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SOME RESULTS ON CHARACTERISTIC FUNCTIONS AND CHARACTERIZATIONS OF THE NORMAL AND GENERALIZED STABLE LAWS

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SUMMARY. A variety of results are presented in this paper. Some results are established concerning how the behaviour of a characteristic function (c.f.) in the neighbourhood of the origin affects the existence of moments of the corresponding distribution function (d.f.); it is shown, *inter alia*, that if two c.f.'s coincide at a sequence of points tending to the origin and one of them corresponds to a d.f. having moments of all orders *and* uniquely determined by its moments, then the two c.f.'s coincide. These results, which find application in the later sections, are presented in Section 2.

We introduce and study a class of distributions which we call "generalized stable (GS) laws", in Section 3. These provide a natural generalization of the stable laws and include the class of semi-stable laws considered by P. Levy.

Our main results concern characterizations (through regression) of the Normal and the GS laws; these are stated in Section 1 and proved in Section 4.

1. MAIN THEOREMS AND CHARACTERIZATIONS OF THE NORMAL AND GS LAWS

In Rao (1967), it was proved that if X_1 and X_2 are independent and identically distributed (i.i.d.) random variables (r.v's) with $EX_1 = 0$, and $0 < \text{var } X_1 < \infty$, then the relation

$$E(a_1X_1+a_2X_2 \mid b_1X_1+b_2X_2) = 0 \text{ almost surely (a.s.)} \qquad \dots \quad (1.1)$$

implies the Normality of X_1. Pathak and Pillai (1968)* have shown that the weaker condition : $a_1b_1+a_2b_2 = 0$, involving only the coefficients in (1.1), implies that var X_1 $< \infty$ and hence that X_1 is either Normal or degenerate. In Section 4, we undertake a fuller investigation of (1.1) and show that, if it holds, then, depending on the coefficients a_1, a_2, b_1 and b_2, X_1 follows a law which is arbitrary, degenerate, Normal or (non-Normal) 'semi-stable' in the sense of Levy (1937). If $a_1 b_1 = 0$ and/or $a_2 b_2 = 0$, the distribution of X_1 is either arbitrary or degenerate. We shall ignore these cases in what follows. We may also assume without loss of generality that $|b_2| \leqslant |b_1|$. We then have the following theorem.

*Published in this issue, see pp. 141–144.

Theorem 1.1 : *Let X_1 and X_2 be i.i.d.r.v.'s with $EX_1 = 0$ satisfying the relation (1.1), with all the coefficients non-zero. Let $\alpha = -a_2/a_1$ and $\beta = b_2/b_1$ with $|\beta| \leqslant 1$.*

(i) *If $\alpha\beta < 0$, then $X_1 = 0$ a.s.*

(ii) *If $\alpha\beta > 0$ and $|\beta| = 1$, then $X_1 = 0$ a.s. if $|\alpha| \neq 1$, and can be arbitrary real-valued/arbitrary if $|\alpha| = 1$.*

(iii) *If $\alpha\beta > 0$ and $|\beta| < 1$, let λ be the unique real solution of the equation $|\alpha|\,|\beta|^{\lambda-1} = 1$; then ($f$ is the c.f. of X_1)*

(a) *$X_1 = 0$ a.s. if $\lambda \leqslant 1$ or if $\lambda > 2$;*

(b) *X_1 is Normal (possibly degenerate) if $\lambda = 2$; and*

(c) *if $1 < \lambda < 2$, then f is an infinitely divisible law, with its d.f. absolutely continuous, having moments of all orders $<\lambda$ but not of orders $\geqslant \lambda$. (f is a 'semi-stable' law in the sense of Levy (1937); for the properties of the Levy representation of such a law, see the proof in Section 4). λ will be called the exponent of the law concerned.*

We also establish a related result placing conditions on the two regressions which restrict the distribution of X_1 to only two possibilities, viz., Normal or degenerate.

Theorem 1.2 : *Let X_1 and X_2 be i.i.d.r.v.'s with $EX_1 = 0$. If*

$$E(a_1X_1+a_2X_2\,|\,b_1X_1+b_2X_2) = 0 = E(b_1X_1+b_2X_2\,|\,a_1X_1+a_2X_2) \quad a.s. \quad \ldots \quad (1.2)$$

where none of the coefficients is zero, then X_1 is Normal (possibly degenerate) iff $a_1b_1+a_2b_2 = 0$ and otherwise degenerate.

Passing from the case of two r.v.'s to more, Rao (1967) also proved that if $X_1, \ldots, X_n$ are i.i.d., and $0 < \text{var}\, X_1 < \infty$, then the condition

$$E(\Sigma\, a_jX_j\,|\,\Sigma b_jX_j) = 0 \quad \text{a.s.}, \quad \ldots \quad (1.3)$$

implies the Normality of X_1 provided the further conditions below are satisfied :

(i) a pair of corresponding coefficients, say a_n and b_n, have the properties :

$$|b_n| > \max\{|b_1|, \ldots, |b_{n-1}|\}\;;\quad a_n \neq 0 \quad \ldots \quad (1.4)$$

and (ii) $(a_jb_j/a_nb_n) < 0$ for $1 \leqslant j \leqslant n-1$.

Here again, Pathak and Pillai (1968) have shown that the weaker condition $\Sigma a_jb_j = 0$ (which depends on the coefficients alone), together with the conditions (i) and (ii) above, implies that var $X_1 < \infty$ and hence that X_1 is Normal or degenerate.

In this paper, we undertake a more exhaustive investigation of the solutions of (1.3). Theorems 1.3 and 1.4 concern such solutions in the presence of certain restrictions on the coefficients including condition (1.4), and Theorems 1.5 to 1.7 concern more general situations. Let us define (for $a_n \neq 0$, $b_n \neq 0$) for $1 \leqslant j \leqslant n-1$:

$$\alpha_j = -a_j/a_n,\; \beta_j = b_j/b_n,\; \partial_j = \begin{cases} \alpha_j/\beta_j & \text{if } \beta_j \neq 0 \\ 0 & \text{if } \beta_j = 0. \end{cases} \quad \ldots \quad (1.5)$$

Suppose k of the β's are distinct, and without loss of generality let these be $\beta_1, \ldots, \beta_k$ ($k \leqslant n-1$). Let

$$\gamma_i = \Sigma\{\partial_j\,|\,\beta_j = \beta_i\},\quad 1 \leqslant i \leqslant k. \quad \ldots \quad (1.6)$$

Again suppose s of the $|\beta|$'s are distinct, so that $s \leqslant k$, and without loss of generality let these be $|\beta_1|, \ldots, |\beta_s|$. Let

$$\varepsilon_i = \Sigma\{\partial_j \mid |\beta_j| = |\beta_i|\}, \; 1 \leqslant i \leqslant s. \quad \ldots \quad (1.7)$$

We are now in a position to enunciate our results.

