

# Essays in Political Economy and Voting

Mihir Bhattacharya



Thesis submitted to the Indian Statistical Institute in partial  
fulfilment of the requirements for the award of the degree of  
Doctor of Philosophy

# Essays in Political Economy and Voting

Mihir Bhattacharya

October 2014

Thesis Supervisor : Professor Arunava Sen



Thesis submitted to the Indian Statistical Institute in partial  
fulfilment of the requirements for the award of the degree of  
Doctor of Philosophy

## Acknowledgement

I am thankful to my supervisor, Professor Arunava Sen, for his constant guidance and support. I feel deeply indebted to him for providing feedback on the thesis at each and every stage. This thesis would not have been in its present shape without his help.

I would like to thank Dr. Parikshit Ghosh for suggestions and comments during the preliminary stages of my doctoral studies.

I am thankful to Dr. Debasis Mishra for useful comments throughout the course of my research.

I would like to thank Professor Jaideep Roy and Dr. Sourav Bhattacharya for taking time to discuss some of the results.

I am thankful to Professor Bhaskar Dutta for suggestions and comments.

I would like to express my gratitude to Professor E. Somanathan, Dr. Tridip Ray, Professor Chetan Ghate and other faculty members of the Economics and Planning Unit for their support and encouragement.

I would like to thank my colleagues and seniors Abdul, Anup, Dushyant, Madhav, Mridu, Sonal, Soumendu, Souvik and Swaprava for their support.

I owe a great deal to my parents for their constant support and encouragement during these years.

Research fellowships from the Indian Statistical Institute are gratefully acknowledged.

*To my parents*

# Contents

<b>1</b>	<b>INTRODUCTION</b>	<b>1</b>
1.1	Multilevel Multidimensional Consistent Aggregators . . . . .	1
1.2	Party Formation as a Network in a Citizen-Candidate Model . . . . .	2
1.3	A Model of Electoral Competition Between National and Regional Parties . . . . .	4
<b>2</b>	<b>MULTILEVEL MULTIDIMENSIONAL CONSISTENT AGGREGATORS</b>	<b>7</b>
2.1	Introduction . . . . .	8
2.2	The Evaluation Model . . . . .	9
2.2.1	Consistency of Aggregators . . . . .	11
2.2.2	Further Axioms . . . . .	12
2.2.3	The Main Result . . . . .	13
2.2.4	Discussion . . . . .	21
2.2.5	Independence of axioms . . . . .	21
2.3	The Finite Case: Aggregating Sets of Alternatives . . . . .	23
2.3.1	The Result . . . . .	26
2.3.2	Independence of axioms . . . . .	28
2.4	Conclusion . . . . .	29
<b>3</b>	<b>PARTY FORMATION AS A NETWORK IN A CITIZEN-CANDIDATE MODEL</b>	<b>31</b>
3.1	Introduction . . . . .	32
3.1.1	Results . . . . .	33
3.2	The Model . . . . .	33
3.2.1	The equilibrium notion . . . . .	38
3.3	Equilibrium Analysis: Populist Parties . . . . .	39
3.3.1	Number of parties . . . . .	39
3.3.2	One-party equilibrium . . . . .	44
3.3.3	Two-party equilibrium . . . . .	46
3.4	Equilibrium Analysis: Internally Democratic Parties . . . . .	49
3.4.1	Number of Parties . . . . .	49
3.4.2	One-party Equilibrium . . . . .	51
3.4.3	Two-party Equilibrium . . . . .	53
3.4.4	Three-party Equilibrium . . . . .	55
3.5	Discussion . . . . .	56
3.6	Conclusion . . . . .	57

<b>4</b>	<b>A MODEL OF ELECTORAL COMPETITION BETWEEN NATIONAL AND REGIONAL PARTIES</b>	<b>59</b>
4.1	Introduction . . . . .	60
4.1.1	Empirical Background . . . . .	60
4.1.2	Results . . . . .	61
4.2	The Model . . . . .	62
4.2.1	Outcome Functions . . . . .	64
4.2.2	Sincere vs Strategic Voting . . . . .	65
4.3	Sincere Voting Equilibrium . . . . .	65
4.3.1	Homogeneous Regions . . . . .	69
4.3.2	Heterogeneous Regions . . . . .	71
4.4	Discussion . . . . .	76
4.5	Conclusion . . . . .	76

# Chapter 1

## INTRODUCTION

This thesis comprises three chapters on issues in political economy and voting. The first chapter considers a multilevel multidimensional aggregation problem in voting. The second chapter considers a model of party formation where citizens propose links to other candidates. The final chapter considers a model of electoral competition between regional and national parties.

We provide a brief description of each chapter below.

### 1.1 MULTILEVEL MULTIDIMENSIONAL CONSISTENT AGGREGATORS

In this chapter we study gerrymander-proof or consistent aggregation rules in different contexts. There are several papers that have studied the structure of consistent voting rules satisfying various versions of consistency. Virtually all these papers have considered models where voters express opinions about a *single* alternative which have to be aggregated into a social opinion about that alternative. Our goal in this paper is to investigate the consistency of voting rules in models where voter opinions over *several* alternatives have to be aggregated.

We consider a model of aggregation where voter opinions have to be aggregated. Each voter submits an *evaluation* for each alternative (or component) indicating the intensity with which she likes the alternative. The set of permissible evaluations for any alternative is the closed unit interval. An *aggregator* considers an *arbitrary* collection of voter evaluations and transforms them into an aggregate opinion.

Voters can be divided into mutually exclusive subgroups. This could be based, for example, on geographical regions/districts or political constituencies. The aggregator generates an aggregate for each subgroup. It can also be used to aggregate subgroup opinions into an opinion for the whole population. Consistency requires the same opinion for the population to emerge (for every possible configuration of voter opinions) irrespective of the way the population is spilt into subgroups. This chapter examines the implications of consistency on aggregators.

We characterize *component-wise  $\alpha$  median* rules. These rules are *separable*, i.e. the

outcome for an alternative depends only on voter opinions for that alternative. Moreover, the outcome for each alternative is the median of the minimum utility (across voters), the maximum utility (across voters) and a fixed but arbitrary number  $\alpha_j$  for each alternative  $j$ .

Consistent voting rules have also been analyzed in [Chambers \(2008\)](#), [Chambers \(2009\)](#) and [Nermuth \(1994\)](#). [Perote-Peña \(2005\)](#), [Bervoets and Merlin \(2012\)](#) and [Plott \(1973\)](#) also analyze models that are similar in spirit to ours with related notions of consistency. Both the [Nermuth](#) and [Chambers](#) papers consider a single alternative voting model.

Our result is a generalization of the result of [Fung and Fu \(1975\)](#) who prove an  $\alpha$ -median characterization result for the one alternative case. There are significant difficulties involved in the extension to the multidimensional case due to its additional richness. However, these are resolved using the same set of axioms as in [Fung and Fu \(1975\)](#) defined suitably for the multidimensional model.

The separability result depends critically on the structure of the model, in particular on the fact that the set of possible evaluations is a continuum. The result no longer holds if the set of evaluations is finite. We investigate this issue in a special “finite” model. This is a model where there are  $m$  alternatives and voter/social opinions pertain to the selection of sets of these candidates. The set of possible evaluations for a candidate is either 0 or 1 indicating disapproval and approval respectively. We characterize a class of separable rules called *Bipartite Rules* by consistency and stronger versions of some of the axioms used for the earlier result. The Bipartite Rule partitions the set of alternatives into two sets (independently of opinions). Alternatives in the first set are assigned value 1 unless all voters disapprove, while alternatives in the second are never selected unless they are unanimously approved.

## 1.2 PARTY FORMATION AS A NETWORK IN A CITIZEN-CANDIDATE MODEL

In this chapter we consider a model of party formation with inter-candidate links. Candidates are situated in a one-dimensional policy space and propose *links* to other candidates in order to form parties. Party formation is modelled in the same way as the formation of *networks*.

The policy space is an interval in the real line. There is a *continuum* of voters whose ideal policy positions are distributed over the policy space. There is a finite set of candidates who also have ideal policy positions. Candidates decide whether or not to participate in elections and also propose links to other candidates in order to form parties. A candidate may also choose to stand as an independent. A profile of proposals leads to the formation of political parties. A party is a set of mutually interlinked candidates; moreover, a candidate cannot belong to two parties. Each party adopts a policy position (we consider two separate ways for this to happen) after which voters vote non-strategically for the party whose policy platform is closest to their own.

Parties allow candidates to commit to a policy position different from their own. In



addition each party standing in an election has to pay a *fixed* cost which may be thought of as the cost of campaigning. These costs are spread out if a candidate joins a party instead of contesting as an independent. On the other hand, there are two types of benefits of joining a party. The winning position is the position of the party which may be different from that of the candidate. Also, there is a fixed benefit/rent from winning which has to be shared among all party members.

A critical element of our model is the assumption regarding the policy position of a party. We assume that this can only be the position of a member of the party. We consider two different assumptions: *populist* and *internally democratic* parties. A populist party chooses the ideal policy position of the member that is closest to the voters' median position. An internally democratic party, on the other hand, chooses the median policy position among its party members.<sup>1</sup>

There are several papers that study party formation such as Riviere (1999), Jackson and Moselle (2002), Levy (2004), Callander (2005) and Osborne and Tourky (2008). However, the paper that bears the closest resemblance to ours is Osborne and Slivinsky (1996). This paper examines the features of electoral competition in the citizen-candidate model without party formation. Our paper can be thought of as model of party (network) formation in the background of the Osborne and Slivinsky (1996) model.

It is well-known that Nash equilibrium is an unsatisfactory equilibrium notion in network formation (Jackson (2008)) models. For instance, the strategy where no candidate offer links is always an equilibrium. We therefore, adopt the *strong stability* notion according to which no *subset* of candidates can jointly deviate profitably from the proposed equilibrium.

We obtain different results for populist and internally democratic parties cases. In the former, there can be at most two parties in equilibrium. In the single party equilibrium, the single party is generically an independent who is the candidate situated closest to the voter median. The two-party equilibrium occurs in the case where the benefits from winning are greater than the cost of participating - a natural assumption. Party positions are equidistant from the voter median. In addition parties are *homogeneous*, i.e. the smallest interval containing the positions of all members of a party are disjoint for the two parties.

Electoral competition is less intense when parties are internally democratic. Consequently, more than two parties can exist in equilibrium as in Osborne and Slivinsky (1996) (with candidates interpreted as parties). We derive conditions for one-party and two-party equilibrium and show the possibility of a three-party equilibrium. The two-party equilibrium in this case also requires electoral benefits to exceed costs. In the three-party equilibrium, benefits must exceed cost of participation.

---

<sup>1</sup>Jackson et al. (2007) consider a model of nomination processes within parties.

### 1.3 A MODEL OF ELECTORAL COMPETITION BETWEEN NATIONAL AND REGIONAL PARTIES

In this Chapter we model the electoral competition between national and regional parties. National parties have the advantage of garnering votes from constituencies across the regions. Regional parties, on the other hand, can contest *only* from one region. The characteristic feature of regional parties is that voters do not consider regional parties of other regions as viable options.

Voters have favorite policy positions on a one-dimensional policy space. The policies are national issues—for instance, the rate of taxation, share of GDP to be spent on education or health etc. We assume that parties have to choose the policy position of a voter from any region.

Parties are *constituency-motivated* i.e they care only about winning the maximum possible number of constituencies. Moreover, parties maximize given the equilibrium strategy of voters. Our objective is to study the equilibrium policies of the parties.

Once the parties have chosen policy positions, voting takes place. A voter in a region can only vote either for the national party or the regional party pertaining to her region. A party wins a constituency if at least a majority of voters vote for it. Outcomes are determined on the basis of constituencies or seats won by the parties.

A key element of our analysis is the *Political Outcome Function* (P.O.F.). This function maps the shares of constituencies into a probability distribution over party policy positions. We use this general formulation in order to capture a wide variety of circumstances. For example in India and the U.K, the party that wins the largest number of constituencies in a plurality vote forms the government and implements its policy position. On the other hand, a coalition government may form where parties share office and one of their policy positions implemented with some probability.

We show that some of these P.O.F.s do not induce sincere behaviour from voters. Under these circumstances, characterizing equilibrium strategic behaviour depends on the exact specification of the P.O.F.s. We avoid these difficulties and directly assume sincere voting behaviour. The consequence of this assumption is that party equilibrium is independent of the P.O.F. provided that the probability of a party's policy position being implemented is increasing in the number of constituencies.

Fix the position of the national party. Since a regional party can only get votes from its region it wants to locate “as close as possible” to the national party's policy as the same side of the region-wide median by the standard Hotelling argument. In view of this behaviour of the regional parties, the national party wants to locate in the interval between the policy positions of the region-wide medians.

The precise location of the national party depends on the structure of *isolation sets*. These sets are constructed from the distribution of voter policy positions and have the following property: by choosing the policy position of a voter in this set, the national party

can “isolate” or “separate” constituencies from their respective region-wide medians. If the voter distribution is *heterogeneous*, there are multiple isolation sets. In *homogeneous* voter distributions, the smallest interval containing all policy positions of constituency medians for one region is disjoint from the smallest interval containing all policy positions of constituency medians of the other region. As a result, the isolation sets are empty. In the heterogeneous case the national party locates in a maximal isolation set. In the homogeneous case, the national party’s policy is the policy position of the region-wide median of the region with the greater number of constituencies.

The main insight of the paper is the following. For a given P.O.F. and a fixed number of constituencies, the national party’s performance improves as the degree of voter heterogeneity increases. In the limit case, where the distribution is homogeneous, the national party can at best do as well as the regional party of the region with the greater number of constituencies. This result is broadly consistent with intuition and empirical evidence.



# Chapter 2

## MULTILEVEL MULTIDIMENSIONAL CONSISTENT AGGREGATORS

## 2.1 INTRODUCTION

It is well known that political parties can manipulate or gerrymander voting results by dividing and redistributing voters among districts. This phenomenon has been observed at regional and national levels in the U.S, Canada, India, the United Kingdom, Germany, Australia and France.<sup>1</sup> An important consideration in the design of voting rules is to ensure that they are immune to such manipulation. The specific property of voting rules or aggregators that guarantees this form of non-manipulability has been called *consistency*. There are several papers that have studied the structure of consistent voting rules satisfying various versions of consistency. Virtually all these papers have considered models where voters express opinions about a *single* alternative which have to be aggregated into a social opinion about that alternative. Our goal in this paper is to investigate the consistency of voting rules in models where voter opinions over *several* alternatives have to be aggregated.

Multidimensional voting models arise naturally in many contexts. Consider the case where there is a finite set of public projects that is under consideration by the Government. Not all projects are feasible because of resource constraints. The Government therefore needs to aggregate the opinions of all voters over all projects by means of a voting rule. We note that multidimensional voting models have been received a great deal of attention in social choice and positive political theory - see [Austen-Smith and Banks \(2000\)](#), [\(2005\)](#) for an extensive review of the literature.

We consider a model of aggregation where voter opinions have to be aggregated over several alternatives. Each voter submits an *evaluation* for each alternative (or component) indicating the intensity with which she likes the alternative.<sup>2</sup> The set of permissible evaluations for any alternative is the closed unit interval. An *aggregator* considers an *arbitrary* collection of voter evaluations and transforms them into an aggregate opinion.

Voters can be divided into mutually exclusive subgroups. This could be based, for example, on geographical regions/districts or political constituencies. The aggregator generates an aggregate for each subgroup. It can also be used to aggregate subgroup opinions into an opinion for the whole population. Consistency requires the same opinion for the population to emerge (for every possible configuration of voter opinions) irrespective of the way the population is split into subgroups. This Chapter examines the implications of consistency on aggregators.

We characterize *component-wise  $\alpha$  median* rules. These rules are *separable*, i.e. the outcome for an alternative depends only on voter opinions for that alternative. Moreover, the outcome for each alternative is the median of the minimum utility (across voters), the maximum utility (across voters) and a fixed but arbitrary number  $\alpha_j$  for each alternative  $j$ .

Consistent voting rules have also been analyzed in [Chambers \(2008\)](#), [Chambers \(2009\)](#)

---

<sup>1</sup>[Katz \(1998\)](#) and [Samuels and Snyder \(2001\)](#) provide empirical evidence of gerrymandering in different electoral systems and countries respectively.

<sup>2</sup>[Macé \(2013\)](#) provides another model of aggregation over evaluations.

and Nermuth (1994).<sup>3</sup> Both the Chambers’ papers consider a different notion of consistency where the sub-group aggregate opinion is replicated the same number of times as the number of voters in that subgroup. This notion is inspired by the electoral college voting system in U.S Presidential elections. Our notion of consistency is similar to that in Nermuth (1994). Both the Nermuth and Chambers papers consider a single alternative voting model.

Our result is a generalization of the result of Fung and Fu (1975) who prove an  $\alpha$ -median characterization result for the one alternative case. There are significant difficulties involved in the extension to the multidimensional case due to its additional richness. However, these are resolved using the same set of axioms as in Fung and Fu (1975) defined suitably for the multidimensional model.

The separability result depends critically on the structure of the model, in particular on the fact that the set of possible evaluations is a continuum. The result no longer holds if the set of evaluations is finite. We investigate this issue in a special “finite” model. This is a model where there are  $m$  alternatives and voter/social opinions pertain to the selection of set of these candidates. The set of possible evaluations for a candidate is either 0 or 1 indicating disapproval and approval respectively. We characterize a class of separable rules called *Bipartite Rules* by consistency and some stronger versions of some of the axioms used for the earlier result. The Bipartite Rule partitions the set of alternatives into two sets (independently of opinions). Alternatives in the first set are assigned value 1 unless all voters disapprove, while alternatives in the second are never selected unless they are unanimously approved.

The paper is organized as follows. We discuss the Evaluations model formally and the notion of consistency in Section 2.2.1. A discussion of the other axioms is contained in Section 2.2.2. Section 2.2.3 presents the component-wise  $\alpha$ -median result and its proof followed by a discussion in Section 2.2.4. Section 2.3 considers the finite set selection model while Section 2.4 concludes.

## 2.2 THE EVALUATION MODEL

The set of components or alternatives is  $X$  with  $|X| = m$ . The set of voters is  $\mathbf{N} = \{1, 2, \dots, n\}$ . Each voter submits an evaluation for each candidate. The set of evaluations is normalized without loss of generality to be the set  $[0, 1]$ . A voter submits  $v_i \in [0, 1]^m$  and we denote the set  $[0, 1]^m$  by  $A$ . A vote *profile*  $v \in A^n$  is the set of voter evaluations  $v = (v_1, \dots, v_n)$ . A component  $v_{ij} \in [0, 1]$  can be interpreted as the evaluation by voter  $i$  for alternative  $j$ .

A district or a group is a non-empty set  $N \subset \mathbf{N}$ . A vote profile is a collection of  $v_i$  for all voters  $i \in N$  such that  $N \subseteq \mathbf{N}$ . A vote profile  $v_S$  is the restriction of  $v$  to the set of voters  $S \subseteq \mathbf{N}$ . An *aggregator* is a function  $f : \cup_{N \in \mathbf{N}} A^N \rightarrow A$  which aggregates vote profiles for

---

<sup>3</sup>Perote-Peña (2005), Bervoets and Merlin (2012) and Plott (1973) analyze models that are similar in spirit to ours with related notions of consistency.

any district or subset  $\mathbf{N}$ . Some examples of aggregators are given below.

- A *constant* aggregator,  $f^c : A^n \rightarrow A$  for every profile  $v$  outputs a fixed set of evaluations  $\mathbf{c}$  in  $A$ ,

$$f^c(v) = \mathbf{c} \quad \forall v \in A^n \quad \forall N \in \mathbf{N}.$$

- A *mean* aggregator,  $f^{\text{mean}} : A^n \rightarrow A$  outputs the arithmetic mean of the evaluation values for each alternative,

$$f_j^{\text{mean}}(v) = \frac{\sum_i v_{ij}}{N} \quad \forall j \in X \quad v \in A^n \quad \forall N \in \mathbf{N}.$$

- The *median* denoted by  $\text{med}(\cdot)$  of a set of  $K$  numbers is the  $\frac{K}{2}$ <sup>th</sup> lowest evaluation when  $K$  is even, or the  $\frac{K+1}{2}$ <sup>th</sup> lowest evaluation if  $K$  is odd. A *median* aggregator selects the median for each component,  $f^{\text{med}} : A^n \rightarrow A$ ,

$$f_j^{\text{med}}(v) = \text{med}_{i \in N}(v_{ij}) \quad \forall j \in X \quad \forall v \in A^n \quad \forall N \in \mathbf{N}.$$

- A *min* aggregator,  $f^{\text{min}} : A^n \rightarrow A$  outputs the minimum evaluation from the set of numbers submitted by the voters for each alternative.

$$f_j^{\text{min}}(v) = \min_{i \in N}(v_{ij}) \quad \forall j \in X \quad \forall v \in A^n \quad \forall N \in \mathbf{N}.$$

A *max* aggregator can be similarly defined.

- An aggregator  $f^\alpha : A^n \rightarrow A$  is *component-wise  $\alpha$ -median* aggregator if  $\exists \alpha \in A$  such that,

$$f_j^\alpha(v) = \text{med}\left(\min_{i \in N}(v_i), \alpha_j, \max_{i \in N}(v_i)\right) \quad \forall j \in X \quad \forall v \in A^n \quad \forall N \in \mathbf{N}.$$

For each alternative  $j$  the aggregator  $f^\alpha$  picks median of the following three numbers—the smallest and greatest among the set of evaluations submitted by all the voters and the  $j^{\text{th}}$  component of  $\alpha$ .

Component-wise  $\alpha$ -median aggregators are generalizations of the min and max aggregators. The min and max rules are  $\alpha$ -median rules with  $\alpha = 0$  and  $\alpha = 1$  respectively.

- Let  $\succ$  be a strict ordering on components. Pick an arbitrary  $v \in A^n$ . The *lexicographic minimum* or *L-min* voter for  $v, N$  is a voter whose evaluation for the  $\succ$ -max alternative is lowest. If there is more than one such voter, break ties by picking a voter whose evaluation for the next-highest alternative according to  $\succ$  is lowest and so on. The L-min rule at  $v, N$  picks the evaluation vector of the L-min voter, i.e.  $f^{\text{L-min}}(v) = v_{\text{L-min}}$ . The L-max rule can be defined analogously.



It is worth drawing attention to the feature of the rules above. The constant, median, mean, min and max aggregators are component-separable rules i.e they aggregate the outcome for each component or alternative separately. The left-aligned and L-min aggregators are not component-separable.

### 2.2.1 CONSISTENCY OF AGGREGATORS

**DEFINITION 1 (Consistency)** An aggregator  $f$  satisfies *consistency* if for all  $N \in \mathbf{N}$ , for all partitions  $\{N_1, N_2, \dots, N_K\}$  of  $N$  and all  $v \in A$ ,

$$f(v) = f(f(v_{N_1}), f(v_{N_2}), \dots, f(v_{N_K})).$$

A vote profile  $v$  can be aggregated directly by  $f$ . It can also be aggregated indirectly as follows. The profile  $v$  can be split into the opinions of subgroups  $(v_{N_1}, \dots, v_{N_K})$ . Since  $f$  is defined for arbitrary collections of opinions,  $f$  can be applied to each sub-collection  $v_{N_1}, \dots, v_{N_K}$ . This yields a  $K$  sized opinion profile on which  $f$  can be applied again. If  $f$  is consistent, the direct and indirect procedures generate the same outcome.

Consistency prevents manipulation by re-assigning voters to subgroups. It is a strong requirement as many of the aggregators described earlier do not satisfy it.

1. **(Median)** The median aggregator violates consistency. Let  $N = \{1, 2, 3\}$  and  $m = 2$ . Considering the partition  $I = \{\{1, 2\}, \{3\}\}$  of  $\mathbf{N}$  we have,

$$\begin{aligned} f^{\text{med}} \left( f^{\text{med}} \left( \begin{pmatrix} 0.4 & 0.1 \\ 0.3 & 0.8 \end{pmatrix}, 0.7 \right), 0.4 \right) &= f^{\text{med}} \left( \begin{pmatrix} 0.1 & 0.7 \\ 0.3 & 0.4 \end{pmatrix} \right) \\ &= \begin{pmatrix} 0.1 \\ 0.3 \end{pmatrix} \\ &\neq \begin{pmatrix} 0.4 \\ 0.4 \end{pmatrix} \\ &= f^{\text{med}} \left( \begin{pmatrix} 0.4 & 0.1 & 0.7 \\ 0.3 & 0.8 & 0.4 \end{pmatrix} \right). \end{aligned}$$

Note that our definition of the median aggregator picks the “lower median evaluation” in societies with an even number of voters. The violation of consistency by the median rule does not depend on this assumption.

2. **(Mean)** The mean aggregator also violates consistency.

Consider the same example and partition as before. We have,

$$\begin{aligned}
f^{\text{mean}} \left( f^{\text{mean}} \left( \begin{pmatrix} 0.4 & 0.1 \\ 0.3 & 0.8 \end{pmatrix}, 0.7 \right), 0.4 \right) &= f^{\text{mean}} \left( \begin{pmatrix} 0.25 & 0.7 \\ 0.55 & 0.4 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0.475 \\ 0.475 \end{pmatrix} \\
&\neq \begin{pmatrix} 0.4 \\ 0.5 \end{pmatrix} \\
&= f^{\text{mean}} \left( \begin{pmatrix} 0.4 & 0.1 & 0.7 \\ 0.3 & 0.8 & 0.4 \end{pmatrix} \right).
\end{aligned}$$

On the other hand, the constant rule, the min rule, the L-min rule and their max counterparts and the component-wise  $\alpha$ -median rules satisfy consistency. The consistency of the constant aggregator is trivial. The consistency of the component-wise  $\alpha$ -median rule is demonstrated in the proof of the theorem. We show the consistency of the min and L-min rule below.

1. (**Min**) Pick an arbitrary profile  $v$  and alternative  $j$ . Suppose that the minimum evaluation for  $j$  in  $v$  is  $\underline{v}_j$ . Let  $\underline{v}_j = v_{ij}$ . Consider an arbitrary partition  $I = \{N_1, \dots, N_K\}$  of  $\mathbf{N}$  and suppose  $i \in N_k$ . Then  $f_j^{\text{min}}(v_{N_k}) = \underline{v}_j$  and  $f_j^{\text{min}}(v_{N_k}) \leq f_j^{\text{min}}(v_{N_{k'}})$  for all  $N_{k'} \in I$ . Therefore,  $f_j^{\text{min}}(f^{\text{min}}(v_{N_1}), \dots, f^{\text{min}}(v_{N_K})) = \min_{N_{k'} \in I} \{f_j^{\text{min}}(v_{N_{k'}})\} = \underline{v}_j = f_j^{\text{min}}(v)$ .

Similarly max rules also satisfy consistency. So do aggregators that pick the minimum for some alternatives and the maximum for others.

2. (**L-min**) Let  $j$  be the alternative that is  $\succ$  maximal. Let  $v$  be an arbitrary profile. The argument for the min rule for alternative  $j$  suffices to show that the L-min aggregator is consistent.

