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On the distribution of burr with applications

Ratan Dasgupta

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Abstract Under certain conditions, the distribution of burr is shown to follow an extreme value distribution. In this context, a result on extremal process based on stationary sequence is proved. Some data sets are analyzed and applications of the results indicated.

Keywords Burr · Ornstein-Uhlenbeck process · Confidence ellipse · Preferred direction

Mathematics Subject Classifications (2010) Primary 62E20; Secondary 62-07 · 60G70 · 60F99

1 Introduction

A burr refers to the raised edge on a job. This occurs when the edge of the job after processing is not at the usual surface level. Burr may be of the form of a fine hairline on the edge of a freshly sharpened tool, or it may be a raised portion on a surface, after being punched by a heavy object.

Unwanted material remaining after a machining operation such as grinding, drilling, milling, or turning form the burr. For example, after perforation of a shaped metallic lametta by a punching machine, a small portion of the metal is found along the rim of the hole on opposite side. Formation of burr in machining causes a significant amount of machining costs.

R. Dasgupta (✉)
Indian Statistical Institute, Theoretical Statistics and Mathematics Unit,
Barnackpur Trunk Road, Kolkata, West Bengal, India
e-mail: ratandasgupta@gmail.com

Burr affects engine performance, reliability, and durability. The cost and time needed to remove burr by subsequent drilling and deburring operations are substantial.

Such raised edge may also occur when the jobs are given a specified shape by aluminum or iron casting, or while manufacturing objects by molding of marble-dust paste, plaster of Paris etc. For example, in some brands of molded tea cups, burr can be seen as a ridge on outer surface.

The maximum reading of surface displacements measured at some specified points on the raised edge of the job from the surface level is a measurement of burr. Thus it is a nonnegative quantity. High degree of burr may lead to malfunction of the job.

In Section 2, we describe a piercing operation on metal sheets and present two sets of observations on burr formed on the rim of pierced holes. The data sets suggest a positively skew distribution of burr. We consider extreme value theory of correlated normal random variables and it is seen that Type 1 extreme value distribution provide a reasonable fit to empirical distribution of burr. In order to account for unequal hole sizes in two data sets on burr, in Section 3 we further model the data by a continuous, strongly Markov, strictly stationary and Gaussian stochastic process, viz., Ornstein-Uhlenbeck Process. After suitably scaling the two data sets, efficiencies of two production processes are compared and the parameters of the common extreme value distribution, combined after scaling are obtained. We consider confidence-ellipse of the parameters and discuss its utility to check stability of the process. Semiparametric testing and data analysis are done to compare the processes. The process stability is checked by bootstrapping the distribution of coefficient of variation under extreme value model and also without this assumption. Confidence intervals for coefficient of variation are obtained. Estimated densities of coefficient of variation seem to be positively skew. In Section 4, we consider the case when burr has a preferred direction. A result on extremal process based on stationary sequence of random variables is proved in Appendix. This may be of independent interest. As an application, we show that an extreme value distribution may as well explain the distribution of burr having a preferred direction.

2 Operation and empirical data

We consider a perforating operation on jobs made of iron sheet. The job is a 100 mm × 150 mm sized L-shaped rectangular sheet and in all, four holes are made, two on each arm.

The operation is done on a 100 ton press, with 250 strokes per hour speed, thus it is a fast operation.

The operation is called piercing. Two holes are made simultaneously at a time. The job is a pierced L-shaped iron sheet, to be used in the chassis of light commercial vehicles or mini-lorries.

The burr is formed circularly surrounding the hole on the metal sheet at the other side of piercing. The high pressure exerted while piercing the metal sheet makes the contact surface unevenly displaced. On the opposite surface, the granules of the metal are sharply and unevenly raised along the rim of the circular hole forming the burr. The properties of metal grains and piercing load affect the magnitude of burr. The grain structure & texture of metals depend on composition, melting point as well as cooling rate, thermal and constitutional under-cooling, and convection, see e.g. Skrotzki et al. (2005).

The burr is removed later by chamfering using a drill. The least count of the instrument, a dial gauge, used to measure burr for the following data sets is 20 micron (μm) or 0.02 mm.

For the first data set of 50 observations on burr (in the unit of millimeter), the hole diameter is 12 mm and the sheet thickness is 3.15 mm. For the second data set of 50 observations, hole diameter and sheet thickness are 9 mm and 2 mm respectively. Hole diameter readings are taken on jobs with respect to one hole, selected and fixed as per a predetermined orientation. The two data sets relate to two different machines under comparison.

Data Set 1
0.04, 0.02, 0.06, 0.12, 0.14, 0.08, 0.22, 0.12, 0.08, 0.26, 0.24, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.28, 0.14, 0.16, 0.24, 0.22, 0.12, 0.18, 0.24, 0.32, 0.16, 0.14, 0.08, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.26, 0.18, 0.16.

Data Set 2
0.06, 0.12, 0.14, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.22, 0.14, 0.06, 0.04, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.04, 0.14, 0.26, 0.18, 0.16.

3 Some observations from empirical data and the model

The followings are the grouped frequency distribution of burr in mm for the two data sets.

The two data sets relate to two different machines for similar operations in the same site of a factory. The Kolmogorov-Smirnov test statistic based on the maximum fluctuation of absolute difference between two empirical distribution functions, for testing the null hypothesis that the two samples arise from same population has the value 0.5. This value is insignificant at 5% level.

Table 1 Frequency distribution for Data Set 1

Burr	.02	.04	.06	.08	.12	.14	.16	.18	.22	.24	.26	.28	.32
Freq.^{n}	2	3	3	4	4	6	7	4	4	6	3	1	3

Table 2 Frequency distribution for Data Set 2

Burr	.02	.04	.06	.08	.12	.14	.16	.18	.22	.24	.26	.28	.32
Freq.^{n}	3	4	5	1	3	8	6	5	7	4	2	0	2

The frequency distribution for the combined sample is given below.

The combined sample indicate the possibility of a heavy right tailed distribution for burr.

We now search for a suitable model based on the following consideration of burr formation. On a particular point i on the circular rim, identified in polar coordinates with suitable reference axis, the net amount of vertical displacement of metal from the surface level is the sum of two positive displacements X_{i1} and X_{i2}, from two sides of the sheet towards the direction of vertical pressure. Thus, total displacement equals to $X_{i1} + X_{i2} = X_i$, say. Typically the displacement of metal on the side of exerting pressure for piercing is more or less homogeneous and of lower order, compared to displacement of metal granules on the other side, which mainly contributes to formation of burr.

