ON SOME CHARACTERIZATIONS OF STATISTICS

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SUMMARY. A statistic T is a mapping of a measurable space $(\boldsymbol{X}, \boldsymbol{A})$ into another measurable space $(\boldsymbol{Y}, \boldsymbol{B})$. Corresponding to each family $\boldsymbol{P}$ of probability measures on $(\boldsymbol{X}, \boldsymbol{A})$ the statistic T defines a family $\boldsymbol{Q}$ of probability measures on $(\boldsymbol{Y}, \boldsymbol{B})$. The classical problem of distribution theory is to find $\boldsymbol{Q}$ for a given $\boldsymbol{P}$ and T. In this paper the authors consider the reverse problem and try to characterize T given $\boldsymbol{P}$ and $\boldsymbol{Q}$. The general problem is unfortunately too complicated and the authors had to restrict themselves to the family of normal measures. For instance, if $\boldsymbol{P}$ be a sufficiently large class of normal measures on the p-dimensional Euclidean space R^p and $\boldsymbol{Q}$ be the class of normal measures on R^1 then T is necessarily linear. A few similar problems are also considered.

1. Introduction

Given two measurable spaces $(\boldsymbol{X}, \boldsymbol{A})$ and $(\boldsymbol{Y}, \boldsymbol{B})$, a statistic T is a measurable mapping of the former into the latter. For each $x \in \boldsymbol{X}$, the statistic T maps x into an element $y = Tx$ in $\boldsymbol{Y}$ and the mapping is done in such a way that for each $B \in \boldsymbol{B}$, the inverse image $T^{-1}B \in \boldsymbol{A}$. For each family $\boldsymbol{P}$ of probability measures on $\boldsymbol{A}$, the statistic T then generates, in a natural manner, a family $\boldsymbol{Q}$ of probability measures on $\boldsymbol{B}$. The classical problem of distribution theory is to determine the family $\boldsymbol{Q}$, given the other six elements of the problem, namely, $\boldsymbol{X}$, $\boldsymbol{A}$, $\boldsymbol{P}$, $\boldsymbol{Y}$, $\boldsymbol{B}$ and the statistic T. For each $P \in \boldsymbol{P}$, the problem is to determine the corresponding $Q = PT^{-1}$ defined by the identity

$$Q(B) = P(T^{-1}B) \quad \text{for all } B \in \boldsymbol{B}.$$

On the other hand one may raise question of the following types :

(i) Given two probability models $(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{P})$ and $(\boldsymbol{Y}, \boldsymbol{B}, \boldsymbol{Q})$, one may enquire about the totality of all mappings (statistics) of $(\boldsymbol{X}, \boldsymbol{A})$ into $(\boldsymbol{Y}, \boldsymbol{B})$, such that each $P \in \boldsymbol{P}$ is carried into some $Q \in \boldsymbol{Q}$.

(ii) Suppose each of the two families $\boldsymbol{P} = \{P_\theta\}$ and $\boldsymbol{Q} = \{Q_\theta\}$ is indexed by a parameter θ. One may enquire about the family of statistics T such that, for each θ

$$P_\theta T^{-1} = Q_\theta.$$

In particular, one may enquire about model-preserving transformations (see Basu, 1968), i.e. about one-to-one bimeasurable transformations $(\boldsymbol{X}, \boldsymbol{A})$ into itself such that, for each $P \in \boldsymbol{P}$,

$$PT^{-1} = P.$$

(iii) Given $(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{P})$, one may enquire about the family of all ancillary statistics (see Basu, 1959), i.e. about measurable transformations of $(\boldsymbol{X}, \boldsymbol{A})$ into some $(\boldsymbol{Y}, \boldsymbol{B})$ such that the family $\boldsymbol{Q}$ of induced measures is a one-element set.

We give below three simple examples of questions of the above three types.

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Example 1 : Let x_1, x_2 be two independent normal variables with equal means μ, $-\infty < \mu < \infty$ and equal variances σ^2, $0 < \sigma^2 < \infty$. Then, every linear mapping of the real plane to the real line (i.e. statistics of the type ax_1+bx_2+c, a, b and c are real numbers) induces a normal distribution on the real line irrespective of μ and σ^2. Is the converse true ? That is, given that the distribution of the real-valued statistic $g(x_1, x_2)$ is normal for all (μ, σ^2), does it follow that the statistic g is linear ? This question falls under type (1) discussed before and the answer is 'no'. But, however, if the class of probability measures is widened, namely, the means of x_1 and x_2 are μ_1, μ_2, $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$, then, under certain conditions, the answer is 'yes'.