Theorem 1.3 : *Let $X_1, \ldots, X_n$ be i.i.d.r.v.'s with $EX_1 = 0$ satisfying the relation* (1.3) *with the coefficients subject to the restrictions* (1.4) *and $\varepsilon_i \geqslant 0$ for all i for which $\beta_i \neq 0$, $1 \leqslant i \leqslant s$ (with strict inequality for at least one i). Let λ be the unique real root of the equation $\Sigma \varepsilon_i |\beta_i|^\lambda = 1$. Then, f being the c.f. of X_1,*

(i) *X_1 is Normal (possibly degenerate) if $\lambda = 2$;*

(ii) *X_1 is degenerate if $\lambda \leqslant 1$ or $\lambda > 2$; and*

(iii) *if $1 < \lambda < 2$, f corresponds to an absolutely continuous distribution having moments of all orders $< \lambda$ but not having moments of orders $\geqslant \lambda$.($|f|^2$ is an i.d. law).*

Theorem 1.4 : *Let $X_1, \ldots, X_n$ be i.i.d.r.v.'s with $EX_1 = 0$, satisfying the relation* (1.3) *with the coefficients subject to the restrictions* (1.4) *and $\gamma_i \geqslant 0$ for all i for which $\beta_i \neq 0$, $1 \leqslant i \leqslant k$ with at least one such $\gamma_i > 0$. Let λ be the unique real root of the equation $\Sigma \gamma_i |\beta_i|^\lambda = 1$. Then*

(i) *X_1 is degenerate if $\lambda \leqslant 1$ or $\lambda > 2$;*

(ii) *X_1 is Normal (possibly degenerate) if $\lambda = 2$; and*

(iii) *if $1 < \lambda < 2$, X_1 follows an infinitely divisible law and the assertions of Theorem* 1.3 (iii) *hold verbatim in this case.*

We pass on to the consideration of the more general situation where not all the γ's are non-negative, with special reference to characterizations of the Normal law. We find that Linnik's remarkable investigation of necessary and/or sufficient conditions in order that the identical distribution of two linear forms in i.i.d.r.v.'s be equivalent to the Normality of those r.v.'s as presented in Linnik (1953a; 1953b), can be applied *mutatis mutandis* to our problem. In fact, let

$$c_j = a_j/b_j \text{ if } b_j \neq 0 \text{ and } c_j = 0 \text{ if } b_j = 0, \text{ and } G(\lambda) = \Sigma\, c_j |b_j|^\lambda.$$

If all the c_j are of the same sign, then X_1 is necessarily degenerate. *In what follows, we shall ignore this case.* We then have the following results.

Theorem 1.5 : *Let $X_1, \ldots, X_n$ be i.i.d.r.v.'s with $EX_1 = 0$, satisfying* (1.3) *and let further (noting that the c_j are taken to be not all of the same sign)*

$$max\,(|b_j| : c_j > 0) \neq max\,(|b_j| : c_j < 0). \quad \ldots \quad (1.8)$$

Then the following set of conditions is necessary and sufficient for X_1 to be Normal (possibly degenerate) :

(i) *all the positive zeros of $G(\lambda)$ which are divisible by 4 are simple zeros;*

(ii) *all the positive zeros of $G(\lambda)$ which are even integers not divisible by 4 are zeros of order 2 at most; if such a zero of order 2 exists, it is unique and the maximum of all the positive zeros of $G(\lambda)$; and*

(iii) *if* $G(\lambda)$ *has a positive zero* γ *which is not an even integer, then it is unique, simple and the maximum of all the positive zeros of* $G(\lambda)$*; and further* $[\gamma/2]$ *is odd—where as usual,* $[x]$ *denotes the largest integer* $\leqslant x$.

Theorem 1.6 : *Let* $X_1, \ldots, X_n$ *be i.i.d.r.v.'s with* $EX_1 = 0$ *satisfying* (1.3) *and* (1.8) *and let* γ *be the largest real zero of* $G(\lambda)$. *(Such a* γ *exists and is necessarily positive). If* X_1 *has finite moment of order* $2m$*, where* $m = [(\gamma+2)/2]$*, then* X_1 *is Normal (possibly degenerate); in particular, this conclusion holds if* X_1 *has moments of all orders.*

The following theorem concerns the situation where the condition (1.8) is not imposed on the coefficients appearing in the relation (1.3).

Theorem 1.7 : *Let* $X_1, \ldots, X_n$ *be i.i.d.r.v.'s with* $EX_1 = 0$ *satisfying* (1.3) *but not necessarily* (1.8). *Suppose* $G(\lambda) \not\equiv 0$ *and let* σ *be the supremum of the real parts of the zeros of* $G(\lambda)$. *If* X_1 *has finite moment of order* $2m$ *where* $m = [(\sigma+2)/2]$*, then* X_1 *is Normal (possibly degenerate). In particular, if* $G(\lambda) \not\equiv 0$*, and* X_1 *has moments of all orders, then* X_1 *is Normal (possibly degenerate).*

In connection with the last two theorems above, we may remark that it is possible to choose the coefficients a_j and b_j and the distribution of X_1 such that $G(\lambda) \not\equiv 0$, X_1 has moment of given even integer order $2m$ but not of order $2m+4$ (so that X_1 is not Normal), while (1.3) is satisfied.

We also establish in Section 4 the following two results concerning the implication of a linear form in n i.i.d.r.v.'s having the same distribution as any of them. Theorem 1.8 is not new and is in fact a special case of Linnik (1953a, Theorem 1), as well as of Lukacs-Laha, vide Lukacs (1968, Theorem 6.2.1). Our principal interest in it is that it is a special case, corresponding to $\lambda = 2$, of Theorem 1.9 which is (believed to be) new.

Theorem 1.8 : *Let* $X_1, \ldots, X_n$ *be non-degenerate i.i.d.r.v.'s and let* $L = \Sigma a_j X_j$ *be a linear form in them with at least two coefficients non-zero. Then the condition : L and* X_1 *have the same distribution, implies that* X_1 *is Normal if and only if* $\Sigma\, a_j^2 = 1$.

Theorem 1.9 : *Let* $X_1, \ldots, X_n$ *be non-degenerate i.i.d.r.v.'s and let* $L = \Sigma a_j X_j$ *be a linear form in them, with at least two of the coefficients non-zero. Then* (i) *the condition : L and* X_1 *have the same distribution implies that* X_1 *follows a (generalized stable) law of the type described in Theorem* 3.2 *below with* λ *as exponent* $(0 < \lambda \leqslant 2)$ *if and only if* $\Sigma |a_j|^\lambda = 1$; (ii) *if further the coefficients are all non-negative, then the condition above implies that* X_1 *follows a (generalized stable) law of the type described in Theorem* 3.1 *below with* λ *as exponent* $(0 < \lambda \leqslant 2)$ *if and only if* $\Sigma a_j^\lambda = 1$.

We conclude this section by remarking that the results of Linnik (1953a, Theorems V and VI) make it possible to enunciate representation theorems, in general valid in a suitable neighbourhood of the origin, for $|f|^2$ if the b_j appearing in (1.3) are not all of the same sign and for f itself if the b_j are all of the same sign; also cf. the last paragraph of Section 3 below.

2. Some results on characteristic functions

In this section, we establish a few results on characteristic functions which are of independent interest, and some of which we shall need later. We use the notation : d.f. for a distribution function on the real line; c.f. for the characteristic function of a d.f.; F, G, etc., will denote d.f.'s and f, g, etc., the corresponding c.f.'s.

The following theorem is essentially contained in the discussion of 'α-decomposition of probability laws' by Linnik (see, for instance, Ramachandran (1966, pp. 133 ff.).

