Sankhyā : The Indian Journal of Statistics
2013, Volume 75-B, Part 1, pp. 1-15

On multivariate folded normal distribution

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Abstract

Folded normal distribution arises when we try to find out the distribution of absolute values of a function of a normal variable. The properties and uses of univariate and bivariate folded normal distribution have been studied by various researchers. We study here the properties of multivariate folded normal distribution and indicate some areas of applications.

AMS (2000) *subject classification.* Primary 62E15; Secondary 62H10.
Keywords and phrases. Folded normal distribution, multivariate distributions, multivariate folded normal distributions, process capability indices.

1 Introduction

Normal distribution has been widely used as the underlying distribution for many quality characteristics used in various industries. However Johnson (1963) has pointed out that, in many situations, even if the underlying distribution is normal, while collecting data, for example, on differences and deviations in measurements, often the algebraic sign of the data is irretrievably lost. The resulting observed variable no more follows a normal distribution- rather it follows a **folded normal distribution** (see Johnson, 1963 and King, 1988). Lin (2004) used folded normal distribution to study the magnitude of deviation of an automobile strut alignment. Leone, Nelson and Nottingham (1961) have mentioned some more applications of folded normal distribution specially when measuring straightness and flatness of any object.

The pdf of univariate folded normal distribution, as proposed by Leone et al. (1961), is given by

$$\begin{aligned} f_X(z) &= h_Z(z) + h_Z(-z) \\ &= \frac{1}{\sigma\sqrt{2\pi}}\left[\exp\left\{-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2\right\} + \exp\left\{-\frac{1}{2}\left(\frac{z+\mu}{\sigma}\right)^2\right\}\right],\ z > 0, \end{aligned}$$

where, 'X' follows univariate folded normal (UFN) distribution with mean μ_f and variance σ_f^2 and $Z \sim \mathrm{N}(\mu, \sigma^2)$ with $h_Z(.)$ being its pdf. Here,

$$\begin{aligned} \mu_f &= \sigma\sqrt{\frac{2}{\pi}}\exp\left(-\frac{\mu^2}{2\sigma^2}\right) + \mu\left[1 - 2\Phi\left(-\frac{\mu}{\sigma}\right)\right], \\ \sigma_f^2 &= \mu^2 + \sigma^2 - \left\{\sigma\sqrt{\frac{2}{\pi}}\exp\left(-\frac{\mu^2}{2\sigma^2}\right) + \mu\left[1 - 2\Phi\left(-\frac{\mu}{\sigma}\right)\right]\right\}^2 \end{aligned}$$

and $\Phi(.)$ is the cdf of univariate standard normal distribution. The subscript 'f' is used to distinguish the mean and variance of a folded normal distribution from that of normal distribution.

Elandt (1961) formulated a general expression for the r^{th} moment of univariate folded normal distribution. The author also proposed two methods of estimating the parameters μ and σ of the 'parent' normal distribution, viz., one based on the first and second raw and central moments of folded normal distribution and the other based on its third and fourth raw and central moments.

Many interesting results of the theory of statistical quality control owe their developments from the folded normal distribution. King (1988) made a thorough research regarding the possible situations where a folded distribution, especially a folded normal distribution, may arise. He put emphasis on some practical consequences encountered during process capability analysis in many common industrial processes. In fact, Lin (2004) has pointed out that folded normal process data is common in mechanical industries. Johnson (1963) discussed the use of CUSUM control chart when the underlying variable follows folded normal distribution while, Liao (2010) has proposed economic tolerance design for the folded normal data in manufacturing industries. Univariate folded normal distribution has also found useful applications in Lin (2004) and Lin (2005) while studying the properties of some univariate process capability indices (PCI).

Vannman (1995) proposed a superstructure of univariate PCIs viz. $C_p(u, v)$, given by

$$C_p(u, v) = \frac{d - u|\mu - M|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}, \quad u \geq 0, v \geq 0$$

where, USL and LSL are respectively the upper and lower specification limits of a process, $d = (USL - LSL)/2$, $M = (USL + LSL)/2$, 'μ', 'σ' and 'T' are the mean, variance and the target of the process and u and v are the scalar constants that can take any non-negative integer value.

However, Taam, Subbaiah and Liddy (1993) pointed out that in most of the practical situations, the manufacturing processes consist of more than one interdependent quality characteristics. As a result, calculation of process capability indices for individual components may yield misleading results. A number of multivariate process capability indices (MPCI) are developed to measure the capability of a process having multiple variables to control. See Taam et al. (1993), Wang and Chen (1998), Polansky (2001), Kirmani and Polansky (2009) and the references there in for further details on multivariate process capability indices (MPCI's). The necessity of constructing the multivariate folded normal distribution was felt by Chatterjee and Chakraborty (2013) while studying the properties of a superstructure of MPCIs, analogous to $C_p(u,v)$, given by

$$C_G(u,v) = \frac{1}{3}\sqrt{\frac{(\mathbf{d}-u\mathbf{D})'\Sigma^{-1}(\mathbf{d}-u\mathbf{D})}{1+v(\boldsymbol{\mu}-\mathbf{T})'\Sigma^{-1}(\boldsymbol{\mu}-\mathbf{T})}}$$

where, USL_i and LSL_i are, respectively, the upper and lower specification limits of the i^{th} quality characteristic, for i = 1(1)p, $\mathbf{D} = (|\mu_1 - M_1|, |\mu_2 - M_2|, \cdots, |\mu_p - M_p|)'$, $\mathbf{d} = ((USL_1 - LSL_1)/2, (USL_2 - LSL_2)/2, \cdots, (USL_p - LSL_p)/2)'$, $\mathbf{T} = (T_1, T_2, \cdots, T_p)'$ and $\mathbf{M} = (M_1, M_2, \cdots, M_p)'$ with $M_i = (USL_i + LSL_i)/2, i = 1(1)p$. Here, T_i is the target value and M_i is the nominal value for the i^{th} characteristic of the item. 'p' denotes the total number of quality characteristics under consideration, Σ is variance—covariance matrix of the vector '$\mathbf{X}$' of the 'p' quality characteristics $X_1, X_2, \cdots, X_p$ and $\boldsymbol{\mu}$ is the mean vector of '$\mathbf{X}$'. Here it is assumed that $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \Sigma)$ and u and v are the scalar constants that can take any non-negative integer value. Note that here we use bold faced letters to denote vectors for the remaining part of the article.

