2: Let $S'(r, n)$, $r \geq 2n$ denote the number of partitions of a set R, $|R| = r$, into n blocks, all of cardinality ≥ 2. In fact, this number is called 2-associated Stirling number of the second kind [6].

Clearly, $S'(r, n)$ represents the number of distributions of r distinct objects into n nondistinct cells with no cells left empty and each cell containing at least two objects, the content of each cell being unordered in nature.

Lemma 1: The following is an algebraically closed relation for $S'(r, n)$:

$$S'(r,n) = \frac{1}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}(n-i-j)^{r-j}\frac{r!}{(r-j)!}.$$

Proof: By definition, $S'(r, n)$ is the number of equivalence relations with n classes on the set $R = \{1, 2, \cdots, r\}$, the cardinality of each class being ≥ 2. Alternatively, the problem can be viewed as finding the number of r-permutations of n distinct cells with repetitions such that each cell is included at least twice in every permutation and then dividing the result by $n!$ in order to take care of the indistinguishability of cells.

The exponential generating function [6]–[8] for $S'(r, n)$ is therefore given by

$$\begin{aligned}\emptyset(x) &= \frac{1}{n!}\{e^x - x - 1\}^n = \sum_r S'(r,n)\frac{x^r}{r!} \\ &= \frac{1}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(e^x - x)^{n-i} \\ &= \frac{1}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}x^j\sum_{p=0}^{\infty}\frac{(n-i-j)^p}{p!}x^p.\end{aligned}$$

A little change in running variables now yields

$$\emptyset(x) = \frac{1}{n!}\sum_{r=0}^{\infty}\frac{x^r}{r!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}(n-i-j)^{r-j}\frac{r!}{(r-j)!}.$$

Since $S'(r, n)$ happens to be the coefficient of $x^r/r!$ in the expansion of $\emptyset(x)$, the lemma follows. Q.E.D.

The formula presented in Lemma 1 for computing $S'(r, n)$ is, however, extremely cumbersome to manipulate. One can circumvent this by innovating a recurrence relation for $S'(r, n)$ which is reflected in Lemma 2.

[1] The ordinary Stirling number of the second kind, commonly denoted by $S(r, n)$ [7], is the number of partitions of a r-element set into n nonempty blocks.

C. Recurrence Relations for $S'(r, n)$

Lemma 2a: The number $S'(r, n)$ satisfies the following triangular recurrence. For $r \geq 2n$, $r \geq 2$,

$$S'(r,n) = nS'(r-1,n) + (r-1)S'(r-2,n-1)$$

with boundary conditions $S'(r, 0) = 0$ for all r; $S'(r, 1) = 1$; $S'(r, r') = 0$ for $r' > \lfloor r/2 \rfloor$; $S'(r, 2) = 2^{r-1} - r - 1$ for $r \geq 4$.

Proof: We will present a combinatorial proof. Clearly, $S'(r, 0) = 0$ since we cannot put r objects into no cells at all. $S'(r, 1) = 1$ because there is only one way of distributing r objects into one cell, and $S'(r, r') = 0$ for $r' > \lfloor r/2 \rfloor$ since in this case it is not possible to distribute r objects into r' cells with each cell containing at least two elements.

To prove $S'(r, 2) = 2^{r-1} - r - 1$, for $r \geq 4$, we can proceed as follows. Let C_1 and C_2 be two cells. The number of ways C_1 can be filled up with objects taken from an r-element set R is the number of all possible subsets of R, which is 2^r. The objects not selected for C_1 can be put in C_2. Removing the distinction between C_1 and C_2 (to take of indistinguishability of cells), we obtain $(2^r/2)$ possible distributions. If we now exclude those cases where any of the cells is either empty or contains exactly one element, we get the number $S'(r, 2)$. Therefore,