For m observations are taken on the circumference of the circle then the magnitude of burr is $\max_{1\le i\le m} X_i = X^*_m$, say. Here the value of m is 100.

Datasets 1 and 2 and subsequent Tables 1, 2 and 3 refer to the observations on burr $X^*_m = X^*$. Burr observations X^* refer to different holes and are assumed to be independent.

Under certain assumptions, the standardized maximum of a set of random variables has an extreme value distribution in the limit, e.g., see Galambos (1987). Therefore a candidate distribution for burr X^*_m may be any of the following three types.

Distribution 3.1 for $X^*_m = \max_{1\le i\le m} X_i$ occur when the tail behavior of the individual random variables X_i are similar to that of normal, lognormal or exponential random variables, among others.

$$F(x) = \exp(-e^{-(x-\mu)/\sigma}), -\infty < x < \infty, \mu \in (-\infty, \infty), \sigma > 0. \tag{3.1}$$

The following type of distributions are appropriate when the tail probability of the random variable X_i s are polynomially decaying, apart from a slowly varying multiplicative factor; if any.

$$\begin{aligned} F(x) &= 0, \qquad\qquad\qquad x < \mu \\ &= \exp(-(\tfrac{x-\mu}{\sigma})^{-\alpha}),\ x \ge \mu,\ \ \mu \in (-\infty, \infty),\ \ \alpha,\ \sigma > 0. \end{aligned}$$

Burr being a nonnegative quantity, one may consider distributions starting from $\mu = 0$; the above then reduces to

$$\begin{aligned} F(x) &= 0, \qquad\qquad\quad x < 0 \\ &= \exp(-(\tfrac{x}{\sigma})^{-\alpha}),\ x \ge 0,\ \alpha,\ \sigma > 0. \end{aligned} \tag{3.2}$$

Table 3 Frequency distribution for combined data

Burr	.02	.04	.06	.08	.12	.14	.16	.18	.22	.24	.26	.28	.32
Freq.^{n}	5	7	8	5	7	14	13	9	11	10	5	1	5

The following type of distributions may occur when the random variables are bounded above.

$$\begin{aligned} F(x) &= \exp(-(\tfrac{\mu-x}{\sigma})^{\alpha}),\ x<\mu,\ \alpha>0 \\ &= 1, \qquad\qquad\qquad x\geq\mu,\ \mu\in(-\infty,\infty),\ \ \alpha,\ \sigma>0. \end{aligned} \tag{3.3}$$

The last type of distributions are negatively skew and the other types are positively skew. The type of distribution appropriate for a particular problem may be found from the skewness of the observed frequency distribution and the probability plot of empirical and theoretical distributions.

Distributions 3.3 are negatively skew and have an upper end point. The magnitude of burr does not have an upper bound in general. The combined sample suggests the possibility of a right tailed distribution for burr. Therefore the model 3.3 is excluded from our consideration. For Eq. 3.1, we have

$$\log(-\log F(x)) = -(x-\mu)/\sigma \tag{3.4}$$

Thus if the model is appropriate then $\log(-\log F_n(x))$ should have an approximate linear relationship with x, where F_n is the sample c.d.f.

Estimation of the parameters of an extreme value distribution by the method of maximum likelihood is computationally involved and requires numerical iteration. One may initially estimate the parameters μ and σ from the intercept and slope of the fitted line via the method of least squares. For the model 3.2, the linear relationship is of the form

$$\log(-\log F(x)) = -\alpha\log x + \alpha\log\sigma \tag{3.5}$$

One may also check the adequacy of the model 3.2 and estimate the parameters, replacing F by F_n in Eq. 3.5. The type of the distribution and the parameters appearing therein are the machine, operator and job characteristics.

The mean displacement of the plane metal-surface towards the direction of pressure, depends on the amount of pressure exerted, thickness of the sheet, texture of the material etc. The actual displacement at a particular point is therefore a random variable fluctuating around the mean displacement. The individual displacements over different points on the rim are correlated in general. Net displacement being sum total of displacements due to several independent causes, the individual fluctuations may be assumed to be normal. The extreme value distribution corresponding to normal random variables may then explain the distribution of burr, which corresponds to maximum of normally distributed fluctuations.

The fluctuations may be weakly correlated for extreme value distribution to hold. From the extreme value theory of correlated normal random variables, we state (without proof) a large sample result; see, e.g., Galambos (1987). Since the autocorrelation function of Ornstein-Uhlenbeck process is exponentially decaying, the limiting distribution stated below may be successfully applied to the observations taken from this process at discrete points of time.

Theorem 1 *Let $Z_n(r)$ be the maximum of a Gaussian stationary sequence $X_1, X_2, \cdots, X_n$ with zero expectation, unit variance, and correlations $r_m = EX_1X_{1+m}$. Let $a_n = \frac{1}{b_n} - \frac{1}{2}b_n(\log\log n + \log 4\pi),\ b_n = (2\log n)^{-1/2}$.*

Then $(Z_n(r) - a_n)/b_n \xrightarrow{d} H_1(x) = \exp(-e^{-x}), -\infty < x < \infty$ *provided,* $r_m \log m \to 0$, *as* $m \to \infty$.

We next plot the cumulative distribution function for Data set 1 and 2 after appropriate transformations to compare with theoretical distributions 3.1 and 3.2. The same procedure is repeated for combined data set as well. In all the cases the distribution 3.1 provides a reasonable fit; see Figs. 1, 2, 3, 4, 5 and 6. The values of R^2 for linear regression are quite high; indicating that the parameter of the distributions may be estimated graphically. The results are summarized in Table 4.

One may compare the magnitude of burr over two data sets taking into consideration the *unequal* hole sizes in two data sets. For the first data set holes have 12 mm diameter, and for the second data set holes are of 9 mm diameter. Thus the perimeter of hole in first and second sets are 12π mm and 9π mm respectively. The displacements of metal over points on rim are assumed to be normal and stationary. The overall displacements may then be modeled by Ornstein-Uhlenbeck process $V(s)$, a stationary continuous Gaussian process.