Example 2 : Let x be a normal variable with zero mean and variance σ^2, $0 < \sigma^2 < \infty$. For what statistics T mapping the real line into the real line, the probability distribution (for all σ^2) of Tx is the same as that of x ? It is shown that a necessary and sufficient condition that T is model-preserving is that for almost all x

$$Tx = \phi(x)\,|x|$$

where $\phi(x)$ is a skew-symmetric function (i.e. $\phi(-x) = -\phi(x)$) taking only the two values -1 and $+1$.

Example 3 : Let x be distributed as in Example 2. What is the class of ancillary statistics, i.e. statistics that induce distributions that are free of the parameter σ^2 ? This question is closely connected with that in Example 2 and the answer is that a necessary and sufficient condition in order that T is a non-trivial ancillary statistic is that T partitions the real line into two sets A and B such that each of them is essentially skew-symmetric about the origin.

(A set E is said to be essentially skew-symmetric about the origin is for almost all x it is true that one and only one of the two numbers x and $-x$ belongs to E).

The above three are typical examples of the kind of questions we want to answer. Two particular methods for solving some isolated questions are given in this paper. In the next section we discuss the above three problems and in the last section, we pose and solve some further problems.

2. Solutions of the examples

We take the three examples, mentioned in Section 1, in the reverse order.

Let x be a normal variable with zero mean and variance σ^2, $0 < \sigma^2 < \infty$ and let A be an ancillary set i.e.

$$P(x \in A\,|\sigma^2)$$

is equal to some constant α for all possible values of σ^2. First of all, we prove that

$$\alpha = 0, \tfrac{1}{2}, 1 \quad \text{only}$$

and in case $\alpha = \frac{1}{2}$, the set A must be skew-symmetric about the origin.

*Excepting possibly for an exceptional set with Lebesgue measure zero.

The argument runs as follows :

(i) Since, for all values of σ^2, the distribution of x is symmetric about the origin, the statistic $|x|$ is sufficient.

(ii) It is easy to check that the family of distributions induced (on the positive part of the real line) by the statistic $|x|$ is complete.

(iii) Since A is an ancillary set, it follows (see Basu, 1959) that A is independent of $|x|$, i.e. the conditional probability

$$P(x \in A \mid |x|, \sigma^2)$$

is equal to the unconditional probability α for almost all values of $|x|$.

(iv) For a given value of $|x|$, say $|x| = u$, the conditional distribution of x is concentrated at the two points $-u$ and $+u$ with equal probabilities $\frac{1}{2}$ and $\frac{1}{2}$.

(v) Hence the conditional probability of the set A given $|x| = u$ can take only one of three values 0, $\frac{1}{2}$ and 1, and since this conditional probability is equal to α for almost all values of u, it follows that $\alpha = 0, \frac{1}{2}$ or 1.

(vi) The two extreme values 0 and 1 for α are trivial in the sense that they can be attained if and only if A is essentially equal to the null set or the whole set.

(vii) Thus, a non-trivial ancillary set A has its constant probability $\alpha = \frac{1}{2}$ and, in this case, it is clear that, for almost all values u, the set A must contain one and only one of the two numbers $-u$ and $+u$. That is, A must be skew-symmetric about the origin.

(viiii) Conversely, it is obvious that every skew-symmetric set is an ancillary set with constant probability $\frac{1}{2}$.

The above chain of arguments apply equally well to a somewhat more general set-up, thus leading to the following lemma.

Lemma 1 : *If the probability distribution of the real random variable x involves a parameter θ in such a manner that, for each value of θ, the distribution of x is continuous and symmetric about the origin, and further, if the class of probability distributions of $|x|$ is boundedly complete, then, every non-trivial ancillary set (similar regions) that could be defined in terms of x must have probability $\frac{1}{2}$ and every such set must be essentially skew-symmetric about the origin.*

(In case, the distribution of x is not continuous we have to add one more clause, namely,

$$P(x = 0 \mid \theta) = 0 \quad \text{for all } \theta).$$

Now, let us turn our attention to the problem raised in Example 2. Here, x is distributed as normal with zero mean and variance σ^2, $0 < \sigma^2 < \infty$, and suppose that we have a statistic $T(x)$ such that $T(x)$ and x are identically distributed for all values of σ^2.