$$S'(r,2) = 2^{r-1} - r - 1.$$

To prove the recurrence we assume that the given r objects are $h_1, h_2, \cdots, h_r$. Consider now a fixed object, say h_1. The number of ways of partitioning the remaining set $(h_2, h_3, \cdots, h_r)$ into n subsets such that each subset contains at least two objects is clearly $S'(r - 1, n)$. We can now reinsert h_1 in each of these n cells thereby getting $nS'(r - 1, n)$ possible configurations. In all these distributions, the cardinality of all subsets containing h_1 would be at least 3. The number of distributions where h_1 belongs to a subset of cardinality exactly 2 can be obtained by picking up another object from the set $(h_2, h_3, \cdots, h_r)$, putting it in some cell together with h_1, and inserting the remaining $(r - 2)$ objects into $(n - 1)$ cells where each of these $(n - 1)$ cells contains at least two objects. Since the partner of h_1 can be chosen from the set $(h_2, h_3, \cdots, h_r)$ in $(r - 1)$ possible ways, the triangular recurrence for $S'(r, n)$ follows. Q.E.D.

Lemma 2b: For $r \geq 2n$, $r \geq 2$, the number $S'(r, n)$ satisfies the following vertical recurrence:

$$S'(r+1,n) = \sum_{p=1}^{r}\binom{r}{p}S'(r-p,n-1)$$

with the same boundary conditions.

Proof: We single out a particular object from $(r + 1)$ objects and place it in some arbitrary cell, say C_1. Let us now choose p other objects from the remaining r objects and place those in C_1. Note that p should be ≥ 1; otherwise, C_1 would contain less than 2 objects. The choice of p objects can be made in $\binom{r}{p}$ possible ways. The remaining $(r - p)$ distinct objects can be distributed in other $(n - 1)$ nondistinct cells with each cell containing ≥ 2 objects in $S'(r - p, n - 1)$ possible ways. Summing over all p's, for $1 \leq p \leq r$, the lemma follows. Q.E.D.

D. Unimodality and Asymptotic Analysis of the Number $S'(r, n)$

Since the number $N_{mg}(k)$, i.e., the number of all multiple group bridging faults, is connected to $S'(r, n)$ as we will see later on, it is interesting to analyze the global behavior of the combinatorial sequence $S'(r, n)$. In this context we recall some definitions.

Definition 3: A sequence $v_0, v_1, \cdots, v_n$ of real numbers is called unimodal if there exists an integer $M \geq 0$ such that [6], [9]

$$v_0 \leq v_1 \leq \cdots \leq v_{M-1} \leq v_M \geq v_{M+1} \geq \cdots \geq v_n.$$

The properties described below reveal some behavioral characteristics of $S'(r, n)$.

Property 1: The sequence $S'(r, n)$ of 2-associated Stirling numbers for fixed r, n-variable is unimodal and if $M(r)$ denotes maximum of $\{n: S'(r, n) \text{ maximum}\}$, then $M(r + 1) = M(r) + \lambda$ where $\lambda = 0$ or 1.

Proof: The proof follows immediately by making an induction on r. For $r = 2, 3$ the proof is self-evident. Now suppose that the claim holds good for $i \leq r$. Then

$$M(i) \leq M(j) \qquad \text{for } 1 \leq i \leq j \leq r.$$

Now let $2 \leq n \leq M(r)$. Then from the triangular recurrence for $S'(r, n)$, we have

$$S'(r + 1, n) - S'(r + 1, n - 1) = n[S'(r, n) - S'(r, n - 1)] + r[S'(r - 1, n - 1) - S'(r - 1, n - 2)] + S'(r, n - 1)$$

which is therefore positive by induction hypothesis. Therefore,

$$S'(r + 1, n) \geq S'(r + 1, n - 1) \qquad \text{for } 2 \leq n \leq M(r).$$

Let us now suppose $M(r) + 2 \leq n \leq \lfloor (r + 1)/2 \rfloor$. From the vertical recurrence, therefore,

$$S'(r + 1, n) - S'(r + 1, n - 1) = \sum_{j=1}^{r} \binom{r}{j} [S'(r - j, n - 1) - S'(r - j, n - 2)]$$

which is negative by the induction hypothesis and $M(j) \leq M(r)$ for $j \leq r$. Hence, $S'(r, n)$ is unimodal with $M(r + 1) = M(r)$ or $M(r + 1) = M(r) + 1$. Q.E.D.