Theorem 2.1. *Let g be the c.f. of a d.f. G which has moments of all orders and is further uniquely determined by its moments (in particular g may be any analytic c.f.). If f be a c.f. such that $f(t) = g(t)$ at a sequence $\{t_n\}$ of values of t tending to zero as $n \to \infty$, then f coincides with g.*

Corollary 2.1 : *Let $f(t) = exp(-ct^2)$ at a sequence $\{t_n\}$ of values t tending to zero, where $c(> 0)$ is a constant. Then $f(t) \equiv exp(-ct^2)$.*

Proof: Let F be the d.f. corresponding to f. Since G has moments of all orders, g has derivatives of all orders. In particular, by Fatou's lemma,

$$\int x^2 dF(x) \leqslant \liminf_{n \to \infty} \frac{2-f(t_n)-f(-t_n)}{t_n^2} = -g''(0)$$

so that the second moment of F exists, and so the first and second order derivatives of f exist for all t. Rolle's theorem successively implies that $f' = g'$ at a sequence of points tending to the origin, and that consequently the same assertion is true of f'' and g''; let then $f''(u_n) = g''(u_n)$ where $u_n \to 0$ as $n \to \infty$. We then have by Fatou's lemma that

$$\int x^4 dF(x) \leqslant \liminf_{n \to \infty} \frac{f''(u_n)+f''(-u_n)-2f''(0)}{u_n^2} = g^{(4)}(0),$$

so that the fourth moment of F exists, and so the third and fourth order derivatives of f exist for all t. Then, by Rolle's theorem, it follows that $f^{(3)}$ and $g^{(3)}$ coincide at a sequence of points tending to the origin, and again the same assertion is true of $f^{(4)}$ and $g^{(4)}$. Thus proceeding, we establish by repeatedly applying Fatou's lemma and Rolle's theorem that F has moments of all even orders and so of all orders, f has derivatives of all orders, and in particular $f^{(k)}(0) = g^{(k)}(0)$ for all positive integers k, so that F and G have the same moments. Since by assumption, G is uniquely determined by its moments, $F \equiv G$.

Theorem 2.2 : *Let f be a c.f. such that $f^{(2n-1)}(t)$ is defined for all t (n a positive integer). If further $[f^{(2n-1)}(t)-f^{(2n-1)}(0)]/t$ is bounded in the deleted neighbourhood of the origin, then the $2n$-th moment of F exists, and conversely.*

Proof: We shall prove the theorem for the case $n = 1$; the general case follows from the fact that if $f^{(2n-1)}(t)$ exists, $\dfrac{f^{(2n-2)}(t)}{f^{(2n-2)}(0)}$ is a c.f. with its first derivative defined for all t.

Let $\phi(t) = [f'(t)-f'(0)]/t$, so that $|\phi(t)| \leqslant c$ for $0 < |t| < \partial$, where c and ∂ are positive constants. By the mean value theorem of the differential calculus,

$$f(t)+f(-t)-2f(0) = tf'[t\theta(t)]-tf'[-t\theta(-t)]$$

where $0 < \theta(\pm t) < 1$, and so

$$= t^2[\theta(t)\phi\{t\theta(t)\}+\theta(-t)\phi\{-t\theta(-t)\}]$$

so that
$$\left|\frac{f(t)+f(-t)-2f(0)}{t^2}\right| \leqslant 2c \text{ if } 0 < |t| < \partial.$$

It then follows from Fatou's lemma that $\int x^2 dF(x)$ exists.

The converse is trivial, since $[f'(t)-f'(0)]/t \to f''(0)$ if the second moment of F exists.

Theorem 2.3 : *Let $\{t_n\}$ denote some sequence of values of t, tending to zero as $n \to \infty$; the sequence is not necessarily one and the same throughout in what follows.*

(a) *If $\log |f(t_n)|/|t_n|^\lambda$ is bounded away from zero for some $\lambda<2$, then F has no (absolute) moments of order $> \lambda$.**

(b) *If $\log |f(t_n)|/|t_n|^\lambda$ is bounded for a sequence $\{t_n\} \to 0$ such that*

(i) *$\Sigma |t_n|^\epsilon < \infty$ for any $\epsilon > 0$, and*

(ii) *$\{t_{n-1}/t_n\}$ is a bounded sequence,*

then F has (absolute) moments of all orders $< \lambda$. In particular, if $\log |f(t)|/|t|^\lambda$ is bounded in the deleted neighbourhood of the origin then F has moments of all orders $< \lambda$.

(c) *If $\log|f(t_n)|/t_n^2$ is bounded, then the second moment of F exists, and conversely.*

(d) *If $\log |f(t_n)|/t_n^2 \to 0$ as $n \to \infty$, then F is degenerate, and conversely.*

Proof: (a) For some $c > 0$, $|f(t_n)|^2 \leqslant \exp(-c|t_n|^\lambda)$. Let $\lambda < \partial \leqslant 2$. Then, if F^* denotes the convolution of F with its conjugate $\bar{F}$, so that the c.f. of F^* is $|f(t)|^2$, let us assume that the moment of order δ exists and set $u_n = t_n/2$. Then

$$\infty > 2\int |x|^\delta\, dF^*(x) = 2\int \limsup_{n\to\infty} \left|\frac{\sin u_n x}{u_n}\right|^\delta dF^*(x) \geqslant 2 \limsup_{n\to\infty} \int \left|\frac{\sin u_n x}{u_n}\right|^\delta dF^*(x)$$

by Fatou's lemma (noting that $|(\sin u_n x)/u_n|^\delta \leqslant |x|^\delta$ assumed integrable),

$$\geqslant 2 \limsup_{n\to\infty} \int \frac{\sin^2 u_n x}{|u_n|^\delta}\, dF^*(x) \text{ since } \delta \leqslant 2,$$

$$= \limsup_{n\to\infty} 2^\delta \left[\frac{1-|f(t_n)|^2}{|t_n|^\delta}\right]$$

$$\geqslant 2^\delta \lim_{n\to\infty} \frac{1-\exp(-c|t_n|^\lambda)}{|t_n|^\delta} = \infty, \text{ since } \partial > \lambda, c > 0.$$

Hence F^*, and so F, has no moments of order $> \lambda$.

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* In such a case, the moment of order λ does not exist either. For the proof, see Ramachandran, B. "On characteristic functions and moments," to appear in *Sankhyā*, Series A.

(b) Assume without loss of generality that $1 \geqslant t_{n-1} > t_n > 0$ for all $n \geqslant 2$. Then, for some $c_1 > 0$,

$$1-|f(t_n)|^2 \leqslant 1-e^{-c_1 t_n^\lambda} \leqslant c_1 t_n^\lambda$$

for all sufficiently large n. Let as before $u_n = t_n/2$, and $c_2, c_3, \ldots$ denote positive constants. Then we have

$$\int \sin^2(u_n x) dF^*(x) \leqslant c_2 u_n^\lambda.$$

Since $\sin^2\theta \geqslant \theta^2.\sin^2 1$ if $0 \leqslant \theta \leqslant 1$, we have, setting $x_n = \frac{1}{u_n}$,

$$\int_{x_{n-1}}^{x_n} x^2 dF^*(x) \leqslant c_3 \int_{x_{n-1}}^{x_n} \frac{\sin^2(u_n x)}{u_n^2} dF^*(x) \leqslant c_4 u_n^{\lambda-2},$$

so that

$$\int_{x_{n-1}}^{x_n} x^\delta . dF^*(x) \leqslant c_4 \frac{x_n^\delta}{x_{n-1}^2} u_n^{\lambda-2} = c_4 \left(\frac{u_{n-1}}{u_n}\right)^2 u_n^{\lambda-\delta} \leqslant c_5 u_n^{\lambda-\delta}$$

since, by assumption, $\left\{\frac{u_{n-1}}{u_n}\right\}$ is a bounded sequence. Further, we have also assumed that $\Sigma u_n^\varepsilon < \infty$ for any $\varepsilon > 0$, so that $\int |x|^\delta \, dF^*(x) < \infty$ for any $\delta < \lambda$, so that F^*, and so F, has moments of all orders $< \lambda$.