Since x and $-x$ have the same distribution for all values of σ^2, it follows that $T(-x)$ has the same distribution as that of $T(x)$ and x. Consider the statistic

$$H(x) = T(x)+T(-x).$$

Clearly, $H(x)$ is a symmetric function of x i.e. $H(x)$ is a function of $|x|$, the mean value of $H(x)$ is zero for all σ^2. Since $|x|$ is a complete sufficient statistic, it follows that $H(x)$ must be zero for almost all values of $|x|$. Thus,

$$T(x) = -T(-x) \text{ for almost all } x.$$

In other words, T must be essentially skew-symmetric about the origin. It follows that $|T(x)|$ is essentially a function of $|x|$.

Now, consider the statistic

$$u(x) = |T(x)| - |x|.$$

Since the mean value of u is identically zero for all values of σ^2 and since $u(x)$ is a function of the complete sufficient statistic $|x|$, we have

$$u(x) = 0 \quad \text{for almost all } x.$$

That is,

$$T(x) = \phi(x)\,|x|$$

where $\phi(x)$ is essentially skew-symmetric and takes only the two values -1 and $+1$.

Conversely, if $\phi(x)$ is an essentially skew-symmetric function taking the two values -1 and $+1$, then $\phi(x)$ is an ancillary statistic taking the two values -1 and $+1$ with equal probabilities and $\phi(x)$ is independent of the complete sufficient statistic $|x|$. And then it follows at once that $\phi(x)\ |x|$ has the same distribution as that of x.

Let us observe, once again, that in the above argument we have nowhere used the normality of the variable x. What we have used is the symmetry (about the origin) of the distribution of x for each σ^2 (which implied the sufficiency $|x|$) and the completeness of the statistic $|x|$. As before, the same arguments, apply to the class of distributions mentioned in Lemma 1.

Lemma 2 : *Let the conditions on the probabiliy distribution of a random variable x be as mentioned in Lemma 1. Then $T(x) = \phi(x)\ |x|$ and x have the same ditributions and conversely, if x and $T(x)$ have the same distributions, then $T(x) = \phi(x)|x|$.*

Now, we are in a position to consider Example 1. In this case, we have two independent random variables x_1 and x_2 each with a normal distribution with mean

μ, $(-\infty < \mu < \infty)$ and variance σ^2, $(0 < \sigma^2 < \infty)$. We demonstrate below that there are plenty of non-linear functions of x_1 and x_2, which induce normal distributions on real line.

We may note that $y_1 = (x_1 - x_2)/\sqrt{2}$ and $y_2 = (x_1 + x_2)/\sqrt{2}$ are both independent normal variates with equal variances σ^2 and that the former has mean zero. Let ϕ be an arbitrary function on the real line such that it is essentially skew-symmetric about the origin and takes the two values -1 and $+1$ only. Then,

$$y_1^* = \phi(y_1)\,|y_1|$$

has the same distribution as that of y_1 and is independent of y_2. The statistic y_1^* is an example of non-linear function which has a normal distribution. Any linear combination of y_1^* and y_2 will also provide another such example. In this case an example of a model-preserving non-linear transformation would be

$$x_1^* = (y_2 - y_1^*)/\sqrt{2} \quad \text{and} \quad x_2^* = (y_2 + y_1^*)/\sqrt{2}.$$

In the next section, we prove a few isolated results on normality-preserving transformations.

3. Normality-preserving transformations

Lemma 3 : *Let* $\boldsymbol{x}' = (x_1, \ldots, x_p)$ *be distributed as multivariate normal with mean vector* $\boldsymbol{\mu}' = (\mu_1, \ldots, \mu_p)$ *and covariance matrix* $\boldsymbol{\Sigma} = (\sigma_{jj'})$, $j, j' = 1, 2, \ldots, p$ *which is positive definite. Let* $T = T(x_1, \ldots, x_p)$ *be a real-valued function in* $x_1, \ldots, x_p$. *Then if* $E(T)$ *and* $V(T)$ *exist,* $E(T)$ *and* $V(T)$ *are infinitely differentiable with respect to the parameters and*

$$\frac{\delta^2}{\delta\mu_j\,\delta\mu_{j'}} E[\exp\,(\sqrt{-1}\;uT)] = \frac{\delta}{\delta\sigma_{jj'}} E\exp\,(\sqrt{-1}\;u\,T) \quad \textit{for } j \neq j' \textit{ and for all real } u$$

$$= 2\,\frac{\delta}{\delta\sigma_{jj}} E\exp(\sqrt{-1}\;uT) \quad \textit{for } j = j' \textit{ and for all real } u.$$