The Ornstein-Uhlenbeck $(O - U)$ process is continuous, strongly Markov, strictly stationary and Gaussian.

Apart from some pathological examples, the above properties *characterize* $O - U$ process.

In some operations metal-sheets are given a finer thickness by pressing these through a series of rollers and then the sheets are given a regular shape by trimming the edges; see Dasgupta (2006) where efficiencies of production processes in terms of minimizing wastage due to trimming are compared by

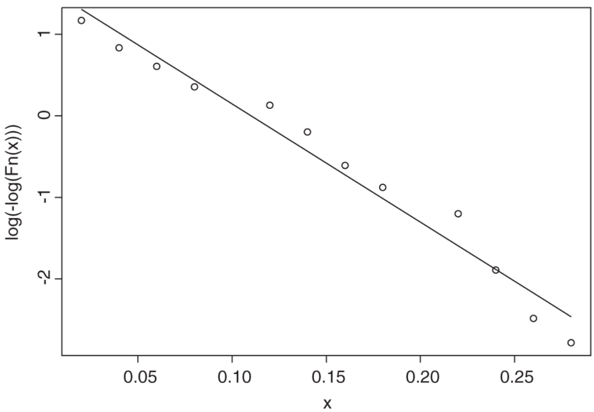


Fig. 1 Extreme value fit Eq. 3.1 of 50 chassis components—data 1

Fig. 2 Extreme value fit Eq. 3.2 of 50 chassis components—data 1

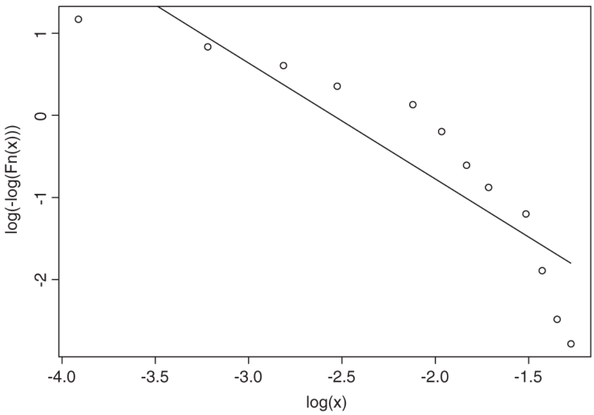


Fig. 3 Extreme value fit Eq. 3.1 of 50 chassis components—data 2

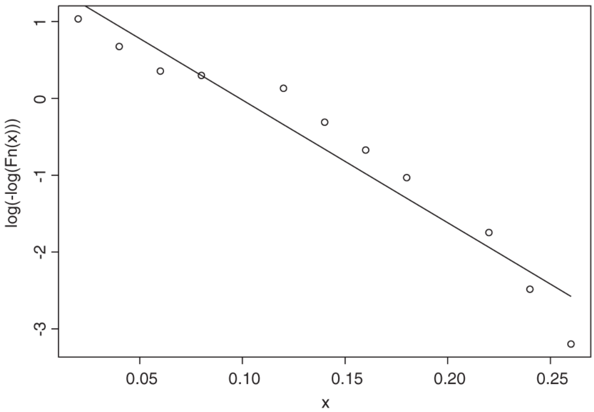
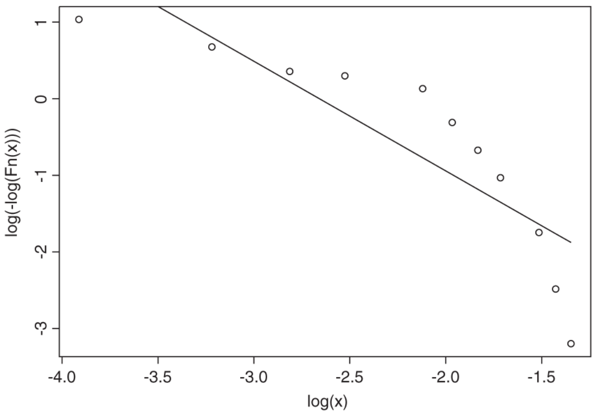


Fig. 4 Extreme value fit Eq. 3.2 of 50 chassis components—data 2



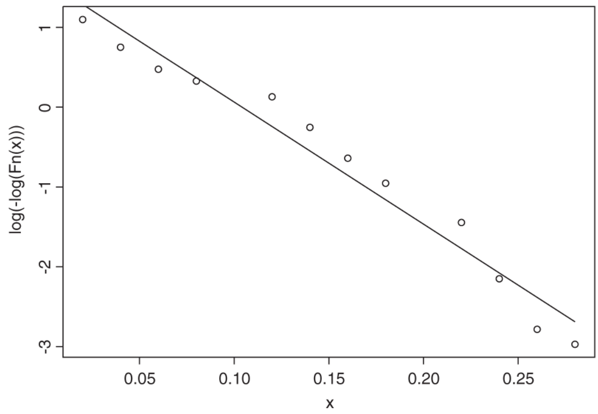


Fig. 5 Extreme value fit Eq. 3.1 of 100 chassis components—data 1,2

this stochastic process. Ornstein-Uhlenbeck process satisfies the following differential equation.

$$dV(s) = -\beta V(s)ds + \gamma dB(s), \ \beta > 0, \ \gamma > 0 \tag{3.6}$$

where $B(s)$ is the standard Brownian motion, β is the drift parameter; $\beta V(s)$ is a restoring force directed towards origin proportional to the distance $V(s)$.