Property 2a: For even r, asymptotically as $r \to \infty$, $S'(r, r/2) \simeq \sqrt{2}\,(r/e)^{r/2}$.

Proof: Let $r = 2t$. Then using Lemma 2a recursively and from the fact that $S'(r, r') = 0$ for $r' > \lfloor r/2 \rfloor$, we obtain

$$\begin{aligned} S'\left(r, \frac{r}{2}\right) &= S'(2t, t) = tS'(2t - 1, t) \\ &\quad + (2t - 1)S'(2t - 2, t - 1) \\ &= (2t - 1)[(2t - 3)S'(2t - 4, t - 2)] \\ &= \cdots \\ &= (2t - 1)(2t - 3)\cdots 5.3.1 \\ &= \frac{(2t)!}{2^t t!} \simeq \sqrt{2}\left(\frac{r}{e}\right)^{r/2} \end{aligned}$$

by Stirling's approximation of the factorials. Q.E.D.

Property 2b: For odd r, i.e., $r = 2t + 1$, asymptotically as $r \to \infty$,

$$S'\left(r, \left\lfloor \frac{r}{2} \right\rfloor\right) \simeq 0.2357(r - 1)^2[(r - 1)/1.4715]^{(r-1)/2}.$$

Proof:

$$\begin{aligned} S'(2t + 1, t) &= tS'(2t, t) + 2tS'(2t - 1, t - 1) \\ &= tS'(2t, t) + 2t[(t - 1)S'(2t - 2, t - 1) \\ &\quad + (2t - 2)S'(2t - 3, t - 2)] \\ &= \cdots \\ &= \frac{t(2t)!}{2^t(t)!} + \frac{2t(t - 1)(2t - 2)!}{2^{t-1}(t - 1)!} \\ &\quad + \frac{2t(2t - 2)(t - 2)(2t - 4)!}{2^{t-2}(t - 2)!} + \cdots \\ &= 2^t \sum_{p=0}^{t} \binom{t}{p} \binom{2t - 2p}{t - p - 1} \frac{p!(t - p + 1)!}{2^{2t-2p}} \\ &= 2^t \sum_{p=0}^{t} \binom{t}{p} \{{}^{(2t-2p)}P_{(t-p+1)} 2^{-(2t-2p)}\} p! \end{aligned}$$

where

$$\begin{aligned} {}^xP_y &= \frac{x!}{(x - y)!} \\ &= 2^t \sum_{p=0}^{t} \binom{t}{p} A_{t-p} B_p \end{aligned}$$

where

$$A_m = {}^{2m}P_{m+1} 2^{-2m} \quad \text{and} \quad B_m = m!.$$

The exponential enumerators for A_m and B_m are $(1/2)x(1 - x)^{-3/2}$ and $(1 - x)^{-1}$, respectively. Hence, by the rule of product of generating functions [7],

$$\sum_{t=0}^{\infty} S'(2t + 1, t)\frac{x^t}{t!} = 2^t \frac{x}{2}(1 - x)^{-3/2}(1 - x)^{-1}.$$

Therefore,

$$\begin{aligned} S'(2t + 1, t) &= \frac{(2t + 2)!\,t!}{3.2^{t+1}(t - 1)!(t + 1)!} \\ &\simeq t^{t+2} e^{-0.3068t} e^{-c} \end{aligned}$$

by Stirling's approximation of the factorials where $c = \log_e 6 - (5/2) \log_e 2$. Putting $t = (r - 1)/2$ and after a little simplification, this yields the desired result. Q.E.D.

From Properties 1, 2a, and 2b the next theorem follows.

Theorem 1: The asymptotic (as $r \to \infty$) maximum value of $S'(r, n)$ for a given r is at least $O((r/e)^{\lfloor r/2 \rfloor})$.

E. 2-Associated Bell Number

Definition 4: We define the number $B'(r)$ of all partitions of the set $R = \{1, 2, \cdots, r\}$, $r \geq 2$ such that each block of a partition is of cardinality ≥ 2, as 2-associated Bell number.[2] Clearly,

$$B'(r) = \sum_{n=1}^{\lfloor r/2 \rfloor} S'(r, n).$$

The following lemmas depict the behavior of $B'(r)$.

