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A Discussion on "Problems of Ruin and Survival in Economics: Applications of Limit Theorems in Probability" by R. Bhattacharya et al

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Abstract

This is a discussion on the paper: Problems of ruin and survival in economics: applications of limit theorems in probability, by Bhattacharya et al. Sankhya Ser. B, Vol. 76. In particular, we indicate some connections with actuarial risk theory.

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In a well written exposition, Professors Rabi Bhattacharya, Mukul Majumdar and Lizhen Lin have provided good insights into certain important problems of economics, making use of limit theorems of probability theory. While Sections 2 and 3 deal with two static problems of development economics, the last section concerns a widely applicable dynamic model of resource management.

Their analysis of a credit system model highlights the dramatic rise of the *probability of collapse due to defaults* as the money is parcelled into larger loans to a smaller number of borrowers. This certainly strengthens the popular perception that huge sums borrowed by "big fish" are at far greater risk. The model considered here may be simple, but the analysis is telling. One hopes this is taken note of in the raging debate on microfinance. It is particularly nice that a lucid account is given along the way of the classical results of Bahadur and Ranga Rao which play a key role.

We will now confine to some comments concerning the Lindley–Spitzer process considered in Section 4. To be more specific, we just indicate how using a dual risk process can help in getting more information on the invariant probability. Let $\{R_n\}, \{c_n\}, \{X_n\}, \{S_N\}, \pi$ be as in Section 4 of Bhattacharya, Majumdar and Lin (2013). Assume $X_0 = 0$. Put $Y_i = -Z_i, i \geq 1$.

So $\{Y_1 + Y_2 + \cdots + Y_n : n \geq 1\}$ is an $\mathbb{R}$-valued random walk. Define

$$\begin{aligned} \tau(u) &= \inf\{n \geq 1 : u + Y_1 + \cdots + Y_n < 0\},\ u \geq 0, \\ \psi(u) &= P(\tau(u) < \infty), \quad u \geq 0. \end{aligned}$$

Note that $\tau(u)$ is a stopping time for each u, and $\tau(u) = \inf\{n \geq 1 : Z_1 + \cdots Z_n > u\}, u \geq 0$. Put

$$M = \max\{0, \sup_{k\geq 1}(Z_1 + \cdots Z_k)\}.$$

For fixed $N > 0$ note that the events $\{\tau(u) \leq N\}$ and $\{X_N > u\}$ coincide. The following is known.

THEOREM 0.1. *Assume $P(Z_1 > 0) > 0$; (this is just (4.5) of Bhattacharya, Majumdar and Lin, 2013). The following are equivalent.*

(i) $E(Z_1) < 0$; (*this is just* (4.4) *of Bhattacharya et al., 2013*);

(ii) $\psi(u) < 1$ *for all* $u \geq 0$;

(iii) X_n *converges to* M *in distribution.*

In such a case,

$$\begin{aligned} 1 - \psi(u) &= \lim_{n\to\infty} P(X_n \leq u) \\ &= \pi((-\infty, u]) = \pi([0, u]),\ u \geq 0. \end{aligned}$$

See Theorems 6.3.1, 6.3.2 on pp. 233–235 of Rolski et al. (1999) for (i) $\Leftrightarrow$ (ii), and Corollary III.3.2 on p. 49 of Asmussen and Albrecher (2010) for other assertions. The connection with actuarial risk theory at the core of the above result is due to Prabhu (1961); see the comment on p. 48 of Asmussen and Albrecher (2010). In the context of risk theory, c_n is the premium income for the insurance company during n-th period; R_n is the claim payment during n-th period; so $u + Y_1 + \cdots + Y_n$ is the current surplus at the end of period n. Hence $\tau(u), \psi(u), 1 - \psi(u)$ are respectively the *ruin time, ruin probability, survival probability* for the insurer with initial capital u. The condition $E(Z_1) < 0$ is called the *net profit condition*, and is a basic requirement in risk models. Thus the survival probability for the risk model $\{Y_n\}$ is the limiting steady state distribution for the Lindley–Spitzer process $\{X_n\}$. From the above result we see that $\psi(x) = \pi((x, \infty)),\ x \geq 0$; thus ruin probability gives the right tail of the distribution π. Also $\pi(\{0\}) = 1 - \psi(0) > 0$.