If, in particular, $\log|f(t)|/|t|^\lambda$ is bounded in the deleted neighbourhood of the origin, the sequence $\{t_n\}$ may be chosen as $\{\beta^n\}$ where $0 < \beta < 1$.

(c) For some $c > 0$, $|f(t_n)|^2 \geqslant \exp(-ct_n^2)$, so that

$$\int x^2 dF^*(x) = 2\int \liminf_{n\to\infty} \left(\frac{1-\cos(t_n x)}{t_n^2}\right) dF^*(x),$$

$$\leqslant 2 \liminf_{n\to\infty} \int \left(\frac{1-\cos(t_n x)}{t_n^2}\right) dF^*(x), \text{ by Fatou's lemma,}$$

$$= 2 \liminf_{n\to\infty} \frac{1-|f(t_n)|^2}{t_n^2}$$

$$\leqslant 2 \liminf_{n\to\infty} \frac{1-\exp(-c\,t_n^2)}{t_n^2} = 2c.$$

Hence F^*, and so F, has the second moment. The converse is trivial.

(d) In this case, by (c), $\int x^2 dF^*(x)$ exists and is $\leqslant c$ for any $c > 0$, and so is zero. Thus F^*, and so F, is degenerate. The converse is trivial, since $\log|f(t)|$ is identically zero if F is degenerate.

3. Generalized stable laws

Definition : A r.v. X will be said to follow a *generalized stable* (GS) *law* if f, its c.f., is non-vanishing and satisfies an equation of the form

$$\prod_{j=1}^{s} [f(\beta_j t)]^{\gamma_j} = \prod_{j=s+1}^{s+k} [f(\beta_j t)]^{\gamma_j} \qquad \ldots (3.1)$$

for all t, where $\gamma_j > 0$ and (without loss of generality) $0 < |\beta_j| \leqslant 1$ for $1 \leqslant j \leqslant s+k$.

We first study two special classes of GS laws, namely those which satisfy for all t an equation of the form

$$f(t) = \prod_{j=1}^{r} [f(\beta_j t)]^{\gamma_j}, \text{ with } \gamma_j > 0 \text{ and } 0 < |\beta_j| < 1 \quad \text{for all } j \qquad \ldots \quad (3.2a)$$

and those which satisfy for all t an equation of the form

$$f(t) = \prod_{j=1}^{r} [f(\beta_j t)]^{\gamma_j}, \text{ with } \gamma_j > 0 \text{ and } 0 < \beta_j < 1 \quad \text{for all } j. \qquad \ldots \quad (3.2b)$$

The particular case $r = 1$ of (3.2b) has been considered by Levy (1937) ; the solutions for this case have been called 'semi-stable' by him.

Theorem 3.1 : *If a c.f. f satisfies (3.2b), it is infinitely divisible. If λ be the unique real solution of the equation $\Sigma \gamma_j \beta_j^{\lambda} = 1$, then*

(a) *$f \equiv 1$ if $\lambda \leqslant 0$ or $\lambda > 2$;*

(b) *if $\lambda = 2$, either f is a Normal c.f., or $f \equiv 1$; and*

(c) *if $0 < \lambda < 2$, then either $f \equiv 1$ or F is non-degenerate except possibly when $\lambda = 1$ (if $\lambda = 1$, f can be the c.f. of any degenerate law). The cases $0 < \lambda < 1$, $1 < \lambda < 2$ and the non-degenerate solutions for $\lambda = 1$ all correspond to absolutely continuous distributions having moments of all orders $< \lambda$ but not having moments of orders $\geqslant \lambda$. (In the non-degenerate cases, the corresponding GS law will be said to have exponent λ).*

(d) *If $L(a, \sigma^2, M, N)$ be the Levy representation of the logarithm of f, then we have :*

(i) $a = 0$ if $\lambda \neq 1$

(ii) *$\sigma = 0$ if $\lambda \neq 2$*

(iii) *$a = 0$, $M \equiv N \equiv 0$ if $\lambda = 2$;*

(iv) *$\sigma = 0$, $M \equiv N \equiv 0$ if $\lambda > 2$ or if $\lambda \leqslant 0$, so that $f \equiv 1$; and*

(v) *if $h(u) = u^{\lambda} N(u)$ and $k(u) = |u|^{\lambda} M(u)$*

where λ is the exponent of the law concerned, then h and k satisfy the functional relations,

$$h(u) = \Sigma \gamma_j \beta_j^{\lambda} h(u/\beta_j); \; k(u) = \Sigma \gamma_j \beta_j^{\lambda} k(u/\beta_j).$$

Proof : f is infinitely divisible. Let ϕ denote that branch of the logarithm of f which is continuous and which vanishes at the origin. Then we have for all t

$$\phi(t) = \gamma_1 \phi(\beta_1 t) + \ldots + \gamma_r \phi(\beta_r t). \qquad \ldots \quad (3.3)$$

On 'iterating' this relation n times, we obtain

$$\phi(t) = \Sigma \frac{n!}{n_1! \, n_2! \ldots n_r!} \gamma_1^{n_1} \ldots \gamma_r^{n_r} \phi\left(\beta_1^{n_1} \ldots \beta_r^{n_r} t\right)$$

$$= \sum_k \gamma_{nk} \, \phi(\beta_{nk} t), \text{ say} \qquad \ldots \quad (3.4)$$

where the summation runs over all distinct r-vectors $(n_1, n_2, \ldots, n_r)$ with non-negative integer elements such that $\sum_{i=1}^{r} n_i = n$. If, as we may assume without loss of generality, $\beta_1 < \beta_2 < \ldots < \beta_r$, then $\beta_1^{n_1} \ldots \beta_r^{n_r} \leqslant \beta_r^n \to 0$ as $n \to \infty$, so that $f(\beta_{nk}\, t) \to 1$ uniformly in k and in any finite interval. [If $|1-f(\beta_r^n\, t)| < \varepsilon$ uniformly for all t in $|t| \leqslant b$ if $n \geqslant N(b, \varepsilon)$, then $|1-f(\beta_{nk}\, t)| < \varepsilon$ for all k, for all such t and n also].