Using the relationship $V(s) = e^{-\beta s} B[\gamma^2(e^{2\beta s} - 1)/2\beta]$, see e.g. Karlin and Taylor (1981), one may write

$$\overline{\lim}_{t\to\infty} \left[\frac{\gamma^2}{\beta}(1 + o(1)) \log t \right]^{-1/2} \sup_{0\le s\le t} \mid V(s) \mid = 1, \ \text{a.s.} \tag{3.7}$$

Thus, the value of $(2 \log t)^{-1/2} \sup_{0\le s\le t} \mid V(s) \mid$ may be taken as an estimate of $\gamma/(2\beta)^{1/2}$, which equals to the standard deviation of the Ornstein-Uhlenbeck process. In the present case, t represents the rim-perimeter of holes, on which the burr is formed. Dissecting the formed hole at a point on the rim and

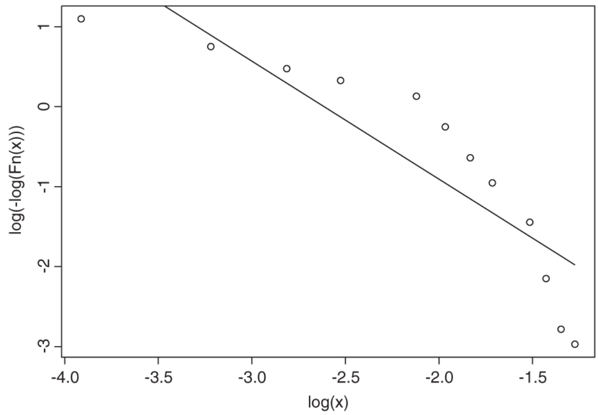


Fig. 6 Extreme value fit Eq. 3.2 of 100 chassis components—data 1,2

Table 4 Fitting the models 3.1 and 3.2 to burr

Data	Model	Parameter	R^2	Residual standard error	No. of distinct points
1,	(3.1)	$(\mu, \sigma) = (0.1101,\ 0.0690)$	0.9684	0.2427	13
1,	(3.2)	$(\alpha, \sigma) = (1.4123,\ 0.0782)$	0.8007	0.6098	13
2,	(3.1)	$(\mu, \sigma) = (0.0987,\ 0.0626)$	0.9364	0.3609	12
2,	(3.2)	$(\alpha, \sigma) = (1.4309,\ 0.0700)$	0.7414	0.7278	12
1& 2,	(3.1)	$(\mu, \sigma) = (0.1041,\ 0.0654)$	0.9619	0.2821	13
1& 2,	(3.2)	$(\alpha, \sigma) = (1.4751,\ 0.0732)$	0.7809	0.6763	13
Scaled 1& 2,	(3.1)	$(\mu, \sigma) = (0.0417,\ 0.0241)$	0.9504	0.3248	25
Scaled 1& 2,	(3.2)	$(\alpha, \sigma) = (1.5303,\ 0.0294)$	0.7586	0.7168	25

then by straightening the perimeter along the X-axis, one may visualize the displacements along Y-axis.

The displacements, caused by high press stroke, over the rim of pierced hole at different points s, $0 \leq s < t$, are modeled by the Gaussian process $V(s)$. When the press is withdrawn, the lower boundary of the curve is pulled back to the surface level of the plane sheet due to elasticity of the material. Leveling the curve at one end does not however change the magnitude of $\sup_{0 \leq s \leq t} \mid V(s) \mid$, measured at the other end. Each observation on burr may then be identified with an independent realization of $\sup_{0 \leq s \leq t} \mid V(s) \mid$. The two sets of process parameters (β_1, γ_1) and (β_2, γ_2) corresponding to two piercing operations may then be compared by the ratio of average burrs $\overline{b}_1$ and $\overline{b}_2$ computed from the two data sets. Thus from Eq. 3.7, one may write

$$\left[\frac{\gamma_2^2}{\beta_2}(1 + o(1)) \log t_2\right]^{-1/2} \Big/ \left[\frac{\gamma_1^2}{\beta_1}(1 + o(1)) \log t_1\right]^{-1/2} \simeq \overline{b}_1/\overline{b}_2. \tag{3.8}$$

Hence,

$$\frac{\gamma_1/(2\beta_1)^{1/2}}{\gamma_2/(2\beta_2)^{1/2}} \simeq \left[\frac{\log t_2}{\log t_1}\right]^{1/2} \frac{\overline{b}_1}{\overline{b}_2}. \tag{3.9}$$

The l.h.s. of Eq. 3.9 represents the ratio of standard deviations of two Ornstein-Uhlenbeck processes associated with two piercing operations and thus it is a measure of efficiency $e_{(2,1)} = e_2/e_1$ of Piercing operation 2 with respect to Piercing operation 1.

For the two data sets under consideration, $t_1 = 12\pi$ mm, $t_2 = 9\pi$ mm and the estimated efficiency, $\hat{e}_{(2,1)} = 1.030 \simeq 1$.

Scaling factor $(2 \log t)^{-1/2}$ of $\sup_{0 \leq s \leq t} \mid V(s) \mid$, used to estimate the parameter $\gamma/(2\beta)^{1/2}$, may lead one to consider the scaled version of two data sets using the scale $(2 \log t)^{-1/2}$. Observe that the computed efficiency $\hat{e}_{(2,1)}$ is near 1 for two scaled data sets. Also, the Kolmogorov-Smirnov test statistic, for testing the null hypothesis that (after scaling) the two samples arise from same population, has the value 0.6 (recall that the previous value of the statistic, corresponding to the original data is 0.5). This value is insignificant at 5% level. The above findings may lead one to estimate the parameters of the common extreme value distribution of burr (per unit length of 1 mm on rim-perimeter) from combined observations; i.e., two samples combined together after scaling, based on the information $t_1 = 12\pi$ mm, $t_2 = 9\pi$ mm; for the first and second data sets respectively.

Extreme value distribution 3.1 and 3.2 probability-plots for scaled and combined data sets with 100 observations are shown in Figs. 7 and 8 respectively. Estimated parameters are shown in Table 4.

For model 3.2, the parameter α refers to the polynomial decay of tail of the distribution of individual displacements $X = X_i$ i.e., $P(X > x) \sim x^{-\alpha} L(x), x \to \infty$, where $L(x)$ is a slowly varying function.

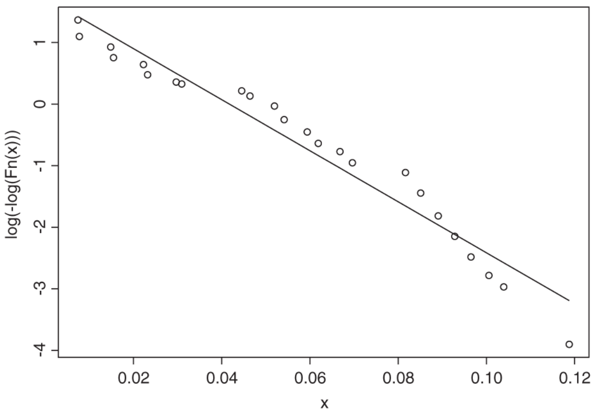
In terms of higher value of R^2 and lower value of 'residual standard error' for straight-line fit in the probability plots of the data points, it is seen that the distribution 3.1, i.e., Type 1 extreme value distribution provide a better fit compared to extreme value distribution 3.2, corresponding to variables that are bounded above; see Table 4. The same feature is revealed in Figs. 1–8. Figures 1, 3, 5, 7 (corresponding to model 3.1), show better linear fit compared to Figs. 2, 4, 6, 8 (corresponding to model 3.2), respectively. We, however, ignore a very small but systematic bias present near the middle of the Figs. 1, 3, 5, 7; (model 3.1). The bias is prominent in the other set of Figs. 2, 4, 6, 8; (model 3.2). This indicates Eq. 3.1 is a better model than Eq. 3.2.