The pdf of a bivariate folded normal (BVFN) distribution was developed by Psarakis and Panaretos (2001). Suppose that $\mathbf{Z} = (Z_1, Z_2)' \sim \mathrm{N}_2(\boldsymbol{\mu}^{(2)}, \Sigma^{(2)})$ for $\boldsymbol{\mu}^{(2)} = (\mu_1, \mu_2)'$ and $\Sigma^{(2)} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$, then, $(|Z_1|, |Z_2|)$ follows BVFN with mean vector $\boldsymbol{\mu}_{\boldsymbol{f}}^{(2)}$ and dispersion matrix $\Sigma_f^{(2)}$, where the superscript '(2)' denotes the dimension of $\mathbf{Z}$. The pdf of $(|Z_1|, |Z_2|)$ can be derived as:

$$f_{|Z_1|,|Z_2|}(z_1, z_2) = \sum_{\substack{u = z_1, -z_1 \\ v = z_2, -z_2}} h_{Z_1,Z_2}(u, v), \text{ for } z_1, z_2 > 0,$$

where $h_{Z_1,Z_2}(.,.)$ denotes the pdf of bivariate normal(BVN) distribution with mean vector $\boldsymbol{\mu}^{(2)}$ and variance-covariance matrix $\Sigma^{(2)}$.

However, Psarakis and Panaretos (2001) have derived the expression of the mgf for the bivariate folded **standard** normal distribution, assuming

$\boldsymbol{\mu}_f^{(2)} = \mathbf{0}$ and $\Sigma_f^{(2)} = I_2$ using Tallis' (1961) formula for mgf of the truncated multi-normal distribution, where only the case of standard multivariate normal distribution is considered. It is to be noted that while deriving the mgf, Psarakis and Paneratos (2001) have decomposed the exponent of the corresponding bivariate normal distribution and as a result, while generalizing the expression for $p \geq 3$, one has to undergo difficult computational procedure. In the present paper, we have developed an expression for multivariate folded normal distribution, following Chakraborty and Chatterjee (2010) and found out its mean vector, dispersion matrix and the mgf. Estimation procedure for the parameters can be presented as a separate article.

In the following section, a few notations used throughout the text, are presented. In Section 3, the pdf of the proposed multivariate folded normal distribution and the forms of the mean vector, dispersion matrix and mgf are developed. This is followed by conclusion in Section 4.

2 Notation

1. Let $\boldsymbol{S(p)} = \{\mathbf{s} : \mathbf{s} = (s_1, s_2, \ldots, s_p), \text{ with } s_i = \pm 1, \forall 1 \leq i \leq p\}$.
2. diag $(s_1, s_2, \ldots, s_p) = \Lambda_s^{(p)}, \boldsymbol{\mu}_s^{(p)} = \Lambda_s^{(p)} \boldsymbol{\mu}^{(p)}, \Sigma_s^{(p)} = \Lambda_s^{(p)} \Sigma^{(p)} \Lambda_s^{(p)'}$
3. For any $\boldsymbol{s} \in \boldsymbol{S(p)}$, let $\mathbf{W_s^{(p)}} = (s_1 Z_1, \ldots, s_p Z_p)' \sim N_p(\boldsymbol{\mu}_s^{(p)}, \Sigma_s^{(p)})$ with $Z_i > 0, \forall i = 1(1)p$ and $\Sigma_s^{(p)} = B_s^{(p)} B_s^{(p)'}$, where $B_s^{(p)}$ is obtained using Choleski's factorization method.
4. Let $(B_s^{(p)})_i^{-1}$ denote the i^{th} row of the $(p \times p)$ matrix $(B_s^{(p)})^{-1}$, $\forall i = 1(1)p$, and $\phi(.)$ denote the pdf of univariate standard normal distribution. Then we define the following:

$$Q_{sj}^{(p)} = \phi\left((B_s^{(p)})_j^{-1} \boldsymbol{\mu}_s^{(p)}\right) \prod_{\substack{i=1 \\ \neq j}}^{p} \Phi\left[(B_s^{(p)})_i^{-1} \boldsymbol{\mu}_s^{(p)}\right], \tag{2.1}$$

$$I_{s_{ii}}^{(p)} = \left\{ \prod_{\substack{j=1 \\ \neq i}}^{p} \Phi\left[(B_s^{(p)})_j^{-1} \boldsymbol{\mu}_s^{(p)}\right] \right\} \times \left\{ \Phi\left[(B_s^{(p)})_i^{-1} \boldsymbol{\mu}_s^{(p)}\right] - (B_s^{(p)})_i^{-1} \boldsymbol{\mu}_s^{(p)} . \phi\left[(B_s^{(p)})_i^{-1} \boldsymbol{\mu}_s^{(p)}\right] \right\}, \tag{2.2}$$

$$I_{s_{ik}}^{(p)} = \phi\left[(B_s^{(p)})_i^{-1}\boldsymbol{\mu}_s^{(p)}\right]\phi\left[(B_s^{(p)})_k^{-1}\boldsymbol{\mu}_s^{(p)}\right] \times \prod_{j=1,\neq i,k}^{p} \Phi\left[(B_s^{(p)})_j^{-1}\boldsymbol{\mu}_s^{(p)}\right], \quad i \neq k. \tag{2.3}$$

It can be noted that since $B_s^{(p)}$ is a $(p \times p)$ matrix, the subscripts i, j and k of $Q_{sj}^{(p)}$, $I_{s_{ii}}^{(p)}$ and $I_{s_{ik}}^{(p)}$ can assume any of the values $1(1)p$ with conditions mentioned above. We can then define $I_{s,p}$ as $I_{s,p} = I_{s_{ij}}^{(p)}$, where $i = 1(1)p, j = 1(1)p$.