Let F_{nk} be the d.f. corresponding to f_{nk}. Then as in the proof of the Central Limit Theorem (cf. Loeve, 1960, pp. 303ff.), we choose and fix a $\tau > 0$ and let

$$a_{nk} = \int_{|x| < \tau} x dF_{nk}(x), \ \overline{F}_{nk}(x) = F_{nk}(x+a_{nk}), \ \overline{f}_{nk} = \text{c.f. of } \overline{F}_{nk} \qquad (3.5)$$

so that $\overline{f}_{nk}(t) = f_{nk}(t) \exp(-it\, a_{nk})$. Since $f(\beta_{nk}\, t) \to 1$ uniformly in k as $n \to \infty$ in any finite t-interval, we have that (cf. Loeve, 1960, p. 303) for any $b > 0$, there exists an $N(b)$ such that

$$|\overline{f}_{nk}(t) - 1| < \frac{1}{2} \text{ for } |t| \leqslant b \text{ if } n \geqslant N(b).$$

It then follows from the elementary relation : $|\log z + 1 - z| \leqslant |1-z|^2$ if $|z-1| < \frac{1}{2}$, that for all $n \geqslant N(b)$, $|t| \leqslant b$, and uniformly in k,

$$|\log \overline{f}_{nk}(t) + 1 - \overline{f}_{nk}(t)| \leqslant |\overline{f}_{nk}(t) - 1|^2. \qquad \ldots \quad (3.6)$$

Again (cf. Loeve, 1960, pp. 304-306, 'Central Inequalities'), for a fixed $b > 0$, there exists a constant $c = c(\tau, b) > 0$ such that for all sufficiently large n, $n \geqslant N(\tau, b)$,

$$\max_{|t| \leqslant b} |\overline{f}_{nk}(t) - 1| \leqslant -c \int_0^b \log |f_{nk}(t)|\, dt$$

so that

$$\sum \gamma_{nk} |\overline{f}_{nk}(t) - 1| \leqslant -c \int_0^b [\sum \gamma_{nk} \log |f_{nk}(t)|]\, dt$$

$$= -c \int_0^b \log |f(t)|\, dt < \infty. \qquad \ldots \quad (3.7)$$

Hence we have for $|t| \leqslant b$ and $n \geqslant \max [N(b), N(\tau, b)]$

$$|\sum \gamma_{nk} \{\log \overline{f}_{nk}(t) + 1 - \overline{f}_{nk}(t)\}| \leqslant \sum \gamma_{nk} |\overline{f}_{nk}(t) - 1|^2 \text{ by } (3.6)$$

$$\leqslant \max_{\substack{k \\ |t| \leqslant b}} |\overline{f}_{nk}(t) - 1| [\sum \gamma_{nk} |\overline{f}_{nk}(t) - 1|]$$

which tends to zero as $n \to \infty$, since the first factor does so and the second factor is bounded according to (3.7). Thus we have a sequence of 'accompanying i.d. laws' the logarithms of whose c.f.'s are given by

$$\phi_n(t) = it(\sum \gamma_{nk}\, a_{nk}) + \sum \gamma_{nk} \int (e^{itx} - 1) d\overline{F}\left(\frac{x}{\beta_{nk}}\right) \qquad \ldots \quad (3.8)$$

so that $\phi_n(t) \to \phi(t)$ as $n \to \infty$, for all t. Thus, f is i.d.

We now pass to the other assertions of the theorem.

(a) If $\lambda \leqslant 0$, equivalently if $\gamma_1 + ... + \gamma_r \leqslant 1$, then it follows from (3.4) that

$$|\phi(t)| \leqslant \Sigma \gamma_{nk} |\phi(\beta_{nk}\, t)| \leqslant (\gamma_1 + ... + \gamma_r)^n \max_k |\phi(\beta_{nk}\, t)| \to 0 \text{ as } n \to \infty$$

so that $\phi \equiv 0$ or F is the d.f. degenerate at the origin. This proves one of the assertions of (a).

Let us consider the case $\lambda > 0$, equivalently $\gamma_1 + ... + \gamma_r > 1$. Let, for $t \neq 0$, $\psi(t) = \log |f(t)| / |t|^\lambda$. Then, with $\gamma_j \beta_j^\lambda = p_j$, (3.3) gives $\psi(t) = \Sigma p_j \psi(\beta_j t)$. Since ψ is continuous and real-valued in $t > 0$ and $\Sigma p_j = 1$, the R.H.S. in this relation, being a weighted average (with positive weights) of the values of ψ at the points $\beta_j t (1 \leqslant j \leqslant r)$, is, by the intermediate value theorem, equal to the value of ψ at some point $\beta(t)$ such that $\beta_1 t \leqslant \beta(t) \leqslant \beta_r t$.

Thus we have for any $t > 0$ a sequence of real values : $\beta^{(0)}(t) = t, \beta^{(1)}(t), ...$ tending to zero as $n \to \infty$ such that for all $n \geqslant 0$

$$\beta_1 \beta^{(n)}(t) \leqslant \beta^{(n+1)}(t) \leqslant \beta_r \beta^{(n)}(t) \qquad ... \quad (3.9)$$

and

$$\psi(t) = \psi[\beta^{(n)}(t)]. \qquad ... \quad (3.10)$$

Choosing $t = 1$ in particular, we see that there exists a sequence $\{t_n\}$ of positive values of t tending to zero such that for all $n \geqslant 0$

$$t_0 = 1; \beta_1 t_n \leqslant t_{n+1} \leqslant \beta_r t_n \qquad ... \quad (3.11)$$

and

$$\psi(t_n) = \psi(1). \qquad ... \quad (3.12)$$

From (3.12) and Theorem 2.3(d), it follows at once that if $\lambda > 2$ then F is degenerate. Further, if $f(t) = e^{iat}, a \neq 0$, then we must have $\Sigma \gamma_j \beta_j = 1$ contradicting the fact that $\Sigma \gamma_j \beta_j^\lambda = 1$ for some $\lambda > 2$. Hence we can only have $a = 0$, or $f \equiv 1$. This completes the proof of (a).

(b) If $\lambda = 2$, then let $c = -2\psi(1) \geqslant 0$. We then have from (3.12) that $|f(t_n)|^2 = \exp(-ct_n^2)$ for all n, so that $|f(t)|^2$ is a Normal c.f. by Theorem 2.1. It follows from the Levy-Cramer theorem that f is itself a Normal c.f. [The use of this powerful result can be avoided by appealing instead directly to the Levy representation for infinitely divisible laws, for which see below]. We note that either F is non-degenerate Normal or $f \equiv 1$ (since $\Sigma \gamma_j \beta_j^2 = 1$ implies that $\Sigma \gamma_j \beta_j$ cannot be equal to one). This proves (b).

(c) $0 < \lambda < 2$. Ignoring the trivial solution $f \equiv 1$ which satisfies (3.2b) whatever be the constants β and γ, we note that $f(t) = e^{iat}, a \neq 0$, can be a solution only if $\lambda = 1$, so that if $0 < \lambda < 1$ or $1 < \lambda < 2$, there are no non-degenerate solutions for (3.2b). Thus degenerate solutions exist only in the case $\lambda = 1$, in which case *any* such c.f. is easily verified to satisfy (3.2b). We shall henceforth consider only the *non-degenerate solutions of* (3.2b).