Proof : It is well known that if $\boldsymbol{x}$ follows an exponential class of distributions with parameters $\boldsymbol{\theta}$, and if $E\beta(\boldsymbol{x})$ exists, then $E\beta(\boldsymbol{x})$ is infinitely differentiable with respect to $\boldsymbol{\theta}$. This proves the first part of Lemma 3. The second part will be obvious, if we can establish

$$\frac{\delta^2}{\delta\mu_j\,\delta\mu_{j'}} f = \frac{\delta}{\delta\sigma_{jj'}} f \qquad \text{for } j \neq j', j, j' = 1, 2, \ldots, p$$

$$= 2\,\frac{\delta}{\delta\sigma_{jj}} f \quad \text{for } j = j' = 1, 2, \ldots, p$$

where $f = f(\boldsymbol{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma})$ is the density function of the normal variates $\boldsymbol{x}$.

This can be proved directly by noting

$$\frac{\delta \log|\Sigma|}{\delta\sigma_{jj'}} = 2\sigma^{jj'} \quad \text{for } j \neq j',\ j, j' = 1, 2, \ldots, p$$
$$= \sigma^{jj} \quad \text{for } j = j' = 1, 2, \ldots, p$$

and

$$\frac{\delta\sigma^{jj'}}{\delta\sigma_{tt'}} = -(\sigma^{jt}\sigma^{tj'} + \sigma^{jt'}\,\sigma^{t'j'}) \quad \text{for } t \neq t', j, j', t, t' = 1, 2, \ldots, p$$
$$= -(\sigma^{jt}\sigma^{tj'}) \quad \text{for } t = t',\ j, j', t = 1, 2, \ldots, p.$$

Alternately, we may prove Lemma 3 by using the unicity property of Fourier transforms, namely,

$$\int \ldots \int e^{\sqrt{-1}\ \boldsymbol{u}'\boldsymbol{x}}\, g(\boldsymbol{x})d\boldsymbol{x} = 0 \text{ for all real } \boldsymbol{u} \Longleftrightarrow g(\boldsymbol{x}) = 0,$$

in

$$\int \ldots \int e^{\sqrt{-1}\ \boldsymbol{u}'\boldsymbol{x}} \left[\frac{\delta^2}{\delta\mu_j\,\delta\mu_{j'}} f - \phi\ \frac{\delta}{\delta\sigma_{jj'}} f \right] dx = 0 \quad \text{for all real } \mu \text{ and } \phi = 1 \text{ if } j \neq j'$$
$$= 2 \text{ if } j = j'.$$

The above expression can be established with the help of the following two relations namely

$$\int \ldots \int e^{\sqrt{-1}\,\boldsymbol{u}'\boldsymbol{x}}\ \frac{\delta^2}{\delta\mu_j\delta\mu_{j'}}\ fd\boldsymbol{x} = \frac{\delta^2}{\delta\mu_j\delta\mu_{j'}}\ E(e^{\sqrt{-1}\,\boldsymbol{u}'\boldsymbol{x}})$$
$$= -u_j u_{j'} E(e^{\sqrt{-1}\ \boldsymbol{u}'\ \boldsymbol{x}})$$

and

$$\phi \int \ldots \int e^{\sqrt{-1}\ \boldsymbol{u}'\ \boldsymbol{x}}\ \frac{\delta}{\delta\sigma_{jj'}} fdx = \phi\ \frac{\delta}{\delta\sigma_{jj'}}\ E(e^{\sqrt{-1}\,\boldsymbol{u}'\boldsymbol{x}})$$
$$= -u_j u_{j'}\ E(e^{\sqrt{-1}\ \boldsymbol{u}'\ \boldsymbol{x}}).$$

Theorem 1 : *Let x be normal with mean μ, $-\infty < \mu < \infty$ and variance one and let $T = T(x)$, a real-valued function, be normal with mean $\nu = \nu(\mu)$ and variance $\psi = \psi(\mu)$. If, moreover, $T(x)$ sets up a one-to-one relation between the domain of x and the range of T, then T must be linear in x.*