3 Multivariate folded normal distribution

The pdf of the multivariate (say, p-variate) folded normal distribution (MVFN) can be written as:

$$\begin{aligned} f_p(z_1, z_2, \ldots, z_p) &= \sum_{(s_1,s_2,\ldots,s_p)\in \boldsymbol{S(p)}} h_p(s_1z_1, s_2z_2, \ldots, s_pz_p) \\ &= \sum_{(s_1,s_2,\ldots,s_p)\in \boldsymbol{S(p)}} h_p(\Lambda_s^{(p)}\mathbf{z}^{(p)}) \quad \text{for each } z_i > 0, \end{aligned} \tag{3.1}$$

where

$$\begin{bmatrix} s_1z_1 \\ s_2z_2 \\ \vdots \\ s_pz_p \end{bmatrix} = \operatorname{diag}(s_1, s_2, \ldots, s_p)\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_p \end{bmatrix} = \Lambda_s^{(p)}\mathbf{z}^{(p)}. \tag{3.2}$$

For $\mathbf{Z}^{(p)} \sim N_p(\boldsymbol{\mu}^{(p)}, \Sigma^{(p)})$, we then have $\boldsymbol{W}_s^{(p)} = \Lambda_s^{(p)}\boldsymbol{Z}^{(p)} \sim N_p(\boldsymbol{\mu}_s^{(p)}, \Sigma_s^{(p)})$ with $\boldsymbol{\mu}_s^{(p)} = \Lambda_s^{(p)}\boldsymbol{\mu}^{(p)}$ and $\Sigma_s^{(p)} = \Lambda_s^{(p)}\Sigma^{(p)}\Lambda_s^{(p)\prime}$.

We first obtain the mean vector, dispersion matrix and the mgf of the bivariate folded normal distribution and then generalize that to get the same for MVFN distribution. It may be noted that the pdf of the BVFN distribution has been reconstructed here as mentioned in Section 1. Accordingly, its mean vector, dispersion matrix and mgf are derived on the basis of the new form of pdf.

3.1. Mean vector. Suppose the mean vector of bivariate folded normal distribution is given by $\boldsymbol{\mu}_f^{(2)}$. Then the expression for $\boldsymbol{\mu}_f^{(2)}$ can be obtained as follows. From (3.1) and (3.2) it can be seen that

$$\begin{aligned}
\boldsymbol{\mu}_f^{(2)} &= E[|\mathbf{Z^{(2)}}| \ ; \ Z^{(2)} \sim N_2(\mu^{(\mathbf{2})}, \boldsymbol{\Sigma}^{(\mathbf{2})})] \\
&= \sum_{(s_1,s_2)\in \boldsymbol{S(2)}} \frac{1}{(\sqrt{2\pi})^2\sqrt{|\ \Sigma_s^{(2)}\ |}} \int_{w_{s_1}=0}^{\infty}\int_{w_{s_2}=0}^{\infty} \begin{pmatrix} w_{s_1} \\ w_{s_2} \end{pmatrix} \\
&\quad \times \exp\left[-\frac{1}{2}(\boldsymbol{w}_s^{(2)} - \boldsymbol{\mu}_s^{(2)})'\Sigma_s^{(2)-1}(\boldsymbol{w}_s^{(2)} - \boldsymbol{\mu}_s^{(2)})\right] dw_{s_1}dw_{s_2}
\end{aligned} \tag{3.3}$$

Now since $\Sigma_s^{(2)}$ is a variance-covariance matrix and hence positive definite, we can express $\Sigma_s^{(2)}$ as $\Sigma_s^{(2)} = B_s^{(2)}B_s^{(2)\prime}, \forall \mathbf{s} \in \boldsymbol{S(2)}$. It may be noted that in general $B_s^{(2)}$ is not unique. However, if we restrict our attention to the subclass of all lower triangular matrices, then $B_s^{(2)}$ is unique (Tong, 1990).

Let us now consider the transformation $\boldsymbol{W}_s^{(2)} \to \boldsymbol{Y}_s^{(2)}$:

$$(\boldsymbol{W}_s^{(2)} - \boldsymbol{\mu}_s^{(2)}) = B_s^{(2)}\boldsymbol{Y_s}^{(2)}, \forall \boldsymbol{s} \in \boldsymbol{S(2)}, \tag{3.4}$$

where $\boldsymbol{Y}_s^{(2)} = \begin{pmatrix} y_{s_1} \\ y_{s_2} \end{pmatrix}$.

Hence from (3.3) and (3.4), we get

$$\begin{aligned}
E[\boldsymbol{W}_s^{(2)}] &= \frac{1}{(\sqrt{2\pi})^2\sqrt{|\Sigma_s^{(2)}|}} \int_{w_{s_2}=0}^{\infty}\int_{w_{s_1}=0}^{\infty} \boldsymbol{w_s}^{(2)} \\
&\quad \times \ \exp\left[-\frac{1}{2}(\boldsymbol{w}_s^{(2)} - \boldsymbol{\mu}_s^{(2)})'\Sigma_s^{(2)-1}(\boldsymbol{w}_s^{(2)} - \boldsymbol{\mu}_s^{(2)})\right] d\boldsymbol{w}_s^{(2)} \\
&= \frac{1}{(\sqrt{2\pi})^2} \int_{y_{s_2}=-(B_s^{(2)})_2^{-1}\boldsymbol{\mu}_s^{(2)}}^{\infty} \int_{y_{s_1}=-(B_s^{(2)})_1^{-1}\boldsymbol{\mu}_s^{(2)}}^{\infty} \left(\boldsymbol{\mu}_s^{(2)} + B_s^{(2)}\boldsymbol{y}_s^{(2)}\right) \\
&\quad \times \ \exp\left[-\frac{1}{2}\boldsymbol{y}_s^{(2)\prime}\boldsymbol{y}_s^{(2)}\right] d\boldsymbol{y}_s^{(2)} \\
&= \frac{\boldsymbol{\mu}_s^{(2)}}{(\sqrt{2\pi})^2} \int_{y_{s_2}=-(B_s^{(2)})_2^{-1}\boldsymbol{\mu}_s^{(2)}}^{\infty} \int_{y_{s_1}=-(B_s^{(2)})_1^{-1}\boldsymbol{\mu}_s^{(2)}}^{\infty} \\
&\quad \times \exp\left[-\frac{1}{2}\boldsymbol{y}_s^{(2)\prime}\boldsymbol{y}_s^{(2)}\right] d\boldsymbol{y}_s^{(2)}
\end{aligned}$$