Lemma 3: The generating function for $B'(r)$ is given by

$$\begin{aligned} \sum_{r=0}^{\infty} B'(r)\frac{x^r}{r!} &= \exp(e^x - x - 1) \\ &= \sum_{r=0}^{\infty} \left(\frac{x^r}{r!}\right) \left\{ r! \sum_{\substack{p_2, p_3, \cdots, p_r \geq 0 \\ 2p_2 + 3p_3 + \cdots + rp_r = r}} \prod_{i=2}^{r} \frac{S_i^{p_i}}{2^{p_i}(p_i)!} \right\} \end{aligned}$$

where $S_i = 1/(i - 1)!$.

The proof of Lemma 3 can be easily obtained from the expansion of $\exp(e^x - x - 1)$.

Lemma 4: The sequence of 2-associated Bell numbers $B'(r)$ satisfies the following recurrence:

$$B'(r + 1) = \sum_{m=1}^{r} \binom{r}{m} B'(r - m) = \sum_{h=0}^{r-1} \binom{r}{h} B'(r)$$

with boundary conditions $B'(0) = 1$, $B'(1) = 0$, $B'(2) = 1$.

Proof: Let R' be the set $\{1, 2, \cdots, r, r + 1\}$ and let $Z(R')$ be the set of all partitions of R' where each block in every partition is of cardinality ≥ 2. We single out an arbitrary element say $y \in R'$ and consider the set $R = R' - \{y\}$. For $M \subset R$, $|M| \geq 1$, let $Z_M(R')$ be the set of all partitions of R' such that the block containing y is $\{y\} \cup M$. Since there is a bijection between $Z(R - M)$ and $Z_M(R')$, we have

$$Z(R') = \bigcup_{\substack{M \subset R \\ |M| \geq 1}} Z_M(R')$$

and therefore

[2]The ordinary Bell number $B(r)$ is the number of all partitions of the set R, $|R| = r$, [6]–[8]. The asymptotic study of $B(r)$ can be found in [11].

$$B'(r+1) = |Z(R')| = \sum_{m=1}^{r} \binom{r}{m} B'(r-m) = \sum_{h=0}^{r-1} \binom{r}{h} B'(h)$$

Q.E.D.

Table I shows values of $B'(r)$ for different r's obtained by using the above recurrence.

F. Enumeration of $N_{mg}(k)$

Theorem 2: The number of all possible multiple group bridging faults $N_{mg}(k)$ in a network with k lines is

$$N_{mg}(k) = B'(k+1) + B'(k) - 1 .$$

Proof: Note that in a network having r lines the multiplicity q of multiple group bridging faults lies within $1 \leq q \leq \lfloor r/2 \rfloor$. The number of ways of placing r distinct objects in $\lfloor r/2 \rfloor$ nondistinct cells with empty cells allowed but each nonempty cell containing at least two objects is now given by

$$S'(r,1) + S'(r,2) + \cdots + S'\left(r, \left\lfloor \frac{r}{2} \right\rfloor\right) = \sum_{n=1}^{\lfloor r/2 \rfloor} S'(r,n) = B'(r).$$

The set of r lines involved in a faulty instance out of k lines may be any of the total of $\binom{k}{r}$ possible choices for $2 \leq r \leq k$. Therefore,

$$\begin{aligned} N_{mg}(k) &= \sum_{r=2}^{k} \binom{k}{r} B'(r) \\ &= \sum_{r=0}^{k-1} \binom{k}{r} B'(r) + \binom{k}{k} B'(k) - \binom{k}{0} B'(0) \\ &= B'(k+1) + B'(k) - 1 . \end{aligned}$$

Q.E.D.

The following example projects some idea about the combinatorial explosion of the number $N_{mg}(k)$ as k gets large.