## 2.2.2 FURTHER AXIOMS

In addition to consistency, we impose certain axioms.

**DEFINITION 2 (Anonymity)** An aggregator  $f$  is *anonymous* if for all  $N \in \mathbf{N}$  for all  $v, v' \in A^N$  and for all bijections  $\Pi_i : N \rightarrow N$ ,

$$[v_i = v'_{\Pi(i)} \text{ for all } i \in N] \Rightarrow [f(v) = f(v')].$$

An aggregator satisfies anonymity if it is invariant with respect to changes in the identities of voters. All the aggregators mentioned above are anonymous. Its easy to construct aggregators that are non-anonymous, for instance, by constructing a ‘‘dictator’’ for every subset of  $\mathbf{N}$ . Consider the case when  $\mathbf{N} = \{1, 2, 3\}$ . Let 1 be the dictator for  $\{1, 2\}$  and  $\{1, 2, 3\}$ , 2 be dictator for  $\{2, 3\}$  and 3 be the dictator for  $\{1, 3\}$ . The outcome at any collection of voter opinions is the evaluation vector of the dictator for that subset of voters.

**DEFINITION 3 (Unanimity)** An aggregator  $f$  is *unanimous* if for all  $N \in \mathbf{N}$  for all  $v \in A^n$  and any  $j \in X$ ,

$$[v_i = \bar{v} \text{ for all } i \in N] \Rightarrow [f(v) = \bar{v}].$$

An aggregator that satisfies unanimity respects consensus. Our notion of unanimity is, therefore, very weak. Note that the unanimity condition does not apply if all voters are unanimous over a *subset* of the alternatives. All the aggregators mentioned earlier except the constant aggregator are unanimous.

**(Continuity)** An aggregator specifies a collection of maps that aggregates arbitrary sets of  $m$ -dimensional voter opinions into an aggregate opinion i.e it is a collection of maps  $f : \mathbb{R}^{ml} \rightarrow \mathbb{R}^m$  where  $l = 1, \dots, n$ . The aggregator satisfies continuity if each of these maps is continuous in the usual sense.

All aggregators discussed earlier except the L-min aggregator satisfy continuity. The violation by L-min is shown below.

Let the set of voters be  $N = \{1, 2\}$  and  $m = 2$ . Let  $v^t$ ,  $t = 2, 3, \dots$  be a sequence of profiles such that  $v_1^t = \begin{pmatrix} 0.7 \\ 0.4 \end{pmatrix}$  and  $v_2^t = \begin{pmatrix} 0.7 - \frac{1}{t} \\ 0.1 \end{pmatrix}$ ,  $t = 2, 3, \dots$ . Clearly,  $v^t \rightarrow \begin{pmatrix} 0.7 & 0.7 \\ 0.4 & 0.1 \end{pmatrix} = \bar{v}$  and  $f^{\text{L-min}}(v^t) = \begin{pmatrix} 0.7 - \frac{1}{t} \\ 0.4 \end{pmatrix}$  for all  $t$ . Therefore,  $f^{\text{L-min}}(v^t) \rightarrow \begin{pmatrix} 0.7 \\ 0.4 \end{pmatrix}$ . However,  $f^{\text{L-min}}(\bar{v}) = \begin{pmatrix} 0.7 \\ 0.1 \end{pmatrix}$ .

The next axiom uses the order structure on the set  $A$ .

**DEFINITION 4 (Monotonicity)** An aggregator  $f$  is *monotonic* if for all  $N \in \mathbf{N}$ , for all  $v, v' \in A^n$ ,

$$[v_{ij} \geq v'_{ij} \text{ for all } i, j] \Rightarrow [f_j(v) \geq f_j(v') \text{ for all } j].$$

Fix an arbitrary collection of voters. Suppose *all* voters in this collection weakly increase their evaluations of *all* alternatives. Then the aggregate opinion outputted by a monotonic aggregator must weakly increase for all alternatives. This is clearly a weak condition and aggregators described earlier, satisfy the axiom. It is of course, easy to construct an aggregator that does not satisfy the axiom.

### 2.2.3 THE MAIN RESULT

The main result shows that the component-wise  $\alpha$ -median aggregators are characterized by the axioms of consistency, unanimity, anonymity, monotonicity and continuity.

**THEOREM 1** An aggregator satisfies *consistency, unanimity, anonymity, monotonicity* and *continuity* if and only if it is a *component-wise  $\alpha$ -median aggregator*.

*Proof:* It is easy to verify that component-wise  $\alpha$ -median aggregators satisfy anonymity, unanimity, continuity and monotonicity. We show that it satisfies consistency.

Consistency: Let  $f^\alpha$  be a component-wise  $\alpha$ -median aggregator. In view of the separability of component-wise aggregators it clearly suffices to show that it satisfies consistency for any arbitrary alternative.

Pick a profile  $v \in A$  and alternative  $j$ . Then  $f_j^\alpha(v) \in \{\min_{i \in N}(v_{ij}), \alpha_j, \max_{i \in N}(v_{ij})\}$ . If  $f_j^\alpha(v) = \min_{i \in N}(v_{ij})$  i.e  $\alpha_j < \min_{i \in N}$ , consistency follows from the same argument used to show that the min aggregator is consistent. Likewise, the arguments used to show that the max aggregator is consistent can be used to show that  $f^\alpha$  is consistent when  $f^\alpha(v) = \max_{i \in N}(v_{ij})$  i.e  $\alpha_j > \max_{i \in N}(v_{ij})$ .

If  $\alpha_j \in [\min_{i \in N}(v_i), \max_{i \in N}(v_i)]$ , then  $f_j^\alpha(v) = \alpha_j$ . Let  $I = \{N_i, \dots, N_K\}$  be any partition of  $N$ . There exists a set  $N_k \in I$  such that for some  $i \in N_k$ ,  $v_{ij} \leq \alpha_j$ . By definition,  $f_j^\alpha(v_{N_k}) \leq \alpha_j$ . Therefore,  $\alpha_j \geq \min_{N_i \in I} (f_j^\alpha(v_{N_i}))$ . Similarly, there exists  $N_{k'} \in I$  such that for some  $i' \in N_{k'}$ ,  $v_{i'j} \geq \alpha_j$ . By definition,  $f_j^\alpha(v_{N_{k'}}) \geq \alpha_j$ . Therefore,  $\alpha_j \leq \max_{N_i \in I} (f_j^\alpha(v_{N_i}))$ . By the above arguments we have,  $\alpha_j \in [\min_{N_i \in I} (f_j^\alpha(v_{N_i})), \max_{N_i \in I} (f_j^\alpha(v_{N_i}))]$ . By definition,  $f_j^\alpha(f^\alpha(v_{N_1}), \dots, f^\alpha(v_{N_K})) = \alpha_j$ . Therefore, component-wise  $\alpha$ -median aggregators are consistent.

Let  $f$  be an aggregator which satisfies consistency, unanimity, anonymity, monotonicity and continuity. Observe that  $f$  is actually a collection of rules  $\{f^k\}$ ,  $k = 1, \dots, |\mathbf{N}|$  where  $f^k$  is an aggregator for any  $k$ -size collection of voter opinions. The next lemma shows that  $f$  can be constructed by a repeated application of the function  $f^2$ .

**LEMMA 1** Let  $N = \{i_1, \dots, i_n\} \subseteq \mathbf{N}$  and let  $v_{i_k} \in A$  for  $k = 1, \dots, n$ . Then  $f^n(v_{i_1}, \dots, v_{i_n}) = f^2(\dots f^2(f^2(v_{i_1}, v_{i_2}), v_{i_3}) \dots v_{i_n})$ .

This lemma follows directly by the application of consistency. For instance, if  $N = \{1, 2, 3, 4\}$ , then

$$f^4(v_1, v_2, v_3, v_4) = f^2(f^3(v_1, v_2, v_3), v_4) = f^2(f^2(f^2(v_1, v_2), v_3), v_4).$$

By Lemma 1 we can restrict attention to  $f^2$ .

Applying Lemma 1 we can restrict attention to the two voter aggregator  $f^2$ . From now onwards, we simply write  $f$  in place of  $f^2$  for simplicity of notation. In some cases, we will revert back to  $f^2$  where necessary. We introduce the notion of two evaluations being *ordered*. Let  $v, v' \in A^n$ ,  $N \in \mathbf{N}$ . If either  $v_j \geq v'_j$  or  $v_j \leq v'_j$  for all  $j \in X$  then  $v$  is ordered with  $v'$ . We define a Box as follows. Let  $v_i, v_k$  be a pair of voter opinions. Then

$$Box(v_i, v_k) = \left\{ v_t \in A^2 \mid v_{ij} \in \left[ \min_{i \in N}(v_{ij}), \max_{i \in N}(v_{kj}) \right] \quad \forall j \in X \right\}.$$

**LEMMA 2** Let  $v_i, v_k \in A$ . Then  $f(v_i, v_k) \in Box(v_i, v_k)$ .

*Proof:* We consider two cases.

- **Case 1:**  $v_i$  and  $v_k$  are ordered. Assume w.l.o.g.  $v_i \geq v_k$ . The case where  $v_k \geq v_i$  can be dealt with by using a symmetric argument. Applying monotonicity,

$$f(v_i, v_k) \geq f(v_k, v_k) = v_k.$$

The last inequality holds due to unanimity. Similarly,

$$f(v_i, v_k) \leq f(v_i, v_i) = v_i.$$

Therefore  $f(v_i, v_k) \in \text{Box}(v_i, v_k)$ .

- **Case 2:** Case 1 does not hold. Let  $\underline{v}$  be such that,

$$\underline{v}_j = \min(v_{ij}, v_{kj}) \quad \forall j \in X.$$

Similarly, let  $\bar{v}$  is such that,

$$\bar{v}_j = \max(v_{ij}, v_{kj}) \quad \forall j \in X.$$

Note that  $\text{Box}(v_i, v_k) = \text{Box}(\underline{v}, \bar{v})$ . By monotonicity,

$$f(v_i, v_k) \geq f(\underline{v}, \underline{v}) = \underline{v}.$$

Similarly,

$$f(v_i, v_k) \leq f(\bar{v}, \bar{v}) = \bar{v}.$$

Therefore,  $f(v_i, v_k) \in \text{Box}(v_i, v_k)$ .

■

The next lemma is illustrated in Figure 2.1.

**LEMMA 3** Let  $v_i, v_k \in A$  be ordered (assume w.l.o.g.  $v_i \leq v_k$ ) and  $f(v_i, v_k) = v_t$ . Then for all  $v_r, v_u \in A$  such that  $v_r \in \text{Box}(v_i, v_t)$  and  $v_u \in \text{Box}(v_t, v_k)$ ,

$$f(v_r, v_u) = v_t, \quad f(v_r, v_t) = v_t, \quad f(v_t, v_u) = v_t .$$

*Proof:* By Lemma 1 and unanimity,

$$\begin{aligned} f(v_i, v_t) &= f^2(v_i, f^2(v_i, v_k)) = f^3(v_i, v_i, v_k) \\ &= f^2(f^2(v_i, v_i), v_k) = f(v_i, v_k) = v_t. \end{aligned}$$

By an analogous argument  $f(v_t, v_k) = v_t$ . By monotonicity,

$$f(v_r, v_u) \leq f(v_t, v_u) \leq f(v_t, v_k) = v_t.$$

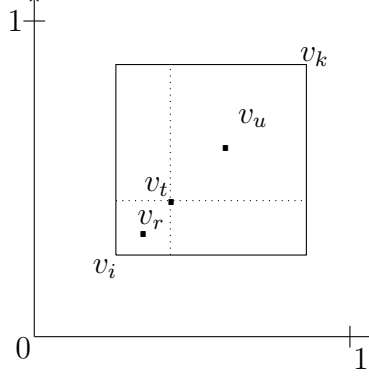


Figure 2.1: Illustration for Lemma 3

Similarly,

$$f(v_r, v_u) \geq f(v_i, v_u) \geq f(v_i, v_t) = v_t.$$

Therefore  $f(v_r, v_u) = v_t$ . Again by monotonicity,

$$f(v_r, v_t) \leq f(v_t, v_k) = v_t.$$

Also,

$$f(v_r, v_t) \geq f(v_i, v_t) = v_t.$$

Therefore  $f(v_r, v_t) = v_t$ . By a similar argument it follows that  $f(v_t, v_u) = v_t$ . ■

**LEMMA 4** Let  $v_i, v_k, v'_i, v'_k$  be such that (i)  $v_i$  is ordered with  $v_k$ ,  $v'_i$  is ordered with  $v'_k$  (ii)  $f^2(v_i, v_k) = v_t \in \text{int}Box(v_i, v_k)$ <sup>4</sup> and (iii)  $f(v'_i, v'_k) = v'_t \in \text{int}Box(v'_i, v'_k)$ . Then  $v'_i < v_t$  and  $v_k > v'_t$  both cannot hold.

*Proof:* We prove this by contradiction. So suppose  $v'_i < v_t$  and  $v_k > v'_t$  hold. Then by applying Lemma 3 on  $Box(v_i, v_k)$  we have  $f(v_t, v'_t) = v_t$  and by applying Lemma 3 on  $Box(v'_i, v'_k)$  we have  $f(v_t, v'_t) = v'_t$ . This is a contradiction. Therefore both  $v'_i < v_t$  and  $v_k > v'_t$  cannot be true. ■

Let  $v_t \in A$ . The box  $MBox(v_t) = Box(\bar{v}_i, \bar{v}_k)$  is a *maximal* box for  $v_t$  if there does not exist  $v'_i < v_i$  and  $v'_k > v_k$  such that  $f(v'_i, v'_k) = v_t$ . Suppose  $v_t$  is in the range of  $f$ . Then  $MBox(v_t)$  exists by the virtue of continuity of  $f$ . Note that a maximal set may not be unique. By similar arguments as in Lemma 3 we can prove the following Lemma.

**LEMMA 5** Let  $MBox(v_t)$  be a maximal box for  $v_t$ . Let  $v_r, v_u \in A$  such that  $v_r \in Box(\bar{v}_i, v_t)$  and  $v_u \in Box(v_t, \bar{v}_k)$ . Then,

---

<sup>4</sup> $\text{int}Box(\cdot)$  denotes the interior of  $Box(\cdot)$ .

- (i)  $f(v_r, v_u) = v_t$ ,  $f(v_r, v_t) = v_t$ ,  $f(v_t, v_u) = v_t$ .  
(ii) Let  $\{v_t^q\}_{q=1}^\infty$  be a sequence such that  $\lim_{n \rightarrow \infty} v_t^q = v_t$ . Then,

$$\lim_{n \rightarrow \infty} MBox(v_t^q) = MBox(v_t).$$

*Proof:* The first part of the proof is proved analogously as in the previous Lemma 3. The second part is an implication of the continuity of  $f^2$ . ■

**LEMMA 6** Let  $v_i, v_k$  and  $v'_i, v'_k, v'_t$  be such that (i)  $v_i$  is ordered with  $v_k$ ,  $v'_i$  is ordered with  $v'_k$  and  $v_t$  is ordered with  $v'_t$  (ii)  $f(v_i, v_k) = v_t \in \text{int}Box(v_i, v_k)$  and (iii)  $f(v'_i, v'_k) = v'_t \in \text{int}Box(v'_i, v'_k)$ . Then  $\exists v''_i, v''_k$  and  $v''_t$  such that (a)  $v''_i, v''_k, v''_t \in Box(v_t, v'_t)$  (b)  $f(v''_i, v''_k) = v''_t$  and  $v''_t \in \text{int}Box(v''_i, v''_k)$  (c)  $v''_t \notin \{v_t, v'_t\}$ .

*Proof:* W.l.o.g. let  $v_i \leq v_k$ ,  $v'_i \leq v'_k$  and  $v_t \leq v'_t$ . By Lemma 4 we have

$$Box(v_i, v_k) \cap Box(v_t, v'_t) \neq \emptyset \text{ and } Box(v_i, v_k)^C \cap Box(v_t, v'_t) \neq \emptyset.^5$$

$$\text{or } Box(v'_i, v'_k) \cap Box(v_t, v'_t) \neq \emptyset \text{ and } Box(v'_i, v'_k)^C \cap Box(v_t, v'_t) \neq \emptyset.$$

Therefore assume w.l.o.g.

$$Box(v'_i, v'_k) \cap Box(v_t, v'_t) \neq \emptyset \text{ and } Box(v'_i, v'_k)^C \cap Box(v_t, v'_t) \neq \emptyset. \quad (\#)$$

Pick  $v_r \in Box(v_i, v_t)$  and  $v_u \in Box(v'_t, v'_k)$ . By applying Lemma 3 to  $Box(v_i, v_k)$  and  $Box(v'_i, v'_k)$  we have  $f(v_r, v_u) \geq f(v_i, v_t) = v_t$  and  $f(v_r, v_u) \leq f(v'_t, v'_k) = v'_t$  respectively. If  $f(v_r, v_u) \notin \{v_t, v'_t\}$  then the Lemma holds with  $v''_i = v_r, v''_k = v_u$  and  $v''_t = f(v_r, v_u)$ . So suppose  $f(v_r, v_u) \in \{v_t, v'_t\}$ . We consider two cases.

**Case 1:**  $f(v_r, v_u) = v_t$ . Consider an increasing sequence  $\{v_r^q\}$  such that  $\lim_{n \rightarrow \infty} v_r^q = v'_t$ . In view of (#) there exists a  $q$  such that  $v_r^q$  is on the boundary of  $Box(v'_i, v'_k)$  and is in  $Box(v_t, v'_t)$ . By continuity  $\lim_{n \rightarrow \infty} f(v_r^q, v'_t) = v'_t$ . By choosing a point  $q'$  sufficiently close to  $v_r^q$  we can satisfy the conditions of the Lemma.

**Case 2:**  $f(v_r, v_u) = v'_t$ . Suppose  $v_k \geq v'_t$ . By applying Lemma 3 to  $Box(v_i, v_k)$  and  $Box(v'_i, v'_k)$  we have  $f(v_r, v_u) = v_t$  and  $f(v_r, v_r) = v'_t$ . This is a contradiction. Hence  $v_k \leq v'_t$ . Now by repeating the arguments in Case 1 the Lemma holds with  $v''_i = v_r, v''_k = v_u$  and  $v''_t = f(v_r, v_u)$ . ■

The next Lemma states that there exists at most one element in the range of  $f$  which is in the interior of its relevant box.

**LEMMA 7** There do not exist  $v_i, v_k, v'_i, v'_k$  such that (i)  $v_i$  is ordered with  $v_k$  and  $v'_i$  is ordered with  $v'_k$  (ii)  $f(v_i, v_k) \in \text{int}Box(v_i, v_k)$  (iii)  $f(v'_i, v'_k) \in \text{int}Box(v'_i, v'_k)$  and (iv)  $f(v_i, v_k) \neq f(v'_i, v'_k)$ .

---

<sup>5</sup> $A^C$  is the complement of set  $A$ .

*Proof:* We prove the Lemma by contradiction i.e there exist  $v_i, v_k, v'_i, v'_k$  as specified in the statement of Lemma 7. Let  $f(v_i, v_k) = v_t$  and  $f(v'_i, v'_k) = v'_t$ .

1. **Case 1:** Suppose  $v_t, v'_t$  are ordered. Assume w.l.o.g.  $v_t \leq v'_t$ . By Lemma 5 there exists  $v''_i, v''_k$  such that  $f(v''_i, v''_k) = v''_t$  and  $v''_t \in \text{int}Box(v''_i, v''_k)$ . In fact, by applying the Lemma repeatedly we can construct a sequence  $\{v_t^q\}_{q=1}^\infty$  such that  $f(v_i^q, v_k^q) = v_t^q \in \text{int}Box(v_i^q, v_k^q)$  for all  $q$  and  $\lim v_t^q = v'_t$ .

Let  $\{\tilde{v}_t^q\}_{q=1}^\infty$ , be a subsequence of  $\{v_t^q\}$  such that  $\tilde{v}_t^q \in MBox(v_t)$  for all  $q$ . Note that  $Box(v_i, v_k) \cap Box(\tilde{v}_i^q, \tilde{v}_k^q) \neq \emptyset$ . We claim that  $\tilde{v}_k^q \geq v'_t$  cannot hold. Suppose contrariwise that  $\tilde{v}_k^q \geq v'_t$ . Pick  $v_r \in Box(v'_i, v'_t) \cap Box(\tilde{v}_i^q, \tilde{v}_k^q)$  and  $v_u \in Box(v'_t, v'_k) \cap Box(\tilde{v}_t^q, \tilde{v}_k^q)$ . Applying Lemma 3 to the boxes  $Box(\tilde{v}_i^q, \tilde{v}_k^q)$  and  $Box(v'_i, v'_k)$  we have  $f(v_r, v_u) = \tilde{v}_t^q$  and  $f(v_r, v_u) = v'_t$  respectively. This is a contradiction. Therefore  $\tilde{v}_k^q \geq v'_t$  cannot hold and we have

$$\lim_{n \rightarrow \infty} \tilde{v}_t^q = v'_t \Rightarrow \lim_{n \rightarrow \infty} \tilde{v}_k^q = v'_t.$$

Since  $\tilde{v}_t^q \rightarrow v'_t$  we know by Lemma 4 that  $\lim_{n \rightarrow \infty} MBox(\tilde{v}_t^q) = MBox(v'_t)$ . Hence  $\lim_{n \rightarrow \infty} MBox(\tilde{v}_t^q) = MBox(v'_t) = Box(\bar{v}_i^q, v'_t)$  where  $\bar{v}_i^q = \lim_{n \rightarrow \infty} \tilde{v}_i^q$ , i.e  $v'_t \notin \text{int}MBox(v'_t) = Box(\bar{v}_i^q, v'_t)$ . However  $v'_t \in \text{int}Box(v'_i, v'_k)$  implies  $v'_t \in \text{int}MBox(v'_t)$  by assumption. Thus we have a contradiction.

2. **Case 2:**  $v_t$  and  $v'_t$  are not ordered. Pick  $v_r \in Box(\mathbf{0}, v_t) \cap Box(\mathbf{0}, v_t)$ .<sup>6</sup> By Case 1  $f(v_r, v_t) \notin \text{int}Box(v_r, v_t)$  and  $f(v_r, v'_t) \notin \text{int}Box(v_r, v'_t)$ .

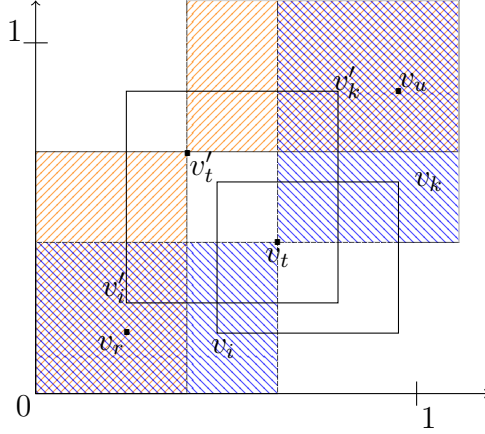


Figure 2.2: Illustration for Case 2

We claim that  $f(v_r, v_t) = v_t$ . Suppose this is false. By virtue of the fact that  $f(v_r, v_t) \notin \text{int}Box(v_r, v_t)$ ,  $f^2(v_r, v_t)$  must lie on the boundary of  $Box(v_r, v_t)$  but not equal to  $v_t$ . By

<sup>6</sup>Recall that  $\mathbf{0} = (0, 0, \dots, 0) \in A$  and  $\mathbf{1} = (1, 1, \dots, 1) \in A$ .



constructing a sequence  $\{v_r^q\}_{q=1}^\infty \rightarrow v_t$  and using arguments from Lemma 5 we obtain a contradiction. Therefore  $f(v_r, v_t) = v_t$ . By an identical argument  $f(v_r, v'_t) = v'_t$ .

Pick  $v_u \in \text{Box}(v_t, \mathbf{1}) \cap \text{Box}(v'_t, \mathbf{1})$ . Using the same arguments as in the previous paragraph, we have  $f(v_u, v_t) = v_t$  and  $f(v_u, v'_t) = v'_t$ . Applying Lemma 3 and monotonicity,

$$f(v_r, v_u) \geq f(v_r, v_t) = v_t.$$

$$f(v_r, v_u) \leq f(v_t, v_u) = v_t.$$

Therefore  $f(v_r, v_u) = v_t$ . However, the same argument with  $v'_t$  substituted for  $v_t$  yields  $f(v_r, v_u) = v'_t$ . We have a contradiction.  $\blacksquare$

**LEMMA 8** Let  $v_i, v_k$  be ordered and  $f(v_i, v_k) = v_t$ . Then

$$[v_r, v_u \in \text{Box}(v_i, v_t), v_r \leq v_u] \Rightarrow [f(v_r, v_u) = v_u].$$

$$[v_r, v_u \in \text{Box}(v_t, v_k), v_r \leq v_u] \Rightarrow [f(v_r, v_u) = v_r].$$

*Proof:* Suppose  $v_r, v_u \in \text{Box}(v_i, v_t)$ ,  $v_r \leq v_u$  and  $f(v_r, v_u) \neq v_u$ . Suppose  $f(v_r, v_u) = v'_t$ . By Lemma 7,  $v'_t \notin \text{intBox}(v_r, v_u)$ . By applying Lemma 3 on  $\text{Box}(v_r, v_u)$  we have  $f(v'_t, v_s) = v'_t$  for all  $v_s \in \text{Box}(v_r, v'_t)$ . Similarly, by applying Lemma 3 on  $\text{Box}(v_i, v_t)$  we have  $f(v_r, v_t) = v_t$ . This implies that there exists  $v'_k \geq v_u$  such that  $f(v_r, v'_k) > v'_t$  and  $f(v_r, v'_k) \in \text{Box}(v_r, v'_t)$ . By applying Lemma 3 on  $\text{Box}(v_r, v'_t)$  we have  $f(f(v_r, v'_k), v'_t) = v'_t$ . However, by Lemma 3 on  $\text{Box}(f(v_t, v'_k), v'_k)$  we have  $f(v'_t, f(v_r, v'_k)) = f(v_r, v'_k)$ . This is a contradiction. Therefore,  $f(v_r, v_u) = v_u$ .

The case where  $v_r \leq v_u$  with  $v_r, v_u \in \text{Box}(v_t, v_k)$  can be proved by an argument similar to the one above.  $\blacksquare$

**LEMMA 9** Pick any  $v_i, v_k \in A$ . Then  $f(v_i, v_k) = f(\underline{v}, \bar{v})$  where  $\underline{v}$  and  $\bar{v}$  are as defined before.