$$
\begin{aligned}
& + \frac{B_s^{(2)}}{(\sqrt{2\pi})^2} \int_{y_{s_2}=-\left(B_s^{(2)}\right)_2^{-1}\boldsymbol{\mu}_s^{(2)}}^{\infty} \int_{y_{s_1}=-\left(B_s^{(2)}\right)_1^{-1}\boldsymbol{\mu}_s^{(2)}}^{\infty} \begin{pmatrix} y_{s_1} \\ y_{s_2} \end{pmatrix} \\
& \times \ \exp\left[-\frac{1}{2}\boldsymbol{y}_s^{(2)\prime}\boldsymbol{y}_s^{(2)}\right] d\boldsymbol{y}_s^{(2)} \\
= \ & \boldsymbol{\mu}_s^{(2)} \prod_{i=1}^{2} \Phi\left[\left(B_s^{(2)}\right)_i^{-1}\boldsymbol{\mu}_s^{(2)}\right] \\
& + B_s^{(2)} \begin{pmatrix} \phi\left(\left(B_s^{(2)}\right)_1^{-1}\boldsymbol{\mu}_s^{(2)}\right) \times \Phi\left[\left(B_s^{(2)}\right)_2^{-1}\boldsymbol{\mu}_s^{(2)}\right] \\ \\ \phi\left(\left(B_s^{(2)}\right)_2^{-1}\boldsymbol{\mu}_s^{(2)}\right) \times \Phi\left[\left(B_s^{(2)}\right)_1^{-1}\boldsymbol{\mu}_s^{(2)}\right] \end{pmatrix}
\end{aligned} \tag{3.5}
$$

From (3.5) the expression for mean of the bivariate folded normal distribution can be obtained as

$$
\begin{aligned}
\boldsymbol{\mu}_f^{(2)} \ = \ & \sum_{(s_1,s_2)\in \boldsymbol{S(2)}} \left\{ \boldsymbol{\mu}_s^{(2)} \prod_{i=1}^{2} \Phi\left[\left(B_s^{(2)}\right)_i^{-1}\boldsymbol{\mu}_s^{(2)}\right] \right. \\
& \left. + B_s^{(2)} \begin{pmatrix} \phi\left(\left(B_s^{(2)}\right)_1^{-1}\boldsymbol{\mu}_s^{(2)}\right) \times \Phi\left[\left(B_s^{(2)}\right)_2^{-1}\boldsymbol{\mu}_s^{(2)}\right] \\ \\ \phi\left(\left(B_s^{(2)}\right)_2^{-1}\boldsymbol{\mu}_s^{(2)}\right) \times \Phi\left[\left(B_s^{(2)}\right)_1^{-1}\boldsymbol{\mu}_s^{(2)}\right] \end{pmatrix} \right\}.
\end{aligned} \tag{3.6}
$$

Thus generalizing (3.6) for p-variate case one can obtain mean vector of MVFN distribution as:

$$
\begin{aligned}
\mu_f^{(p)} \ = \ & \sum_{\boldsymbol{s}\in \boldsymbol{S(p)}} \boldsymbol{\mu}_s^{(p)} \prod_{i=1}^{p} \Phi\left[\left(B_s^{(p)}\right)_i^{-1}\boldsymbol{\mu}_s^{(p)}\right] \\
& + \sum_{\boldsymbol{s}\in \boldsymbol{S(p)}} B_s^{(p)} \begin{pmatrix} \phi\left(\left(B_s^{(p)}\right)_1^{-1}\boldsymbol{\mu}_s^{(p)}\right) \times \prod_{i=2}^{p} \Phi\left[\left(B_s^{(p)}\right)_i^{-1}\boldsymbol{\mu}_s^{(p)}\right] \\ \vdots \\ \phi\left(\left(B_s^{(p)}\right)_j^{-1}\boldsymbol{\mu}_s^{(p)}\right) \times \prod^{p}_{\substack{i=1 \\ \neq j}} \Phi\left[\left(B_s^{(p)}\right)_i^{-1}\boldsymbol{\mu}_s^{(p)}\right] \\ \vdots \\ \phi\left(\left(B_s^{(p)}\right)_p^{-1}\boldsymbol{\mu}_s^{(p)}\right) \times \prod_{i=1}^{p-1} \Phi\left[\left(B_s^{(p)}\right)_i^{-1}\boldsymbol{\mu}_s^{(p)}\right] \end{pmatrix}
\end{aligned}
$$

$$= \sum_{\boldsymbol{s}\in \boldsymbol{S(p)}} \boldsymbol{\mu}_s^{(p)} \prod_{i=1}^{p} \Phi[(B_s^{(p)})_i^{-1} \boldsymbol{\mu}_s^{(p)}] + \sum_{\boldsymbol{s}\in \boldsymbol{S(p)}} B_s^{(p)} \boldsymbol{Q}_s^{(p)}, \tag{3.7}$$

where $\boldsymbol{Q}_s^{(p)}$ is a $(p \times 1)$ vector whose j^{th} element $Q_{s_j}^{(p)}$ is defined in (2.1).

It can be easily verified that for $p = 1$, the expression in (3.7) reduces to the same expression for the mean of univariate folded normal distribution as given by Leone et al. (1961).

3.2. Dispersion matrix. Suppose the dispersion matrix of a BVFN distribution is given by $\Sigma_f^{(2)}$. Then the expression for $\Sigma_f^{(2)}$ can be obtained as follows:

$$\begin{aligned}
\Sigma_f^{(2)} &= E\left[\left(\boldsymbol{X}^{(2)} - \boldsymbol{\mu}_f^{(2)}\right)\left(\boldsymbol{X}^{(2)} - \boldsymbol{\mu}_f^{(2)}\right)' : \right. \\
\boldsymbol{X}^{(2)} &= \left. \mid \boldsymbol{Z}^{(2)} \mid, \boldsymbol{Z}^{(2)} \sim N_2(\boldsymbol{\mu}^{(2)}, \Sigma^{(2)})\right] \\
&= E[\boldsymbol{X}^{(2)}\boldsymbol{X}^{(2)'}] - \boldsymbol{\mu}_f^{(2)}\boldsymbol{\mu}_f^{(2)'} \\
&= \sum_{(s_1,s_2)\in \boldsymbol{S(2)}} E[\boldsymbol{W}_s^{(2)}\boldsymbol{W}_s^{(2)'}] - \boldsymbol{\mu}_f^{(2)}\boldsymbol{\mu}_f^{(2)'}, \quad \text{(say)}.
\end{aligned} \tag{3.8}$$