In all the above cases, distributions 3.1 provide a satisfactory fit to the data compared to distributions 3.2, indicating that the individual displacements at different points over the rim may possibly be modeled by (correlated) normal random variables.

A kernel density estimate of burr for the combined data after scaling indicates a right tailed distribution with more mass concentration for values to the left of mode, see Fig. 9.

We did not modify the density estimate for burr towards lower tail, as $-\infty < x < \infty$ in the proposed model 3.1; much like a normal model is considered appropriate for nonnegative random variables like height or weight.

Fig. 7 Extreme value fit Eq. 3.1 of 100 components—scaled data 1,2



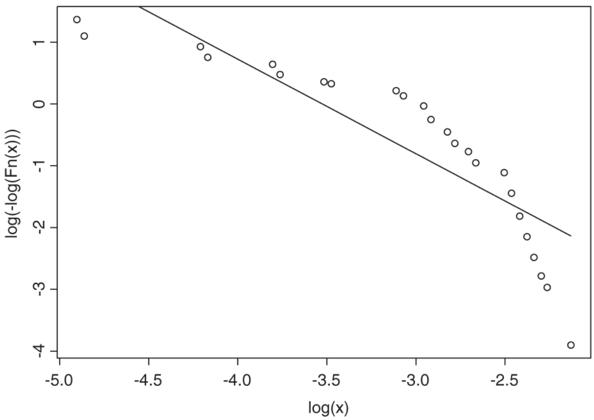


Fig. 8 Extreme value fit Eq. 3.2 of 100 components—scaled data 1,2

We shall further observe that fit to model 3.1 is better when the parameters of the burr distribution are estimated via method of moments, vide Fig. 10.

Likelihood equations for model 3.1 do not admit explicit solutions and hence require numerical iterative methods. The m.l.e. of μ and σ are given below.

$$\hat{\mu} = -\hat{\sigma} \log \left\{ \frac{1}{n} \sum_{i=1}^{n} e^{-X_i/\hat{\sigma}} \right\}, \quad \hat{\sigma} = \overline{X} - \frac{1}{n} \sum_{i=1}^{n} X_i e^{-(X_i - \hat{\mu})/\hat{\sigma}}$$

One may estimate $e^{-(X_i-\hat{\mu})/\hat{\sigma}}$ appearing in the above expression by $-\log \hat{F}_X(X_i)$, where $\hat{F}$ is taken as empirical c.d.f. calculated from available data. Based on the scaled data set 1 & 2 combined, consisting of 100 observations, the maximum likelihood estimates thus obtained are as follows.

$$\hat{\mu} = 0.051398, \quad \hat{\sigma} = 0.052583$$

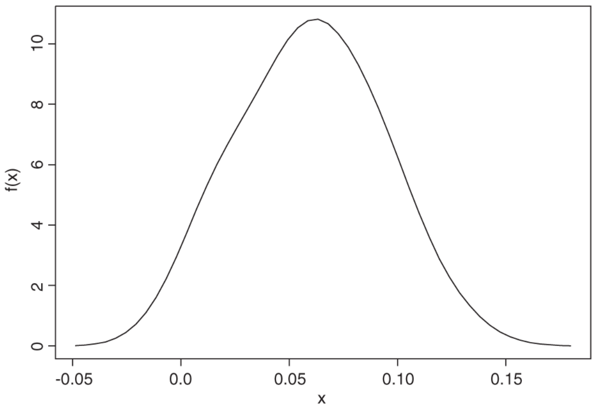


Fig. 9 Kernel density of scaled burr, combined data (width = .075)

Figure 10 describes the closeness of empirical distribution function of combined data on scaled burr with proposed model 3.1 where the parameters are estimated by

(i) method of regression: $(\hat{\mu}, \hat{\sigma}) = (0.0417,\ 0.0241)$
(ii) method of maximum likelihood: $(\hat{\mu}, \hat{\sigma}) = (0.051398,\ 0.052583)$
(iii) method of moments: $(\hat{\mu}, \hat{\sigma}) = (0.04614798,\ 0.02344912)$.

A robust procedure of estimating parameters by fitting a regression line via minimising the L_1 distance provides the following estimates. L_1 distance assigns less weight to the outliers compares to squared error loss.

(iv) method of L_1 regression (M-estimate) : $(\hat{\mu}, \hat{\sigma}) = (0.03991528,$ $0.0270572)$

Distribution 3.1 has mean $(\mu + \gamma\sigma)$ and variance $\pi^2\sigma^2/6$, where $\gamma =$ .5772... is Euler's constant; these were equated with observed mean (= .05968281) and variance (= .000904485) of scaled burr with combined data of 100 observations to estimate of parameters via method of moments.

From Fig. 10 it is clear that the method of moments provides a better fit of the model to the empirical distribution, amongst the four above mentioned procedures.