*Proof:* There is nothing to prove in the case where  $v_i$  and  $v_k$  are ordered. Therefore assume that  $v_i, v_k$  are not ordered. Let  $f(\underline{v}, \bar{v}) = v_t$ . For each  $j \in X$  we have  $v_t^j, v_t'^j$  such that

$$(i) \ v_t^j = v_{tj}, \ v_{tj'}^j = \min(v_{ij'}, v_{kj'}) \ \forall j' \in X.$$

$$(ii) \ v_t'^j = v_{tj}, \ v_{tj'}'^j = \max(v_{ij'}, v_{kj'}) \ \forall j' \in X.$$

Note that  $v_t^j \in \text{Box}(\underline{v}, v_t)$  and  $v_t'^j \in \text{Box}(v_t, \bar{v})$  for all  $j$ . By applying Lemma 8 to  $\text{Box}(\underline{v}, v_t)$  and  $\text{Box}(v_t, \bar{v})$  and using monotonicity we have

$$f(v_i, v_k) \geq f(v_t^j, v_t') = v_t^j.$$

$$f(v_i, v_k) \leq f(v_t'^j, v_t') = v_t'^j.$$

This implies  $f(v_i, v_k) = f(\underline{v}, \bar{v}) = v_t$ .  $\blacksquare$

As an implication of Lemma 9 we can restrict attention to any ordered pair  $v_i, v_k$ . Our final Lemma proves the theorem.

LEMMA 10 There exists  $\alpha \in A$  such that for all  $v_i, v_k \in A$

$$f_j(v_i, v_k) = \text{med}(\min_{i \in N} \{v_{ij}\}, \max_{i \in N} \{v_{ij}\}, \alpha_j) \quad \forall j \in X.$$

*Proof:* Let  $f(\mathbf{0}, \mathbf{1}) = v_t^*$ . We show that  $f$  is an  $\alpha$ -median rule with  $\alpha = v_t^*$ . Let  $v_i, v_k \in A^n$ . By Lemma 9 we only need to consider the case where they are ordered. W.l.o.g. assume  $v_i \leq v_k$ .

1. **Case 1:** Suppose  $v_i, v_k$  are both ordered with respect to  $v_t^*$ . We show that  $f$  is an  $\alpha$ -median rule with  $\alpha = v_t^*$ . By Lemma 3,  $f(v_i, v_k) = v_t^*$  for all  $v_i \in \text{Box}(\mathbf{0}, v_t^*)$  and  $v_k \in \text{Box}(v_t^*, \mathbf{1})$ . By Lemma 8,  $f(v_k, \mathbf{1}) = v_k$  for all  $v_k \in \text{Box}(v_t^*, \mathbf{1})$  and  $f(\mathbf{0}, v_i) = v_i$  for all  $v_i \in \text{Box}(\mathbf{0}, v_t^*)$ . By Lemma 8 and 9,

$$f(v_i, v_k) = f(\underline{v}, \bar{v}) = \bar{v} \quad \forall v_i, v_k \in \text{Box}(\mathbf{0}, v_t^*).$$

$$f^2(v_i, v_k) = f^2(\underline{v}, \bar{v}) = \underline{v} \quad \forall v_i, v_k \in \text{Box}(v_t^*, \mathbf{1}).$$

Therefore  $v_t^*$  is the  $\alpha$ -median for all  $v_i$  and  $v_k$  ordered such that either  $v_i, v_k \in \text{Box}(\mathbf{0}, v_t^*)$  or  $v_i, v_k \in \text{Box}(v_t^*, \mathbf{1})$ . If  $v_i, v_k \in \text{Box}(\mathbf{0}, v_t^*)$  are not ordered then by Lemma 8 and 9,  $f^2(v_i, v_k) = \bar{v}$ . Similarly if  $v_i, v_k \in \text{Box}(v_t^*, \mathbf{1})$  are not ordered then by applying Lemma 8 and 9,  $f^2(v_i, v_k) = \underline{v}$ . Therefore, in both the cases  $f$  picks the component-wise  $\alpha$ -median for  $j \in X$  with  $\alpha = v_t^*$ .

2. **Case 2:** Suppose  $v_i$  is ordered with  $v_t^*$  but  $v_k$  is not ordered with  $v_t^*$ . Pick  $v_t^\gamma \in \text{Box}(v_i, v_k) \cap \text{Box}(\mathbf{0}, v_t^*)$  such that  $v_{tj}^\gamma = \text{med}(v_{ij}, v_{kj}, \alpha_j)$  for all  $j \in X$ . By Lemma 3 and 8 and monotonicity,  $f(\mathbf{0}, v_t^*) = v_t^\gamma \leq f(v_i, v_k)$  and  $f(v_t^*, \mathbf{1}) = v_t^* \leq f(v_i, v_k)$ . This implies  $f(v_i, v_k) = v_t^\gamma$ . The same arguments hold for the case when  $v_i$  is not ordered with  $v_t^*$  but  $v_k$  is ordered with  $v_t^*$ .
3. **Case 3:** Neither  $v_i$  nor  $v_k$  is ordered with respect to  $v_t^*$ . Pick  $\underline{v}_i, \underline{v}_k, \bar{v}_i, \bar{v}_k$  such that  $\underline{v}_{ij} = \min(v_{ij}, v_{tj}^*)$ ,  $\underline{v}_{kj} = \min(v_{kj}, v_{tj}^*)$ ,  $\bar{v}_{ij} = \max(v_{ij}, v_{tj}^*)$  and  $\bar{v}_{kj} = \max(v_{kj}, v_{tj}^*)$ . By applying Lemma 8 to  $\text{Box}(\mathbf{0}, v_t^*)$  and  $\text{Box}(v_t^*, \mathbf{1})$  and using monotonicity ,

$$f(\underline{v}_i, \underline{v}_k) = \underline{v}_k \leq f(v_i, v_k).$$

$$f(\bar{v}_i, \bar{v}_k) = \bar{v}_i \geq f(v_i, v_k).$$

This implies  $f(v_i, v_k) = \text{med}(v_{ij}, v_{kj}, \alpha_j)$ . ■

Let  $f(\mathbf{0}, \mathbf{1}) = v_t^*$  such that  $v_t^* \in A$ . We have proved that  $f^2$  is a component-wise  $\alpha$ -median aggregator with  $\alpha = v_t^*$ . Note that  $f^k$  is also a component-wise aggregator i.e the aggregation over an alternative is independent of the opinions over other alternatives. We show that  $f^k$  is a component-wise  $\alpha$ -median rule for  $k = 1, 2, \dots, n$ .

Let  $v \in A^k$ ,  $k \in N$  be a profile. We show that

$$f_j(v) = \text{med}\left(\min_{i=1,\dots,k} v_{ij}, \max_{i=1,\dots,k} v_{ij}, \alpha_j\right)$$

for all  $j \in X$ . There are several cases to consider. Pick  $j \in X$ . Suppose  $v_{ij} \leq \alpha_j$  for all  $i \in N$ . Since  $f^2$  is a component-wise  $\alpha$ -median aggregator  $f^2(v_{ij}, v_{i'j}) = \max(v_{ij}, v_{i'j})$  for all  $i, i'$ . Therefore,

$$\begin{aligned} f^k(v_{1j}, \dots, v_{kj}) &= f^2(\dots f^2(f^2(v_{1j}, v_{2j}), \dots, v_{kj})) \\ &= \max(\dots \max(\max(v_{1j}, v_{2j}), \dots, v_{kj})) \\ &= \max(v_{1j}, \dots, v_{kj}) \\ &= f^k(v_{1j}, \dots, v_{kj}) \\ &= \text{med}(\min_i(v_{ij}), \max_i(v_{ij}), \alpha_j). \end{aligned}$$

Suppose  $v_{ij} \geq \alpha_j$  for all  $i \in N$ . An argument analogous to the previous one gives  $f^k(v_{1j}, \dots, v_{kj}) = \min(v_{1j}, \dots, v_{kj}) = \text{med}(\min_i(v_{ij}), \max_i(v_{ij}), \alpha_j)$ .

Finally consider the case where  $\alpha_j \in (\min_i(v_{ij}), \max_i(v_{ij}))$ . Let

$$\begin{aligned} f^2(v_{1j}, v_{2j}) &= z^1. \\ f^2(f^2(v_{1j}, v_{2j}), v_{3j}) &= z^2. \\ &\vdots \\ f^2(\dots(f^2(v_{1j}, v_{2j}), \dots, v_{kj})) &= z^{k-1}. \end{aligned}$$

In view of the nature of  $f^2$  there must exist  $q$  such that  $z^q = \alpha_j$  and  $z^{q'} = \alpha_j$  for all  $q' \geq q$ . Therefore  $f^k(v_{1j}, \dots, v_{kj}) = \text{med}(\min_i(v_{ij}), \max_i(v_{ij}), \alpha_j)$ . This completes the proof.  $\blacksquare$

## 2.2.4 DISCUSSION

Theorem 1 generalizes the [Fung and Fu \(1975\)](#) result from the one dimensional to the multidimensional case. The structure of the proof broadly follows that of [Fung and Fu \(1975\)](#). However, the generalization of specific arguments is not straightforward since several “new” cases can arise regarding the location of the evaluation vectors chosen for aggregation.

## 2.2.5 INDEPENDENCE OF AXIOMS

We show that the axioms used in Theorem 1 are independent. We consider each axiom in turn and show that there exists an aggregator that satisfies the other axioms.

**Consistency:** The median aggregator satisfies all the axioms except consistency.

**Unanimity:** Constant aggregators satisfy all the axioms except unanimity.

**Anonymity:** We define an aggregator that specifies a dictator for every subset of the voters and outputs the vector of evaluations of the dictator for all profile. We proceed as follows. Let  $\underline{i}(N) = \min_{i \in N} \#i$ . Then  $f^D$  is a sequential dictator aggregator if  $f^D = v_{\underline{i}(N)}$  for all  $N \in \mathbf{N}$  for all  $v \in A^n$ .

The aggregator is consistent as we show below. Consider a profile  $v \in A^n$ . Then by definition of the aggregator,  $f^D(v_1, \dots, v_n) = v_1$ . Consider any partition  $I = \{N_1, \dots, N_K\}$ . By applying the rule to the sub-groups we have,

$$f^D(f^D(v_{N_1}), \dots, f^D(v_{N_K})) = f(v_{\underline{i}(N_1)}, \dots, v_{\underline{i}(N_K)}) = v_{\underline{i}(N)} = f^D(v) = v_1.$$

The sequential dictatorship clearly violates anonymity.

**Continuity:** We have shown earlier that the L-min aggregator satisfies all the axioms other than continuity.

**Monotonicity:** We define an aggregator for the case when the number of alternatives is two. The construction can be easily generalized to an arbitrary number of alternatives.

Define  $f^2$  as follows. Pick  $\bar{v} \in A$  with  $\bar{v}_2 > 0$ . The aggregator will be separable. For the first component,  $f^2$  picks the smaller of the first component of the two voter evaluations, i.e  $f_1(v_i, v_k) = \min(v_{i1}, v_{k1})$  for all  $v_i, v_k \in A$ . For the second component, there are three cases:

- (i)  $\max(v_{i2}, v_{k2}) \leq \bar{v}_2$ . Then  $f_2(v_i, v_k) = \max(v_{i2}, v_{k2})$ .
- (ii)  $\min(v_{i2}, v_{k2}) \geq \bar{v}_2$ . Then  $f_2(v_i, v_k) = \min(v_{i2}, v_{k2})$ .
- (iii)  $\min(v_{i2}, v_{k2}) < \bar{v}_2$  and  $\max(v_{i2}, v_{k2}) > \bar{v}_2$ . Then

$$f_1(v_i, v_k) = \max\left(\min(v_{i2}, v_{k2}), \bar{v}_2 - |\bar{v}_2 - \max(v_{i2}, v_{k2})|\right).$$

The aggregator  $f^k$ ,  $k \in \{1, \dots, n\}$  can be obtained from  $f^2$  in the following way. For any  $v \in A^k$ ,

$$f(v_1, \dots, v_k) = \max\left(\min_i v_{i2}, \bar{v}_2 - |\bar{v}_2 - \max_i v_{i2}|\right).$$

We show that the rule is not monotonic. In Figure 2.3  $v_r \in A$  satisfies  $v_{rj} < \bar{v}_j$ ,  $j \in \{1, 2\}$ . For the profile  $v = (v_r, \bar{v})$  we have  $f(v_r, \bar{v}) = \bar{v}$ . Pick  $v_u$  such that  $v_{uj} > \bar{v}_j$ ,  $j \in \{1, 2\}$  and  $f(v_r, v_u) = v_t$  where  $v_{t2} < \bar{v}_2$ . Therefore, the rule violates monotonicity.

The aggregator is consistent for any profile  $v \in A^n$ . Suppose  $\bar{i}$  such that  $v_{\bar{i}2} = \max_{i \in N} v_{i2}$ . Consider a partition  $I = \{N_1, \dots, N_K\}$ . Suppose  $\bar{i} \in N_k$  for some  $k \in \{1, \dots, K\}$ . Note that  $\min_{i \in N_k} v_{i2} \geq \min_{i \in N_k} v_{i2}$ . Therefore,

$$\max\left(\min_{i \in N_k} v_{i2}, \bar{v}_2 - |\bar{v}_2 - \max_{i \in N_k} v_{i2}|\right) \geq \max\left(\min_{i \in N_{k'}} v_{i2}, \bar{v}_2 - |\bar{v}_2 - \max_{i \in N_{k'}} v_{i2}|\right)$$

for all  $N_{k'} \in I$ . Therefore,

$$f(f(v_{N_1}, \dots, f(v_{N_K})) = \max\left(\min_{i \in N} v_{i2}, \bar{v}_2 - |\bar{v}_2 - v_{\bar{i}2}|\right) = f(v).$$

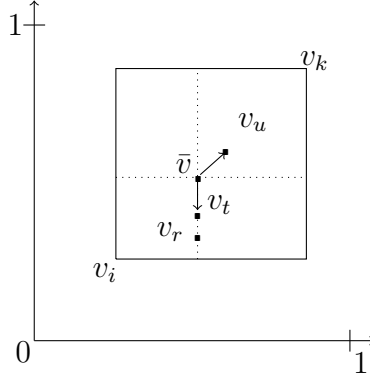


Figure 2.3: Violation of monotonicity

Figure 2.4 shows the continuity of the aggregator. Continuity is an issue only for sequences of the following kind:  $\{(v_1^q, v_2^q)\}$ ,  $q = 1, 2, \dots$  such that (i)  $v_1^q = v_{rj} \leq \hat{v}_j$  for all  $q$  and for  $j \in \{1, 2\}$  and (ii)  $\{v_2^q\} \rightarrow \hat{v}$ . In this case,  $f(v_1^q, v_2^q) \rightarrow v_t$  and  $f(v_r, \hat{v}) = v_t$  so that  $f$  is continuous.

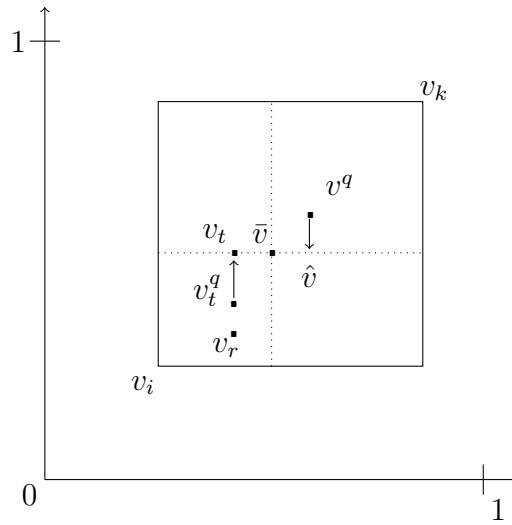


Figure 2.4: Continuity of the aggregator

## 2.3 THE FINITE CASE: AGGREGATING SETS OF ALTERNATIVES

In the previous model, voters submitted a utility number for each alternative with utilities normalized to lie in the set  $[0, 1]$ . In this section we depart radically from this model and consider a model where voters have *binary* choices over each alternative. They can either declare 0 for an alternative indicating disapproval or 1 indicating approval. The aggregation

rule takes tuples of voter opinions as inputs and outputs an aggregate binary opinion for each candidate. This is therefore a model of the aggregation of sets. Our goal is to study the role of consistency in this framework. <sup>7</sup>

Our first observation is that Theorem 1 no longer holds in this setting. For example, the L-min rule satisfies all the axioms of Theorem 1 since continuity holds vacuously. In particular, separability across components in the aggregation rule is no longer guaranteed. We shall impose further axioms that are natural in this context to show that the aggregation rule must be constant over a large class of profiles. We show that an aggregator satisfies consistency, *component unanimity* and *component anonymity* if and only if it is a *Bipartite Rule*. These aggregators pick the same set of alternatives for “almost” all vote profiles. These aggregators pick a fixed set of alternatives unless voters unanimously approve that alternative and always reject an alternative unless voters unanimously reject its selection. We proceed to details.

The set of candidates or alternatives is  $X$  with  $|X| = m$ . The set of voters is  $\mathbf{N} = \{1, 2, \dots, n\}$ . A voter submits  $v_i \in \{0, 1\}^m$  and we denote the set  $\{0, 1\}^m$  by  $A$ . A component  $v_{ij} = 0$  indicates that voter does not approve of  $j$  while a value of 1 indicates approval.

A district or a group is a non-empty set  $N \subset \mathbf{N}$ . A vote profile is a collection of  $v_i$  for all voters  $i \in N$  such that  $N \subseteq \mathbf{N}$ . A vote profile  $v_S$  is the restriction of  $v$  to a vote profile for voters in  $S \subseteq \mathbf{N}$ . An *aggregator* is a function  $f : \cup_{N \in \mathbf{N}} A^N \rightarrow A$  which aggregates voter profiles for any district or subset  $\mathbf{N}$ .

Several aggregators introduced in Section 2 are not well-defined in this model. These include the median and the mean aggregators. Component-wise  $\alpha$ -medians rules are also not well-defined unless  $\alpha_j$  is either 0 or 1. The min., left-aligned, constant and L-min. aggregators are well-defined in this setting.

We now turn to axioms. The main axiom as before will be consistency which is defined exactly as before. Monotonicity is no longer required and continuity holds vacuously. However, some new axioms are introduced.

**DEFINITION 5 (Component unanimity)** An aggregator  $f$  satisfies *component unanimity* if for all  $j \in X$ ,  $N \in \mathbf{N}$  and  $v \in A^N$ ,

$$[v_{ij} = \bar{v}_j \quad \forall i \in N] \Rightarrow [f_j(v) = \bar{v}_j.]$$

The axiom requires the aggregator to select alternatives that have been approved unanimously and reject alternatives that have been rejected unanimously. Aggregators that satisfy component unanimity are the min, max and L-min. Constant rules violate this condition.

**DEFINITION 6 (Component anonymity)** An aggregator  $f$  satisfies *component anonymity* if for all  $N \in \mathbf{N}$  for all bijections  $\sigma_{ij} : N \times K \rightarrow N$  and all  $j \in X$   $v, v' \in A$ ,

$$[v_{ij} = v'_{\sigma_{ij}j} \quad \text{for all } i \in N] \Rightarrow [f_j(v) = f_j(v')].$$

---

<sup>7</sup>There is a fairly extensive literature on the aggregation of sets of alternatives - see for instance, [Barberà et al. \(1991\)](#), [Plott \(1973\)](#), [Goodin and List \(2006\)](#), [Kasher and Rubinstein \(1997\)](#).

Component anonymity requires the component outcome to be invariant to permutations of opinions on an alternative  $j$ . The min, max and constant aggregators satisfy this condition. The following piece of notation will be used for the next definition. Let  $W(v) = \{j \mid v_{ij} = 1 \text{ for all } i \in N\}$  and  $L(v) = \{j \mid v_{ij} = 0 \text{ for all } i \in N\}$ .

**DEFINITION 7 (Bipartite Rule)** An aggregator  $f^{\text{BR}}$  is a *Bipartite Rule* if there exists a partition  $\{F, F^c\}$  of  $X$  such that

- (i)  $\left[ j \in F \right] \Rightarrow \left[ f_j^{\text{BR}}(v) = 1 \text{ for all } v \text{ such that } j \notin L(v) \right]$ .
- (ii)  $\left[ j \in F^c \right] \Rightarrow \left[ f_j^{\text{BR}}(v) = 0 \text{ for all } v \text{ such that } j \notin W(v) \right]$ .

Bipartite Rule divides the set of alternatives  $X$  into favoured ( $F$ ) and non-favoured sets ( $F^c$ ). Alternatives in the favoured set are always selected by the aggregator unless all voters reject it. An alternative in the non-favoured set does not get selected unless all voters approve.

Bipartite Rules satisfy component unanimity and component anonymity. These aggregators are consistent and separable. To see that they are consistent suppose  $v \in A^n$  and  $I = \{N_1, \dots, N_k\}$  is a partition of  $N$ . Let  $j \in X$  be any alternative. There exists a set  $N_k \in I$  such that if there is no unanimous decision over  $j$  in the profile for  $n$  voters then there is no unanimity over  $j$  in  $v_{N_k}$ . This implies  $f_j^{\text{BR}}(v_{N_k}) = f_j^{\text{BR}}(v)$ . Therefore,  $f(f(v_{N_1}), \dots, f(v_{N_k})) = f(v)$ .

Bipartite Rules are constant over a “large” number of vote profiles. If the number of voters is large, the set of profiles where voters are unanimous over a component is “small”. Consequently, a Bipartite Rule will be “nearly” constant.

**Remark.** Note that Bipartite Rules are a type of component-wise  $\alpha$ -median rule with  $\alpha_j = 1$  or  $0$  for each alternative.

**EXAMPLE 1** The set of voters  $N = \{1, 2, 3\}$  and set of alternatives  $X = \{a, b, c, d\}$ . Let  $f^{\text{Q}}$  be a Bipartite Rule with the set of favoured alternative  $F = \{a, c\}$  and the set of non-favoured alternatives be  $F^c = \{b, d\}$ . Then,

$$f^{\text{BR}} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Our next result is a characterization of Bipartite Rules.

### 2.3.1 THE RESULT

**THEOREM 2** An aggregator satisfies *consistency*, *component unanimity* and *component anonymity* if and only if it is a *Bipartite Rule*.

*Proof:* Suppose an aggregator satisfies consistency, component unanimity and component anonymity. Define the order  $(\preceq)$  on  $A^n$  as follows.

$$v_i \preceq v_k \text{ if } f(v_i, v_k) = v_i \text{ for all } v_i, v_k \in A.$$

We show that the order  $(\preceq)$  is a partial order i.e it satisfies the following three properties.

- (i) Reflexivity: Pick any  $v_i \in A$ . By component unanimity,  $f(v_i, v_i) = v_i$ . Therefore,  $v_i \preceq v_i$  for all  $v_i \in A$ .
- (ii) Anti-symmetry: Suppose  $v_i, v_k \in A$  such that  $v_i \preceq v_k$  and  $v_k \preceq v_i$ . Then by definition,  $f(v_i, v_k) = v_i$  and  $f(v_k, v_i) = v_k$ . By component anonymity,  $f(v_i, v_k) = f(v_k, v_i) = v_i = v_k$ .
- (iii) Transitivity: Suppose  $v_i, v_k, v_t \in A$  such that  $v_i \preceq v_k$  and  $v_k \preceq v_t$ . By definition,  $f(v_i, v_k) = v_i$  and  $f(v_k, v_t) = v_k$ . Therefore, by consistency and component unanimity,

$$\begin{aligned} f(v_i, v_t) &= f^2(f(v_i, v_k), f(v_k, v_t)) = f^4(v_i, v_k, v_k, v_t). \\ &= f^3(v_i, f(v_k, v_k), v_t) = f^3(v_i, v_k, v_t) = f^2(v_i, f(v_k, v_t)) = f(v_i, v_k) = v_t. \end{aligned}$$

Therefore, the ordering  $(\preceq)$  is a partial order. We claim the following. Suppose  $v_i \preceq v_k$  for some  $v_i, v_k \in A^n$ . Then  $f(v_i, v_t) \preceq f(v_k, v_t)$  for all  $v_t \in A^2$ .

By consistency and component unanimity we have,

$$\begin{aligned} f^2(f(v_i, v_k), f(v_k, v_t)) &= f^4(v_i, v_k, v_k, v_t) = f^3(v_i, v_k, v_t). \\ &= f^2(f(v_i, v_k), v_t) = f(v_i, v_t). \end{aligned}$$

Therefore, the aggregator is increasing in the order  $(\preceq)$ . We claim that  $f(v_i, v_k) \preceq v_i$  and  $f(v_i, v_k) \preceq v_k$ . By consistency, component anonymity and component unanimity,

$$f^2(f(v_i, v_k), v_i) = f^3(v_i, v_k, v_i) = f^2(f(v_i, v_i), v_k) = f(v_i, v_k).$$

Therefore, by the definition of  $(\preceq)$  we have  $f(v_i, v_k) \preceq v_i$ . Similarly, we can show that  $f(v_i, v_k) \preceq v_k$ . Therefore, the aggregator outputs a vector of evaluations which is a lower bound according to  $(\preceq)$ . We finally show that the aggregator must select the unique greatest lower bound vector of opinions for any pair of voter opinions.



Suppose  $f(v_i, v_k) = v_t$ . We have shown that  $v_t$  must be an upper bound of  $v_i$  and  $v_k$ . We claim that  $v_t$  is the unique greatest lower bound. We prove this by contradiction. Suppose  $v'_t$  is another lower bound. By definition,

$$f(v_i, v'_t) = v'_t \text{ and } f(v_k, v'_t) = v'_t.$$

Therefore, by consistency,

$$\begin{aligned} f(v_i, v'_t) &= f^2(v_i, f(v_k, v'_t)) = f^3(v_i, v_k, v'_t). \\ &= f^2(f(v_i, v_k), v'_t) = f(v_t, v'_t). \end{aligned}$$

Since  $f(v_i, v'_t) = v'_t$  we have  $f(v_t, v'_t) = v'_t$ . Therefore,  $v'_t \preceq v_t = f(v_i, v_k)$ . Therefore,  $f(v_i, v_k)$  is the unique greatest lower bound of  $v_i$  and  $v_k$ .

We show that the aggregator is invariant to permutations of opinions over an alternative.<sup>8</sup> We claim the following. Let  $\pi : N \times X \rightarrow N$  be a bijection. Suppose  $v, v' \in A^2$  such that  $v'_{ij} = v_{\pi(ij)}$  for some  $j \in X$  and  $v'_{i'j'} = v_{i'j'}$  for all  $j' \in X, j' \neq j$ . Then  $f(v) = f(v')$ .

We prove the above claim by contradiction. Consider a profile  $v \in A^2$  and an alternative  $j \in X$ . The claim is trivially true if  $j \in L(v) \cup W(v)$ . Suppose  $j \notin L(v) \cup W(v)$ . Let  $v = (v_i, v_k)$  and  $v' = (v'_i, v'_k)$  such that  $v_{ij} = v'_{kj}, v_{kj} = v'_{ij}, v'_{i'j'} = v_{i'j'}$  and  $v'_{k'j'} = v_{k'j'}$  for all  $j' \neq j, f_j(v) = f_j(v')$  and  $f_{j'}(v) \neq f_{j'}(v')$ . Therefore, the bijection  $\pi$  is such that  $\pi(i, j) = k$  and  $\pi(k, j) = i$  and  $\pi(i', j') = i'$  for all  $i' \in N$  and  $j' \in X, j' \neq j$ .