In collective risk theory (that is, non-life insurance), continuous time *renewal/Sparre Andersen models* have been considered; it is generally assumed that (a) the interarrival times of claims are i.i.d. random variables, (b) claim sizes are also i.i.d. random variables, and (c) these families are independent. Also the premium rate is taken to be a positive constant; an implicit assumption is that settlement of claims is instantaneous. A net profit condition is necessary; else, ruin is certain however large the initial capital may be. Study of ruin problems for such models forms a major part. In these models, ruin can occur only at a claim arrival time. The risk process observed only at claim arrival times constitutes a random walk on $\mathbb{R}$; hence random walks like $\{Y_n\}$ above have attracted a lot of attention. While Asmussen and Albrecher (2010), Embrechts, Kluppelberg and Mikosch (1997) and Rolski et al. (1999) are encyclopaedic treatises on the subject, Ramasubramanian (2009) is an introductory account.

We shall assume $P(Z_1 > 0) > 0$ and $E(Z_1) < 0$. Let G denote the common distribution function of R_i; then the distribution on $[0, \infty)$ given by

$$dG_I(x) = \frac{1}{E(R_1)}(1 - G(x))dx$$

is the *integrated tail/stationary excess distribution* corresponding to G.

A main key to the analysis is the *Pollaczek–Khinchin formula* for ruin/survival probability. This says

$$1 - \psi(x) = \sum_{k=0}^{\infty}(1-p)p^k G_0^{(*)k}(x), \ x \geq 0, \tag{1}$$

where G_0 is the *conditional ascending ladder height distribution* given by $G_0(x) = G^+(x)/G^+(\infty), \ x \geq 0$, and $p = G^+(\infty)$, with G^+ denoting the ascending ladder height distribution corresponding to the random walk $\{S_n\}$. See Chap. 6 of Rolski et al. (1999) for definitions and a detailed exposition.

The classical *Cramer–Lundberg risk model* has been investigated extensively. Here, $\{c_n\}$ is a sequence of i.i.d. random variables having an exponential distribution with parameter $\lambda > 0$; we assume that the premium rate is 1. In this case, claims arrive according to a homogeneous Poisson process with arrival rate λ. So c_n denotes the premium income for insurance company between $(n-1)$-th and n-th claim arrivals. In the Cramer–Lundberg model, the conditional ladder distribution G_0 coincides with the integrated tail distribution G_I. Thus π is a geometric compound of G_I.

Note that $\rho = E(c_1)/E(R_1) - 1 > 0$ by the net profit condition. Using the above theorem and (1) in the Cramer–Lundberg model it can be shown that

$$\pi(\{0\}) = \frac{\rho}{(1+\rho)} = 1 - \lambda E(R_1).$$

See Section 5.3 of Ramasubramanian (2009).

We say the claim size distribution G (distribution of R_i in our notation) is *matrix exponential* if its density can be written as $\mathbf{a}e^{\mathbf{T}x}\mathbf{t}$, $x \geq 0$ for some row vector $\mathbf{a}$, square matrix $\mathbf{T}$, and some column vector $\mathbf{t}$. If $\mathbf{T}$ is a subintensity matrix and $\mathbf{t} = -\mathbf{T1}$, where $\mathbf{1}$ is the column vector with all components one, the distribution is said to be of *phase-type.* Besides the exponential distribution, the latter class includes hyperexponential, Erlangian, and Coxian distributions; the class is closed under finite mixtures, finite convolutions, and geometric compounds. For the Cramer–Lundberg model with phase-type (or even matrix-exponential) claim size distributions, explicit or computationally tractable expressions for ruin probability are known, of course, assuming net profit condition. See Section 8.3 of Rolski et al. (1999), and Chapter IX of Asmussen and Albrecher (2010). Clearly Example 4.1 of Bhattacharya et al. (2013) is a special case.

Let R_i have finite moment generating function in a neighbourhood $(-h_0, h_0)$ of 0. Suppose there is $\nu \in (0, h_0)$ such that

$$\int_0^\infty e^{\nu x} \frac{1}{(1+\rho)} dG_I(x) = 1.$$

Note the above gives an *exponentially tilted transformation* of the subprobability distribution $\frac{1}{(1+\rho)} dG_I(x)$. Also $\{e^{\nu S_n} : n \geq 1\}$ is a martingale, where $S_n = Z_1 + \cdots + Z_n$. Then a classical result due to Cramer and Lundberg states that

$$\lim_{t\to\infty} e^{\nu t}\psi(t) = \beta, \tag{2}$$

where the constant β can be explicitly determined. In risk theory $\nu > 0$ is called *Lundberg coefficient.* In this case it can also be shown that ν is the unique strictly positive solution to

$$\hat{m}_Z(s) = 1, \tag{3}$$

where $\hat{m}_Z$ is the moment generating function of Z_1. See Asmussen and Albrecher (2010), Embrechts et al. (1997), Ramasubramanian (2009) and Rolski et al. (1999). So π decays like $e^{-\nu t}$ in this case.