Now,

$$\begin{aligned}
E[\boldsymbol{W}_s^{(2)}\boldsymbol{W}_s^{(2)'}] &= \frac{1}{(\sqrt{2\pi})^2\sqrt{|\Sigma_s^{(2)}|}} \int_{w_{s_2}=0}^{\infty}\int_{w_{s_1}=0}^{\infty} \boldsymbol{w}_s^{(2)}\boldsymbol{w}_s^{(2)'} \\
&\quad \times \exp\left[-\frac{1}{2}(\boldsymbol{w}_s^{(2)} - \boldsymbol{\mu}_s^{(2)})'\Sigma_s^{(2)-1}(\boldsymbol{w}_s^{(2)} - \boldsymbol{\mu}_s^{(2)})\right] d\boldsymbol{w}_s^{(2)} \\
&= \frac{1}{(\sqrt{2\pi})^2} \int_{y_{s_2}=-(B_s^{(2)})_2^{-1}\boldsymbol{\mu}_s^{(2)}}^{\infty} \int_{y_{s_1}=-(B_s^{(2)})_1^{-1}\boldsymbol{\mu}_s^{(2)}}^{\infty} (\boldsymbol{\mu}_s^{(2)} + B_s^{(2)}\boldsymbol{y}_s^{(2)}) \\
&\quad \times (\boldsymbol{\mu}_s^{(2)} + B_s^{(2)}\boldsymbol{y}_s^{(2)})' \times \exp\left[-\frac{1}{2}\boldsymbol{y}_s^{(2)'}\boldsymbol{y}_s^{(2)}\right] d\boldsymbol{y}_s^{(2)},
\end{aligned} \tag{3.9}$$

by using the transformation (3.4). Hence from (3.9) we have

$$\begin{aligned}
E[\boldsymbol{W}_s^{(2)}\boldsymbol{W}_s^{(2)'}] &= \frac{1}{(\sqrt{2\pi})^2} \int_{y_{s_2}=-(B_s^{(2)})_2^{-1}\boldsymbol{\mu}_s^{(2)}}^{\infty} \int_{y_{s_1}=-(B_s^{(2)})_1^{-1}\boldsymbol{\mu}_s^{(2)}}^{\infty} \\
&\quad \times \left[\boldsymbol{\mu}_s^{(2)}\boldsymbol{\mu}_s^{(2)'} + B_s^{(2)}Y_s^{(2)}\boldsymbol{\mu}_s^{(2)'} + \boldsymbol{\mu}_s^{(2)}\boldsymbol{Y}_s^{(2)'}B_s^{(2)'} \right. \\
&\quad \left. + B_s^{(2)}\boldsymbol{Y}_s^{(2)}\boldsymbol{Y}_s^{(2)'}B_s^{(2)'}\right]
\end{aligned}$$

$$\times \exp\left[-\frac{1}{2}\sum_{i=1}^{2} y_{s_i}^2\right] dy_{s1}dy_{s2}$$
$$= \sum_{i=1}^{4} I_{s_i}^{(2)}, \quad \text{(say)}, \tag{3.10}$$

where

$$I_{s_1}^{(2)} = \frac{\boldsymbol{\mu}_s^{(2)}\boldsymbol{\mu}_s^{(2)\prime}}{(\sqrt{2\pi})^2} \int_{y_{s_2}=-(B_s^{(2)})_2^{-1}\boldsymbol{\mu}_s^{(2)}}^{\infty} \int_{y_{s_1}=-(B_s^{(2)})_1^{-1}\boldsymbol{\mu}_s^{(2)}}^{\infty} \times \exp\left[-\frac{1}{2}\sum_{i=1}^{2} y_{s_i}^2\right] dy_{s_1}dy_{s_2}$$
$$= \boldsymbol{\mu}_s^{(2)}\boldsymbol{\mu}_s^{(2)\prime} \prod_{i=1}^{2} \Phi[(B_s^{(2)})_i^{-1}\boldsymbol{\mu}_s^{(2)}] \tag{3.11}$$

$$I_{s_2}^{(2)} = \frac{B_s^{(2)}}{(\sqrt{2\pi})^2} \left(\int_{y_{s_2}} \int_{y_{s_1}} \boldsymbol{y}_s^{(2)} \exp\left[-\frac{1}{2}\sum_{i=1}^{2} y_{s_i}^2\right] dy_{s_1}dy_{s_2} \right) \boldsymbol{\mu}_s^{(2)\prime}$$
$$= B_s^{(2)} \begin{pmatrix} \Phi\left[(B_s^{(2)})_2^{-1}\boldsymbol{\mu}_s^{(2)}\right] \phi\left[(B_s^{(2)})_1^{-1}\boldsymbol{\mu}_s^{(2)}\right] \\ \\ \Phi\left[(B_s^{(2)})_1^{-1}\boldsymbol{\mu}_s^{(2)}\right] \phi\left[(B_s^{(2)})_2^{-1}\boldsymbol{\mu}_s^{(2)}\right] \end{pmatrix} \boldsymbol{\mu}_s^{(2)\prime} \tag{3.12}$$

Similar to (3.12), it can be shown that,

$$I_{s_3}^{(2)} = \boldsymbol{\mu}_s^{(2)} \begin{pmatrix} \Phi\left[(B_s^{(2)})_2^{-1}\boldsymbol{\mu}_s^{(2)}\right] \phi\left[(B_s^{(2)})_1^{-1}\boldsymbol{\mu}_s^{(2)}\right] \\ \\ \Phi\left[(B_s^{(2)})_1^{-1}\boldsymbol{\mu}_s^{(2)}\right] \phi\left[(B_s^{(2)})_2^{-1}\boldsymbol{\mu}_s^{(2)}\right] \end{pmatrix}^{\prime} B_s^{(2)\prime}, \tag{3.13}$$