We claim that  $f(v)$  must be ordered with  $f(v')$ . Suppose contrariwise, that  $f(v)$  is not ordered with  $f(v')$ . Then  $f(f(v), f(v')) = v''$  where  $v'' \notin \{f(v), f(v')\}$ . W.l.o.g assume that  $v''_{j'} = f_{j'}(v)$ . By definition of  $(\preceq)$ , we have  $f_{j'}(v_i, v'_i) = f_{j'}(v')$ . This is a violation of component unanimity. Therefore,  $f(v)$  is ordered with  $f(v')$ .

W.l.o.g suppose  $f(v) \preceq f(v')$ . By the definition of  $(\preceq)$ , we have  $f_{j'}(v_i, f(v')) = f_{j'}(v)$ . This is a contradiction to component anonymity since by our construction  $f_{j'}(v) = f_{j'}(v') = 1 - f_j(v)$ . Similar arguments can be made when  $f(v') \preceq f(v)$ . The final claim proves separability.

We claim the following. For all  $v, v' \in A^2, [v_j = v'_j] \Rightarrow [f_j(v) = f_j(v')]$  for all  $j \in X$ .

Let  $\bar{v} = (v_1, v_2) \in A^2$  be a profile such that  $\bar{v}_{1j} + \bar{v}_{2j} = 1$ . To prove the claim it is sufficient to show that for all  $v \in A^2, [v_{1j} + v_{2j} = 1] \Rightarrow [f_j(v) = f_j(\bar{v})]$  for all  $j \in X$ . So pick any  $j \in X$  and  $v \in A^2$  such that  $v_{1j} + v_{2j} = 1$ . By definition  $f^4(\bar{v}, \bar{v}) = f^2(f^2(\bar{v}), f^2(\bar{v}))$ . By the property of  $\bar{v}$  there exists a profile  $\hat{v} = (\hat{v}_1, \hat{v}_2) \in A^2$  such that  $f^4(\bar{v}, \bar{v}) = f^4(v, \hat{v})$ . We construct  $\hat{v}$  as follows: (i)  $\hat{v}_{ij} = v_{ij}$  (ii)  $\hat{v}_{i'j'} = v_{i'j'}$  for all  $j' \notin L(v) \cup W(v), j' \neq j$  (iii)  $\hat{v}_{i'j'} = 1 - v_{i'j'}$  for all  $j' \in L(v) \cup W(v), j' \neq j$  for  $i \in \{1, 2\}$ . Therefore,  $(v, \hat{v})$  is constructed by permutations of component values in the profile  $(\bar{v}, \bar{v})$ .

By our previous claim and consistency, we have  $f^4(\bar{v}, \bar{v}) = f^4(v, \hat{v}) = f^4(f(v), f(v), \hat{v})$ . Now, we construct a profile  $\tilde{v} = (\tilde{v}_1, \tilde{v}_2) \in A^2$  such that  $f^4(v, \hat{v}) = f^4(\tilde{v}, v)$ . We construct  $\tilde{v}$

---

<sup>8</sup>Recall that component anonymity only states that the aggregator is invariant *only* over the alternative for which the opinions are permuted.

as follows: (1)  $\tilde{v}_{ij} = f_j(v)$  (ii)  $\tilde{v}_{ij'} = v_{ij'}$  for all  $j' \notin L(v) \cup W(v)$ ,  $j' \neq j$  (iii)  $\tilde{v}_{ij'} = 1 - v_{ij'}$  for all  $j' \in L(v) \cup W(v)$ ,  $j' \neq j$  for  $i \in \{1, 2\}$ . Therefore,  $(\tilde{v}, v)$  is constructed by permutations of component values in the profile  $(f(v), f(v), \hat{v})$ .

By our previous claim, we have  $f(v, \hat{v}) = f(\tilde{v}, v)$ . Also, note that  $f_j(\tilde{v}) = f_j(v)$ . By consistency, component anonymity and component unanimity, we have  $f_j^4(\tilde{v}, \tilde{v}) = f_j^4(v, \hat{v}) = f_j^3(f(v), \hat{v}) = f_j^4(f(v), f(v), \hat{v}) = f_j^4(\tilde{v}, v) = f_j^3(f_j(v), v) = f_j^2(v)$ .

Therefore, our claim is true and  $f^2$  is a Bipartite Rule where an alternative  $j \in X$  is in the favoured set  $F \subset X$  if  $f_j(0, 1) = 1$  and it is in the non-favoured set  $F^C$  if  $f_j(0, 1) = 0$ .

We show that if  $f^2$  is a Bipartite Rule then  $f^k$  is a Bipartite Rule,  $k \in \{1, \dots, N\}$ . To see this, take any profile  $v \in A^K$ . Then we have  $f(v) = f^2(\dots f^2(f^2(v_1, v_2), v_3), \dots, v_K)$ . Since  $f^2$  is separable, we can focus our attention to any arbitrary alternative  $j$ . Suppose  $j \in L(v)$ . Then by component unanimity  $f_j(v) = 0$ . Similarly  $f_j(v) = 1$  if  $W(v)$ . Suppose  $j \notin L(v) \cup W(v)$ . Let

$$\begin{aligned} f^2(v_{1j}, v_{2j}) &= z^1. \\ f^2(f^2(v_{1j}, v_{2j}), v_{3j}) &= z^2. \\ &\vdots \\ f^2(\dots(f^2(v_{1j}, v_{2j}), \dots v_{kj})) &= z^{k-1}. \end{aligned}$$

In view of the nature of  $f^2$  there must exist  $q$  such that  $z^q \in \{0, 1\}$  such that  $z^{q'} = z^q$  for all  $q' \geq q$ . Therefore,  $f^K$  is a Bipartite Rule with  $j \in X$  in the favoured set  $F$  if  $f_j(0, 1) = 1$  or  $j$  in the non-favoured set  $F^C$  if  $f_j(0, 1) = 0$ . This completes the proof.  $\blacksquare$

Theorem 2 implies that the result of the previous model holds in this setting but with a stronger set of axioms. These aggregators are also similar to Unanimity Rules described in [Bervoets and Merlin \(2012\)](#).

### 2.3.2 INDEPENDENCE OF AXIOMS

We show the independence of the axioms below.

**Component unanimity:** Constant Rules satisfy all axioms except component unanimity.

**Component anonymity:** L-min aggregators satisfy all axioms except component anonymity.

**Consistency:** The following aggregator satisfies all the axioms except consistency. An aggregator  $f^P$  is a *Parity aggregator* if for any profile  $v \in A^n$ ,  $N \in \mathbf{N}$ : (i)  $f_j^P(v) = \min_{i \in N}(v_{ij})$  if  $N$  is odd and (ii)  $f_j^P(v) = \max_{i \in N}(v_{ij})$  if  $N$  is even. This aggregator satisfies the other component unanimity and component anonymity but is not consistent. We show that it violates consistency. Note that the aggregator is separable so it is sufficient to show its violation of consistency for some arbitrary alternative  $j$ . Suppose  $v_j = (0, 1, 1)$  is the vector of opinions

of voters 1, 2 and 3 for an alternative  $j$ . By definition we have  $f_j^P(v_j) = \min(0, 1, 1) = 0$ . Consider the partition  $I = \{\{1, 2\}, \{3\}\}$ . By applying the aggregator to the subgroups we have  $f_j^P(f_j^P(0, 1), 1) = f^P(1, 1) = 1$ . Therefore,  $f_j^P(v_j) \neq f_j^P(f_j^P(v_{\{1,2\}}), f_j^P(v_3))$ .

None of the axioms can be weakened to give separability of the aggregator. To see this note that the L-min aggregator satisfies consistency, component unanimity and anonymity but not component anonymity. Moreover, the L-min aggregator is not separable. Therefore, component anonymity plays a vital role in characterizing separable aggregators.

## 2.4 CONCLUSION

This Chapter examines the structure of consistent, multidimensional, multilevel aggregators in two distinct models. We characterize a class of separable rules called component-wise  $\alpha$ -median rules and generalize the one-dimensional results of [Fung and Fu \(1975\)](#). These can also be seen as component-wise  $\alpha$ -median aggregators. If the set of evaluations is finite, separability is no longer guaranteed. In addition to consistency, stronger notions of unanimity and anonymity are required to characterize a class of separable rules called Bipartite Rules.



## Chapter 3

# PARTY FORMATION AS A NETWORK IN A CITIZEN-CANDIDATE MODEL

### 3.1 INTRODUCTION

Party formation is an integral part of the functioning of modern democracies. Our objective in this chapter is to study this phenomenon in a model where candidates are situated in a one-dimensional policy space and propose *links* to other candidates in order to form parties. Party formation is modelled in the same way as the formation of *networks* which have recently received a great deal of attention recently (Jackson (2008), Goyal (2012)). We believe that this formulation offers several fresh insights on a classical issue in political theory.

We consider a voting model where the policy space is an interval in the real line. There is a *continuum* of voters whose ideal policy positions (henceforth policy positions) are distributed over the policy space. There is a *finite* set of candidates who also have policy positions. Candidates decide whether or not to participate in elections and also propose links to other candidates in order to form parties. In case a candidate chooses not to stand, she becomes a voter. Since there is a continuum of voters, this decision does not affect the distribution of policy positions. A candidate may also choose to stand as an independent. A profile of proposals leads to the formation of political parties. A party is a set of mutually interlinked candidates; moreover a candidate cannot belong to two parties. Each party adopts a policy position (we consider two separate ways for this to happen) after which voters vote non-strategically for the party whose policy platform is closest to their own.

There are two reasons why parties emerge in this model. The first is that it allows candidates to commit to a policy position (the party's) which is not their own. In addition each party standing in an election has to pay a *fixed* cost which may be thought of as the cost of campaigning. These costs are spread out if a candidate joins a party instead of contesting as an independent. On the other hand, there are two types of costs of joining a party. The winning position is the position of the party which may be different from that of the candidate. Also, there is a fixed benefit/rent from winning which has to be shared among all party members.

A critical element of our model is the assumption regarding the policy position of a party. We assume that this can only be the position of a member of the party. We consider two different assumptions: *populist* and *internally democratic* parties. A populist party chooses the policy position of the member that is closest to the voters' median position. An internally democratic party, on the other hand, chooses the median policy position among its party members.<sup>1</sup>

There are several papers that study party formation such as Riviere (1999), Jackson and Moselle (2002), Levy (2004), Callander (2005) and Osborne and Tourky (2008).<sup>2</sup> However, the paper that bears the closest resemblance to ours is Osborne and Slivinsky (1996). This paper examines the features of electoral competition in the citizen-candidate model without

---

<sup>1</sup>Jackson et al. (2007) consider a model of nomination processes within parties.

<sup>2</sup>Dhillon (2005) provides an extensive survey on party and coalition formation. Bhattacharya (2014) considers a model of group formation similar to ours.

party formation. Our model can be thought of as a model of party (network) formation in the background of the [Osborne and Slivinsky \(1996\)](#) model.

### 3.1.1 RESULTS

It is well-known that Nash equilibrium is an unsatisfactory equilibrium notion in network formation models ([Jackson \(2008\)](#)). For instance, the strategy where no candidate offer links is always an equilibrium. We therefore, adopt the *strong stability* notion also described in [Jackson \(2008\)](#). According to which no *subset* of candidates can jointly deviate profitably from the proposed equilibrium.

We obtain different results for populist and internally democratic parties cases. In the former, there can be at most two parties in equilibrium. This stands in contrast to [Osborne and Slivinsky \(1996\)](#) where multiple candidate equilibria (greater than two) exist. In the single party equilibrium, the single party is generically an independent who is the candidate situated closest to the voter median. Several conditions are required for the existence of two-party equilibrium. In equilibrium party positions are equidistant from the voter median. Moreover parties are *homogeneous*, i.e. the smallest interval containing the positions of all members of a party are disjoint for the two parties.

Electoral competition is less intense when parties are internally democratic. Consequently, more than two parties can exist in equilibrium as in [Osborne and Slivinsky \(1996\)](#) (with candidates interpreted as parties). We derive necessary and sufficient conditions for one-party equilibrium. We derive necessary conditions for two-party equilibrium. We show by means of examples that two-party and three-party equilibrium can exist.

This chapter is organized as follows. Section 3.2 sets out the model and the equilibrium concept. Sections 3.3 and 3.4 contain the analysis for the populist parties and internally democratic parties, respectively. Section 3.5 contains a discussion on some aspects of the results while the final section concludes.

## 3.2 THE MODEL

The policy space  $X$  is a subset of the real line  $\mathbb{R}$ . A unit mass of voters have policy positions that are distributed over  $X$ . The cumulative distribution of voter policy positions  $F(x)$  is continuous and assumed to have a unique median  $x_M$ .<sup>3</sup>

The set of candidates is  $N = \{1, 2, \dots, n\}$ . Each candidate  $i$  has a policy position  $x_i \in X$ . The vector  $(x_1, \dots, x_n)$  will remain fixed through the analysis. We assume for simplicity that no two candidates have the same policy position, i.e.  $x_i \neq x_j$  for all  $i, j \in N$ . Voters and candidates have Euclidean preferences on  $X$  i.e. voter/candidate  $i$  derives utility  $v_i(x) = -|x - x_i|$  from the policy position  $x$ .

---

<sup>3</sup>The median  $x_M$  of  $F(x)$  satisfies the following:  $F(x_M) = 1 - F(x_M) = \frac{1}{2}$ .

The game unfolds as follows. Each candidate  $i$  either decides to withdraw from political competition, indicated by choosing “0” or offering a set of “links” to the other candidates (including herself) in order to form a political party.<sup>4</sup>

A strategy  $s_i$  for  $i$ , is either  $\{0\}$  or a non-empty subset of  $N$  that includes  $i$ . The requirement that the candidate must offer a link to herself if she chooses to participate indicates that a candidate is willing to contest the election as an independent if a larger party is not formed. We believe this assumption is natural. It also simplifies the analysis considerably.

Let  $S_i$  denote the set of strategies for candidate  $i$  and  $S = S_1 \times \dots \times S_n$  denote the product strategy space. Let  $s \in S$ . A *link* between two candidates  $i$  and  $j$  is formed if  $j \in s_i$  and  $i \in s_j$ , i.e. if *both* candidates agree to be linked. Let  $L(s)$  denote the collection of all links formed in  $s$ , i.e.  $(i, j) \in L(s)$  if  $i \in s_j$  and  $j \in s_i$ .

We say that there is a path from  $i$  to  $j$  in  $L(s)$  if there exists a sequence  $\{i_1, \dots, i_m\}$  such that (i)  $i_1 = i$  (ii)  $i_m = j$  and (iii)  $(i_k, i_{k+1}) \in L(s)$  for all  $k = 1, \dots, m - 1$ . In other words, a path exists between  $i$  and  $j$  in  $L(s)$  if there exists a sequence of candidates beginning in  $i$  and ending in  $j$  such that consecutive candidates have a link.

It is convenient to think of the graph induced by  $s$ . We denote this graph by  $G(s)$ . The nodes in this graph are the candidates and the set of edges is the set of links  $L(s)$ . A *component* of  $G(s)$  is a set of nodes  $C$  such that

- (i)  $[i, j \in C] \Rightarrow [\text{there is a path from } i \text{ to } j]$ .
- (ii)  $[i \in C \text{ and } j \notin C] \Rightarrow [\text{there is no path from } i \text{ to } j]$ .

Clearly, all nodes in  $G(s)$  can be partitioned uniquely into components. A *clique* is a set of nodes  $C$  in  $G(s)$  satisfying the property  $[i, j \in C] \Rightarrow [(i, j) \in L(s)]$ . In other words, a clique is a complete subgraph of  $G(s)$ .

A *party*  $P_k$  is a subset of candidates. A *party structure*  $\mathcal{P} = \{P_1, \dots, P_K\}$  is a collection of parties.

A strategy profile  $s$  induces a party structure  $\mathcal{P}(s)$  in the following manner:

- (a) Every component in  $G(s)$  is a party if it is a clique.
- (b) If a component in  $G(s)$  is not a clique, then all candidates in the component participate as independents.

It is possible that no political party forms at  $s$ . We illustrate party formation with an example.

---

<sup>4</sup>Note that if a candidate chooses not to stand in election and becomes a voter, the voter distribution remains unchanged.



EXAMPLE 2 Let  $N = \{1, 2, \dots, 10\}$ . Consider the strategy profile  $s$  where  $s_i = \{1, 2, 3\}$  for all  $i \in \{1, 2, 3\}$ ,  $s_4 = \{4, 5, 6, 7\}$ ,  $s_5 = \{4, 5, 6, 7, 8\}$ ,  $s_6 = \{4, 5, 6, 7, 8\}$ ,  $s_7 = \{4, 5, 6, 7\}$ ,  $s_8 = \{5, 8\}$ ,  $s_9 = \{9, 10\}$  and  $s_{10} = \{10\}$ . The set of links is  $L(s) = \{(1, 2), (1, 3), (2, 3), (4, 5), (4, 6), (4, 7), (5, 6), (5, 7), (5, 8), (6, 7)\}$ .

There are four components:  $\{1, 2, 3\}$ ,  $\{4, 5, 6, 7, 8\}$ ,  $\{9\}$  and  $\{10\}$ . However, only three of them are cliques:  $\{1, 2, 3\}$ ,  $\{9\}$  and  $\{10\}$ . Accordingly, the set parties of is  $\{\{1, 2, 3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{8\}, \{10\}\}$ . This is shown in the figure below.

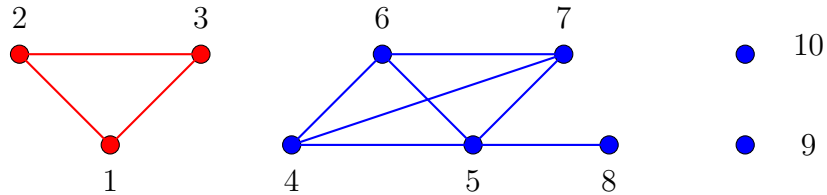


Figure 3.1: Party formation

Let  $P_k(s)$  be a political party. The *party policy position* is denoted by  $x(P_k(s)) \in X$  of party  $P_k(s)$  and is the policy position of a member of the party. Similar assumptions have been made in Jackson et al. (2007) and Levy (2002). We consider two types of parties:

- (i) Populist parties: The party policy position  $P_k(s)$  is the policy position of the member of  $P_k(s)$  who is closest to the median of the voter distribution  $x_M$ . If there is more than one policy position at the same distance from  $x_M$ , the policy position closest to the median of the constituent policy positions in the party is chosen. In the case where two policy positions are closest to  $x_M$  and each one is a median position in the party, each policy position is chosen with equal probability. One can think of the party policy position party as the position of a “populist” leader.
- (ii) Internally democratic parties: The party policy position  $P_k(s)$  is the median policy position of the members in the party. If the number of members in the party is odd, the party policy position is the unique median policy position in the party. If the number of members in the party is even, the party policy position is the policy position of the member who is closest to  $x_M$ . In case both median policy positions are equidistant from  $x_M$ , each policy position is chosen with equal probability. The party policy position of an internally democratic party emerges as an outcome of intra-party voting. Since preferences are Euclidean, the median policy position will defeat any other member position in a pairwise vote (the median-voter theorem). Party policy positions of this nature have been considered in Jackson et al. (2007) and Teorell (1999).

Voters (including candidates who have chosen not to participate) vote for parties on the basis of their party policy position. They are non-strategic and vote for the policy platform closest to their own policy position. The party that wins the most votes is elected and its policy position implemented. More than one political party may receive the most votes in which case each of the winners has its party policy position chosen with equal probability. These features of the political competition model are borrowed directly from [Osborne and Slivinsky \(1996\)](#).

We assume that each party has to pay a fixed cost  $c > 0$  for participating in elections. This may be thought of as the minimum cost necessary for campaigning and bringing the party policy position to the attention of voters. We assume that these costs are shared equally amongst the members of a party.

The winning party receives benefits or rents over and above the utility of having its party policy position implemented. These can be interpreted as “spoils of victory” borrowing the term from [Osborne and Slivinsky \(1996\)](#). This positive rent  $r > 0$  is shared equally amongst all members of the winning political party.

Suppose political parties  $\{P_k\}_{k=1}^W$  with policy positions  $\{x(P_k)\}_{k=1}^W$  obtain highest vote shares in the elections. By our assumption, each of the party policy positions is implemented with probability  $\frac{1}{W}$ . Consider candidate  $i$ 's with policy position  $x_i$ . Her expected payoff from the outcome of the elections is  $\frac{1}{W}(-|x_i - x(P_1)| - \dots - |x_i - x(P_W)|) = -|x_i - \hat{x}|$  where  $\hat{x} = \frac{1}{W}(x(P_1) + \dots + x(P_W))$ . This follows from the assumption of Euclidean preferences. We shall refer to  $\hat{x}$  as the *certainty-equivalent* outcome of the lottery where each of the positions  $\{x(P_k)\}_{k=1}^W$  is implemented with probability  $\frac{1}{W}$ .

We can now describe the payoffs to candidate  $i$  at strategy profile  $s$ . As before, let  $\mathcal{P}(s)$  be the set of political parties formed at  $s$  and let  $\mathcal{W}(s) = \{P_k(s)\}_{k=1}^W$  be the set of winning political parties at  $s$ . Candidate  $i$ 's payoff  $\pi_i(s)$  is given by

$$\pi_i(s) = \begin{cases} \frac{r}{W|P_k(s)|} - |x_i - \hat{x}| - \frac{c}{|P_k(s)|} & \text{if } i \in P_k(s) \text{ and } P_k(s) \in \mathcal{W}(s) \\ -|x_i - \hat{x}| - \frac{c}{|P_k(s)|} & \text{if } i \in P_{k'}(s) \text{ and } P_{k'}(s) \notin \mathcal{W}(s) \\ -|x_i - \hat{x}| & \text{if } s_i = 0 \text{ and } \mathcal{P}(s) \neq \emptyset \\ -\infty & \text{if no candidate participates i.e. } \mathcal{P}(s) = \emptyset. \end{cases}$$

Following [Osborne and Slivinsky \(1996\)](#) we assume that all candidates obtain a payoff of  $-\infty$  if no party forms i.e. the political process breaks down. Observe also that the distribution  $F$  of voters' policy positions is unaffected by a candidate's participation decision. This is a consequence of the fact that there finite number of candidates and a continuum of voters. Therefore, the participation decisions by candidates do not affect calculations regarding winning parties and winning party policy positions.

The main features of the game are summarized below:

- (i) The distribution of voters' policy position is fixed and common knowledge.
- (ii) Candidates decide whether or not to participate in electoral competition.

- (iii) Participating candidates specify candidates they choose to form links with. This leads to party formation and party policy platforms. All parties pay a fixed cost which is distributed equally amongst party members.
- (iv) Voters vote sincerely for parties based on the party policy platforms.
- (v) The party that wins the election implements its policy platform.
- (vi) Winning political parties enjoy added rents that are equally distributed amongst its members.

We provide an illustrative example below.

**EXAMPLE 3** The policy space is  $[0, 1]$  and the median of the voter distribution is  $x_M$ . The set of candidates is  $N = \{1, 2, \dots, 8\}$ . Policy positions  $\{x_i\}_{i=1}^8$  are at equal distances from each other as shown in Figure 3.2.

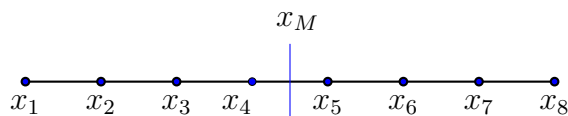


Figure 3.2: Distribution of voter policy positions

Let  $s_1 = \{2, 3, 4\}$ ,  $s_2 = \{1, 3, 4\}$ ,  $s_3 = \{1, 2, 4\}$ ,  $s_4 = \{1, 2, 3\}$ ,  $s_i = \{i\}$  for  $i \in \{5, 6\}$  and  $s_i = \{0\}$  for  $i \in \{7, 8\}$ . Then  $L(s_1, \dots, s_8) = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$ . Therefore,  $\mathcal{P}(s) = \{\{1, 2, 3, 4\}, \{5\}, \{6\}\}$  i.e. there is a political party comprising 1, 2, 3 and 4 and independent candidates 5 and 6. Candidates 7 and 8 become voters. Let  $P_1(s)$ ,  $P_2(s)$  and  $P_3(s)$  denote the parties  $\{1, 2, 3, 4\}$ ,  $\{5\}$  and  $\{6\}$  respectively.

Consider the case where parties are populist. The policy positions of the various parties are  $x(P_1(s)) = x_4$ ,  $x(P_2(s)) = x_5$  and  $x(P_3(s)) = x_6$ . For convenience suppose  $x_M$  lies halfway between  $x_4$  and  $x_5$ . Then party  $P_1(s)$  obtains at least half the votes and  $P_2(s)$  and  $P_3(s)$  obtain strictly less than half the votes. Therefore, party  $P_1(s)$  wins the election and implements its policy position  $x_4$ . This is illustrated in Figure 3.3.

The payoffs of the candidates are,

$$\pi_i(s) = \begin{cases} \frac{r-c}{3} - |x_i - x_4| & \text{for all } i \in P_1(s) \\ -|x_i - x_4| - c & \text{for } i \in \{5, 6\} \\ -|x_i - x_4| & \text{for } i \in \{7, 8\} \end{cases}$$

Consider the case where parties are internally democratic. The policy positions of the various parties are  $x(P_1(s)) = x_3$ ,  $x(P_2(s)) = x_5$  and  $x(P_3(s)) = x_6$ . The outcome of the

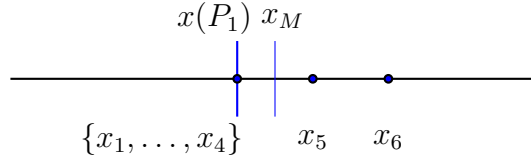


Figure 3.3: Party positions

election depends on the distribution of voters' policy positions. Any of the parties can win depending on  $F$ . For instance, let be the probability density function  $f$  (derived from  $F$ ) be of the form shown in Figure 3.4. The shaded regions I, II and III indicate the votes obtained by parties  $P_1$ ,  $P_2$  and  $P_3$  respectively. Since Region II has the greatest area,  $P_2$  obtains most votes. Therefore, the policy position  $x_4$  is implemented and the payoffs of candidates are same as before.

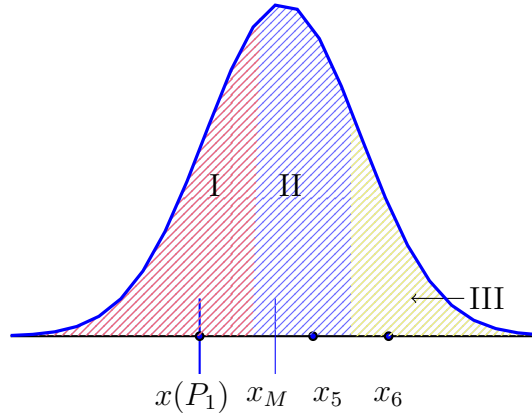


Figure 3.4: Candidate 5 wins

### 3.2.1 THE EQUILIBRIUM NOTION

It is well-known that the notion of Nash equilibrium applied to network formation is very weak and leads to unsatisfactory predictions. For example, no candidate offering any links is always a Nash equilibrium. Following Jackson (2008) we use the notion of strong stability. According to this notion, no subset of agents can jointly deviate and improve upon the proposed equilibrium.