For testing $H_o : F = F_o$, a robust nonparametric test viz., the Kolmogorov-Smirnov test statistic $D_n = \sqrt{n}\,\sup_x |F_n(x) - F_o(x)|$ may be used. However, note that we have estimated the parameters of the model from the data and one needs to use adjusted KS statistic, adjustment obtained mainly via simulation, e.g., see Marks (2007), where the leading term of approximations is D_n in various models F, under which the relevant parameters are estimated; the next term of the approximation is $O(n^{-1/2})$. In view of that we may use the critical values of D_n for the present purpose as a first approximation, n being 100 for the combined burr data. D_n has the values 1.90, 2.66, 1.30 and 1.97 for cases (i), (ii), (iii) and (iv) respectively. The third value is insignificant at 5% level (with

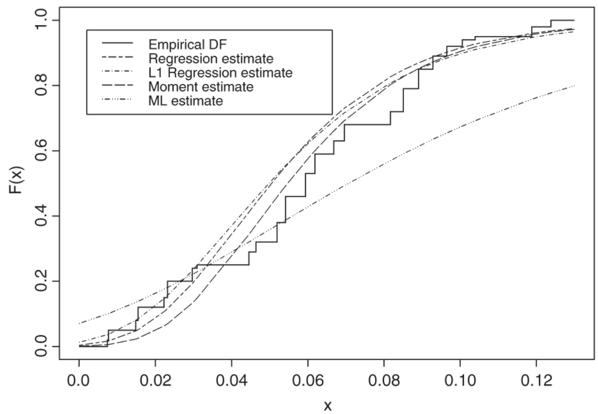


Fig. 10 Empirical and fitted distributions of scaled burr, combined data

critical value 1.36). Other three values are significant even at a much lower level of 0.50% (with critical value 1.73).

Thus we accept the model 3.1 with estimated parameters given in (iii).

This incidentally reveals an interesting feature of the data. Sometimes the method of moments provides a better fit in comparison to the method of maximum likelihood, with initial estimate taken from probability plot.

It may be mentioned that Jureckova (2003), Beirlant et al. (2006) and Koning and Peng(2007) studied nonparametric statistical methods for comparing within a class of Gumbel type extreme value distribution: $H_o : F \in D(G_\gamma)$, where $G_\gamma(x) = \exp\{-(1+\gamma x)^{-1/\gamma}\}$ with $\gamma > 0$; see Hüsler and Peng (2008) for these and related references. Distributions 3.1 are of different class.

The extreme value distribution 3.1 may be used to monitor the stability of the production process. From the long-term data collected at a steady state, suppose the process parameters are found to be stable at (μ_o, σ_o). One may also assign the specified target values for (μ_o, σ_o).

To check the stability/meeting the specification of the current production, one may estimate the parameters (μ, σ) on batches of sample sizes n and compute the maximum likelihood estimator $(\hat{\mu}, \hat{\sigma})$. The asymptotic dispersion matrix of the m.l.e. $(\hat{\mu}, \hat{\sigma})$ is given by

$$\Sigma = n^{-1} \begin{pmatrix} 1.10867\sigma^2 & .25696\sigma^2 \\ .25696\sigma^2 & .60793\sigma^2 \end{pmatrix}. \tag{3.10}$$

See e.g., (22.65) and (22.110) of Johnson et al. (1995).

Thus, $(\hat{\mu} - \mu_o, \hat{\sigma} - \sigma_o)\Sigma^{-1}(\hat{\mu} - \mu_o, \hat{\sigma} - \sigma_o)'$ is an approximate χ^2 random variable. One may then compute the confidence ellipse of (μ, σ) with approximate confidence coefficient α as

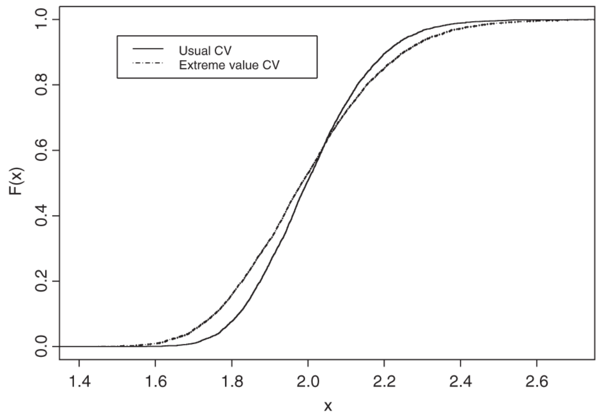
$$\{(\breve{\mu}, \breve{\sigma}) : (\breve{\mu} - \mu_o, \breve{\sigma} - \sigma_o)\hat{\Sigma}^{-1}(\breve{\mu} - \mu_o, \breve{\sigma} - \sigma_o)' \le \chi^2_{\alpha,n-3}\} \tag{3.11}$$

Here $(\breve{\mu}, \breve{\sigma})$ represents a particular element of confidence ellipse. $\hat{\Sigma}$ is the estimated value of Σ from Eq. 3.10 via estimated value of σ^2 from samples. Values of $(\hat{\mu}, \hat{\sigma})$ when plotted for several batches of samples on two dimensional confidence-ellipse (Eq. 3.11) may indicate whether the process is stable, or there is a systematic preferred direction present, as evident from the possible cluster of plotted points.

Other robust nonparametric based procedures, like L_1 estimates may also be used to check for process control. As already seen, in this particular data set performance of L_1 estimates seems to be poor compared to moment estimates.

We may further analyze the data with less model assumptions. The coefficient of variation (C.V.) is the ratio of standard deviation to mean, and this is considered to be an index of process stability. Simple random sample with replacement of size 100 were selected out of 100 data points observed and the bootstrap sample C.V. calculated; and the whole procedure were independently repeated 5,000 times (bootstrap simulation) to approximate the distribution of C.V. The Empirical Distribution Function (EDF) of bootstrapped C.V. is plotted in Fig. 11.

Fig. 11 EDF of bootstrapped CV (usual and extreme value)

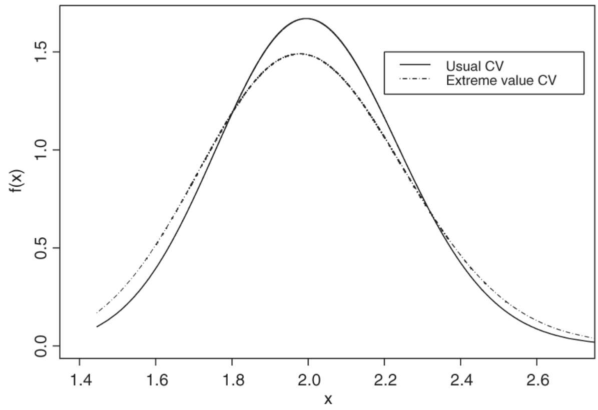


Also note that for a random variable X with extreme value distribution 3.1 $E(X) = \mu + \gamma\sigma,\ V(X) = \pi^2\sigma^2/6$, where $\gamma \approx 0.5772$. Thus the ratio of the parameters $\sigma/\mu = C.V_{\text{ext}}$, say, after estimating σ and μ via the above relations from data, may be considered as an alternative index of stability for the fitted distribution 3.1. That is, take $\hat{E}(X) = \overline{x},\ \hat{V}(X) = s_x^2$; the sample mean and variance calculated from data and obtain an estimate of σ and μ by solving above two equations, then obtain an estimate of the ratio $\sigma/\mu = C.V_{\text{ext}}$. The EDF of bootstrapped $C.V_{\text{ext}}$ is also shown in Fig. 11.