$$I_{s_4}^{(2)} = \frac{B_s^{(2)}}{(\sqrt{2\pi})^2} \left\{ \int_{y_{s_2}} \int_{y_{s_1}} \boldsymbol{y}_s^{(2)}\boldsymbol{y}_s^{(2)\prime} \exp\left[-\frac{1}{2}\sum_{i=1}^{2} y_{s_i}^2\right] dy_{s_1}dy_{s_2} \right\} B_s^{(2)\prime}$$
$$= B_s^{(2)} \begin{bmatrix} I_{s_{11}}^{(2)} & I_{s_{12}}^{(2)} \\ I_{s_{21}}^{(2)} & I_{s_{22}}^{(2)} \end{bmatrix} B_s^{(2)\prime}, \tag{3.14}$$

where

$$
\begin{aligned}
I^{(2)}_{s_{11}} &= \frac{1}{(\sqrt{2\pi})^2}\int_{y_{s_2}}\int_{y_{s_1}} y^2_{s_1}\exp\left[-\frac{1}{2}\sum_{i=1}^{2} y^2_{s_i}\right]\ dy_{s_1}dy_{s_2} \\
&= \Phi\left[(B^{(2)}_s)_2^{-1}\boldsymbol{\mu}^{(2)}_s\right]\Phi\left[(B^{(2)}_s)_1^{-1}\boldsymbol{\mu}^{(2)}_s\right] \\
&\quad -\left\{(B^{(2)}_s)_1^{-1}\boldsymbol{\mu}^{(2)}_s\right\}\phi\left[(B^{(2)}_s)_1^{-1}\boldsymbol{\mu}^{(2)}_s\right],
\end{aligned}
$$

$$
\begin{aligned}
I^{(2)}_{s_{22}} &= \Phi\left[(B^{(2)}_s)_1^{-1}\boldsymbol{\mu}^{(2)}_s\right]\Phi\left[(B^{(2)}_s)_2^{-1}\boldsymbol{\mu}^{(2)}_s\right] \\
&\quad -(B^{(2)}_s)_2^{-1}\boldsymbol{\mu}^{(2)}_s\times\phi\left[(B^{(2)}_s)_2^{-1}\boldsymbol{\mu}^{(2)}_s\right],
\end{aligned}
$$

$$
\begin{aligned}
I^{(2)}_{s_{12}} &= \frac{1}{(\sqrt{2\pi})^2}\int_{y_{s_2}}\int_{y_{s_1}} y_{s_1}y_{s_2}\exp\left[-\frac{1}{2}\sum_{i=1}^{2} y^2_{s_i}\right]\ dy_{s_1}dy_{s_2} \\
&= \prod_{i=1}^{2}\phi[(B^{(2)}_s)_i^{-1}\boldsymbol{\mu}^{(2)}_s] \\
&= I^{(2)}_{s_{21}}.
\end{aligned}
$$

Using (3.11) to (3.14), we can obtain the expression for $E[W^{(2)}_s W^{(2)'}_s]$ from (3.10).

As such from (3.8), the expression for dispersion matrix of bivariate folded normal distribution can be obtained as:

$$
\begin{aligned}
\Sigma^{(2)}_f &= \sum_{s\in S(2)} E[\boldsymbol{W}^{(2)}_s\boldsymbol{W}^{(2)'}_s \quad |\boldsymbol{W}^{(2)}_s=\Lambda^{(2)}_s\boldsymbol{Z}^{(2)}]-\boldsymbol{\mu}^{(2)}_f\boldsymbol{\mu}^{(2)'}_f \\
&= \sum_{\mathbf{s}\in\boldsymbol{S(2)}}\boldsymbol{\mu}^{(2)}_s\boldsymbol{\mu}^{(2)'}_s\prod_{i=1}^{2}\Phi\left[(B^{(2)}_s)_i^{-1}\boldsymbol{\mu}^{(2)}_s\right] \\
&\quad +\sum_{\mathbf{s}\in\boldsymbol{S(2)}} B^{(2)}_s\begin{pmatrix}\Phi\left[(B^{(2)}_s)_2^{-1}\boldsymbol{\mu}^{(2)}_s\right]\phi\left[(B^{(2)}_s)_1^{-1}\boldsymbol{\mu}^{(2)}_s\right] \\ \\ \Phi\left[(B^{(2)}_s)_1^{-1}\boldsymbol{\mu}^{(2)}_s\right]\phi\left[(B^{(2)}_s)_2^{-1}\boldsymbol{\mu}^{(2)}_s\right]\end{pmatrix}\boldsymbol{\mu}^{(2)'}_s
\end{aligned}
$$

$$+\sum_{\mathbf{s}\in \boldsymbol{S}(\mathbf{2})} \boldsymbol{\mu}_s^{(2)} \begin{pmatrix} \Phi\left[(B_s^{(2)})_2^{-1}\boldsymbol{\mu}_s^{(2)}\right]\phi\left[(B_s^{(2)})_1^{-1}\boldsymbol{\mu}_s^{(2)}\right] \\ \\ \Phi\left[(B_s^{(2)})_1^{-1}\boldsymbol{\mu}_s^{(2)}\right]\phi\left[(B_s^{(2)})_2^{-1}\boldsymbol{\mu}_s^{(2)}\right] \end{pmatrix}' B_s^{(2)'}$$

$$+\sum_{\mathbf{s}=(s_1,s_2)\in \boldsymbol{S}(\mathbf{2})} B_s^{(2)}\begin{bmatrix} I_{s_{11}}^{(2)} & I_{s_{12}}^{(2)} \\ I_{s_{21}}^{(2)} & I_{s_{22}}^{(2)} \end{bmatrix} B_s^{(2)'} - \boldsymbol{\mu}_f^{(2)}\boldsymbol{\mu}_f^{(2)'}. \qquad (3.15)$$

We are now in a position to generalize (3.15) for multivariate (p-variate, say) case. With the help of the matrix $I_{s,p} = ((I_{sij}^{(p)})), i,j = 1(1)p$, which is defined in (2.2) and (2.3), the dispersion matrix of p-variate folded normal distribution can be obtained as:

$$\Sigma_f^{(p)} = \sum_{\boldsymbol{s}\in \boldsymbol{S}(\boldsymbol{p})} \left\{ \boldsymbol{\mu}_s^{(p)}\boldsymbol{\mu}_s^{(p)'} \prod_{i=1}^{p}\left[{B_s^{(p)}}_i^{-1}\boldsymbol{\mu}_s^{(p)}\right] + \boldsymbol{\mu}_s^{(p)}\boldsymbol{Q}_s^{(p)'}B_s^{(p)'} \right.$$
$$\left. + B_s^{(p)}\boldsymbol{Q}_s^{(p)}\boldsymbol{\mu}_s^{(p')} + B_s^{(p)} I_{s,p} B_s^{(p)'} \right\} - \boldsymbol{\mu}_f^{(p)}\boldsymbol{\mu}_f^{(p)'},$$

where $\boldsymbol{s} = (s_1, s_2, \ldots, s_p)$.