Let  $T \subset N$ . A set of links  $L(s')$  is  $T$ -reachable from  $L(s)$  if,

- (i)  $[(i, j) \in L(s') \text{ and } (i, j) \notin L(s)] \Rightarrow [\{i, j\} \subseteq T]$ .
- (ii)  $[(i, j) \in L(s) \text{ and } (i, j) \notin L(s')] \Rightarrow [\{i, j\} \cap T \neq \emptyset]$ .

A set of links is  $T$ -reachable by a subset  $T$  of candidates if (a) a new link is formed and both the candidates who form the link are in  $T$  (b) a link is broken and at least one of the candidates who breaks a link is in  $T$ . This is illustrated in the example below.

**EXAMPLE 4** Let  $\mathbf{N} = \{1, 2, \dots, 5\}$  and let  $s$  be a strategy such that  $L(s) = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$ . The set of links  $L(s') = \{(1, 2), (1, 3), (1, 4), (4, 5)\}$  is not  $T$ -reachable from  $L(s)$  if  $T = \{1, 2, 3\}$  since a new link  $(1, 4)$  is formed for which the approval of 4 is required. Similarly,  $L(s')$  is not  $T$ -reachable from  $L(s)$  by  $T = \{1, 2, 4\}$  since the link  $(2, 3)$  requires the approval of 3. The set of links  $L(s')$  is  $T$ -reachable by the set of candidates  $T = \{1, 2, 3, 4\}$ .

**DEFINITION 8 (Strong Stability)** The strategy profile  $s^*$  is *strongly stable* if for any  $T \subset N$  and any  $L(s')$  that is  $T$ -reachable from  $L(s^*)$ ,

$$[\pi_i(s') > \pi_i(s^*) \text{ for some } i \in T] \Rightarrow [\exists j \in T \text{ s.t. } \pi_j(s') < \pi_j(s^*)].$$

Suppose  $s^*$  is strongly stable. Consider a deviation by a set of candidates  $T$ . If this deviation makes some member of  $T$  strictly better-off, there must exist another candidate in  $T$  who is made strictly worse-off.

### 3.3 EQUILIBRIUM ANALYSIS: POPULIST PARTIES

In this section we examine equilibria when parties are populist. Our main result is that at most two parties can form in equilibrium.

We characterize one-party equilibrium. Two-party equilibria are hard to characterize. We provide some necessary conditions that must hold in such equilibria. We also provide numerical examples of such equilibria.

#### 3.3.1 NUMBER OF PARTIES

**PROPOSITION 1 (Number of parties in equilibrium)** Suppose  $s^*$  is an equilibrium. Then  $|\mathcal{P}(s^*)| \leq 2$ .

*Proof:* We prove by contradiction. Let  $s^*$  be an equilibrium and suppose  $|\mathcal{P}(s^*)| > 2$ . We proceed in steps. According to the next Lemma the “extreme” parties must be winning.

**LEMMA 11** Let  $P_1(s^*), P_K(s^*) \in \mathcal{P}(s^*)$  be such that  $x(P_1(s^*)) \leq x(P_k(s^*))$  for all  $k \in \mathcal{P}(s^*)$  and  $x(P_K(s^*)) \geq x(P_k(s^*))$  for all  $k \in \mathcal{P}(s^*)$ . Then  $P_1(s^*), P_K(s^*) \in \mathcal{W}(s^*)$  i.e. both parties must win the election with positive probability.

*Proof:* Suppose the Lemma is false. Let  $x(P_1(s^*)) \leq x(P_k(s^*))$  for all  $k \in \mathcal{P}(s^*)$ . Suppose  $P_1(s^*) \notin \mathcal{W}(s^*)$ . We consider two cases separately:

Case 1:  $|P_1(s^*)| > 1$ . Since parties are populist, there exists a candidate say  $i$  who is a member of  $P_1(s^*)$  and  $x_i \neq x(P_1(s^*))$ . The payoff of the candidate is  $\frac{c}{|P_1(s^*)|} - |x_i - \hat{x}(s)|$ . Consider a deviation by candidate  $i$  to non-participation i.e.  $s'_i = \{0\}$ . Notice that by definition the new party is  $P_1(s'_i, s^*_{-i}) = P_1(s^*) \setminus \{i\}$  and  $x(P_1(s'_i, s^*_{-i})) = x(P_1(s^*))$ . Therefore,  $\hat{x}(s'_i, s^*_{-i}) = \hat{x}(s^*)$  and  $P_1 \setminus \{i\}$ . We have  $\pi_i(s'_i, s^*_{-i}) = |x_i - \hat{x}(s^*)| > -|x_i - \hat{x}(s^*)| - \frac{c}{|P_1(s^*)|} = \pi_i(s^*)$ . Hence, the deviation is beneficial.

Case 2:  $|P_1(s^*)| = 1$  i.e.  $i$  is an independent candidate. Once again, consider a deviation  $s'_i = \{0\}$ . Therefore,  $\hat{x}(s'_i, s^*_{-i}) \leq \hat{x}(s^*)$ . Hence,  $\pi_i(s'_i, s^*_{-i}) = -|x_i - \hat{x}(s'_i, s^*_{-i})| > -|x_i - \hat{x}(s^*)| - \frac{c}{|P_1(s^*)|} = \pi_i(s^*)$ . Therefore,  $i$  deviates.  $\blacksquare$

Since  $|\mathcal{P}(s^*)| \geq 3$  and candidate positions are all distinct there must exist two parties say  $P_1(s^*)$  and  $P_2(s^*)$  such that either  $x(P_1(s^*)), x(P_2(s^*)) \leq \hat{x}(s^*)$  or  $x(P_1(s^*)), x(P_2(s^*)) \geq \hat{x}(s^*)$ . Assume w.l.o.g. that the former is true and also assume w.l.o.g. that  $x(P_1(s^*)) < x(P_2(s^*)) < x(P_k(s^*))$  for all  $k \notin \{1, 2\}$ . By Lemma 11  $P_1(s^*) \in \mathcal{W}(s^*)$ .

Let  $i_1 \in P_1(s^*)$  and  $i_2 \in P_2(s^*)$  be such that  $x_{i_1} = x(P_1(s^*))$  and  $x_{i_2} = x(P_2(s^*))$ . Let  $|\mathcal{W}(s^*)| = W^*$  be the number of winning parties in  $s^*$ . We consider several cases:

Case 1:  $r \geq c$ . There are four sub-cases to consider:

Case 1(a):  $|P_1(s^*)| \leq |P_2(s^*)|$  and  $P_2(s^*) \in \mathcal{W}(s^*)$ .

Case 1(b):  $|P_1(s^*)| \leq |P_2(s^*)|$  and  $P_2(s^*) \notin \mathcal{W}(s^*)$ .

Case 1(c):  $|P_1(s^*)| > |P_2(s^*)|$  and  $P_2(s^*) \in \mathcal{W}(s^*)$ .

Case 1(d):  $|P_1(s^*)| > |P_2(s^*)|$  and  $P_2(s^*) \notin \mathcal{W}(s^*)$ .

We consider each case separately.

Case 1(a) : Let  $i_3 \in P_2(s^*)$  be a candidate such that her policy position  $x_{i_3}$  is closest to  $x(P_1(s^*))$  and  $x_{i_3} > x(P_1(s^*))$ . Let  $T = \{i \in P_1(s^*) \text{ s.t. } x_i \leq x(P_1(s^*))\} \cup \{i_3\}$ . Consider the following deviation  $s'$ :  $s'_i = T$  for all  $i \in T$ . The set of links  $L(s')$  is  $T$ -reachable from  $L(s^*)$ . Moreover, the set of candidates  $T$  form a clique and component i.e. candidates in  $T$  form a new party. Let this party be denoted by  $P_3(s')$ . By construction, the remaining candidates in  $P_1(s^*)$  and  $P_2(s^*)$  continue to be parties which we denote by  $P_1(s')$  and  $P_2(s')$  respectively.

Suppose  $i_3 \neq i_2$ . Then, by the definition of populist parties we have,  $x(P_1(s^*)) < x(P_3(s')) \leq x(P_2(s^*)) = x(P_2(s'))$ . Some votes for  $P_2(s^*)$  switch to  $P_3(s')$  and all the votes of  $P_1(s^*)$  also shift to  $P_3(s')$ . Since  $P_1(s^*) \in \mathcal{W}(s^*)$  we have  $P_3(s') = \mathcal{W}(s^*)$ . Therefore,  $\hat{x}(s') = x(P_3(s')) = x_{i_3}$ .

Suppose  $i_3 = i_2$ . Then, by the definition of populist party policy position we have, either (i)  $x(P_2(s')) < x(P_1(s^*)) < x(P_3(s')) = x(P_2(s^*))$  or (ii)  $x(P_1(s^*)) < x(P_3(s')) = x(P_2(s^*)) < \hat{x}(s^*) < x(P_2(s'))$ . Since  $P_2(s^*) \in \mathcal{W}(s^*)$ , we have  $P_3(s') \in \mathcal{W}(s')$ . Therefore,  $\hat{x}(s') = x(P_3(s')) = x_{i_3}$ .

Note that  $x_i \leq x_{i_3} < \hat{x}(s^*)$  for all  $i \in P_3(s')$ . Hence, each candidate  $i \in P_3(s')$  strictly prefers  $\hat{x}(s')$  to  $\hat{x}(s^*)$ . We show that  $\pi_i(s') > \pi_i(s^*)$  for all  $i \in P_3(s') = T$ . Let  $\Delta G_i(s', s^*)$  denote the change in rents and costs for candidate  $i$  when the strategy changes from  $s^*$  to  $s'$ .<sup>5</sup> Let  $|P_1(s^*)| = n_1$ ,  $|P_2(s^*)| = n_2$  and  $|P_3(s')| = n_3$ . Then, for all  $i \in P_3(s') \setminus \{i_3\}$ ,  $\Delta G_i(s', s^*)$  is given by

$$\begin{aligned}\Delta G_i(s', s^*) &= \frac{r}{n_3} - \frac{c}{n_3} - \frac{r}{Wn_1} + \frac{c}{n_1} \\ &= \frac{Wn_1r - Wn_1c - n_3r + Wn_3c}{Wn_1n_3}\end{aligned}$$

The numerator is non-negative since

$$Wn_1(r - c) - n_3(r - Wc) \geq 0.$$

Therefore,  $\Delta G_i(s', s^*) \geq 0$  for all  $i \in P_3(s') \setminus \{i_3\}$ . Since each member of  $P_3(s')$  prefers  $\hat{x}(s')$  to  $\hat{x}(s^*)$ , we have  $\pi_i(s') > \pi_i(s^*)$  for all  $i \in P_3(s') \setminus \{i_3\}$ . We show that  $\pi_{i_3}(s') > \pi_{i_3}(s^*)$ . We have

$$\begin{aligned}\Delta G_{i_3}(s', s^*) &= \frac{r}{n_3} - \frac{c}{n_3} - \frac{r}{Wn_2} + \frac{c}{n_2} \\ &= \frac{Wn_2r - Wn_2c - rn_3 + Wn_3c}{Wn_1n_3}\end{aligned}$$

The numerator is non-negative since

$$Wn_2(r - c) - n_3(r - c) \geq 0.$$

Therefore,  $\Delta G_{i_3}(s', s^*) \geq 0$ . Since  $i_3$  strictly prefers  $\hat{x}(s')$  to  $\hat{x}(s^*)$ , we have  $\pi_{i_3}(s') > \pi_{i_3}(s^*)$ . Hence, the deviation to  $s'$  is beneficial for  $T$ .

Case 1(b): Suppose  $P_2(s^*)$  is losing with certainty. We claim that  $|P_2(s^*)| = 1$ . To see this, consider any candidate  $i \neq i_2$  and the following deviation by  $i$ :  $s'_i = \{0\}$ . The remaining members of  $P_2(s^*)$  is a party i.e.  $P_1(s'_i, s^*_{-i}) = P_1(s^*) \setminus \{i\}$ . By definition  $x(P_1(s^*)) = x(P_1(s'_i, s^*_{-i}))$ . Therefore,  $\hat{x}(s') = \hat{x}(s^*)$ . We have,  $\pi_i(s'_i, s^*_{-i}) = -|x_i - \hat{x}(s')| > -|x_i - \hat{x}(s^*)| - \frac{c}{|P_2(s^*)|} = \pi_i(s^*)$ . Therefore,  $i$  deviates. Hence,  $|P_2(s^*)| = 1$ . Since  $|P_1(s^*)| \leq |P_2(s^*)|$ , we have  $|P_1(s^*)| = 1$ . Let the candidates  $i_1$  and  $i_2$  be as defined before i.e.  $x(P_1(s^*)) = x_{i_1}$  and  $x(P_2(s^*)) = x_{i_2}$ .

Let  $T = \{i_1\} \cup \{i_2\}$ . Consider the deviation  $s'$  by  $T$ :  $s'_i = T$  for all  $i \in T$ . The set of links  $L(s')$  is  $T$ -reachable from  $L(s^*)$ . Moreover, the candidates in  $T$  form a clique and a component. Therefore, the candidates in  $T$  form a party. Let this party be denoted by  $P_3(s')$ . By the definition of populist parties, we have  $x(P_1(s^*)) < x(P_3(s')) = x(P_2(s^*)) < \hat{x}(s^*) < x(P_1(s'))$ . The votes for party  $P_1(s^*)$  and  $P_2(s^*)$  switch to  $P_3(s')$ . Therefore,  $\hat{x}(s') = x(P_3(s')) = x_{i_2}$ .

---

<sup>5</sup>We shall use this notation extensively in the rest of the Chapter.

Both candidates in  $P_3(s')$  strictly prefer  $\hat{x}(s')$  to  $\hat{x}(s^*)$ . We show that  $\pi_i(s') > \pi_i(s^*)$  for all  $i \in P_3(s')$ . Let  $|P_1(s^*)| = n_1$ ,  $|P_2(s^*)| = n_2$  and  $|P_3(s')| = n_3$ . Then  $\Delta G_i(s', s^*)$  for all  $i_1 \in P_3(s')$  is given by

$$\begin{aligned}\Delta G_{i_1}(s', s^*) &= \frac{r}{n_3} - \frac{c}{n_3} - \frac{r}{Wn_1} + \frac{c}{n_1} \\ &= \frac{Wn_1(r - c) - n_3r + Wn_3c}{Wn_1n_3}\end{aligned}$$

The numerator is non-negative since,

$$Wn_1(r - c) - n_3(r - Wc) \geq 0.$$

Therefore,  $\Delta G_{i_1}(s', s^*) \geq 0$ . Hence,  $\pi_{i_1}(s') > \pi_{i_1}(s^*)$ .

We show that  $\pi_{i_2}(s') > \pi_{i_2}(s^*)$ . We have

$$\Delta G_{i_2}(s', s^*) = \frac{r}{n_3} - \frac{c}{n_3} + c$$

The numerator is non-negative i.e.

$$r + (n_3 - 1)c \geq 0.$$

Therefore  $\Delta G_{i_2}(s', s^*) \geq 0$  and hence  $\pi_{i_2}(s') > \pi_{i_2}(s^*)$ . Therefore,  $T$  deviates.

Case 1(c): Let  $i_1$  and  $i_2$  be as defined earlier. Let  $T = \{i \in P_2(s^*) \text{ s.t. } x_i \leq x(P_2(s^*))\} \cup \{i_1\}$  and consider the deviation  $s'$ :  $s'_i = T$  for all  $i \in T$ . Clearly,  $T$  forms a party. Let this new party be denoted by  $P_3(s')$ . The remaining members in  $P_1(s^*)$  and  $P_2(s^*)$  continue to be parties, which we denote by  $P_1(s')$  and  $P_2(s')$ . By definition, either (i)  $x(P_1(s')) < x(P_3(s')) = x(P_2(s^*))$  or (ii)  $x(P_3(s')) = x(P_2(s^*)) < \hat{x}(s^*) < x(P_1(s'))$ . Some votes for party  $P_1(s^*)$  and all the votes for party  $P_2(s^*)$  switch to  $P_3(s')$ . Therefore,  $P_3(s') = \mathcal{W}(s')$  and hence  $\hat{x}(s') = x(P_3(s'))$ . Each candidate in  $P_3(s')$  strictly prefers  $\hat{x}(s')$  to  $\hat{x}(s^*)$ . We show that  $\pi_i(s') > \pi_i(s^*)$  for all  $i \in P_3(s')$ .

Let  $|P_1(s^*)| = n_1$ ,  $|P_2(s^*)| = n_2$  and  $|P_3(s')| = n_3$ . Then  $\Delta G_i(s', s^*)$  for all candidates  $i \in P_3(s') \setminus \{i_1\}$  is given by,

$$\begin{aligned}\Delta G_i(s', s^*) &= \frac{r}{n_3} - \frac{c}{n_3} - \frac{r}{Wn_2} + \frac{c}{n_2} \\ &= \frac{Wn_2r - Wn_2c - n_3r + Wn_3c}{Wn_1n_3}.\end{aligned}$$

The numerator is non-negative since

$$Wn_2(r - c) - n_3(r - Wc) \geq 0.$$

Therefore,  $\Delta G_i(s', s^*) \geq 0$  and  $\pi_i(s') > \pi_i(s^*)$  for all  $i \in P_3(s') \setminus \{i_1\}$ .



We show that  $\pi_{i_2}(s') > \pi_{i_2}(s^*)$ . We have

$$\begin{aligned}\Delta G_{i_1}(s') &= \frac{r}{n_3} - \frac{c}{n_3} - \frac{r}{Wn_1} + \frac{c}{n_1} \\ &= \frac{Wn_1r - Wn_1c - n_3r + Wn_3c}{Wn_1n_3}.\end{aligned}$$

The numerator is non-negative since  $Wn_1(r-c) - n_3(r-Wc) \geq 0$ . Therefore  $\Delta G_{i_1}(s', s^*) \geq 0$  and  $\pi_{i_1}(s') > \pi_{i_1}(s^*)$ . Hence  $T$  deviates to  $s'$ .

Case 1(d): We can use the arguments in Case 1(b) to show that  $|P_2(s^*)| = 1$ . Let this candidate be  $i_2$  i.e.  $x(P_2(s^*)) = x_{i_2}$ . Let  $i_1$  be as defined earlier. Consider  $T = \{i \in P_1(s^*) \text{ s.t. } x_i \leq x(P_1(s^*))\} \cup \{i_2\}$  and the deviation  $s'$ :  $s'_i = T$  for all  $i \in T$ . The candidates in  $T$  form a new party which we denote by  $P_3(s')$ . The remaining members in party  $P_1(s^*)$  form a party which we denote by  $P_1(s')$ . By definition, we have  $x(P_1(s^*)) < x(P_3(s')) = x(P_2(s^*))$ . The votes for party  $P_1(s^*)$  and  $P_2(s^*)$  switch to  $P_3(s')$ . Therefore  $\{P_3(s')\} = \mathcal{P}(s')$  and  $\hat{x}(s') = x(P_3(s'))$ .

Each candidate in  $P_3(s')$  strictly prefers  $\hat{x}(s')$  to  $\hat{x}(s^*)$ . Let  $|P_1(s^*)| = n_1$ ,  $|P_2(s^*)| = n_2$  and  $|P_3(s')| = n_3$ . Then  $\Delta G_i(s', s^*)$  for all  $i \in P_3(s') \setminus \{i_2\}$  is given by

$$\begin{aligned}\Delta G_i(s', s^*) &= \frac{r}{n_3} - \frac{c}{n_3} - \frac{r}{Wn_1} + \frac{c}{n_1} \\ &= \frac{Wrn_1 - Wcn_1 - rn_3 + Wcn_3}{Wn_1n_3}.\end{aligned}$$

The numerator is non-negative since

$$Wn_1(r-c) - n_3(r-Wc) \geq 0.$$

Therefore  $\Delta G_i(s', s^*) \geq 0$  for all  $i \in P_3(s') \setminus \{i_2\}$  and hence  $\pi_i(s') > \pi_i(s^*)$  for all  $i \in P_3(s') \setminus \{i_2\}$ .

We show that  $\pi_{i_2}(s') > \pi_{i_2}(s^*)$ . We have

$$\Delta G_{i_2}(s', s^*) = \frac{r}{n_3} - \frac{c}{n_3} + c = \frac{r + (n_3 - 1)c}{n_3} \geq 0.$$

Therefore,  $\Delta G_i(s', s^*) \geq 0$  and  $\pi_{i_2}(s') > \pi_{i_2}(s^*)$ . Hence,  $T$  deviates.

Case 2: Suppose  $r < c$ . We claim that  $|P_k(s^*)| = 1$  for  $k \in \{1, 2\}$ . Let  $i_k \in P_k(s^*)$  be such that  $x_{i_k} = x(P_k(s^*))$ . There exists a candidate  $i \neq i_k$  who can deviate to non-participation i.e.  $s'_i = \{0\}$ . As a result, the remaining members in  $P_k(s^*)$  forms a party i.e.  $P_1(s'_i, s^*_{-i}) = P_k(s^*) \setminus \{i\}$ . By definition  $x(P_k(s'_i, s^*_{-i})) = x(P_1(s^*))$ . Therefore  $\hat{x}(s') = \hat{x}(s^*)$ . The change in payoffs for candidate  $i$  is given by  $\pi_i(s') = -|x_i - \hat{x}(s^*)| > -|x_i - \hat{x}(s^*)| - \frac{c}{|P_k(s^*)|} = \pi_i(s^*)$ . Therefore,  $i$  deviates. Hence  $|P_k(s^*)| = 1$  for  $k \in \{1, 2\}$ .

Let  $i_1$  and  $i_2$  be as defined earlier. By the earlier arguments we have  $P_1(s^*) = \{i_1\}$  and  $P_2(s^*) = \{i_2\}$ . Consider the set of candidates  $T = \{i_1, i_2\}$  and the deviation  $s'$ :  $s'_i = T$  for  $i \in T$ . The candidates  $T$  form a party. Let this party be denoted by  $P_3(s')$ . By definition, we have  $x(P_3(s')) = x_{i_2}$ . Votes for  $P_1(s^*)$  and  $P_2(s^*)$  switch to  $P_3(s')$ . Therefore,  $\hat{x}(s') = x(P_3(s'))$ . Both candidates in  $P_3(s')$  strictly prefer  $\hat{x}(s')$  to  $\hat{x}(s^*)$ . We show that  $\pi_i(s') > \pi_i(s^*)$  for  $i \in P_3(s')$ . We have

$$\Delta G_i(s', s^*) = \frac{r}{2} - \frac{c}{2} - \frac{r}{W} + c = \frac{W(r - c) - 2(r - Wc)}{2W} \geq 0.$$

Therefore,  $\pi_i(s') > \pi_i(s^*)$  for  $i \in P_3(s')$ . Hence  $T$  deviates. ■

According to Proposition 1 all equilibria have at most two parties.

### 3.3.2 ONE-PARTY EQUILIBRIUM

In this subsection, we characterize equilibria where only one party participates.

**PROPOSITION 2 (One-party equilibrium)** Let  $s^*$  be an equilibrium with  $\mathcal{P}(s^*) = \{P_k(s^*)\}$ . Then there are three cases:

- (i)  $r > c$ . If there is a unique candidate closest to  $x_M$  then  $P_k(s^*)$  consists only of this candidate. If there are two candidates that are closest to  $x_M$  then there are two further possibilities :
  - (a) Party  $P_k(s^*)$  consists of the two candidates closest to  $x_M$ .
  - (b) If the condition  $|x(P_k(s^*)) - x_M| \leq c - \frac{r}{2}$  holds, then either of the two candidates participating independently is an equilibrium.
- (ii)  $r < c$ . If there is a unique candidate closest to  $x_M$  then  $P_k(s^*)$  consists only of this candidate. If there are two candidates that are closest to  $x_M$ . Then there are three further possibilities :
  - (a) Party  $P_k(s^*)$  consists of the two candidates closest to  $x_M$ .
  - (b) If the condition  $|x(P_k(s^*)) - x_M| \leq c - \frac{r}{2}$  holds, then either of the two candidates participating independently is an equilibrium.
  - (c) If the condition  $|x_i - x_M| \leq \frac{c-r}{2n}$  holds, then  $P_k(s^*)$  consists of *any* candidate  $i \in N$ .
- (iii)  $r = c$ .  $P_k(s^*)$  consists of the candidate closest to  $x_M$  together with any subset of the remaining candidates.

*Proof:* Let  $s^*$  be an equilibrium with  $\mathcal{P}(s^*) = \{P_k(s^*)\}$ .

- (i) We first show that there exists a strategy profile which supports a one-party equilibrium of the type described. Let  $i$  be the (unique) candidate who is closest to  $x_M$ . Consider the strategy  $s^*$ :  $s_i^* = \{i\}$  and  $s_{i'}^* = \{0\}$  for all  $i' \neq i$ . We claim that  $s^*$  is an equilibrium. Firstly, note that no subset of candidates  $T$  such that  $i \notin T$  can deviate jointly to enter and win. If  $T$  deviates to  $s'$ , then the outcome after the deviation is  $\hat{x}(s') = \hat{x}(s^*) = x_i$ . Therefore, their payoff difference is  $\pi_j(s') - \pi_j(s^*) = -|x_j - x_i| - \frac{c}{|T|} + |x_j - x_i| = -\frac{c}{|T|} < 0$ . Therefore,  $T$  will not deviate if  $i \notin T$ .

Suppose  $T \subset N$  such that  $i \in T$  deviates to  $s'$  to form a new party. We have  $\hat{x}(s') = \hat{x}(s^*) = x_i$ . Then  $i$ 's payoff is  $\pi_i(s') = -|x_i - \hat{x}(s')| + \frac{r-c}{|T|} < -|x_i - \hat{x}(s^*)| + r - c = \pi_i(s^*)$ . Hence,  $i$  does not deviate and  $s^*$  is an equilibrium.

- (a) Let  $i$  be a candidate closest to  $x_M$ . We show that  $i \in P_k(s^*)$ . Suppose not. Then candidate  $i$  can deviate to  $s'_i = \{i\}$  and win the election. Since  $\hat{x}(s') = x_i$  her payoff difference is  $\pi_i(s') - \pi_i(s^*) = -|x_i - x_i| + r - c + |x_i - x(P_k(s^*))| = r - c + |x_i - x(P_k(s^*))| > 0$ . Therefore,  $i$  deviates, leading to a contradiction. Hence  $i \in P_k(s^*)$ .

Suppose  $j \in P_k(s^*)$  and  $j$  is not a candidate who is closest to  $x_M$ . Consider the deviation  $s'_i = P_k(s^*) \setminus \{j\}$  for all  $i \in P_k(s^*) \setminus \{j\}$ . Since the party  $P_k(s^*)$  contains the candidate closest to  $x_M$ , the outcome after the deviation is the same i.e.  $\hat{x}(s') = \hat{x}(s^*)$ . Therefore, the payoff difference for all  $i \in P_k(s^*) \setminus \{j\}$  is  $\pi_i(s') - \pi_i(s^*) = -|x_i - \hat{x}(s')| + \frac{r-c}{|P_k(s^*)|-1} + |x_i - \hat{x}(s^*)| - \frac{r-c}{|P_k(s^*)|} = \frac{r-c}{|P_k(s^*)|-1} - \frac{r-c}{|P_k(s^*)|} > 0$ . Therefore, all the deviating candidates are better-off. The same arguments can be made for the case when there are two candidates who are closest to  $x_M$ .