Proof : By Lemma 3, $\nu(\mu)$ and $\psi(\mu)$ are infinitely differentiable with respect to μ. Hence, taking differentiation under the integral sign, we get

$$\frac{d\nu}{d\mu} = \frac{d}{d\mu} ET(x) = \text{cov}\,(x, T(x))$$

$$\frac{d^2\nu}{d\mu^2} = \text{cov}\,[(x-\mu)^2, T(x)],\ \frac{d\psi}{d\mu} = \text{cov}\,[x, (T(x)-\nu)^2]$$

and

$$\frac{d^2\psi}{d\mu^2} + 2\left(\frac{d\nu}{d\mu}\right)^2 = \text{cov}\,[(x-\mu)^2, (T-\nu)^2] \quad \ldots \quad (1)$$

Now, since $T(x)$ and x are one-to-one functions, we have (writing $y = T(x)$)

$$x = T^{-1}(y).$$

Hence
$$\mu = E(x) = \frac{1}{\sqrt{2\pi\psi}} \int_{-\infty}^{\infty} [T^{-1}(y)] e^{-\frac{1}{2}\left(\frac{y-\nu}{\sqrt{\psi}}\right)^2} dy, \qquad \ldots (2)$$

and
$$1 = \frac{1}{\sqrt{2\pi\psi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{1}{2}\left(\frac{y-\nu}{\sqrt{\psi}}\right)^2} dy. \qquad \ldots (3)$$

Differentiating (2) and (3) with respect to μ and using (1), we get

$$2\psi^2 = 2\psi \left(\frac{d\nu}{d\mu}\right)^2 + \left(\frac{d\psi}{d\mu}\right)^2 \qquad \ldots (4)$$

and

$$\frac{d\psi}{d\mu}\left[\frac{d^2\psi}{d\mu^2} + 2\left(\frac{d\nu}{d\mu}\right)^2\right] + 2\psi\left(\frac{d\nu}{d\mu}\right)\left(\frac{d^2\nu}{d\mu^2}\right) = 0. \qquad \ldots (5)$$

Differentiating (4) with respect to μ and using (5), we get

$$\frac{d\psi}{d\mu}\left[2\psi + \left(\frac{d\nu}{d\mu}\right)^2\right] = 0$$

i.e.
$$\frac{d\psi}{d\mu} = 0 \text{ i.e. } \psi = a, \text{ which is a constant.} \qquad \ldots (6)$$

Using this in (4), we get

$$\left(\frac{d\nu}{d\mu}\right)^2 = \psi = a > 0 \quad \text{and so} \quad \nu = c\mu + b \qquad \ldots (7)$$

where $c^2 = a$, and b is constant. Now

$$E(T(x) - cx - b] = 0 \quad \text{for all } \mu$$

and since x is a complete sufficient statistic for μ, we get

$$T(x) = cx + b \quad \text{almost everywhere}$$

The above proof is somewhat lengthy. We are thankful to the referee for supplying us the following alternative shorter proof :

Let N stand for the distribution function of a standardised normal random variable. Let T, ν, ψ be as in Theorem 1 and assume without loss of generality that $\nu(0) = 0$ and $\psi(0) = 1$. Then, if $\boldsymbol{B}$ is the Borel field, we get

$$\exp\left[-\frac{1}{2\psi}(T(x)-\nu)^2 + \frac{1}{2}\{T(x)\}^2\right] dN(x) = E_N\left[\exp\left(-\frac{1}{2}\mu^2 + \mu x\right) \Big| T^{-1}\boldsymbol{B}\right] dN(x) \text{ a.e.}$$

$$= \exp\left(-\frac{1}{2}\mu^2 + \mu x\right) dN(x) \text{ a.e. since } T^{-1}\boldsymbol{B} = \boldsymbol{B}.$$

Moreover, we may note that if we delete the condition on $T(x)$, in Theorem 3, to be distributed as normal, we shall need the following assumptions :

(i) The first four moments of $T(x)$ exist and

(ii) $E(T-\nu)^2(x_j-\mu_j) = 0$ for $j = 1, 2, \ldots, p$.

The authors have come to notice an unpublished work as an abstract of Lehmann and Stein (1953) in which they consider problems of a similar nature. For instance, if in Theorem 1 we assume that the variance of T is constant in μ, then the linearity of T would follow from the result stated in the above abstract.

References

BASU, D. (1959) : The family of ancillary statistics. *Sankhyā*, **21**, 247-256.

——— (1968) : On sufficiency and invariance, (To be published in Professor S. N. Roy Memorial Volume).