It may be noted that here also the univariate analogue of this variance-covariance matrix is the same as that of the expression given by Leone et al. (1961).

3.3. Moment generating function (mgf). Suppose the mgf of a BVFN distribution is denoted by $M_X^{(2)}(t)$. Then the expression for $M_X^{(2)}(t)$ can be obtained as follows:

$$\begin{aligned} M_X^{(2)}(t) &= E[e^{\boldsymbol{t}'\boldsymbol{X}^{(2)}} \mid \boldsymbol{X}^{(2)} = \mid \boldsymbol{Z}^{(2)} \mid, \boldsymbol{Z}^{(2)} \sim N_2(\boldsymbol{\mu}^{(2)}, \Sigma^{(2)})] \\ &= \sum_{\boldsymbol{s}\in \boldsymbol{S}(\mathbf{2})} E[e^{\boldsymbol{t}'\boldsymbol{W}_s^{(2)}}] \\ &= \sum_{\boldsymbol{s}\in \boldsymbol{S}(\mathbf{2})} \frac{1}{(\sqrt{2\pi})^2\sqrt{|\Sigma_s^{(2)}|}} \int_{w_{s2}=0}^{\infty}\int_{w_{s1}=0}^{\infty} \exp\left[\boldsymbol{t}'\boldsymbol{w}_s^{(2)}\right] \\ &\quad \times \exp\left[-\frac{1}{2}(\boldsymbol{w}_s^{(2)} - \boldsymbol{\mu}_s^{(2)})'\Sigma_s^{(2)-1}(\boldsymbol{w}_s^{(2)} - \boldsymbol{\mu}_s^{(2)})\right] d\boldsymbol{w}_s^{(2)} \end{aligned}$$

Using the transformation $\boldsymbol{W}_s^{(2)} \to \boldsymbol{Y}_s^{(2)}$ as given in (3.4), we have

$$\begin{aligned} M_X^{(2)}(\boldsymbol{t}) &= \sum_{\boldsymbol{s}\in\boldsymbol{S(2)}} \frac{1}{(\sqrt{2\pi})^2} \int_{y_{s_2}} \int_{y_{s_1}} \\ &\quad \times \exp\left[\boldsymbol{t}'(\boldsymbol{\mu}_s^{(2)} + B_s^{(2)}\boldsymbol{y}_s^{(2)}) - \frac{1}{2}\sum_{i=1}^{2} y_{s_i}^2\right] dy_{s_1} dy_{s_2} \\ &= \frac{1}{(\sqrt{2\pi})^2} \sum_{s\in S(2)} \exp\left[\boldsymbol{t}'\boldsymbol{\mu}_s^{(2)} + \frac{1}{2}\boldsymbol{t}' B_s^{(2)} B_s^{(2)'} \boldsymbol{t}\right] \\ &\quad \times \int_{y_{s_2}}^{\infty} \int_{y_{s_1}}^{\infty} \exp\left[\boldsymbol{y}_s^{(2)'}\boldsymbol{y}_s^{(2)} - 2\boldsymbol{t}' B_s^{(2)}\boldsymbol{y}_s^{(2)} + \boldsymbol{t}' B_s^{(2)} B_s^{(2)'}\boldsymbol{t}\right] dy_{s_1} dy_{s_2} \end{aligned} \tag{3.16}$$

Now, let

$$\begin{aligned} \boldsymbol{t}_s &= B_s^{(2)'}\mathbf{t} \\ &= \begin{pmatrix} (B_s^{(2)'})_1 \\ (B_s^{(2)'})_2 \end{pmatrix} \boldsymbol{t} \\ &= \begin{pmatrix} (B_s^{(2)'})_1 \boldsymbol{t} \\ (B_s^{(2)'})_2 \boldsymbol{t} \end{pmatrix} \\ &= \begin{pmatrix} t_{s_1} \\ t_{s_2} \end{pmatrix}, \end{aligned}$$

where $(B_s^{(2)'})_i = i^{th}$ row of $B_s^{(2)'}, \forall i = 1, 2.$ Thus from (3.16) mgf of BVFN can be obtained as

$$\begin{aligned} M_X^{(2)}(\mathbf{t}) &= \sum_{\boldsymbol{s}\in\boldsymbol{S(2)}} \frac{1}{(\sqrt{2\pi})^2} \int_{y_{s_2}}^{\infty} \int_{y_{s1}}^{\infty} \exp\left[-\frac{1}{2}\sum_{i=1}^{2}(y_{s_i} - t_{s_i})^2\right] dy_{s_1} dy_{s_2} \\ &= \sum_{\boldsymbol{s}\in\boldsymbol{S(2)}} \left\{ \exp\left[\boldsymbol{t}'\boldsymbol{\mu}_s^{(2)} + \frac{1}{2}\boldsymbol{t}'\Sigma_s^{(2)}\boldsymbol{t}\right] \prod_{i=1}^{2} \Phi\left[(B_s^{(2)'})_i\boldsymbol{t} + (B_s^{(2)})_i^{-1}\boldsymbol{\mu}_s^{(2)}\right] \right\} \end{aligned} \tag{3.17}$$