- (b) Suppose there are two candidates  $i_1$  and  $i_2$  who are closest to  $x_M$  and  $P_k(s^*) = \{i_1\}$ . Consider the deviation  $s'_{i_2} = \{i_2\}$ . The outcome after the deviation is  $\hat{x}(s') = \frac{x_{i_1} + x_{i_2}}{2} = x_M$ . The payoff difference for  $i_2$  is  $\pi_{i_2}(s') - \pi_{i_1}(s^*) = -|x_{i_2} - x_M| + \frac{r}{2} - c + |x_{i_2} - x_{i_1}| = \frac{r-2c}{2} + |x_{i_2} - x_M|$ . Therefore, candidate  $i_2$  does not deviate if  $|x_{i_2} - x_M| \leq c - \frac{r}{2}$ . Since  $|x_{i_2} - x_M| = |x(P_k(s^*)) - x_M|$ , we have  $|x(P_k(s^*)) - x_M| \leq c - \frac{r}{2}$ .

- (ii) We prove part (a). Part (b) can be proved similarly. Suppose there is a unique candidate closest to  $x_M$  and the claim is false. We show that  $|P_k(s^*)| = 1$ . Let  $i_k \in P_k(s^*)$  be such that  $x_{i_k} = x(P_k(s^*))$ . Then there exists a candidate  $i' \neq i_k$ ,  $i' \in P_k(s^*)$ . Consider the deviation  $s$ :  $s_i = \{0\}$ . By definition,  $\hat{x}(s') = \hat{x}(s^*)$ . Moreover, candidate  $i$ 's payoff is  $\pi_i(s') = -|x_i - \hat{x}(s')| > \frac{r-c}{|P_k(s^*)|} - |x_i - \hat{x}(s^*)| = \pi_i(s^*)$ . Therefore, candidate  $i$  deviates. This is a contradiction. Hence,  $|P_k(s^*)| = 1$ . If there are two candidates in the party who are equally close to  $x_M$  we can use similar arguments to show that  $|P_k(s^*)| = 2$ . Therefore,  $|P_k(s^*)| \leq 2$ .

- (c) We show that any candidate  $i$  such that  $2|x_i - x_M| < \frac{c-r}{n}$  participating independently is an equilibrium. Consider the strategy  $s^*$ :  $s_i^* = \{i\}$  and  $s_{i'}^* = \{0\}$  for all

$i' \neq i$ . We claim that  $s^*$  is an equilibrium. Any subset of candidates  $T$  that cannot win will not enter. Suppose  $T$  can deviate to  $s'$  and win as a party. Let this party be denoted by  $P_1(s')$ . Let  $j \in T$  such that  $x_j = x(P_1(s'))$ . We have,  $\pi_j(s') = \frac{r-c}{|T|}$ . The payoff of candidate  $j$  in  $s^*$  is  $\pi_j(s^*) = -|x_j - x_i|$ . The candidate  $j$  can obtain at most  $-2|x_i - x_M|$  by deviating to  $s'$ . If  $|x_j - x_M| > 2|x_i - x_M|$ , then  $P_1(s')$  will lose the election. Therefore  $|x_j - x_M| < 2|x_i - x_M|$ . Hence  $j$  does not deviate if  $\frac{c-r}{|T|} > 2|x_i - x_M|$ . Hence, any set of candidates  $T$  will not deviate.

- (iii) Let  $T = P_k(s^*)$ . We claim that the candidate closest to  $x_M$  belongs to  $T$ . Suppose  $i$  is the candidate closest to  $x_M$  and  $i \notin P_k(s^*)$ . Then candidate  $i$  can deviate to  $s'_i = \{i\}$ . The outcome of the election is  $\hat{x}(s') = x_i$ . The payoff difference for  $i$  is  $\pi_i(s') - \pi_i(s^*) = r - c + |x_i - x(P_k(s^*))| > 0$ . Therefore,  $i$  deviates. Hence  $i \in P_k(s^*)$ .

Suppose  $s^*$  is a strategy where  $T$  forms a party i.e.  $T = P_k(s^*)$  and  $T$  consists of the candidate(s) closest to  $x_M$ . We show that this is an equilibrium. Suppose a set of candidates  $T'$  deviates to  $s'$  and form a party. If their party wins then  $\hat{x}(s') = \hat{x}(s^*)$ . The payoff difference for a candidate  $i \in T'$ ,  $i \notin T$  is  $\pi_i(s') - \pi_i(s^*) = -|x_i - \hat{x}(s')| - \frac{r-c}{|T'|} + |x_i - \hat{x}(s^*)| = 0$ . Similarly, if  $i \in T' \cap T$ , then the payoff difference is  $\pi_i(s') - \pi_i(s^*) = -|x_i - \hat{x}(s')| - \frac{r-c}{|T'|} + |x_i - \hat{x}(s^*)| - \frac{r-c}{|P_k(s^*)|} = 0$ .

■

According to Proposition 2 “most” one party equilibria consists of candidate closest to the voter median. In the special case when benefits equal costs, there are multiple equilibria. The party consists of the candidate closest to the voter median together with an arbitrary subset of the other candidates.

### 3.3.3 TWO-PARTY EQUILIBRIUM

The structure of equilibrium is considerably more complex than the one-party equilibrium case. We are unable to provide a complete description of such equilibria. However, we are able to show by means of an example that such equilibria exist. We are also able to identify some important features of such equilibria.

Our model does not preclude the formation of parties that are heterogeneous in the following sense. There are four candidates located at 0.1, 0.2, 0.3 and 0.4. The candidates located at 0.1 and 0.3 form one party and the candidates located at 0.2 and 0.4 form another. One of the nice features of our model is that this phenomenon cannot occur in equilibrium.

We say that party  $P_k \in \mathcal{P}$  is *homogeneous* if the *smallest* interval that contains the policy positions of *all* the members of  $P_k$  does not contain a policy position of a member of another party  $P_{k'}(s) \in \mathcal{P}$ ,  $P_{k'} \neq P_k$ . A *heterogeneous* party is a party that is not homogeneous. The example below provides further clarification.

**EXAMPLE 5**  $N = \{1, 2, 3, 4, 5, 6\}$ . Let  $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$ . If  $P_1 = \{1, 3, 5\}$  and  $P_2 = \{2, 4, 6\}$ , neither party is homogeneous. The smallest interval that contains all the policy positions of members in  $P_1$  is  $[x_1, x_5]$ . However,  $x_2, x_4 \in [x_1, x_5]$  and  $3, 5 \in P_2$ .

On the other hand, if  $P_1 = \{1, 2, 3\}$  and  $P_2 = \{4, 5, 6\}$ , then both parties are homogeneous.

The next Proposition highlights some key features of all two-party equilibria.

**PROPOSITION 3 (Two-party equilibrium)** Let  $s^*$  be an equilibrium with  $\mathcal{P}(s^*) = \{P_1(s^*), P_2(s^*)\}$  and  $x(P_1(s^*)) < x(P_2(s^*))$ . Then

- (i)  $\max\{|P_1(s^*)|, |P_2(s^*)|\} > 1$ .
- (ii)  $r \geq 2c$ .
- (iii) There exists  $\epsilon > 0$  such that  $x(P_1(s^*)) = x_M - \epsilon$  and  $x(P_2(s^*)) = x_M + \epsilon$ .
- (iv) Both the parties are homogeneous.

*Proof:* Let  $s^*$  is an equilibrium as specified in the statement of Proposition 3.

- (i) Suppose contrariwise that  $|P_1(s^*)| = |P_2(s^*)| = 1$ . Let  $i_1$  and  $i_2$  be as defined earlier. Consider the deviation  $s'$ :  $s_i = \{i_1, i_2\}$  for  $i \in \{i_1, i_2\}$ . Let the new party be denoted by  $P_1(s')$ . Then  $\hat{x}(s') = \hat{x}(s^*)$ . Observe that for  $i = i_1, i_2$ ,  $\pi_i(s') - \pi_i(s^*) = -|x_i - \hat{x}(s')| + \frac{r-c}{2} + |x_i - \hat{x}(s^*)| - \frac{r}{2} + c = \frac{r-c}{2} - \frac{r}{2} + c = \frac{c}{2}$ . Therefore  $\{i_1, i_2\}$  deviate which leads to a contradiction. Hence, either  $|P_1(s^*)| > 1$  or  $|P_2(s^*)| > 1$ .
- (ii) Suppose  $|P_1(s^*)| > 1$ . Consider a candidate  $i \in P_1(s^*)$ ,  $i \neq i_1$ . Candidate  $i$  can deviate to  $s'_i = \{0\}$ . By definition, the remaining members of  $P_1(s^*)$  continue to be a party. Therefore,  $\hat{x}(s') = \hat{x}(s^*)$ . Therefore  $\pi_i(s^*) = \frac{r}{2|P_1(s^*)|} - \frac{c}{|P_1(s^*)|} - |x_i - \hat{x}(s^*)|$  and  $\pi_i(s') = -|x_i - \hat{x}(s')| = -|x_i - \hat{x}(s^*)|$ . Candidate  $i$  does not deviate to  $s'$  if  $\pi_i(s^*) - \pi_i(s') = \frac{r}{2|P_1(s^*)|} - \frac{c}{|P_1(s^*)|} \geq 0$ . Therefore,  $r \geq 2c$ .
- (iii) Applying Lemma 11,  $P_k(s^*) \in \mathcal{W}(s^*)$  for  $k \in \{1, 2\}$ . This immediately implies that there exists  $\epsilon > 0$  such that  $x(P_1(s^*)) = x_M - \epsilon$  and  $x(P_2(s^*)) = x_M + \epsilon$ .
- (iv) If  $P_1(s^*)$  is homogeneous then so is  $P_2(s^*)$ . Suppose contrariwise that neither party is homogeneous. Therefore there exists a candidate  $i \in P_1(s^*)$  such that  $x_i \geq x(P_2(s^*))$ . Consider the set of players  $T = P_1(s^*) \setminus \{i\}$ . Let the strategy profile  $s'$  be such that  $s'_j = T$  for all  $j \in T$ . Let this new party be  $P'_1(s)$ . Let the remaining parties be  $P'_2(s) = P_2(s^*)$  and  $P'_3(s) = \{i\}$ . We show that every candidate in  $T$  is better-off after the deviation.

The party positions satisfy  $x(P'_1(s)) < x(P'_2(s)) < x(P'_3(s))$ . Therefore, the new outcome is  $\hat{x}(s') = x(P'_1(s))$ . All the candidates in  $P_1(s^*)$  strictly prefer  $\hat{x}(s')$  to  $\hat{x}(s^*)$ .

We have

$$\Delta G_i(s', s^*) = \frac{r-c}{|P_1(s^*)| - 1} - \frac{r}{2|P_1(s^*)|} + \frac{c}{|P_1(s^*)|}$$

$$= \frac{2|P_1(s^*)|(r-c) - (|P_1(s^*)| - 1)(r-2c)}{2|P_1(s^*)|(|P_1(s^*)| - 1)}.$$

From (ii)  $r \geq c$ . This implies that

$$2|P_1(s^*)|(r-c) - (|P_1(s^*)| - 1)(r-2c) \geq 0$$

i.e. the numerator is non-negative. Hence  $\Delta G_i(s', s^*) \geq 0$ . Since the policy outcome strictly improves,  $\pi_i(s') > \pi_i(s^*)$ . Thus  $s^*$  is not an equilibrium contrary to our assumption. ■

The conditions above are only necessary and not sufficient for the existence of equilibrium. Suppose for example there exists a candidate  $i$  located at  $x_M$ . Consider the deviation by  $i$  to  $s'_i = \{i\}$ . If  $i$  wins then  $\hat{x}(s') = x_i$ . The payoff difference for  $i$  is given by  $\pi_i(s') - \pi_i(s^*) = -|x_i - \hat{x}(s')| + r - c + |x_i - \hat{x}(s^*)|$ . We have already shown that  $r \geq 2c$ . Therefore, if  $i$  wins then  $\pi_i(s') - \pi_i(s^*) > 0$ . In order to sustain  $s^*$  as an equilibrium  $i$  should not be able to win. This would require  $F\left(\frac{2x_M - \epsilon}{2}\right) \geq F\left(\frac{2x_M + \epsilon}{2}\right) - F\left(\frac{2x_M - \epsilon}{2}\right)$  and  $1 - F\left(\frac{2x_M + \epsilon}{2}\right) \geq F\left(\frac{2x_M + \epsilon}{2}\right) - F\left(\frac{2x_M - \epsilon}{2}\right)$ . In other words, the party policy positions of the winning parties cannot be “too far” away from the median.

Of course, there may not be a candidate located at the median. But one has to ensure that any candidate in the interval  $(x(P_1(s^*)), x(P_2(s^*)))$  cannot form an independent party and win. Whether or not this candidate wins depends on the specification of  $F$ . The conditions that ensures this are  $F\left(\frac{x_i + x_M - \epsilon}{2}\right) \geq F\left(\frac{x_i + x_M + \epsilon}{2}\right) - F\left(\frac{x_i + x_M - \epsilon}{2}\right)$  and  $1 - F\left(\frac{x_i + x_M + \epsilon}{2}\right) \geq F\left(\frac{x_i + x_M + \epsilon}{2}\right) - F\left(\frac{x_i + x_M - \epsilon}{2}\right)$ .

Several other potential deviations must also be taken care of. For instance, one has to ensure that the leader in one of the winning parties does not deviate to form a party with a subset of candidates. To ensure that none of these deviations are beneficial is tedious. Instead of enumerating these conditions separately, we provide an example to show that such an equilibrium can exist.

**EXAMPLE 6** Let  $F$  be the uniform distribution on the policy space  $X = [0, 1]$ . The set of candidates is  $N = \{1, \dots, 5\}$  with  $x_1 = 0.43$ ,  $x_2 = 0.45$ ,  $x_3 = 0.55$ ,  $x_4 = 0.56$  and  $x_5 = 0.56$ . Suppose  $r = 0.06$  and  $c = 0.01$ . Consider a set of strategies  $s^*$  such that  $P_1(s^*) = \{1, 2\}$  and  $P_2(s^*) = \{3, 4, 5\}$ .

We claim that  $s^*$  is an equilibrium. In order to confirm this, we consider deviations of all possible types and show that they are not profitable.

(i) Candidate  $i \in P_k(s^*)$ ,  $k \in \{1, 2\}$  deviates to  $s'_i = \{i\}$ . She loses the election with certainty. Moreover, the outcome  $\hat{x}(s')$  will be worse than  $\hat{x}(s^*)$ . Therefore  $\Delta G_i(s', s^*) = 0 - \frac{r-c}{|P_k(s^*)|} \leq 0$ . Therefore,  $i$  will not deviate.

(ii) Candidates 2 and 3 can deviate to  $s'$  such that:  $s'_i = \{2, 3\}$  for  $i \in \{2, 3\}$ . Then the party  $P_1(s') = \{2, 3\}$  is formed. Party  $P_1(s')$  does not win since: (i)  $2F(\frac{x_2+x_4}{2}) = 1.01 < 1 + F(\frac{x_2+x_2}{2}) = 1.44$  and (ii)  $2F(\frac{x_3+x_1}{2}) = 0.98 < 1 - F(\frac{x_3+x_4}{2}) = 1.555$ . Therefore, the conditions imply that party  $P_1(s') \notin \mathcal{W}(s')$ .

(iii)  $s'_i = \{2, 3, 4, 5\}$  for all  $i \in \{2, 3, 4, 5\}$ . The new party is  $P_2(s') = \{2, 3, 4, 5\}$ . By definition  $\hat{x}(s') = x(P_2(s')) = x_3$ . We have  $\pi_2(s') - \pi_2(s^*) = -|0.2 - 0.3| + \frac{0.06-0.01}{4} - (-|0.2 - 0.25| + \frac{0.06}{4} - \frac{0.01}{2}) = -0.1 + 0.0125 + 0.05 - 0.15 + 0.05 = -0.1825$ . Therefore, 2 does not deviate.

(iv)  $s'_i = \{1, 2, 3\}$  for all  $i \in \{1, 2, 3\}$ . The new party is  $P_1(s') = \{1, 2, 3\}$ . By definition  $\hat{x}(s') = x(P_1(s')) = x_2$ . We have  $\pi_3(s') - \pi_3(s^*) = -|0.45 - 0.55| + \frac{0.06-0.01}{3} - (-|0.55 - 0.5| + \frac{0.06}{6} - \frac{0.01}{3}) = -0.1 + 0.0167 + 0.05 - 0.01 + 0.033 = -0.0103$ . Therefore, 3 does not deviate.

(v)  $s'_i = \{1, 2, 3, 4, 5\}$  for all  $i \in \{1, 2, 3, 4, 5\}$ . The new party  $P_1(s') = \{1, 2, 3, 4, 5\}$ . By definition  $\hat{x}(s') = x(P_1(s')) = 0.3$ . We have  $\pi_2(s') - \pi_2(s^*) = -|0.45 - 0.55| + \frac{0.06-0.01}{5} - (-|0.45 - 0.5| + \frac{0.06}{4} - \frac{0.01}{2}) = -0.1 + 0.01 + 0.05 - 0.015 + 0.005 = -0.05$ . Therefore, 2 does not deviate. Conditions (i)-(v) establish that  $s^*$  is an equilibrium.

This example can be generalized to provide appropriate sufficient conditions for the existence of two-party equilibria. However, these conditions are cumbersome and we choose not to present them.

### 3.4 EQUILIBRIUM ANALYSIS: INTERNALLY DEMOCRATIC PARTIES

When parties are populist, a three-party equilibrium is not possible since two candidates from adjacent extreme parties can jointly deviate to form a winning party. The candidate closest to the voter median could ensure that her policy is chosen after the deviation, thus making every deviating candidate better-off. This deviation is not beneficial when parties are internally democratic since the new party, formed by the deviating candidates, does not win.

#### 3.4.1 NUMBER OF PARTIES

We show the existence of a three-party equilibrium by the following example.

**EXAMPLE 7** Let  $F$  be the uniform distribution over  $[0, 1]$ . The set of candidates is  $N = \{1, 2, 3, 4, 5, 6, 7\}$  with policy positions  $x_1 = 0.12, x_2 = 0.166, x_3 = 0.45, x_4 = 0.5, x_5 = 0.55, x_6 = 0.834$  and  $x_7 = 0.88$ . Let  $s^*$  be a strategy profile such that  $s_i^* = \{1, 2\}$  for all  $i \in \{1, 2\}$ ,  $s_i^* = \{3, 4, 5\}$  for all  $i \in \{3, 4, 5\}$  and  $s_i^* = \{6, 7\}$  for all  $i \in \{6, 7\}$ . Therefore, there are three parties  $P_1(s^*) = \{1, 2\}$ ,  $P_2(s^*) = \{3, 4, 5\}$  and  $P_3(s^*) = \{6, 7\}$ . Let  $r = 0.04$  and  $c = 0.01$ .

The party positions are  $x(P_1(s^*)) = 0.16$ ,  $x(P_2(s^*)) = 0.5$  and  $x(P_3(s^*)) = 0.81$ . Therefore, each party gets equal votes since  $F(\frac{0.166+0.5}{2}) = F(\frac{0.834+0.5}{2}) - F(\frac{0.5+0.166}{2}) = 1 - F(\frac{0.834+0.5}{2}) = 0.33$ . Hence,  $\hat{x}(s^*) = \frac{1}{3}(0.166 + 0.5 + 0.834) = 0.5$ . We show that  $s^*$  is an equilibrium.

There are three types of deviations:

(i)  $s'_i = \{0\}$ : Candidate 1 deviates to non-participation. The policy positions of the remaining parties remain the same and so does the outcome. Therefore, candidate 1 loses since  $\pi_1(s^*) - \pi_1(s') = \frac{r}{6} - \frac{c}{2} > 0$  and the deviation is not beneficial. Similar arguments can be made to show that candidate 3, 5, 7 will not deviate to non-participation.

Candidate 2 deviates to  $s'_2 = \{0\}$ . The outcome is  $\hat{x}(s') = x_5 = \hat{x}(s^*)$  since party  $P_2(s^*)$  wins. As before, candidate 2 loses since  $\pi_2(s^*) - \pi_2(s') = \frac{r}{6} - \frac{c}{2} > 0$ . Similarly, it can be shown that candidate 4 or 6 will not deviate to non-participation.

(ii)  $s'_i = \{i\}$ . Candidate 1 deviates to  $s'_1 = \{1\}$ . Both parties  $P_2(s^*)$  and  $P_3(s^*)$  get the most votes. Therefore, the outcome is  $\hat{x}(s') = \frac{x_4+x_6}{2} > \hat{x}(s^*)$ . Candidate 1 prefers  $\hat{x}(s^*)$  to  $\hat{x}(s')$ . Moreover, she loses  $\frac{r}{6} - \frac{c}{2} > 0$ . Hence, she is worse-off. Similar arguments can be made to show that no other candidate deviates to non-participation.

(iii) Deviation to join another party. Consider the set  $T = \{2, 3, 4, 5\}$ . Let  $s'$  be strategy profile such that  $s'_i = T$  for all  $i \in T$ . Let the party formed by  $T$  be denoted by  $P_2(s')$ . By definition,  $x(P_2(s')) = x_5$ . Let the remaining parties be  $P_1(s') = \{1\}$  and  $P_3(s') = \{6, 7\}$ . By definition,  $x(P_1(s')) = x_1$  and  $x(P_3(s')) = x_6$ . The outcome is  $\hat{x}(s') = x_4 = \hat{x}(s^*)$ . Therefore,  $\pi_2(s^*) - \pi_2(s') = -|x_2 - x_4| - \frac{r-c}{4} + |x_2 - \hat{x}(s')| - \frac{r}{6} + \frac{c}{2} = \frac{r-c}{4} - \frac{r-3c}{12} = \frac{r-3c}{12} = 0.008 > 0$ .

Similar arguments can be made to show that candidate 6 does not deviate to join party  $P_2(s^*)$ . A deviation by candidate 1 to join party  $P_2(s^*)$  will also not be beneficial. The outcome of such a deviation will remain unchanged. Therefore  $\pi_1(s^*) - \pi_1(s') = -|x_1 - \hat{x}(s')| + \frac{r}{12} - \frac{c}{4} + |x_1 - \hat{x}(s^*)| - \frac{r}{12} + \frac{c}{4} = \frac{r-3c}{6} - \frac{r-3c}{12} > 0$ . Similarly, candidate 7 will not join  $P_2(s^*)$ . Similarly, no candidate in  $P_1(s^*)$  will form a party with candidates in  $P_3(s^*)$ .

Consider the set of candidates  $T = \{3, 4, 5, 6, 7\}$ . Let the strategy profile  $s'$  be such that  $s'_i = T$  for all  $i \in T$ . After the deviation  $x(P_1(s')) = x_2$  and  $x(P_2(s')) = x_5$ . Therefore,  $\hat{x}(s') = x_5$ . Hence  $\pi_3(s') - \pi_3(s^*) = -|x_3 - x_5| + \frac{r-c}{5} + |x_3 - \hat{x}(s^*)| - \frac{r}{6} + \frac{c}{3} = -0.043$ . Therefore, candidate 3 does not deviate.

A final deviation is the case where  $T = \{4, 5, 6, 7\}$  forms a party and wins. We show that candidate 4 is worse-off after the deviation. Let the deviation  $s'_i = T$  for all  $i \in T$ . The outcome is  $\hat{x}(s') = x_5$  and  $\pi_4(s') - \pi_4(s^*) = -|x_4 - \hat{x}(s')| + \frac{r-c}{4} + |x_4 - \hat{x}(s^*)| - \frac{r}{6} + \frac{c}{3} = -0.05 + 0.0042 = -0.46$ . Therefore, candidate 4 does not deviate. Similarly, we can show that the set of candidates  $T = \{1, 2, 3\}$  or  $T = \{1, 2, 3, 4\}$  cannot deviate beneficially. It follows that parties  $\{1, 2\}$ ,  $\{3, 4, 5\}$  and  $\{6, 7\}$  form in equilibrium.



### 3.4.2 ONE-PARTY EQUILIBRIUM

In this section we will characterize equilibria where only one party participates.

We introduce some notation. Let  $i_k \in P_k(s)$  be such that  $x_{i_k} = x(P_k(s))$ . Candidates  $i_{k-1}, i_{k+1} \in P_k(s^*)$  be such that (i)  $x_{i_{k-1}} < x_{i_k} < x_{i_{k+1}}$  (ii) there does not exist  $i \in P_k(s)$ ,  $i \neq i_k$  such that  $x_i \in (x_{i_{k-1}}, x_{i_{k+1}})$ . Therefore,  $i_{k-1}$  and  $i_{k+1}$  are members of the party whose policy positions are adjacent to  $x_{i_k}$ , on the left and right of  $x(P_k(s^*))$  respectively.

**PROPOSITION 4 (One-party equilibrium)** Let  $s^*$  be an equilibrium with  $P(s^*) = \{P_k(s^*)\}$ . We consider three cases separately.

(i)  $r > c$ . If there exists a unique candidate closest to  $x_M$  then  $P_k(s^*)$  consists only of this candidate. If there are two candidates that are closest to  $x_M$  then there are two further possibilities :

(a) Party  $P_k(s^*)$  consists of the two candidates closest to  $x_M$ .

(b) If the condition  $|x(P_k(s^*)) - x_M| \leq c - \frac{r}{2}$  holds, then either of the two candidates participating independently is an equilibrium.

(ii)  $r < c$ . If there is a unique candidate closest to  $x_M$  then  $P_k(s^*)$  consists of this candidate and any two other candidates, i.e.  $|P_k(s^*)| \leq 3$ . If there are two candidates closest to  $x_M$  then there are two further possibilities :

(a)  $P_k(s^*)$  consists of both candidates closest to  $x_M$ .

(b) If the condition  $|x(P_k(s^*)) - x_M| \leq c - \frac{r}{2}$  holds, then  $P_k(s^*)$  consists of one of these candidates.

In either case  $P_k(s^*)$  can have three other candidates i.e.  $|P_k(s^*)| \leq 4$ . The conditions (i)  $c \leq |P_k(s^*)| \max\{|x(P_k(s^*)) - x_{i_{k+1}}|, |x(P_k(s^*)) - x_{i_{k-1}}|\} + r$  and (ii)  $c > n|x(P_k(s^*)) - x_M| + r$  are necessary for both (a) and (b).

(iii)  $r = c$ .  $P_k(s^*)$  consists of the candidate closest to  $x_M$  together with any subset of the remaining candidates. The conditions stated in (ii) are necessary.

*Proof:*

- (i) Suppose there is a unique candidate  $i$  closest to  $x_M$ . We show that  $i \in P_k(s^*)$ . Suppose not. Then candidate  $i$  can deviate to  $s'_i = \{i\}$  and win the election. Since  $\hat{x}(s') = x_i$  her payoff difference is  $\pi_i(s') - \pi_i(s^*) = -|x_i - x_i| + r - c + |x_i - x(P_k(s^*))| = r - c + |x_i - x(P_k(s^*))| > 0$ . Therefore,  $i$  deviates leading to a contradiction. Hence  $i \in P_k(s^*)$ .