LEHMANN, E. L. and Stein, C. (1953) : The totality of transformations leaving a family of normal distributions invariant (abstract). *Ann. Math. Stat.*, **24**, 142.

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Equating the coefficient of u^4 in (6), we get

$$\frac{\delta\psi}{\delta\mu_j} = 0 \quad \text{for } j = 1, 2, \ldots, p$$

i.e. ψ is a function of $\sigma_{11}, \ldots, \sigma_{pp}$ only. ... (9)

Using this in (8) and equating the coefficient of u^2, we get

$$\left(\frac{\delta\nu}{\delta\mu_j}\right)^2 = \frac{\delta\psi}{\delta\sigma_{jj}} \quad \text{for } j = 1, 2, \ldots, p.$$

i.e. $\frac{\delta\nu}{\delta\mu_j}$, $j = 1, 2, \ldots, p$, are functions of $\sigma_{11}, \ldots, \sigma_{pp}$ only. ... (10)

Using (8) in (6), we get

$$\frac{\delta\nu}{\delta\sigma_{jj}} = 0 \quad \text{for } j = 1, 2, \ldots, p. \qquad \ldots (11)$$

i.e. ν is a function of $\mu_1, \ldots, \mu_p$ only.

Noting (10) and (11), we get

$$\nu = \sum_{j=1}^{p} a_j \mu_j + b \qquad \ldots (12)$$

where $a_1, a_2, \ldots, a_p$ and b are constants.

Hence, we get

$$E(T(\boldsymbol{x}) - \sum_{j=1}^{p} a_j x_j - b) = 0 \quad \text{for all } \boldsymbol{\mu} \text{ and } \sigma_{jj},\ j = 1, 2, \ldots, p$$

and since the family of distribution for $\boldsymbol{\mu}$ is complete, we get

$$T(x) = \sum_{j=1}^{p} a_j x_j + b, \text{ almost everywhere.}$$

Corollary : *Let $\boldsymbol{x}' = (x_1, \ldots, x_p)$ be distributed as multivariate normal with mean vector $\boldsymbol{\mu}' = (\mu_1, \ldots, \mu_p)$, $-\infty < \mu_i < \infty$, $i = 1, 2, \ldots, p$ and covariance matrix $\boldsymbol{\Sigma} = (\sigma_{jj'})$, $0 < \sigma_{jj} < \infty$ which is positive definite, and $\rho_{jj'} = \sigma_{jj'}/(\sigma_{jj}\sigma_{j'j'})^{1/2}$ for $j \neq j'$, being given constant values. Let $T_t(\boldsymbol{x})$ be real-valued functions of $\boldsymbol{x}$, $t = 1, 2, \ldots, k$,. If $\boldsymbol{T}' = (T_1(x), \ldots, T_k(x))$ is distributed as multivariate normal with mean vector $\boldsymbol{\nu}' = (\nu_1, \ldots, \nu_k)$ and covariance matrix $\psi = (\psi_{tt'})$, $t, t' = 1, 2, \ldots, k$, then*

$$\boldsymbol{T} = \boldsymbol{A}x + \boldsymbol{B} \text{ almost everywhere}$$

where $\boldsymbol{A} : k \times p$ and $\boldsymbol{B} : k \times 1$ are matrices not depending on μ_i and σ_{ii}, $i = 1, 2, \ldots, p$.

Now, we remark that if in corrollary, we take $k = p$ and make the assumption that the transformations $\boldsymbol{T}(\boldsymbol{x})$ and $\boldsymbol{x}$ are one-to-one, we can possibly narrow down the probability measures of x by taking $\boldsymbol{\Sigma}$, a particular matrix for the class of positive definite matrices. This result is similar to that of Theorem 1.

Moreover, we may note that if we delete the condition on $T(x)$, in Theorem 3, to be distributed as normal, we shall need the following assumptions :

(i) The first four moments of $T(x)$ exist and

(ii) $E(T-\nu)^2(x_j-\mu_j) = 0$ for $j = 1, 2, \ldots, p$.

The authors have come to notice an unpublished work as an abstract of Lehmann and Stein (1953) in which they consider problems of a similar nature. For instance, if in Theorem 1 we assume that the variance of T is constant in μ, then the linearity of T would follow from the result stated in the above abstract.

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——— (1968): On sufficiency and invariance. (To be published in Professor S. N. Roy Memorial Volume).

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