It is worth mentioning here that, for bivariate folded standard normal distribution, i.e. for $\boldsymbol{\mu}^{(2)} = \mathbf{0}$, $\Sigma_1^{(2)} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ and $\Sigma_2^{(2)} = \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$, the expression in (3.17) can be simplified as

$$\begin{aligned} M_X^{(2)}(\mathbf{t}) &= 2\ e^{\frac{1}{2}\mathbf{t}'\Sigma_1^{(2)}\boldsymbol{t}} \prod_{i=1}^{2} \Phi\left[(B_1^{(2)'})_i\mathbf{t}\right] + 2\ e^{\frac{1}{2}\mathbf{t}'\Sigma_2^{(2)}\mathbf{t}} \prod_{i=1}^{2} \Phi\left[(B_2^{(2)'})_i\mathbf{t}\right] \\ &= 2e^{T}\Phi_2(\mathbf{b_s}; R) + 2e^{T'}\Phi_2(\mathbf{b_s}; R'), \end{aligned} \tag{3.18}$$

where, $T = (1/2)\mathbf{t}'\Sigma_1^{(2)}\mathbf{t}$, $T' = (1/2)\mathbf{t}'\Sigma_2^{(2)}\mathbf{t}$, $\mathbf{b_s} = R\mathbf{t}$ (or $R'\mathbf{t}$ as the case may be), $\Phi_2(\mathbf{b_s}; R)$ is the cdf of the bivariate standard normal distribution with correlation matrix $R = \Sigma_1^{(2)}$ and $\Phi_2(\mathbf{b_s}; R')$ is that with correlation matrix $R' = \Sigma_2^{(2)}$.

The mgf of bivariate folded standard normal distribution, proposed by Psarakis and Panaretos (2001), matches with the expression given in (3.18). Thus their expression is a special case of $M_X^{(2)}(\mathbf{t})$. In fact, the assumption of $\boldsymbol{\mu}^{(2)} = \mathbf{0}$ would fail to discriminate a folded normal distribution from a half normal distribution as has been discussed by Leone et al. (1961).

We now generalize (3.17) for multivariate (p-variate, say) case. Thus mgf of MVFN can be obtained as

$$\begin{aligned} M_X^{(p)}(\mathbf{t}) = \sum_{(s_1,s_2,\ldots,s_p)\in \boldsymbol{S(p)}} \Bigg\{ & \exp\left[\mathbf{t}'\boldsymbol{\mu}_s^{(p)} + \frac{1}{2}\mathbf{t}'\Sigma_s^{(p)}\mathbf{t}\right] \\ & \times \prod_{i=1}^{p} \Phi\left[(B_s^{(p)'})_i\mathbf{t} + (B_s^{(p)})_i^{-1}\boldsymbol{\mu}_s^{(p)}\right] \Bigg\} \end{aligned} \tag{3.19}$$

It is interesting to note that for $p = 1, M_X^{(p)}(\mathbf{t})$ in (3.19) gives the mean and variance of the univariate folded normal distribution as obtained from Leone et al. (1961).

4 Conclusion

Univariate folded normal distribution was developed by Leone et al. (1961). In the present paper, we have developed its multivariate counterpart. Although, Psarakis and Panaretos (2001) had already proposed bivariate folded normal distribution, they constructed the mgf of only the bivariate folded standard normal distribution. On the contrary, we have constructed a more general form of the distribution for 'p'-variate case with a general form of the mean vector and the dispersion matrix of the corresponding p-variate normal distribution. We have also derived the expressions of the mean vector, dispersion matrix and the mgf of the MVFN distribution. It can be shown that the corresponding expressions obtained by Leone et al. (1961) and Psarakis and Panaretos (2001) can be derived as special cases of ours. We have also made a brief discussion on the possible application of MVFN distribution in multivariate process capability analysis which is one of the core areas of the theory of statistical quality control. However, we have not dealt with the estimation procedure of the parameters involved in MVFN which is required for exploring the distribution further. Given the complicated form of the distribution itself, such estimation will definitely be challenging yet interesting.

Acknowledgement. The authors acknowledge some very useful discussions with Prof. Shibdas Bandapadhyay, Applied Statistics Unit, Indian Statistical Institute, Kolkata and the anonymous referee(s) for their valuable comments to make the paper better.

References

Chakraborty, A.K. and Chatterjee, M. (2010). On Multivariate Folded Normal Distribution. Technical Report No. SQCOR - 2010-02, SQC & OR Unit. Indian Statistical Institute, Kolkata.

Chatterjee, M. and Chakraborty, A.K. (2013). Some Properties Of $C_G(u, v)$. In *Proceedings of International Conference On Quality and Reliability Engineering, Bangalore, India*. To be published.

Elandt, R.C. (1961). The folded normal distribution: two methods of estimating parameters from moments. *Technometrics*, **3**, 551–562.

Johnson, N.L. (1963). Cumulative sum control charts for the folded normal distribution. *Technometrics*, **5**, 451–458.

King, J.R. (1988). When is a normal variable not a normal variable? *Qual. Eng.*, **1**, 173–178.

Kirmani, S. and Polansky, A.M. (2009). Multivariate process capability via lowner ordering. *Linear Algebra Appl.*, **430**, 2681–2689.

Leone, F.C., Nelson, L.S. and Nottingham, R.B. (1961). The folded normal distribution, *Technometrics*, **3**, 543–550.

Liao, M.Y. (2010). Economic tolerance design for folded normal data. *Int. J. Prod. Res.*, **18**, 4123–4137.

Lin, H.C. (2004). The measurement of a process capability for folded normal process data. *Int. J. Adv. Manuf. Technol.*, **24**, 223–228.

Lin, P.C. (2005). Application of the generalized folded-normal distribution to the process capability measures, *Int. J. Adv. Manuf. Technol.*, **26**, 825–830.

Polansky, A.M. (2001). A smooth nonparametric approach to multivariate process capability, *Technometrics*, **43**, 199–211.

psarakis, s. and panaretos, j. (2001). On some bivariate extensions of the folded normal and the folded T distributions. *J. Appl. Statist. Sci.*, **10**, 119–136.

tallis, g.m. (1961). The moment generating function of the truncated multi-normal distribution. *J. R. Stat. Soc. Ser. B Stat. Methodol.*, **23**, 223–229.

taam, w., subbaiah, p. and liddy, j.w. (1993). A note on multivariate capability indices. *J. Appl. Stat.*, **20**, 339–351.

tong, y.l. (1990). *The multivariate normal distribution.* Springer-Verlag, New York.

vannman, k. (1995). A unified approach to capability indices. *Statist. Sinica*, **5**, 805–820.

wang, f.k. and chen, j.c. (1998). Capability index using principal components analysis. *Qual. Eng.*, **11**, 21–27.

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Paper received: 14 April 2011; revised: 7 March 2013.