Suppose there is a candidate  $j \in P_k(s^*)$  such that  $j \neq i$ . Consider the deviation  $s'_i = \{i\}$ . The outcome after the deviation is  $\hat{x}(s') = x_{i_1}$ . Therefore, the payoff difference is  $\pi_i(s') - \pi_i(s^*) = -|x_i - \hat{x}(s')| + r - c + |x_i - \hat{x}(s^*)| - \frac{r-c}{|P_k(s^*)|} = r - c - \frac{r-c}{|P_k(s^*)|} > 0$ . Therefore,  $i$  deviates leading to a contradiction. Therefore,  $P_k(s^*)$  consists only of the candidate closest to  $x_M$ . Similar arguments can be made when there are two candidates who are closest to  $x_M$ .

Suppose there are two candidates  $i_1$  and  $i_2$  closest to  $x_M$  and  $P_k(s^*) = \{i_1\}$ . Consider the deviation  $s'_{i_2} = \{i_2\}$ . The outcome after the deviation is  $\hat{x}(s') = \frac{x_{i_1} + x_{i_2}}{2} = x_M$ . The payoff difference for  $i_2$  is  $\pi_{i_2}(s') - \pi_{i_1}(s^*) = -|x_{i_2} - x_M| + \frac{r}{2} - c + |x_{i_2} - x_{i_1}| = \frac{r-2c}{2} + |x_{i_2} - x_M|$ . Therefore, candidate  $i_2$  does not deviate if  $|x_{i_2} - x_M| \leq c - \frac{r}{2}$ . Since  $|x_{i_2} - x_M| = |x(P_k(s^*)) - x_M|$ , we have  $|x(P_k(s^*)) - x_M| \leq c - \frac{r}{2}$ .

- (ii) Suppose there exists a unique candidate  $i$  closest to  $x_M$ . We show that  $|P_k(s^*)| \leq 3$ . Suppose not. Assume w.l.o.g.  $x(P_k(s^*)) \leq x_M$ . There exist two candidates  $i, j$  such that  $x_i < x(P_k(s^*))$  and  $x_j > x(P_k(s^*))$ . Consider the deviation  $s'_i = s'_j = \{0\}$ . The outcome after the deviation is  $\hat{x}(s') = \hat{x}(s^*)$ . Therefore  $\pi_i(s') - \pi_i(s^*) = -|x_i - \hat{x}(s')| + |x_i - \hat{x}(s^*)| - \frac{r-c}{|P_k(s^*)|} = -\frac{r-c}{|P_k(s^*)|} > 0$ . Similarly,  $\pi_j(s') - \pi_j(s^*) > 0$ . Therefore  $\{i, j\}$  deviates, leading to a contradiction. Hence,  $|P_k(s^*)| \leq 3$ . Similarly, we can show that  $|P_k(s^*)| \leq 4$  when there are two candidates in  $P_k(s^*)$  both closest to  $x_M$ .

Let  $i_k$  be such that  $x_{i_k} = x(P_k(s^*))$ . Take a candidate  $i'$  such that  $x_{i'} > x(P_k(s^*))$ . Consider the deviation  $s'_{i'} = \{0\}$ . The outcome after the deviation is  $\hat{x}(s') = x(P_k(s')) = x_{i_k-1}$ . Therefore,  $\pi_{i'}(s') - \pi_{i'}(s^*) = -|x_{i'} - x_{i_k-1}| - \frac{r-c}{|P_k(s^*)|} + |x_{i'} - \hat{x}(s')|$ . The deviation is not beneficial if  $c \leq |P_k(s^*)||x(P_k(s^*)) - x_{i_k-1}| + r$ . Similarly, we can show that  $c \leq |P_k(s^*)||x(P_k(s^*)) - x_{i_k+1}| + r$ .

Suppose there exists a candidate  $i$  whose policy position is closer to  $x_M$  than  $x(P_k(s^*))$ . Consider the deviation  $s'_i = \{i\}$ . Then  $\hat{x}(s') = x_i$  and her payoff from rents and policy outcome would increase by at most  $r + 2|x(P_k(s^*)) - x_M|$ . Her cost would be  $c$ . Suppose there exists a set of candidates  $T$ . Consider the deviation  $s'$  such that  $s'_j = T$  for all  $j \in T$ . The candidates  $T$  form a party. Let this party be denoted by  $P_1(s')$ . Suppose  $\hat{x}(s') = x(P_1(s'))$ . Then the payoff of candidate  $j \in T$  would increase by at most  $\frac{r}{|T|} + 2|x(P_k(s^*)) - x_M|$  and her cost would be  $\frac{c}{|T|}$ . In either of the above two cases the deviation is not beneficial if  $c > n|x(P_k(s^*)) - x_M| + r$ .

- (iii) Suppose  $i$  is the unique candidate closest to  $x_M$  and  $i \notin P_k(s^*)$ . Consider the deviation

$s'_i = \{i\}$ . The outcome is  $\hat{x}(s') = x_i$ . The payoff difference for  $i$  is  $\pi_i(s') - \pi_i(s^*) = -|x_i - x_i| + |x_i - \hat{x}(s^*)| = |x_i - \hat{x}(s^*)| > 0$ . This is a contradiction. Therefore,  $i \in P_k(s^*)$ .

We show that for any  $P_k(s^*)$  no two candidates can leave the party beneficially as was the case in (ii). Suppose  $i, j \in P_k(s^*)$  such that  $x_i < x(P_k(s^*))$  and  $x_j > x(P_k(s^*))$ . Consider the deviation  $s_i = s_j = \{0\}$  by two candidates in the party  $P_k(s^*)$ . The outcome after the deviation is  $\hat{x}(s') = \hat{x}(s^*)$ . The payoff difference is  $\pi_i(s') - \pi_i(s^*) = -|x_i - \hat{x}(s')| - \frac{r-c}{|P_k(s^*)|} + |x_i - \hat{x}(s^*)| = 0$ . Therefore,  $\{i, j\}$  does not deviate. Similar arguments as the ones made in (ii) can be used to show that the conditions in (ii) are necessary. ■

The one-party equilibrium when benefits exceed costs is identical in the populist and internally democratic cases. The party consists only of the candidates closest to the voter median. However, when costs exceed benefits, equilibrium party structures differ. In every case, the candidate closest to the median is a member of the party. There may be two (or three in the case where there are two candidates closest to the voter median) others in the party. Party size in the internally democratic case is greater than the case with populist parties. This is due to the fact that candidates in internally democratic parties can influence party policies more than in the populist case. Candidates who are not closest to  $x_M$  can nevertheless influence party policies by shifting the party median towards themselves.

### 3.4.3 TWO-PARTY EQUILIBRIUM

Two-party equilibria exist as in the populist case. However, the exact characterization is difficult. As in the populist case, we will identify some key features of the equilibrium and construct an example to demonstrate its existence.

**PROPOSITION 5 (Two-party equilibrium)** Let  $s^*$  be an equilibrium with  $\mathcal{P}(s^*) = \{P_1(s^*), P_2(s^*)\}$  and  $x(P_1(s^*)) < x(P_2(s^*))$ . Then

- (i)  $\max\{|P_1(s^*)|, |P_2(s^*)|\} > 1$ .
- (ii)  $r \geq c$ . If  $\max\{|P_1(s^*)|, |P_2(s^*)|\} \geq 3$  then  $r \geq 2c$ .
- (iii) There exists  $\epsilon > 0$  such that  $x_M$  i.e.  $x(P_1(s^*)) = x_M - \epsilon$  and  $x(P_2(s^*)) = x_M + \epsilon$ .

*Proof:* Suppose  $s^*$  is an equilibrium as specified in the statement of Proposition 5. Arguments identical to those in the populist parties case can now be used to prove parts (i) and (iii).

(ii) Suppose  $r < c$ . We first claim that  $|P_k(s^*)| = 1$  for  $k \in \{1, 2\}$ . We show this in steps. First we rule out the case where  $|P_k(s^*)| \geq 3$  for some  $k \in \{1, 2\}$ . Then we show that  $|P_k(s^*)| \neq 2$ .

Suppose  $|P_k(s^*)| > 2$  for some  $k \in \{1, 2\}$ . There exist two candidates  $i, j \in P_k(s^*)$  such that  $x_i < x(P_1(s^*))$  and  $x_j > x(P_1(s^*))$ . Consider the deviation to a strategy  $s'$  such that  $s'_i = s'_j = \{0\}$ . Let the party  $P_1(s') = P_1(s^*) \setminus \{i, j\}$ . The outcome  $\hat{x}(s') = \hat{x}(s^*)$ . Moreover,  $x_i < \hat{x}(s') < \hat{x}(s^*)$ . Therefore,  $i$  strictly prefers  $\hat{x}(s')$  to  $\hat{x}(s^*)$ . We have  $\pi_i(s') - \pi_i(s^*) = -|x_i - \hat{x}(s')| + |x_i - \hat{x}(s^*)| - \frac{r}{2|P_k(s^*)|} + \frac{c}{|P_k(s^*)|} > 0$ . Therefore,  $i$  and  $j$  deviate i.e.  $|P_k(s^*)| < 3$  for  $k \in \{1, 2\}$ . Assume w.l.o.g.  $|P_1(s^*)| = 2$ . Let  $i \in P_1(s^*)$  be such that  $x_i < x(P_1(s^*))$ . Consider the deviation by  $i$ :  $s'_i = \{0\}$ . After the deviation the outcome  $\hat{x}(s') = \hat{x}(s^*)$ . The payoffs of  $i$  are such that  $\pi_i(s') - \pi_i(s^*) = -|x_i - \hat{x}(s')| + |x_i - \hat{x}(s^*)| - \frac{r}{2|P_k(s^*)|} + \frac{c}{|P_k(s^*)|} > 0$ . Therefore,  $i$  deviates. Similarly we can arrive at a contradiction if  $|P_2(s^*)| = 2$ . Hence,  $|P_k(s^*)| = 1$  for  $k \in \{1, 2\}$ .

We show that there is a beneficial deviation where both the independent candidates form a party. Let  $i_1$  and  $i_2$  be such that  $x(P_1(s^*)) = x_{i_1}$  and  $x(P_2(s^*)) = x_{i_2}$ . Consider the deviation  $s'$  such that  $s'_i = \{i_1, i_2\}$  for  $i \in \{i_1, i_2\}$ . Then,  $\hat{x}(s') = \hat{x}(s^*)$ . For any  $i \in \{i_1, i_2\}$ ,  $\pi_i(s') - \pi_i(s^*) = -|x_i - \hat{x}(s')| + \frac{r-c}{2} + |x_i - \hat{x}(s^*)| - \frac{r}{2} + c = \frac{r-c}{2} - \frac{r-2c}{2} = \frac{c}{2} > 0$ . Therefore, the set of candidates  $\{i_1, i_2\}$  deviate. Hence,  $r \geq c$ .

Suppose there exists a party  $P_k(s^*)$  such that  $|P_k(s^*)| \geq 3$ . Then there exist two candidates  $i, j \in P_k(s^*)$  such that  $x_i < x(P_k(s^*))$  and  $x_j > x(P_k(s^*))$ . Consider the deviation  $s'_i = s'_j = \{0\}$ . The outcome after the deviation is  $\hat{x}(s') = \hat{x}(s^*)$ . Therefore,  $\pi_i(s') - \pi_i(s^*) = -|x_i - \hat{x}(s')| + |x_i - \hat{x}(s^*)| - \frac{2r}{|P_k(s^*)|} + \frac{c}{|P_k(s^*)|}$ . The deviation is not beneficial only if  $r \geq 2c$ . ■

The structure of two-party equilibria in the populist and internally democratic cases is similar though not identical. An important difference is that the arguments used to establish homogeneity in the populist parties case can no longer be used. We are unable to show that parties are homogeneous. We conjecture that heterogeneity might exist in equilibrium.

We demonstrate the existence of two-party equilibrium by means of the example below.

**EXAMPLE 8** Let  $F$  be the uniform distribution of voters policy positions over the policy space  $X = [0, 1]$ . Let  $N = \{1, \dots, 5\}$  with  $x_1 = 0.28$ ,  $x_2 = 0.33$ ,  $x_3 = 0.62$ ,  $x_4 = 0.67$  and  $x_5 = 0.72$ . Let  $r = 1$  and  $c = 0.01$ .

Let the strategy  $s^*$  be such that  $P_1(s^*) = \{1, 2\}$  and  $P_2(s^*) = \{3, 4, 5\}$ . Then  $x(P_1(s^*)) = x_2 = 0.33$  and  $x(P_2(s^*)) = x_5 = 0.67$ . Each party gets votes equal to  $F(\frac{0.38+0.67}{2}) = 1 - F(\frac{0.38+0.67}{2}) = 0.5$ . The expected outcome is  $\hat{x}(s^*) = \frac{0.38+0.67}{2} = 0.5$ . We show that  $s^*$  is a two-party equilibrium. We consider all possible types of deviation and show that none of them are profitable.

(i)  $s'_i = \{0\}$ . Clearly, candidate 1 does not benefit by deviating to  $s'_1 = \{0\}$ . If candidate 1 deviates to  $s'_1 = \{0\}$ , the set of candidates  $\{4, 5\}$  continue to be a party. The outcome of the election is same as before. Moreover, candidate 1 loses  $\frac{r}{4} - \frac{c}{2} = 0.245$ . Therefore, candidate 1 does not deviate. Similar arguments can be made to show that 2 does not deviate to non-participation.

Suppose candidate 5 deviates to  $s'_5 = \{0\}$ . Let the party  $P_2(s') = \{3, 4\}$ . Let  $P_1(s') = \{1, 2\}$ . Using the properties of uniform distribution, the outcome of the election is  $\hat{x}(s') = x(P_2(s')) = 0.62$ . We have  $\pi_5(s') - \pi_5(s^*) = -|0.72 - 0.62| + |0.72 - 0.5| - \frac{r}{6} + \frac{c}{3} = -0.1 + 0.22 - 0.1667 + 0.0033 = -0.0434$ . Similar arguments can be made if candidate 3 deviates to  $s'_3 = \{0\}$  or if candidate 4 deviates to  $s'_4 = \{0\}$ .

(ii)  $s'_i = \{i\}$ . Suppose candidate 1 deviates to  $s'_1 = \{1\}$ . The policy positions of the parties are  $\{0.28, 0.33, 0.67\}$ . Using the properties of uniform distribution, party  $P_2(s') = P_2(s^*)$  will win the election. Therefore,  $\hat{x}(s') = 0.67$  and the payoff difference for candidate 1 is  $\pi_1(s') - \pi_1(s^*) = -|0.28 - 0.67| + |0.28 - 0.5| - \frac{r}{4} + \frac{c}{2} = -0.39 + 0.22 - 0.25 + 0.005 = -0.415$ . Therefore, 1 does not deviate. Similar arguments can be made for the other candidates deviating to non-participation.

(iii) Formation of a new party. Consider the set of candidates  $T = \{2, 3, 4, 5\}$ . Suppose  $T$  deviates to a strategy  $s'$  such that  $s'_i = T$  for all  $i \in T$ . Let the parties be  $P_1(s') = \{1\}$  and  $P_2(s') = \{2, 3, 4, 5\}$ . Party  $P_2(s')$  wins the election and  $\hat{x}(s') = x(P_2(s')) = 0.62$ . The difference in payoffs for candidate 2 is  $\pi_2(s') - \pi_2(s^*) = -|0.33 - 0.62| + \frac{r-c}{4} + |0.33 - 0.5| - \frac{r}{4} + \frac{c}{2} = -0.29 + 0.2475 + 0.17 - 0.25 + 0.005 = -0.1175$ . Similar arguments can be made to show that  $T = \{1, 2, 3, 4\}$  or  $T = \{1, 4, 5\}$  will not deviate to form a new party.

Finally, suppose  $T = \{1, 2, 3, 4, 5\}$  deviates to  $s'$ :  $s'_i = T$  for all  $i \in T$ . Then, there is only one party  $P_1(s') = \{1, 2, 3, 4, 5\}$ . Hence,  $\hat{x}(s') = 0.62$ . The payoffs difference for candidate 1 is  $\pi_1(s') - \pi_1(s^*) = -|0.28 - 0.62| + \frac{r-c}{5} + |0.28 - 0.5| - \frac{r}{4} + \frac{c}{2} = -0.34 + 0.198 + 0.22 - 0.25 + 0.05 = -0.167$ . Therefore, candidate 1 does not benefit from the deviation. This completes the argument.

Unfortunately, parties are homogeneous in Example 8 so the question of heterogeneity remains open.

### 3.4.4 THREE-PARTY EQUILIBRIUM

We have already shown the existence three-party equilibrium. These equilibria are even harder to characterize. Of course, equidistance from the median is not applicable and we are not able to settle the issue of homogeneity. However, Proposition 6 shows that there cannot be an equilibrium where three candidates participate independently.

**PROPOSITION 6 (Three-party equilibrium)** Suppose  $s^*$  is an equilibrium such that  $\mathcal{P}(s^*) = \{P_1(s^*), P_2(s^*), P_3(s^*)\}$  with  $x(P_1(s^*)) < x(P_2(s^*)) < x(P_3(s^*))$ . Then

$$(i) \max\{|P_1(s^*)|, |P_2(s^*)|, |P_3(s^*)|\} > 1.$$

$$(ii) r \geq c.$$

*Proof:* Let  $s^*$  be the strategy that satisfies the conditions in the statement of the Proposition. Let  $\hat{x}(s^*)$  be the policy outcome. Then either  $x(P_1(s^*)) < x(P_2(s^*)) \leq \hat{x}(s^*)$  or  $x(P_2(s^*)) > x(P_3(s^*)) \geq \hat{x}(s^*)$ . Assume w.l.o.g. that the former is true.

(i) Suppose contrariwise that  $|P_k(s^*)| = 1$  for all  $k \in \{1, 2, 3\}$ . Let  $i_1$  and  $i_2$  be such that  $x(P_1(s^*)) = x_{i_1}$  and  $x(P_2(s^*)) = x_{i_2}$ . Consider the deviation  $s'_i = \{i_1, i_2\}$  for  $i \in \{i_1, i_2\}$ . The new party  $P_2(s') = \{i_1, i_2\}$  is such that  $x(P_2(s')) = x_{i_2}$ . The votes of  $P_1(s^*)$  switch to  $P_2(s')$ . Therefore,  $P_2(s^*) \in \mathcal{W}(s^*)$ . Hence  $\hat{x}(s') = x(P_2(s'))$ . Both the deviating candidates prefer  $\hat{x}(s')$  to  $\hat{x}(s^*)$ . Assume w.l.o.g.  $P_2(s^*) \in \mathcal{W}(s^*)$ . Then, for  $i \in \{i_1, i_2\}$  we have  $\Delta G_i(s', s^*) = \frac{r-c}{2} - \frac{r}{2} + c = \frac{c}{2} > 0$ . Therefore  $\pi_i(s') - \pi_i(s^*) > 0$  for  $i \in \{i_1, i_2\}$ .

(ii) Suppose contrariwise that  $r < c$ . Consider the set of candidates  $T = \{i \in P_1(s^*) \text{ s.t. } x_i \leq x(P_2(s^*))\}$ . All the members of party  $P_1(s^*)$  deviate to non-participation. This makes party  $P_2(s^*)$  win the election i.e.  $\hat{x}(s') = x(P_2(s^*))$ . Therefore, the each candidate in  $T$  is better-off in terms of policy outcome. Moreover  $\Delta G_i(s', s^*) = \frac{r}{W|P_1(s^*)|} + \frac{c}{|P_1(s^*)|} = \frac{Wc-r}{|P_1(s^*)|} > 0$  i.e.  $s^*$  is not an equilibrium. Therefore  $r \geq c$ .

■

### 3.5 DISCUSSION

The results in the populist parties, internally democratic parties and the Osborne and Slivinsky (1996) model are summarized in Table 3.1.

	Populist	Internally Democratic	Osborne and Slivinsky (1996)
<b>One-party</b>			
Existence	Yes	Yes	Yes
$r > c$	$ P_k  = 1$	$ P_k  = 1$	NA
$r < c$	$ P_k  \leq 2$	$ P_k  \leq 4$	$r \leq 2c$
$r = c$	$ P_k  \geq 1$	$ P_k  \geq 1$	NA
<b>Two-party</b>			
Existence	Yes	Yes	Yes
$\max_k  P_k  > 1$	Yes	Yes	NA
$r \geq c$	Yes $r \geq 2c$	Yes. If $\max_k  P_k  > 2$ then $r \geq 2c$	Yes. $r \geq 2(c - \epsilon)$ s.t $\epsilon > 0$
$r < c$	No	No	Yes. $r \geq 2(c - \epsilon)$ s.t $\epsilon > 0$
Homogeneity	Yes	?	NA
<b>Three-party</b>			
Existence	No	Yes	Yes
$r \geq c$	NA	Yes	Yes. $r \geq 3c$
$r < c$	NA	No	No
$\max_k  P_k  > 1$	NA	Yes	NA

Table 3.1: Comparison of the results

### 3.6 CONCLUSION

In this Chapter we have shown that a network formation model can be fruitfully employed to gain insights into the process of political party formation. We feel that it would be interesting to consider alternative assumptions concerning party policy positions and to model electoral competition where different parties have different rules for forming policy positions.





## Chapter 4

# A MODEL OF ELECTORAL COMPETITION BETWEEN NATIONAL AND REGIONAL PARTIES

## 4.1 INTRODUCTION

In this Chapter we model the electoral competition between national and regional parties. National parties have the advantage of garnering votes from constituencies across the regions. Regional parties, on the other hand, can contest *only* from one region. The characteristic feature of regional parties is that voters do not consider regional parties of other regions as viable options.

Voters have favorite policy positions on a one-dimensional policy space. The policies are national issues—for instance, the rate of taxation, share of GDP to be spent on education or health etc. We assume that parties have to choose the policy position of a voter from any region.<sup>1</sup>

Parties are *constituency-motivated* i.e they care only about winning the maximum possible number of constituencies.<sup>2</sup> Moreover, parties maximize given the equilibrium strategy of voters. Our objective is to study the equilibrium policies of the parties.

We provide a brief empirical background to our results.

### 4.1.1 EMPIRICAL BACKGROUND

Countries like India, the United Kingdom and Canada have witnessed strong regional parties. Some of them have affected national politics in a significant way. We note the following facts.

1. In India, multiple regional parties exist like the Trinamool Congress in West Bengal, AIADMK and DMK in Tamil Nadu, BJD in Orissa, etc. These parties play an important role in forming coalitions at the national level (Heath et al. (2005)). India also has several national parties like the Congress Party and the BJP.
2. Canada has regional parties like the Bloc Québécois which fields candidates from Québec, while the Reform Party only runs from regions other than Québec (Massicotte (2005)). It also has national parties such as the Conservative Party of Canada and the New Democratic Party.
3. The United Kingdom has regional parties in Northern Ireland, Scotland, and Wales which compete with national parties. Even the national parties are mostly effective in particular regions like the Conservative Party in the ‘South’ and rural areas and the Labour party in the ‘North’ and urban constituencies (Gallagher and Mitchell, 2005).

---

<sup>1</sup>This assumption is only made to avoid delicate issues of equilibrium existence. If additional conditions are imposed on the distribution of voters to guarantee existence, the same results would hold.

<sup>2</sup>There are numerous works which model electoral competition with *policy-motivated candidates* (See Wittman (1983), Duggan and Fey (2005), Peress (2010) and Casamatta and De Donder (2005)).

## 4.1.2 RESULTS

There are two regions. Both regions are divided into constituencies. Each voter belongs uniquely to a constituency which uniquely belongs to a region. The distribution of voter policy positions is given and fixed throughout the analysis.

Once the parties have chosen policy positions, voting takes place. A voter in a region can only vote either for the national party or the regional party pertaining to her region. A party wins a constituency if at least a majority of voters vote for it. Outcomes are determined on the basis of constituencies or seats won by the parties.

A key element of our analysis is the *Political Outcome Function* (P.O.F.). This function maps the shares of constituencies into a probability distribution over party policy positions. We use this general formulation in order to capture a wide variety of circumstances. For example in India and the U.K, the party that wins the largest number of constituencies in a plurality vote forms the government and implements its policy position. On the other hand, a coalition government may form where parties share office and one of their policy positions implemented with some probability.

We show that these P.O.F.s do not induce sincere behaviour from voters. Under these circumstances, characterizing equilibrium strategic behaviour depends on the exact specification of the P.O.F.s. We avoid these difficulties and directly assume sincere voting behaviour. The consequence of this assumption is that party equilibrium is independent of the P.O.F. provided that the probability of a party's policy position being implemented is increasing in the number of constituencies.

Fix the position of the national party. Since a regional party can only get votes from its region, it wants to locate "as close as possible" to the national party's policy as the same side of the region-wide median by the standard Hotelling argument. In view of this behaviour of the regional parties, the national party wants to locate in the interval between the policy positions of the region-wide medians.

The precise location of the national party depends on the structure of *isolation sets*. These sets are constructed from the distribution of voter policy positions and have the following property: by choosing the policy position of a voter in this set, the national party can "isolate" or "separate" constituencies from their respective region-wide medians. If the voter distribution is *heterogeneous*, there are multiple isolation sets. In *homogeneous* voter distributions, the smallest interval containing all policy positions of constituency medians for the two regions are disjoint. As a result, the isolation sets are empty. In the heterogeneous case the national party locates in a maximal isolation set. In the homogeneous case, the national party's policy is the policy position of the region-wide median of the region with the greater number of constituencies.

The main insight of the paper is the following. For a given P.O.F. and a fixed number of constituencies, the national party's performance improves as the degree of voter heterogeneity increases. In the limit case, where the distribution is homogeneous, the national party can at

best do as well as the regional party of the region with the greater number of constituencies. This result is broadly consistent with intuition and empirical evidence (Bailey and Brady (1998), Gerber and Lewis (2004)).

There are other papers which consider electoral competition with multiple districts-Austen-Smith (1984), Austen-Smith (1987), Eyster and Kittsteiner (2007), and Callander (2005). In most of these papers, parties are implicitly assumed to be national. Our model, therefore, offers some insights on the influence of regional parties on policy outcomes.

The paper is organised as follows. We describe the model and give definitions in Section 4.2. In Section 4.3 we present the results for sincere voting equilibrium. In Section 4.4 we present the result and proof in two parts. We conclude with a discussion in Section 4.4.