Let then f be a non-degenerate solution of (3.2b) in the case $0 < \lambda < 2$. Then in any closed interval $[a, b]$, $0 < a < b$, $\log |f(t)|$ is bounded away from zero; for, if for a sequence $\{\tau_n\}$ of points in $[a, b]$, $|f(\tau_n)| \to 1$, then by the Bolzano-Wierstrass theorem and the continuity of f, there exists a τ_0 in $[a, b]$ such that $|f(\tau_0)| = 1$, and then, by (3.10), $\log|f(t)| = 0$ at a sequence of positive values of t tending to zero, so that f corresponds to a degenerate d.f., contrary to assumption. Let then $a > 0$ and let $0 < m \leqslant |\psi(t)| = \frac{-\log |f(t)|}{t^{\lambda}} \leqslant M$ for $t \epsilon \left[a, \frac{a}{\beta_1}\right]$ Then by (3.9) and (3.10), it is easily seen that for all $t \geqslant a$, the same inequalities hold; for, for any such t, $\beta^{(n)}(t)$ belongs to the above interval for some value of n (depending on t), in view of (3.9). Thus, if $|t| \geqslant a$, we have

$$\exp(-M|t|) \leqslant |f(t)| \leqslant \exp(-m|t|^{\lambda}). \qquad \dots \quad (3.13)$$

The right inequality shows that f *belongs to an absolutely continuous distribution* (with a continuous version of the density given by $p(x) = \frac{1}{2\pi} \int e^{-itx} f(t)dt$). We now pass to the consideration of the moments of F. [It is of some interest to note that the two inequalities in (3.13) do not enable us to infer anything about the existence of the moments of F, which appear to be determined more by the behaviour of the c.f. in the neighbourhood of the origin (as indicated by Theorem 2.3) than by that at infinity. Thus, $g(t) = \exp(-|t|^{\mu})$ and $h(t) = \max[\exp(-|t|^{\mu}), \exp(-|t|^{\nu})]$, where $0 < \mu < \nu \leqslant 1$ are both c.f.'s of Polya's type (being convex functions in $t > 0$) and coincide for $|t| \geqslant 1$. But, by Theorems 2.3(a) and (b), G has moments of all orders $< \mu$ but no moments of order $> \mu$, while H has moments of all orders $< \nu$ but no moments of order $> \nu$.] Since $\psi(1) \neq 0$, it follows immediately from (3.12) that the sequence $\psi(t_n)$ is bounded as well as bounded away from zero as $n \to \infty$ (being in fact equal to a non-zero constant for all n) so that Theorem 2.3(a) guarantees that F has no moments of order $> \lambda$. To apply Theorem 2.3(b), we notice that by (3.11), $\frac{t_n}{t_{n+1}} \leqslant \frac{1}{\beta_1}$ and that, since $t_n \leqslant \beta_r^{n+1}$, Σt_n^{ε} converges for any $\varepsilon > 0$, so that all the conditions of that theorem are satisfied and hence F has moments of all orders $< \lambda$. We shall show below that the moment of order λ does not exist (a proof of this fact using only relations (3.9)-(3.12) is desirable); this will complete the proof of Part (c).

It is worth noting at this stage, however, that the solutions of (3.2b) having finite first moment are all degenerate if $\lambda \leqslant 1$; more precisely, $f \equiv 1$ if $\lambda < 1$; and $f(t) = e^{iat}$, where a can be any real number, if $\lambda = 1$. To see this, we note that if the first moment exists, then $f'(t)$ is defined for all t; denoting $\gamma_j\beta_j$ by α_j and $[f'(t)/f(t)] - f'(0)$ by $\xi(t)$, we have the relation,

$$\xi(t) = \Sigma\, \alpha_j \xi(\beta_j t)$$

which, after n iterations, gives us the inequality

$$|\xi(t)| \leqslant (\alpha_1 + \dots + \alpha_r)^n \max [|\xi(\beta_1^{n_1} \dots \beta_r^{n_r} t)|],$$

This follows from (3.12): see the foot note on p. 130.

the maximum being taken over all distinct r-vectors $(n_1, n_2, \ldots, n_r)$ with non-negative integer elements such that $\sum_1^r n_i = n$, so that if $\Sigma\, \alpha_j \leqslant 1$, then $\xi(t) = 0$ for all t. Hence f is constant. If then $f(t) = e^{iat}$, $a = 0$ if $\Sigma \gamma_i \beta_i \neq 1$ and a can be any real number if $\Sigma\, \gamma_i \beta_i = 1$, if f were to satisfy (3.2b). This proves our above assertion.

(d) We pass to the Levy representation $L(a, \sigma^2, M, N)$ of the logarithm of f. (3.2) implies that we must have :

$$a(\gamma_1\beta_1 + \ldots + \gamma_r\beta_r - 1) = 0$$
$$\sigma^2(\gamma_1\beta_1^2 + \ldots + \gamma_r\beta_r^2 - 1) = 0$$
$$\Sigma \gamma_j N(u/\beta_j) = N(u);\ \Sigma\, \gamma_j M(u/\beta_j) = M(u).$$

If λ be the exponent of the GS law concerned, it is clear that $a = 0$ if $\lambda \neq 1$, and $\sigma^2 = 0$ if $\lambda \neq 2$, as a consequence of the above relations. This proves assertions (i) and (ii). Also if $h(u) = u^\lambda N(u)$, $k(u) = |u|^\lambda M(u)$ and $\gamma_j \beta_j^\lambda = p_j$, we have assertion (v) :

$$h(u) = \Sigma p_j h(u/\beta_j);\ k(u) = \Sigma p_j k(u/\beta_j). \qquad \ldots \quad (3.14)$$

Thus $h(u)$ is a weighted average (with positive weights) of the values of h at the points $u/\beta_j (1 \leqslant j \leqslant r)$. If $N(u) \neq 0$, then let $b > 0$ be any point such that $N(b) \neq 0$, and let $a = b\beta_1$ (where, we recall, $0 < \beta_1 < \ldots < \beta_r < 1$). It is clear that if $c = |N(b)| a^a$ and $d = |N(a)| b^a$, then $0 < c \leqslant |h(u)| \leqslant d$ for all u in $[a, b]$. If now $u \epsilon [a\, \beta_r, a]$, then, for $1 \leqslant j \leqslant r$, every u/β_j lies in $[a, b]$ so that by (3.14), the same bounds for $|h|$ hold in the interval $[a\, \beta_r, a]$ also. Similarly, the same bounds are valid in all the intervals $[a\, \beta_r^k, a\beta_r^{k-1}]$ also, $k = 2, 3, \ldots$, successively, so that, for all $u \,\epsilon\, (0, b)$, we have $c \leqslant |h(u)| \leqslant d$. Let now $m > 1$ be so chosen that $c.m^\lambda > d$. Then $\int_0^a u^2 dN(u) < \infty$ implies that

$$u^2[|(N(u)| - |N(mu)|] \leqslant \int_u^{mu} x^2 dN(x) \to 0 \ \text{ as } \ u \to 0+,$$

so that we must have $\lambda < 2$ if $N(u) \not\equiv 0$. Similarly, if $M(u) \not\equiv 0$, then $\lambda < 2$. Thus, in particular, if $\lambda = 2$, then $a = 0$, $M \equiv 0$, $N \equiv 0$ so that we can only have a (possibly degenerate) Normal law as the solution of (3.2b) in this case. If $\lambda > 2$, then $a = \sigma^2 = 0$, $M \equiv N \equiv 0$, so that $f \equiv 1$; as already noted, $f \equiv 1$ if $\lambda \leqslant 0$ as well. Thus assertions (iii) and (iv) are proved.