The 95% bootstrapped confidence interval for C.V. and $C.V_{\text{ext}}$ turn out to be (1.649, 2.408) and (1.736, 2.328) respectively.

In Fig. 12 the kernel density estimate of the bootstrapped C.V. and $C.V_{\text{ext}}$ are shown. Figures are drawn by automatic bandwidth selection with selected kernel as Gaussian using the software SPLUS. The densities are drawn on the range of data and these indicate positively skew densities for C.V.s.

Fig. 12 Estimated density of bootstrapped CV (usual and extreme value)



4 Preferred direction and weak convergence

In some operations, the burr formed has a preferred direction. As a result, the measurements taken around that location provide readings nearly the maximum value of displacement. The operator or machine may exert more force towards a particular region of the job and less force in the remaining region. For example, while sharpening a knife the burr is formed near the edge; the far end near the tip is processed last and force exerted at the tip may vary compared to other regions. Such preferred direction of burr may also occur when the material processed is of non-homogeneous texture. One may assume the displacements at a point i on rim-surface from its original level of sheet, to be a scaled amount $c_i X_i;\ 0 < c_i < 1,$ where $c_i \uparrow 1$, towards the preferred direction of burr. The magnitude of burr Y is then $\max_{1\le i\le m} c_i X_i$. However, due to randomly located non-homogeneous material texture on the job, there may be intermittently full displacements X_i at soft regions, in between the scaled down displacements. One may then assume the displacements over a point i is either (1) the full amount X_i, or (2) a scaled down amount $c_i X_i;\ c_i \uparrow 1, 0 < c_i < 1$. In other words, we are considering a model where *some* of the displacements are dampened by scale factors less than 1. Then the magnitude of burr Y has the following bound.

$$\max_{1\le i\le m} c_i X_i \le Y_m \le \max_{1\le i\le m} X_i. \tag{4.1}$$

Let the random variables X_i be stationary, although these may be dependent. One may deduce the limit law of standardized Y_m to be an extreme value distribution under certain conditions. Zeevi and Glynn (2004) consider estimating tail decay along with characterization of dependence assumptions required for almost-sure limit of normalized extreme values in stationary sequences, see also the references therein.

The following theorem proved in Appendix provides an approximation bound of the sequence Y_m in terms of maximum of the stationary sequence X_i. This extends A3 of Dasgupta et al. (1981). The result holds irrespective of weak limit existing for standardized maximum of the sequence X_i.

Theorem 2 *Let $(X_i, i \ge 1)$ be a sequence of stationary random variables and there exist constants $a_m,\ b_m$ such that the standardized maximum,*

$$Z_m = b_m^{-1}(\max_{1\le i\le m} X_i - a_m) = O_p(1), \tag{4.2}$$

i.e., the distribution of $\{Z_m, m \ge 1\}$ is tight. Also let $(c_i;\ 0 < c_i < 1,\ c_i \to 1)$, be a sequence of non-decreasing constants and let $i_o = i_o(n)$ be a sequence of positive integers satisfying

$$n^{-1} i_o(n) \to 0, \quad c(i_o) = c(i_o(n)) = 1 - o(|\, a_n^{-1} b_n|);\ n \ge 1. \tag{4.3}$$

Then for any Y_m satisfying Eq. 4.1*, i.e.,* $\max_{1\le i\le m} c_i X_i \le Y_m \le \max_{1\le i\le m} X_i$*; one has,*

$$b_m^{-1}(Y_m - \max_{1\le i\le m} X_i) \to 0, \textit{ in probability.} \tag{4.4}$$

Further if,

$$f(i) = |\ a_i b_i^{-1}|\ \ (\to \infty) \tag{4.5}$$

is non-decreasing in i and $\overline{\lim}_{i\to\infty} \frac{f(i\delta)}{f(i)} < \infty$, *for every fixed* $\delta > 0$, *then the condition* 4.3 *on c is implied by* $c(i) = 1 - o(|a_i^{-1} b_i|)$.

Remark 1 Variables $\{X_i, i \geq 1\}$ need not be continuous or independent for Theorem 2 to hold. Since the distribution of standardized $\max_{1\leq i\leq m} X_i$, i.e., Z_m is tight; it follows that the distribution of standardized Y_m is also tight, with possibilities of weak limit(s) by Prokhorov's theorem. However a unique weak limit may not exist, when the variables are not continuous or independent. Thus approximation 4.4 of Theorem 2 is not merely an extension of weak convergence result, but provides useful bounds for burr Y in terms of X, even in absence of a unique limit in a broad setup.

Condition 4.3 is weaker than Eq. 4.5. Theorem 2 is applicable even if Eq. 4.5 does not hold, e.g., if $f(i) = |\ a_i b_i^{-1}|$ is bounded above; as for example, in the case of normal random variables the (milder) assumption 4.3 requires $c(i_o) = c(i_o(n)) = 1 - o(|\ a_n^{-1} b_n|) = 1 - o(1) \to 1;\ n \geq 1$, and one may consider $i_o = i_o(n) = [\log(e + n)] \to \infty,\ \ n^{-1} i_o(n) \to 0$.

Remark 2 Instead of a single sequence $\{c_i\}$ of dampening factor affecting the burr, there may be a number of sequences of dampening factors $\{c_{i1}\}, , \cdots, \{c_{ip}\}$; where each sequence satisfies the conditions of Theorem 2. These sequences of factors may be of different magnitudes, depending on different types of soft regions located in the job. In such a situation, let the displacement at a point i contributing to burr be either dampened by one of the sequence of c's depending on that *particular* point i, or let it be the full displacement X_i. The resultant burr Y then satisfies the following.

$$\min_{j=1,\cdots,p} \max_{1\leq i\leq m} c_{ij} X_i \leq Y_m \leq \max_{1\leq i\leq m} X_i. \tag{4.6}$$

Towards the end of Appendix, a modified proof is outlined extending Theorem 2 for Y_m satisfying the bound Eq. 4.6.