Example 1: Consider the enumeration of $N_{mg}(k)$ in circuits having i) 5 lines, ii) 10 lines, and iii) 15 lines.

i) $k = 5; N_{mg}(5) = B'(6) + B'(5) - 1 = 51$.

ii) $k = 10; N_{mg}(10) = B'(11) + B'(10) - 1 = 98\,253 + 17\,722 - 1 = 115\,974$.

iii) $k = 15; N_{mg}(15) = B'(16) + B'(15) - 1 = 1\,216\,070\,380 + 166\,888\,165 - 1 = 1\,382\,958\,544$.

In a practical situation, however, the occurrence of short-circuit faults in all possible ways is highly improbable, and in most cases, fault instances remain confined to adjacent lines in the network. The computation of multiple group bridging faults in such an environment is considered in the next section.

III. Combinatorial Formulation of the Problem in a Practical Environment

The most realistic model of short-circuit faults in a physical network is based on the assumption that all lines which are likely to be involved in a faulty situation are geometrically contiguous (for example, in parallel tracks of a printed circuit board). In such an environment we assume that the network has altogether k lines lying side by side and a fault instance can only affect an r-subset of contiguous lines for $2 \leq r \leq k$. The multiplicity q of any multiple group bridging fault lies within the range $1 \leq q \leq \lfloor r/2 \rfloor$ as before.

Combinatorially, the number of possible multiple group bridging faults involving r adjacent lines is equivalent to the number of partitioning the integer r into exactly q parts where order counts, for all values of q and where no partition contains any number less than 2.

Note that for a given q, there is a bijection between the set of all ordered partitions of r into q parts with each part being ≥ 2 and the set of all ordered partitions of the integer $(r - q)$ in q parts, each being ≥ 1. The cardinality of the latter set is [8]

$$\binom{r-q-1}{q-1}.$$

Therefore, the number of all multiple group bridging faults involving r contiguous lines will be (assuming each of these r lines is involved in every fault instance)

$$\sum_{q=1}^{\lfloor r/2 \rfloor} \binom{r-q-1}{q-1}.$$

Since we can choose r contiguous lines from a set of k contiguous lines in $(k - r - 1)$ possible ways, the total number of multiple group bridging faults $N'_{mg}(k)$ in a set of k contiguous lines where all lines affected in every fault instance are contiguous, is given by[3]

$$N'_{mg}(k) = \sum_{r=2}^{k} (k - r + 1) \sum_{q=1}^{\lfloor r/2 \rfloor} \binom{r-q-1}{q-1}.$$

The relation can be simplified by using some results of Fibonacci numbers [7], [8], which are defined by the well-known recurrence

$$F_k = F_{k-1} + F_{k-2} \qquad \text{with } F_0 = 0,\ F_1 = 1 .$$

Lemma 5:

$$\sum_{q=1}^{\lfloor r/2 \rfloor} \binom{r-q-1}{q-1} = F_{r-1} .$$

The proof immediately follows from the relation [10]

$$F_{r+2} = 1 + \sum_{j \geq 1} \binom{r-j+1}{j}.$$

Lemma 6:

$$\sum_{r=1}^{n} rF_r = nF_{n+2} - F_{n+3} + 2 .$$

Proof: Let S be the required sum. Then,

$$\begin{aligned} S &= F_1 + 2F_2 + 3F_3 + \cdots + nF_n \\ &= n \sum_{i=1}^{n} F_i - \sum_{r=1}^{n-1} \sum_{j=1}^{r} F_j = nF_{n+2} - F_{n+3} + 2, \end{aligned}$$

using a well-known relation [8]

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 .$$

Q.E.D.

Theorem 3:

$$N'_{mg}(k) = F_{k+3} - (k+2) .$$

Proof: We have

$$\begin{aligned} N'_{mg}(k) &= \sum_{r=2}^{k} (k - r + 1) \sum_{q=1}^{\lfloor r/2 \rfloor} \binom{r-q-1}{q-1} \\ &= \sum_{r=2}^{k} (k - r + 1) F_{r-1} \text{ (from Lemma 5)} \\ &= k(F_1 + F_2 + \cdots + F_{k-1}) \\ &\quad - (F_1 + 2F_2 + \cdots + (k-1)F_{k-1}) \\ &= k(F_{k+1} - 1) - [(k-1)F_{k+1} - F_{k+2} + 2] \\ &\quad \text{(from Lemma 6)} \\ &= F_{k+3} - (k+2) . \end{aligned}$$

Q.E.D.