We shall now show that, if $0 < \lambda < 2$, for a non-degenerate solution of (3.2b) with exponent λ, the absolute moment of order λ does not exist. (The only degenerate solutions for values of λ in this range occur when $\lambda = 1$, and in that case f can be the c.f. of *any* degenerate law). We have already noted that $\sigma^2 = 0$ for all such λ and $a = 0$ if $\lambda \neq 1$.

Let F^* denote the convolution of F with its conjugate d.f. F, so that F^* has $|f(t)|^2$ as its c.f. Then, we have the analogue of (3.8) for log $\{|f(t)|^2\}$, namely,

$$2 \log |f(t)| = \Sigma \gamma_{nk} \int (\cos tx - 1) dF^*(x/\beta_{nk}).$$

[This is also an immediate consequence of the fact that, for all sufficiently large n,

$$\Sigma\, \gamma_{nk}[1 - |f(\beta_{nk}\, t)|^2] \leqslant -2\, \Sigma\, \gamma_{nk} \log |\, f(\beta_{nk}\, t)|$$
$$\leqslant \Sigma\, \gamma_{nk}\{[1 - |\, f(\beta_{nk}\, t)|^2]/|f(\beta_{nk}\, t)|^2\}$$

whence it follows that as $n \to \infty$

$$\Sigma \gamma_{nk}[1-|f(\beta_{nk} t)|^2] \to -2 \log |f(t)|.]$$

It then follows from Gnedenko and Kolmogorov (1954, p. 88, Theorem 2), that (M and N denoting the Levy functions appearing in the representation for $\log|f|^2$)

$$\Sigma \gamma_{nk} \left[F^*\left(\frac{u}{\beta_{nk}}\right)-1\right] \Longrightarrow N(u) \qquad \text{for } u > 0$$

$$\Sigma \gamma_{nk} F^*\left(\frac{u}{\beta_{nk}}\right) \Longrightarrow M(u)\{= -N(-u)\} \qquad \text{for } u < 0$$

where $\Longrightarrow$ signifies convergence to a function at all continuity points thereof. We have already proved that F and hence F^* is a continuous (indeed an absolutely continuous) d.f. Hence, if b be any continuity point of N, it follows that

$$\Sigma \gamma_{nk} \beta_{nk}^{\lambda} \left\{\frac{1-F^*(b/\beta_{nk})}{\beta_{nk}^{\lambda}}\right\} \to |N(b)|.$$

Noting that $\Sigma \gamma_{nk} \beta_{nk}^{\lambda} = (\Sigma \gamma_j \beta_j^{\lambda})^n = 1$, we see that the expression on the LHS of the above relation is a weighted average (with positive weights) of the values at the points β_{nk} of the continuous function $\{1-F^*(b/\theta)\}/\theta^{\lambda}$ of the positive variable θ, so that the LHS $= \{1-F^*(b/\theta_n)\}/\theta_n^{\lambda}$, where $\beta_1^n \leqslant \theta_n \leqslant \beta_r^n$. Hence, if $N(u) \not\equiv 0$, then there exists a sequence $\{u_n\}$ of positive real numbers tending to infinity such that $u_n^{\lambda}[1-F^*(u_n)]$ tends to a finite positive limit as $n \to \infty$. Hence F^* and so F does not have a finite absolute moment of order λ. Similarly, if $M(u) \not\equiv 0$, then also the same conclusion holds. This completes the proof of Theorem 3.1.

We then easily have the following result concerning c.f.'s which satisfy relation (3.2a).

Theorem 3.2 : *Let f be the c.f. of a non-degenerate d.f. satisfying* (3.2a). *Then it is infinitely divisible, and if λ be the unique real solution of the equation* $\Sigma \gamma_j |\beta_j|^{\lambda} = 1$, *then*

(a) $0 < \lambda \leqslant 2$,

(b) *if $\lambda = 2$, f is a Normal c.f., and*

(c) *if $0 < \lambda < 2$, then F is an absolutely continuous d.f. having moments of all orders $< \lambda$ but not having moments of orders $\geqslant \lambda$.*

Proof: We obtain the infinite divisibility of f by proceeding along the lines of the corresponding part of the proof of Theorem 3.1. The other assertions follow from Theorem 3.1 and the fact that $g = |f|^2$ satisfies a relation of the form (3.2b), namely,

$$g(t) = \prod_{j=1}^{j=r} [g(|\beta_j| t)]^{\gamma_j}$$

To prove (b), we also need the Levy-Cramer theorem.

It is also possible to give general representation theorems for GS laws following Linnik (1953a, Theorems V and VI). The representations would be true for $|f|^2$ in case f satisfies an equation of the form (3.1) where the β's are not necessarily of the same sign, and for f itself if the β's are all of the same sign. (Incidentally, the representation (2.2) of Theorem V in Linnik (1953a) appears to be valid, not for all $u > 0$ but for $0 < u < 1$).

4. Proofs of the main results

In this section, we prove the main results of the paper, stated in Section 1.

Proof of Theorem 1.1 : The relation (1.1) is equivalent to the following relation, valid for all real $t(\alpha = -a_2/a_1$ and $\beta = b_2/b_1)$:

$$E\Big[(X_1-\alpha X_2)\, e^{it(X_1+\beta X_2)}\Big] = 0.$$

If f be the c.f. of X_1 and I is the largest interval around the origin in which f is non-vanishing (I is necessarily symmetric about the origin), then we have from the above that

$$\phi(t) = \alpha\phi(\beta t) \qquad \text{for} \quad t\epsilon I \qquad \ldots \quad (4.1)$$

where $\phi(t) = f'(t)/f(t)$, and on integration, with $\gamma = \alpha/\beta$,

$$f(t) = [f(\beta t)]^\gamma \quad \text{for} \quad t\epsilon I. \qquad \ldots \quad (4.2)$$

If $\alpha\beta < 0$, i.e., $\gamma < 0$, then (4.2) implies that for $t\epsilon I$, $|f(t)| \geqslant 1$ since $|f(\beta t)| \leqslant 1$ and hence $|f(t)| = 1$ for such t. Hence $|f| \equiv 1$ or $X_1 = 0$ a.s. This proves assertion (i) of the theorem.

Let then $\alpha\beta > 0$, and let us consider the case $|\beta| = 1$. From (4.1), we have if $\beta = 1$ and $\alpha = 1$, then f can be arbitrary; if $\beta = -1$, then $|\alpha| = 1 \Longrightarrow \alpha = -1$ and so f can be any arbitrary real-valued c.f.; if $|\alpha| \neq 1$, then $\phi(t) = \alpha^2\phi(\beta^2 t) = \alpha^2\phi(t)$ implies that $\phi(t) \equiv 0$ for $t\epsilon I$ and so $X_1 = 0$ a.s. This proves (ii).