Cramer–Lundberg model with heavy tailed claim size distributions have also been studied by many authors. A distribution G supported on $(0, \infty)$ is said to be *subexponential*, denoted $G \in \mathcal{S}$, if

$$\lim_{t\to\infty} \frac{1 - (G * G)(t)}{1 - G(t)} = 2.$$

It can be shown that $G \in \mathcal{S}$ is equivalent to partial sums and partial maxima of $\{R_i\}$ having the same tail behaviour. This corresponds to actuaries' notion of dangerous risk. This class of heavy tailed distributions includes the class $\mathcal{R}_{-\alpha}$ of *regularly varying distributions with tail index* $\alpha > 1$, (hence, Pareto (α) with $\alpha > 1$), lognormal, heavy tailed Weibul distribution, etc. An introductory account is given in Ramasubramanian (2009), and one may look into Embrechts et al. (1997), Rolski et al. (1999) and references therein for more information on subexponential distributions. A landmark result, due to Embrechts and Veraverbeke, states if claim size distribution G is such that $G_I \in \mathcal{S}$ then

$$\lim_{t \to \infty} \frac{\psi(t)}{(1 - G_I(t))} = \rho^{-1}. \tag{4}$$

For the heavy tailed distributions mentioned above $G_I \in \mathcal{S}$, and hence the above asymptotic result is applicable. In such cases π and G_I have the same tail behaviour.

The case when $c_n = c > 0$ is a constant is a special case of the renewal model when claims arrive at regular intervals of length c; Examples 4.2, 4.3 and the subsequent discussion towards the end of Section 4 of Bhattacharya et al. (2013) fall under this category. Section 5.1 of Rolski et al. (1999) gives the probability generating function of M in terms of that of claim size distribution, when claims arrive at regular intervals and claim sizes are integer valued. Corollary 6.5.2 of Rolski et al. (1999) gives an explicit expression for ψ for the general renewal risk model with claim size distribution being Exp (δ); in this case

$$\psi(u) = \left(1 - \frac{\nu}{\delta}\right) e^{-\nu u}, \quad u \geq 0,$$

where the Lundberg coefficient $\nu > 0$ is defined by (3) whenever it exists.

In the renewal risk model, conditional ladder height distribution and the integrated tail distribution may not coincide in general; in fact, G_0 may not even be known. If Z_1 has finite moment generating function in a neighbourhood of 0, then Lundberg coefficient is once again defined by (3) whenever it exists. Theorem 6.5.7 of Rolski et al. (1999) gives the analogue of (2) for general renewal risk model whenever Lundberg coefficient exists. So π decays exponentially in this case as well.

For a large class of heavy-tailed claim size distributions, there is an interesting asymptotic result in the renewal risk set up. Let the claim size distribution G be such that $G_I \in \mathcal{S}$, that is, its integrated tail distribution is subexponential. Then Theorem 6.5.11 of Rolski et al. (1999) states that (4) holds in this case as well. This means that the asymptotic behaviour of

$\psi(\cdot)$ depends on the interarrival times only through $E(c_1)$. The interesting observation is that one needs to consider only the integrated tail distribution of the generic increment $(Z_n)^+$ rather than that of the more cumbersome ladder height distribution. See Rolski et al. (1999) for details.

Recall that $G \in \mathcal{R}_{-\alpha}$ if $1 - G(x) = x^{-\alpha}L(x),\ x > 0$, where L is a slowly varying function (at ∞.) If $G \in \mathcal{R}_{-\alpha}$ with $\alpha > 1$, then $G_I \in \mathcal{R}_{-(\alpha-1)}$. In particular, if G is Pareto (α) then G_I is Pareto $(\alpha - 1)$. So G_I is far more heavy tailed than G. Consequently by our analysis, it follows that the tail of π can be far heavier than that of R_i. In the context of design/management of water reservoirs, this means if rainfall is believed to follow a heavy tailed distribution, then the reservoir must be able to withstand an inflow of much higher order of magnitude; moreover, the results above give a handle on the order of magnitude.

Note that, at least in the Cramer–Lundberg model, (1) implies that the tail behaviour of π is the same as that of G_I from the first time the random walk $Z_1 + \cdots + Z_n$ overshoots level 0. For $G \in \mathcal{S}$ such that $G_I \in \mathcal{S}$, this tail behaviour is the same as that of $\max\{0, Z_1, \cdots, Z_n\}$. Thus, even though the rate of convergence to π may be quite slow, a crisis point could be sooner.

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