Corollary 1 *Let* $b_m^{-1}(\max_{1\leq i\leq m} X_i - a_m)$ *converge weakly to an extreme value distribution. Then under the conditions of Theorem 2, the standardized* Y_m *of Eqs.* 4.1 *and* 4.6 *converge weakly to the above mentioned extreme value distribution with same standardization.*

Corollary 2 *If the random variables* X_i *satisfy the assumptions of Theorems 1 and 2, then the standardized* Y_m *of Eqs.* 4.1 *and* 4.6 *converge weakly to the extreme value distribution* 3.1.

5 Concluding remarks

The distributions of burr in industrial products are investigated from empirical and theoretical considerations. Extreme value distribution of type 1 is seen to provide a reasonable fit to the distribution of burr. Individual displacements on the raised edge over different points may be modeled by an Ornstein-Uhlenbeck process $V(s)$. Based on maximum fluctuation of $|V(s)|$, realized as burr, the parameters of two different production processes and hence efficiency of the productions are estimated. These reflect the characteristic of the job-material, particular job production under consideration, and the machine. Confidence interval and density estimates of coefficient of variation, a measure of stability for production process are obtained by bootstrap. Some general results on extremal process based on a stationary sequence of random variables are obtained, which may be of independent interest.

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Appendix

Here we prove Theorem 2, extending A3 of Dasgupta et al. (1981). The earlier proof of Dasgupta et al. (1981) has to be suitably modified so as to extend the stated results from 'iid continuous random variables' to 'stationary random variables'. We provide a modified proof elaborated below.

Assume that the distribution of the stationary sequence X_i's are nondegenerate as the proof is trivial otherwise.

Consider a fixed sequence $i_o = i_o(m) = o(m)$. If the value of $\max_{1\le j\le m} X_j$ is attained at a single index of X, then from stationarity of the random variables, $P\{\cup_{i=1}^{i_o}(X_i = \max_{1\le j\le m} X_j)\} = \frac{i_o}{m}$; since the index of X, attaining the maximum value of X, is uniformly distributed on the set $\{1, \cdots, m\}$.

In a similar fashion, if the maximum is attained for two distinct values of the index, then the probability that both the indices are in the set $\{1, \cdots, i_o\}$, is $\frac{i_o(i_o-1)}{m(m-1)} \le \frac{i_o}{m}$, etc.

Thus, in general, the probability that at least one index with maximum value of X will lie within the set $\{i_o + 1, \cdots, m\}$ is at least $(1 - \frac{i_o}{m})$. In other words,

$$P\{\cup_{i=i_o+1}^{m}(X_i = \max_{1\le j\le m} X_j)\} \ge 1 - \frac{i_o}{m} \to 1, \text{ as } m \to \infty.$$

Now using the fact that $\{c_i\}$ is a nondecreasing sequence, one gets

$$P\{\cup_{i=i_o+1}^{m}(c_i X_i = \max_{1\le j\le m} c_j X_j)\} \ge 1 - \frac{i_o}{m} \to 1.$$

On the intersection of above two sets, we have from Eq. 4.1

$$c_{i_o} \max_{1\le i\le m} X_i \le Y_m \le \max_{1\le i\le m} X_i.$$

Thus it suffices to show that

(1) $b_m^{-1}(1 - c_{i_o}) \max_{1\le i\le m} X_i \to 0, \quad$ in distribution.
Now from Eq. 4.2, $Z_m = b_m^{-1}(\max_{1\le i\le m} X_i - a_m)$ is bounded in probability. Write,

$$b_m^{-1}(1 - c_{i_o}) \max_{1\le i\le m} X_i = (1 - c_{i_o})(a_m b_m^{-1} + Z_m).$$

Since $c_{i_o} \to 1$, it is sufficient to show that,

$$(1 - c_{i_o})|a_m b_m^{-1}| \to 0,$$

i.e., $c(i_o(m)) = 1 - o(|\, a_m^{-1} b_m|)$. Hence the first part of the theorem.
To prove the second part, note that $(1 - c(k)) f(k) = o(1)$, there exists a sequence $h(k) \to \infty$ such that

(2) $(1 - c(k)) f(k) h(k) = o(1)$. (e.g., if $((1 - c(k)) f(k) \le \epsilon_k \to 0$, then one may take $h(k) = \epsilon_k^{-1/2}$.)
Let $n = n(k)$ be such that

(3) $f(n(k)) < f(k-1)h(k-1) \le f(n(k)+1)$.
Such a choice of $n = n(k)$ is possible as $f(i) \to \infty$, as $i \to \infty$.
(For a fixed k the middle term of the above is finite and as $f(n(k)) \to \infty$, $n(k) \to \infty$; (3) is ensured.)
From the assumption $\overline{\lim}_{i\to\infty} \frac{f(i\delta)}{f(i)} < \infty$, for every fixed $\delta > 0$, it follows that

(4) $k^{-1} n(k) \to \infty$.
To see this, suppose $n(k) << k$, divide the terms of (3) by $f(k)$; then both r.h.s. and l.h.s. terms of (3) are bounded above in view of $\overline{\lim}_{i\to\infty} \frac{f(i\delta)}{f(i)} < \infty$, whereas the middle term $\to \infty$, leading to a contradiction.
Next, for $m = 1, 2, \cdots$ define, $k(m) = k$ if $n(k) \le m < n(k+1)$. Then

(5) $f(m) < f(n(k+1)) < f(k)h(k)$ see (3). Now from (2)

(6) $(1 - c_{k(m)}) f(m) \to 0$, where $\frac{k(m)}{m} \le \frac{k(m)}{n(k(m))} \to 0$, from (4).

Hence the second part.

We have a different bound for burr Y given in Eq. 4.6. Proof of Theorem 2 remains valid for every fixed $j = 1, \cdots, p$; as the sequence of constants $\{c_{1j}, \cdots, c_{mj}\}$ satisfy the conditions therein. Theorem 2 therefore, holds for the burr Y satisfying the bound of Eq. 4.6, as the minimum is taken over j, a finite p number of terms.

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