We pass on to the only non-trivial case : $\alpha\beta > 0$ and $|\beta| < 1$. Let λ be the unique solution of the equation $|\alpha|\,|\beta|^{\lambda-1} = 1$, i.e., $\gamma|\beta|^\lambda = 1$. If $\lambda \leqslant 1$, then $|\alpha| \leqslant 1$, and from (4.1), we have that, for $t\epsilon I$,

$$\phi(t) = \alpha\phi(\beta t) = \ldots = \alpha^n\phi(\beta^n t) \to 0 \text{ as } n \to \infty$$

since $\phi(\beta^n t) \to \phi(0) = 0$, so that $f \equiv 1$. If $\lambda > 2$, then, from (4.2), for any fixed $t\epsilon I$,

$$\frac{\log|f(t)|}{|t|^\lambda} = \frac{\log|f(\beta t)|}{|\beta t|^\lambda} = \ldots = \frac{\log|f(\beta^n t)|}{|\beta^n t|^\lambda} = \ldots$$

for all positive integers n, so that the sequence $\dfrac{\log|f(\beta^n t)|}{(\beta^n t)^2} \to 0$ as $n \to \infty$ and consequently, by Theorem 2.3(d), $X_1 = 0$ a.s. This proves assertion (iiia).

If $\lambda = 2$, then the sequence $\dfrac{\log f(\beta^n t)}{(\beta^n t)^2}$ remains constant, so that, by Corollary 2.1, f is a Normal c.f. (possibly degenerate). This proves (iiib).

Let $1 < \lambda < 2$. We first note that f is non-vanishing. For, if not, let $\pm t_0$, $t_0 > 0$, be the zeros of f nearest to the origin. Then, in $I : (-t_0, t_0)$, we have the relation : $|f(t)| = |f(\beta t)|^\gamma$. Letting $t \to t_0 - 0$ through values in I, we obtain by the continuity of $|f|$ that $|f(\beta t_0)| = 0$ also, contrary to assumption. Hence *relation* (4.2) *is valid for all real t*, being valid in any interval around the origin in which f does not vanish.

Since, for all t and all positive integers n, $f(t) = [f(\beta t)]^\gamma = [f(\beta^n t)]^{\gamma^n}$, where $\gamma = \frac{1}{|\beta|^\lambda} > 1$, we see that f^{1/γ^n} is a c.f. for every n and hence f is infinitely divisible. f is in fact a 'semi-stable' law in the sense of P. Levy (1937), having finite expectation. We omit the proofs of the other statements of Theorem 1.1 (iiic), since they follow from our proofs of Theorems 3.1 and 3.2. We may, however, remark that if (using an obvious notation) the Levy representation of the logarithm of f be $L(a, \sigma^2, M, N)$, and λ be the exponent of f, then defining $h(u) = u^\lambda N(u)$ and $k(u) = |u|^\lambda M(u)$, we have the relations :

$$h(u) = h(\beta u) \text{ and } k(u) = k(\beta u) \text{ if } \beta > 0;$$

$$h(u) = k(\beta u) \text{ and } k(u) = h(\beta u) \text{ if } \beta < 0.$$

Proof of Theorem 1.2 : Let us assume as before that $|\beta| \leqslant 1$ (without loss of generality). Then, in any interval I around the origin in which f is non-vanishing, we have $\phi(t) = \alpha\phi(\beta t)$, $\phi = f'/f$, from the first of the relations (1.2). Also, in a sub-interval of I where $f(\alpha t)$ does not vanish, we have from the second of the relations (1.2) that $\phi(t) = \beta\phi(\alpha t)$, so that $\phi(t) = \alpha\beta\phi(\alpha\beta t)$ there. This implies that if $|\alpha\beta| \neq 1$, then $\phi(t)$ vanishes in that sub-interval and so $f \equiv 1$; if $|\alpha\beta| = 1$, then, by Theorem 1.1 (Sections (iiib) and (i) respectively), f belongs to a Normal (possibly degenerate) d.f. if $\alpha\beta = 1$, and $f = 1$ if $\alpha\beta = -1$.

Proof of Theorem 1.3 : In view of the condition (1.4) and the definitions (1.5), the two linear statistics may be written as :

$$L_1 = -\alpha_1 X_1 - \ldots - \alpha_{n-1} X_{n-1} + X_n$$
$$L_2 = \beta_1 X_1 + \ldots + \beta_{n-1} X_{n-1} + X_n$$

where $|\beta_i| < 1, 1 \leqslant i \leqslant n-1$. The condition $E(L_1 | L_2) = 0$ a.s., implies that, in any interval I around the origin where f does not vanish,

$$\phi(t) = \sum_{j=1}^{n-1} \alpha_j \phi(\beta_j t), \text{ where } \phi = f'/f,$$

and, on integration,

$$\log f(t) = \sum_{j=1}^{n-1} \partial_j \log f(\beta_j t), \quad t \epsilon I. \qquad \ldots (4.3)$$

If $g(t) = \log(|f(t)|^2)$ for $t\epsilon I$, then, recalling the definition (1.7) of the ϵ's and setting $\theta_i = |\beta_i|$, we have

$$g(t) = \sum_{i=1}^{s} \epsilon_i g(\theta_i t), \ t\epsilon I.$$

Since the $\varepsilon_i \geqslant 0$, with $\varepsilon_i > 0$ for at least one i, by assumption, there is a unique real root of the equation $\Sigma \varepsilon_i \theta_i^\lambda = 1$. Denoting this root by λ, the assertions of the theorem follow from Theorem 3.1, since $|f(t)|^2$ satisfies a relation of the form (3.2b). We note further that $|f|^2$ is an infinitely divisible c.f., and that the Levy-Cramer theorem on the Normal law has to be invoked to prove assertion (ii) of the theorem.

Proof of Theorem 1.4 : Assertion (4.3) holds under the conditions of the present theorem as well, and we have

$$\log f(t) = \sum_{i=1}^{k} \gamma_i \log f(\beta_i t), \quad t \in I,$$

where $\beta_1, \ldots, \beta_k$ are the distinct β's. Since $\gamma_i \geqslant 0$ for all i and $\gamma_i > 0$ for at least one i, there is a unique real value of λ satisfying the equation $\Sigma \gamma_i |\beta_i|^\lambda = 1$. All the assertions of the theorem then follow from Theorem 3.2.

Theorems 1.5-1.7 : We need only consider here the remark about the case where the c_j's are all non-negative. Relation (1.3) yields : $\Sigma c_j \log f(b_j t) = 0$ for all t in some neighbourhood of the origin so that $\pi |f(b_j t)|^{c_j} = 1$ there. If every $c_j \geqslant 0$, then this implies that $f(t) \equiv 1$. The proofs of these three theorems follow from the work of Linnik (1953a, 1953b), as pointed out in Section 1.

Proof of Theorems 1.8 *and* 1.9 : Suppose $L = \Sigma a_j X_j$ and X_1 have the same distribution, where at least two a_j's are non-zero. Then, for all t,

$$f(t) = f(a_1 t) \ldots f(a_n t)$$

so that $|f(t)| \leqslant |f(a_1 t)| \leqslant |f(a_1^{2m} t)|$ for all positive integers m. This shows that if $a_1 \neq 0$, then $|a_1| \leqslant 1$, if f belongs to a non-degenerate d.f. Also if $|a_1| = 1$, if a_2 be any other non-zero coefficient, then

$$|f(a_2 t)|\,|f(a_3 t)| \ldots |f(a_n t)| = 1 \text{ implies that } |f(a_2 t)| = 1$$

which is impossible since f does not belong to a non-degenerate d.f. Hence every $|a_j| < 1$. The assertions of Theorems 1.8 and 1.9 then follow from Theorems 3.1 and 3.2.

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