[3]In this context note that in a set of k lines lying side by side with the above restriction that each "multiple bridging fault" can only involve contiguous lines but with the freedom that "all lines" involved in a "multiple group bridging fault" need not be contiguous, the number of all multiple group bridging faults comes out very easily to be $(2^{k-1} - 1)$.

TABLE I

r	0	1	2	3	4	5	6	7	8	9	10
$B'(r)$	1	0	1	1	4	11	41	162	715	3425	17 722

r	11	12	13	14	15	16
$B'(r)$	98 253	580 317	3 633 280	24 011 157	166 888 165	1 216 070 380

It is well known that [8] the Fibonacci number F_i is given by

$$F_i = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^i - \left(\frac{1-\sqrt{5}}{2}\right)^i\right]$$

$$\simeq \frac{1}{\sqrt{5}}(1.618)^i$$

as i gets large. Therefore,

$$N'_{mg}(k) \simeq \frac{1}{\sqrt{5}}(1.618)^{k+3} - (k + 2)$$

$$\leqslant 2^{k-1}$$

as k becomes large.

Example 2: Consider a circuit with 10 lines lying side by side. Assume any fault instance can only involve adjacent lines. The number of all multiple group bridging faults in this context $N'_{mg}(k)$ will therefore be

$$N'_{mg}(10) = F_{13} - (10 + 2) = 233 - 12 = 221.$$

IV. Conclusion

In this correspondence a measure of the number of all possible multiple bridging faults in a logic circuit is given. It has been shown that this number is related to the 2-associated Bell number $B'(k)$ which grows up more rapidly than that of $(k/e)^{\lfloor k/2 \rfloor}$. However, if fault instances are assumed to be confined within sets of adjacent lines then their numerical complexity comes out to be on the order of the $(k + 3)$th Fibonacci number F_{k+3}, which increases less than that of 2^{k-1}. Moreover, it is known that some undetectable bridging fault in a network can invalidate a valid stuck-at fault test set [13], which further aggravates the test generation problem. With large-scale integration, particularly in MOS LSI networks, the occurrence of bridging faults has got a substantial probability [14], and the stuck-at fault model becomes less and less sound. The inherent numerical complexity embedded in the short-circuit fault-detection problem will possibly render the test generation approach for bridging faults an infeasible one. The best alternative to cope with the testing problem of bridging faults will therefore be designing easily testable networks assuming a wider class of fault model including both stuck-at and bridging faults.

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Semisystolic Array Implementation of Circular, Skew Circular, and Linear Convolutions

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***Abstract* — Semisystolic array implementation of circular and linear convolutions in one and multidimensions are discussed. The common feature of the various architectures studied is the broadcasting of the input sequence to the cells of the array. In the case of circular convolutions, there is also circular communication between the cells. A circular convolution of period N can be calculated in N time steps whereas the response time for the computation of N outputs of linear convolution with finite weight and data vectors is also N time steps without initial delay.**

***Index Terms* — Convolution, FFT algorithms, parallel processing, semisystolic arrays, Toeplitz forms, VLSI.**

I. Introduction

Systolic and semisystolic array implementations of signal processing tasks promise to be of great significance in digital and optical signal processing because of simplicity, regularity, and parallelism [1], [2]. Circular and skew circular convolutions (CC and SCC) are basic building blocks in the computation of FIR filters, convolutions, and fast Fourier transforms [3]–[5]. They can be written as

$$\text{CC: } y(n) = \sum_{k=0}^{N-1} x(n - k)h(k) \bmod N \tag{1}$$

$$\text{SCC: } y(n) = \sum_{k=0}^{N-1} \operatorname{sgn}(k - n)x(n - k)h(k) \bmod N \tag{2}$$

where

$$\operatorname{sgn}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0. \end{cases} \tag{3}$$

Linear convolution is obtained if reduction modulo N in (1) is skipped.

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