## 4.2 THE MODEL

The set of voters is  $\mathbf{N} = \{1, 2, \dots, n\}$ . The policy space is an interval  $X \subset \mathbb{R}$ . Voter  $i$  has ideal policy position (henceforth policy position)  $x_i \in X$ . We assume for simplicity that the policy positions are distinct for all voters. The set of regions is  $R = \{R_1, R_2\}$ . Each region has a finite number of constituencies. For convenience we assume an odd number of constituencies in both regions. The set of parties is denoted by  $\mathcal{P}$ . Let the set of voters' policy positions be denoted by  $\bar{X}$ . A party  $l \in \mathcal{P}$  can choose a policy position  $x(l) \in \bar{X}$ .<sup>3</sup>

The set of constituencies of Region  $j$  is denoted by  $R_j$ . The set of voters in a constituency  $k \in R_j$  is denoted by  $R_j^k$ . For any two distinct regions  $j$  and  $j'$ ,  $R_j \cap R_{j'} = \emptyset$  and for any two distinct constituencies  $k$  and  $k'$ ,  $R_j^k \cap R_j^{k'} = \emptyset$  for any  $j, j' \in R$ . The constituencies form a region i.e.  $\bigcup_{k \in j} R_j^k = R_j$  for all  $j \in R$  and regions form the country as a whole i.e.  $\bigcup_{j \in R} R_j = R$ . Each voter belongs to a constituency i.e.  $\bigcup_{j \in R} (\bigcup_{k \in j} R_j^k) = \mathbf{N}$ .

There are two regional parties:  $S_j$ , the regional party of Region  $R_j$ ,  $j \in \{1, 2\}$  and one national party  $N$ . The set of parties a voter can vote for is denoted by  $I_i$  i.e.  $I_i = \{S_j, N\}$  if  $i \in R_j$  for  $j \in \{1, 2\}$ . Voters do not consider voting for the regional party from the other region. Therefore, regional parties can only win votes from their respective regions.

Voters have *single-peaked Euclidean* preferences represented by utility functions  $u_i : X \rightarrow \mathbb{R}$ . If a policy  $x$  is implemented then  $u_i(x) = -|x - x_i|$  for all  $i \in \mathbf{N}$ . Therefore, a voter can get a maximum utility of zero (if her own policy position is implemented). The strategy of voter  $i$ ,  $v_i \in I_i$  is a vote for a party  $l \in \mathcal{P}$ .

We use the plurality rule at the constituency level. The party with the most votes in a constituency wins that constituency. The total number of constituencies won by party  $l$  is  $V_l \in \{0, 1, \dots, |R_1| + |R_2|\}$ . Let  $V$  denote the tuple of seat shares i.e.  $V = \{V_l\}_{l \in \mathcal{P}}$ . Let  $\bar{V} = \{0, 1, \dots, |R_1| + |R_2|\}$ . A political outcome function is a probability distribution  $\{P_l(V)\}_{l \in \mathcal{P}}$  over the set of policy positions chosen by the parties.

---

<sup>3</sup>The results do not depend on this assumption. Parties can be allowed to choose any policy in  $X$ . However, certain additional assumptions need to be made on the distribution of voter policy positions to guarantee existence of equilibrium.

**DEFINITION 9 (Political outcome function(P.O.F.))** The *P.O.F.* is a function  $P_l : \bar{V}^3 \rightarrow [0, 1]$  such that for all  $l \in \mathcal{P}$  and  $V \in \bar{V}^3$

(i)  $P_l(V) \geq 0$ .

(ii)  $\sum_{l \in \mathcal{P}} P_l(V) = 1$ .

We consider different outcome functions in this Chapter. These will be described later in section 4.2.1. Voters choose  $v_i$  to maximize their expected payoff

$$\pi_i(v_i, v_{-i}) = - \sum_{l \in \mathcal{P}} P_l(V(v_i, v_{-i})) |x(l) - x_i| \quad \text{for all } i \in \mathbf{N}.$$

Suppose voter  $i$  is indifferent between voting for either of the two parties in  $I_i$ . We assume that  $i$  votes for either of the two parties with equal probability.

Voters are *sincere* if they vote for the party whose policy position is closest to their policy position according to the Euclidean distance norm,  $d_i(x(l)) = |x_i - x(l)|$ . Voters are *strategic* if they are not sincere.

**DEFINITION 10 (Voter equilibrium)** A voter strategy  $v^*(x)$  is a *voter equilibrium* if for all  $i \in N$  for all  $v'_i \in I_i$

$$\pi_i(v_i^*, v_{-i}^*) \geq \pi_i(v'_i, v_{-i}^*).$$

We assume that parties are aware of the equilibrium strategies of the voters for a given tuple  $x \in \bar{X}^3$ . For any set of policy positions  $x$  the parties calculate their seat shares from the function  $v^*(x)$ . Therefore, parties assume that players will play Nash equilibrium strategies for every policy tuple. Let  $\mathbf{E}(\cdot)$  denote the expectation function. The payoff of party  $l$  is given by

$$\Pi_l(x) = \mathbf{E}(V_l(v^*(x))) \quad \text{for all } l \in \mathcal{P}.$$

Therefore, parties choose policies to maximize the expected number of seats.<sup>4</sup>

**DEFINITION 11 (Party equilibrium)** A party strategy  $x^*$  is a *party equilibrium* if

$$\mathbf{E}(V_l(v^*(x^*))) \geq \mathbf{E}(V_l(v^*(x(l)', x(-l)^*))) \quad \text{for all } x(l)' \in \bar{X} \text{ for all } l \in \mathcal{P}.$$

A party equilibrium is a tuple  $x^*$  from which no party can deviate and get a higher expected payoff.

A *political equilibrium* is a strategy profile  $(v^*(x^*), x^*)$  such that  $v^*(x^*)$  is a voter equilibrium and  $x^*$  is a party equilibrium.

---

<sup>4</sup>This type of formulation is standard in the literature- See [Besley and Coate \(1997\)](#) for example. [Laslier \(2005\)](#) considers a model with different party objectives.

## 4.2.1 OUTCOME FUNCTIONS

We give two natural examples of outcome functions in this Chapter - maximal and coalitional. We define these below.

Under the maximal outcome function the set of winning parties is the party with the most number of constituencies. Let the set of winning parties be  $W^{\max}(V) = \{l \in \mathcal{P} \mid V_l = \max_{l' \in \mathcal{P}} V_{l'}\}$  and  $|W^{\max}(V)| = W$ .

The P.O.F. is *maximal* if

$$P_l^{\max}(V) = \begin{cases} \frac{1}{W} & \text{if } l \in W^{\max}(V) \\ 0 & \text{otherwise} \end{cases}$$

for all  $V \in \bar{V}$  for all  $l \in \mathcal{P}$ .

A *minimal winning coalition* or an m.w.c. is a collection of parties  $C$  such that (i)  $\sum_{l \in C} V_l \geq \frac{|R_1| + |R_2|}{2}$  and (ii)  $\sum_{l \in C, l' \neq l} V_{l'} < \frac{|R_1| + |R_2|}{2}$  for all  $l' \in C$ .<sup>5</sup> It is a coalition of parties that wins at least half the total number of constituencies. In addition, if a coalition partner drops out, the remaining parties win strictly less than half the total number of constituencies.

The coalition P.O.F. assumes (i) that each m.w.c. can form with equal probabilities and (ii) the policies of a member of a m.w.c. is implemented with the probability equal to the share of its constituencies in the total number of constituencies won by the m.w.c.<sup>6</sup>

Let  $\mathbb{C}$  be the set of all possible m.w.c.s. For every  $C \in \mathbb{C}$ , let  $V_C = \sum_{l \in C} V_l$ . A P.O.F. is *coalitional* if

$$P_l^c(V) = \frac{1}{|\mathbb{C}|} \sum_{\{C \in \mathbb{C} \mid l \in C\}} \frac{V_l}{V_C}$$

for all  $V \in \bar{V}$  for all  $l \in \mathcal{P}$ .

It is easy to show that  $\sum_{l \in \mathcal{P}} P_l^c(V) = 1$ .

We abuse notation slightly by using  $\Pi_1, \Pi_2, \Pi_N$  to denote the payoffs and  $V_1, V_2, V_N$  to denote the seat shares of parties  $S_1, S_2$  and  $N$  respectively.

Suppose  $V_1 = 3, V_2 = 3, V_N = 4$ . There are three m.w.c.s:  $\{S_1, S_2\}, \{S_2, N\}$  and  $\{S_1, N\}$ . Then

$$P_1^C(V) = P_2^C(V) = \frac{1}{3} \left( \frac{3}{6} + \frac{3}{7} \right) = \frac{13}{42} \text{ and } P_N^C(V) = \frac{1}{3} \left( \frac{4}{7} + \frac{4}{7} \right) = \frac{8}{21}.$$

**The game:** The distribution of voters' policy positions is fixed through the analysis. Parties choose policies from the set  $\bar{X}$ . Voters vote to maximize expected utility given the policy positions of parties and actions of other voters. The outcome is determined by the outcome function. Parties win a constituency if a strict majority of voters vote for it. The outcome function determines which parties win. The winning party's policy position is implemented.

<sup>5</sup>The concept was introduced formally by Riker (1962).

<sup>6</sup>This is an implication of Gamson's Law. See (Browne and Franklin (1973) and Warwick and Druckman (2001)).

## 4.2.2 SINCERE VS STRATEGIC VOTING

We show that voters need not be sincere under either of the two P.O.F.s described earlier. Let  $x^{\max}(v)$  and  $x^c(v)$  represent the policy outcomes under maximal and coalition outcome functions respectively.

**EXAMPLE 9** Let  $x(S_1) = 0.1$ ,  $x(N) = 0.3$  and  $x(S_2) = 0.9$ . Suppose voter  $i \in R_1$  has policy position  $x_i = 0.25$  i.e.  $x(N) \succ_i x(S_1) \succ_i x(S_2)$ . Let  $v_{-i}$  be such that  $V_1 = V_2 = V_N$  when  $i$  votes sincerely.

Then  $P_1^{\max}(V) = P_2^{\max}(V) = P_N^{\max}(V)$  and the expected policy outcome  $x^{\max}(v) = \frac{0.1+0.3+0.9}{3} = 0.433$ . If  $i$  votes strategically for  $S_1$ , then  $P_1^{\max}(V') = 1 > P_2^{\max}(V') = P_N^{\max}(V') = 0$  i.e.  $x^{\max}(v') = x(S_1) = 0.1$ . Voter  $i$  benefits by voting strategically since  $d_i(x^{\max}(v)) > d_i(x^{\max}(v'))$ .

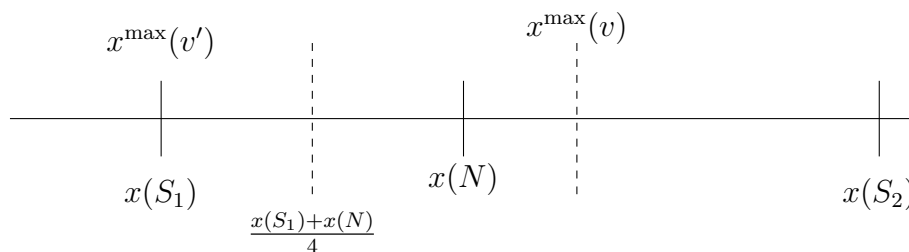


Figure 4.1: Strategic voting

Now consider the coalition outcome function. Suppose  $V_l = 3$  for  $l \in \mathcal{P}$  i.e.  $P_l^c(V) = 2 \times \frac{1}{2} \times \frac{V_l}{V_l + V_{-l}} = 2 \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{3}$  for all  $l \in \mathcal{P}$ . As before  $x^c(v) = \frac{0.1+0.3+0.9}{3} = 0.433$ . This is illustrated in Figure 4.1

If  $i$  votes strategically for  $S_1$ ,  $V_1' = 4, V_2' = 3, V_N' = 2$  i.e.  $P_1^c(V') = \frac{1}{3} \left( \frac{4}{7} + \frac{4}{6} \right) = \frac{1}{3}(0.571 + 0.667) = 0.412$ ,  $P_2^c(V') = \frac{1}{3} \left( \frac{3}{7} + \frac{3}{5} \right) = \frac{1}{3}(0.429 + 0.6) = 0.343$  and  $P_N^c(V') = \frac{1}{3} \left( \frac{2}{6} + \frac{2}{5} \right) = \frac{1}{3}(0.33 + 0.4) = 0.243$ . Therefore,  $x^c(v') = 0.412 \times 0.1 + 0.343 \times 0.9 + 0.243 \times 0.3 = 0.0412 + 0.3087 + 0.0729 = 0.4228$ . Once again, voter  $i$  benefits by voting strategically since  $d_i(x^c(v)) > d_i(x^c(v'))$ .

The example suggests that voters will typically vote strategically. Characterizing equilibrium behaviour in these circumstances is known to be difficult and may involve the use of mixed strategies. We choose to avoid these difficulties by assuming for the rest of the Chapter that voters vote sincerely. This assumption has a significant benefit. The precise specification of the P.O.F. is no longer important for our results as long as probabilities are increasing in vote shares.

## 4.3 SINCERE VOTING EQUILIBRIUM

A sincere voting equilibrium is a political equilibrium where all voters vote sincerely.

We denote the voter with the median policy position in constituency  $R_j^k$  by  $m_j^k$  and her ideal policy position as  $x(m_j^k)$ . The median policy among the constituency medians  $\{x(m_j^k)\}_{k \in R_j}$  is denoted by  $x(m_j)$  for both Regions  $R_1$  and  $R_2$ .<sup>7</sup> We call  $m_j$  the region-wide median of  $R_j$  for  $j \in \{1, 2\}$ . We provide some necessary conditions for equilibrium.

**PROPOSITION 7** Suppose  $x^*$  is a sincere voting equilibrium. Then

(i) For  $j \in \{1, 2\}$  we have

$$(a) [x(N)^* < x(m_j)] \Rightarrow [x(N)^* < x(S_j)^*].$$

$$(b) [x(N)^* > x(m_j)] \Rightarrow [x(N)^* > x(S_j)^*].$$

(ii)  $x(N)^* \in [x(m_1), x(m_2)]$ .

(iii)  $V_j \geq \frac{|R_j|}{2}$  for  $j \in \{1, 2\}$ .

*Proof:* Suppose  $x^*$  is a sincere voting equilibrium.

(i) Assume w.l.o.g. that  $x(m_1) < x(m_2)$ . We prove the claim by contradiction. Suppose  $x(N)^* < x(m_1)$  and  $x(S_1)^* < x(N)^*$ . A majority of voters in a majority of constituencies in Region 1 prefer  $x(N)^*$  to  $x(S_1)^*$ . Therefore  $\Pi_1(V) < \frac{|R_1|}{2}$ . Consider the following deviation by  $S_1$ :  $x(S_1)' = x(N)^*$ . Then  $S_1$  wins at least  $\frac{|R_1|}{2}$  constituencies. Therefore  $\Pi_1(V') - \Pi_1(V) > 0$ . Similarly we can show that  $[x(N)^* > x(m_1)] \Rightarrow [x(S_1)^* < x(N)^*]$ .

(ii) We prove the claim by contradiction. Suppose  $x(N)^* < x(m_1)$ . Using the arguments in (i),  $x(S_1)^* > x(N)^*$  and  $x(S_2)^* > x(N)^*$ . Moreover,  $x(S_1)^* = x(N)^* + \epsilon$  for some small  $\epsilon > 0$ . To see this, suppose  $\epsilon$  is large. Then voter  $m_1$  will prefer  $x(N)^*$  to  $x(S_1)$  and  $S_1$  will win less than  $\frac{|R_1|}{2}$  constituencies. Therefore, for small  $\epsilon$ ,  $S_1$  wins at least a majority of constituencies in region  $R_1$ . Hence,  $V_1 > \frac{|R_1|}{2}$ . This implies that  $N$  wins strictly less than  $\frac{|R_1|}{2}$  constituencies from  $R_1$ .

Consider the following deviation by  $N$ :  $x(N)' = \min\{x(S_1)^*, x(S_2)^*\}$ . Assume w.l.o.g.  $x(S_1)^* = \min\{x(S_1)^*, x(S_2)^*\}$ . The national party wins  $\frac{|R_1|}{2}$  constituencies from  $R_1$ . All median voters  $m_2^k \in R_2$  with  $x(m_2^k) \leq x(N)^*$  vote for  $N$ . Therefore  $\Pi_N(V') - \Pi_N(V) > 0$ . Hence,  $x(N)^* \geq x(m_1)$ . Using similar arguments we can show that  $x(N)^* \leq x(m_2)$ . Therefore,  $x(N)^* \in [x(m_1), x(m_2)]$ .

(iii) Suppose contrariwise that  $V_j < \frac{|R_2|}{2}$ . Consider the deviation  $x(S_j)' = x(N)^*$ . All the voters in region  $R_j$  are indifferent between voting for  $S_j$  and  $N$ . Indifferent voters vote with equal probability for both parties. Therefore, the payoff difference is  $\Pi_1(V') - \Pi_1(V) > 0$ . Hence  $V_j \geq \frac{|R_2|}{2}$ .

---

<sup>7</sup>Recall that we assume an odd number of voters in each constituency and an odd number of constituencies in each region. Therefore, the median voters are unique.



■

In any equilibrium where voters are sincere, the national party locates in the interval of the two region-wide medians. The regional parties on the other hand stay on the same side of the national party's policy as their respective region-wide medians. Moreover, the regional parties win at least half the number of constituencies in expectation.

We introduce the notion of *isolation sets* to describe the equilibrium. W.l.o.g. assume  $x_{M_1} \leq x_{M_2}$ . Let  $\{s_1, \dots, s_{|R_1|}\}$  and  $\{t_1, \dots, t_{|R_2|}\}$  be two sets of indices. The policy positions of medians of the constituencies of Regions 1 and 2 can be arranged in the following order:

$$x(m_1^{s_1}) < x(m_1^{s_2}) < \dots < x(m_1^{s_{|R_1|}}).$$

$$x(m_2^{t_1}) < x(m_2^{t_2}) < \dots < x(m_2^{t_{|R_2|}}).$$

A set  $A(x_1, \dots, x_n)$  is an *isolation set* for  $(x_1, \dots, x_n)$  if there exist  $s_q \in \{s_1, \dots, s_{|R_1|}\}$  and  $t_p \in \{t_1, \dots, t_{|R_2|}\}$  such that (i)  $A(x_1, \dots, x_n) = [x(m_2^{t_p}), x(m_1^{s_q})]$  (ii)  $x(m_1^{s_q}) \geq x(m_1)$  and (iii)  $x(m_2^{t_p}) \leq x(m_2)$ .

There may be several or no isolation sets. If the latter is true we abuse notation slightly to say that the isolation set is empty and write  $A(x_1, \dots, x_n) = \emptyset$ . We clarify these notions with some examples.

**EXAMPLE 10** Let  $R_1 = \{R_1^1, R_1^2, R_1^3\}$  and  $R_2 = \{R_2^1, R_2^2, R_2^3\}$ . The location of the constituency medians are shown in the figure below.

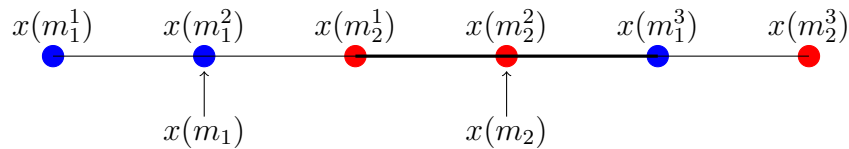


Figure 4.2: There are two isolation sets

In Figure 4.2, there are two isolation sets: (i)  $[x(m_2^1), x(m_1^3)]$  and (ii)  $[x(m_2^2), x(m_1^3)]$ .

**EXAMPLE 11** Let  $R_1 = \{R_1^1, R_1^2, R_1^3\}$  and  $R_2 = \{R_2^1, R_2^2, R_2^3\}$ . The location of the constituency medians are shown in Figure 4.3.

In Figure 4.3, there are three isolation sets: (i)  $[x(m_2^1), x(m_1^2)]$ , (ii)  $[x(m_2^1), x(m_1^3)]$  and (iii)  $[x(m_2^2), x(m_1^3)]$ .

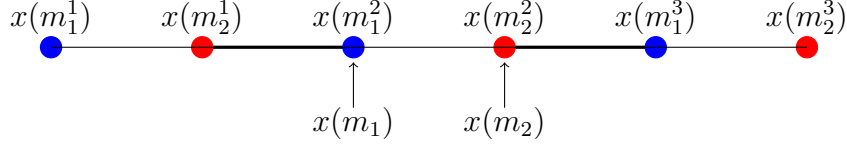


Figure 4.3: There are three isolation sets

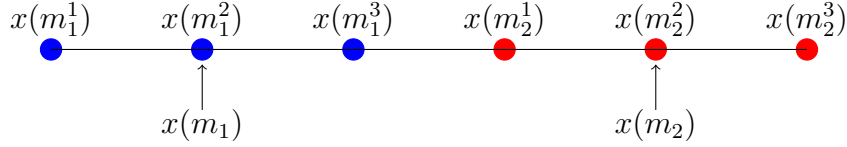


Figure 4.4: There are no isolation sets

**EXAMPLE 12** Let  $R_1 = \{R_1^1, R_1^2, R_1^3\}$  and  $R_2 = \{R_2^1, R_2^2, R_2^3\}$ . The location of the constituency medians are shown in the figure below.

In Figure 4.4, there are no isolation sets.

A *maximal isolation set*  $\bar{A}(x_1, \dots, x_n)$  is an isolation set  $[x(m_2^{t_p}), x(m_1^{s_q})]$  with the property that  $|R_1| - s_q + 1 + t_p \geq |R_1| - s'_q + 1 + t'_p$  for all  $s'_q \in \{1, \dots, |R_1|\}$  and  $t'_p \in \{1, \dots, |R_2|\}$ . We illustrate the notion of maximal isolation set with an example.

**EXAMPLE 13** Let  $R_1 = \{R_1^1, \dots, R_1^5\}$  and  $R_2 = \{R_2^1, \dots, R_2^7\}$ . The location of the constituency medians are shown in the figure below.

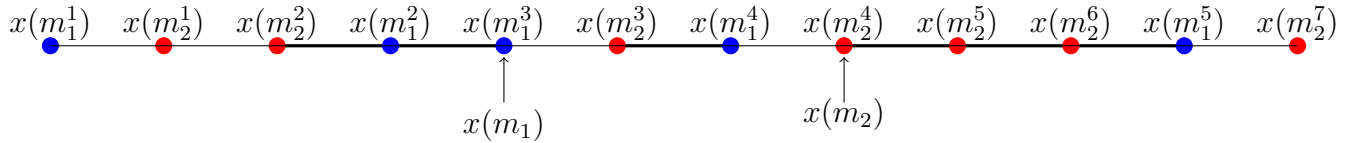


Figure 4.5: There are three maximal isolation sets

There are three maximal isolation sets: (i)  $[x(m_2^2), x(m_1^3)]$  (ii)  $[x(m_2^3), x(m_1^4)]$  and (iii)  $[x(m_2^4), x(m_1^5)]$ .

A voter policy position distribution  $(x_1, \dots, x_n)$  is *homogeneous* if  $A(x_1, \dots, x_n) = \emptyset$ . An equivalent definition of homogeneity is the following: the smallest interval that contains the policy positions of medians of Region 1 is disjoint from the smallest interval that contains the

policy positions of medians for Region 2. The voter distribution is homogeneous in Example 12.

If the voter distribution is not homogeneous, it is *heterogeneous*. The distributions in Examples 10, 11 and 13 are heterogeneous.

### 4.3.1 HOMOGENEOUS REGIONS

In this subsection we describe the equilibrium for homogeneous voter distributions.

**PROPOSITION 8** Suppose the following conditions hold:

- (i)  $x(m_1^k) < \frac{3x(m_1) + x(m_2)}{4}$  for all  $k \in R_1$ .
- (ii)  $x(m_2^k) > \frac{x(m_1) + 3x(m_2)}{4}$  for all  $k \in R_2$ .

Then the following strategy  $x^*$  is a sincere voting equilibrium:  $x(N)^* = x(m_j)$  if  $|R_j| \geq |R_{j'}|$  and  $x(S_j^*) = x(m_j)$  for all  $j \in \{1, 2\}$ .

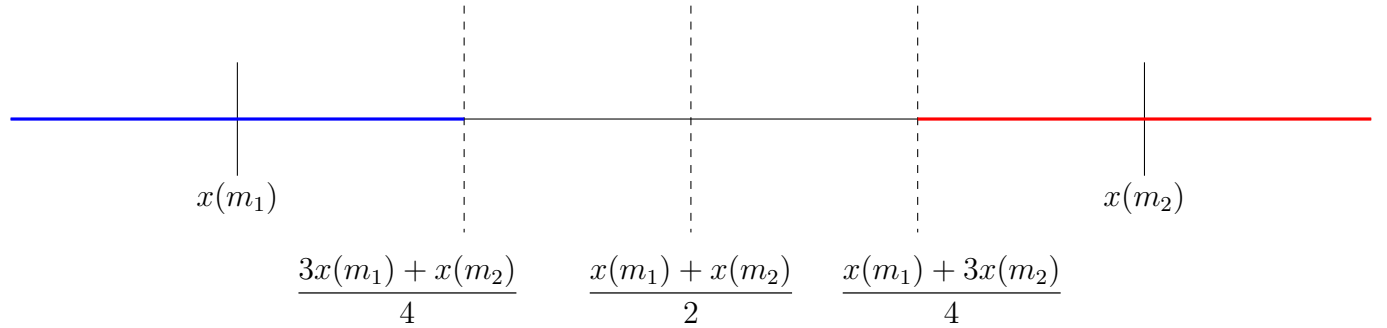


Figure 4.6: Illustration for Proposition 8

*Proof:* Suppose  $x^*$  is an equilibrium which satisfies the conditions in the statement of the Proposition. Then  $\bar{A}(x_1, \dots, x_n) = \emptyset$ . Assume w.l.o.g. that  $|R_1| \geq |R_2|$ . We show that the following strategy is an equilibrium:  $x(N)^* = x(S_1)^* = x(m_1)$  and  $x(S_2)^* = x(m_2)$ .

Suppose  $S_1$  deviates to  $x(S_1)'$ . If  $x(S_1)' < x(m_1)$ , then a majority of voters in a majority of constituencies in  $R_1$  will prefer  $x(N)^*$  to  $x(S_1)'$ . After the deviation  $S_2$  wins at most  $\frac{|R_1|}{2}$  seats. The payoff difference for  $S_1$  is  $\Pi_1(x(S_1)') - \Pi_1(x^*) = 0$ . Similar arguments can be made to show that a deviation to  $x(S_1)' \geq x(S_1)^* = x(m_1)$  is not profitable. Therefore,  $S_1$  does not deviate.

The national party  $N$  wins  $\frac{|R_1|}{2}$  number of constituencies from Region 1. Since  $x(S_1)^* = x(m_1)$  any deviation by  $N$  leads to its winning less than  $\frac{|R_1|}{2}$  of seats. Suppose  $N$  deviates to  $x(N)' > x(N)^*$ . By condition (ii),  $N$  cannot win any constituencies from  $R_2$  if  $x'_N < \frac{x(m_1)+x(m_2)}{2}$ . Suppose  $x(N)' > \frac{x(m_1)+x(m_2)}{2}$ . From condition (i)  $N$  cannot gain any constituencies from  $R_1$ . Since  $|R_1| \geq |R_2|$  the payoff difference is  $\Pi_N(x(N)') - \Pi_N(x^*) \leq 0$ . Therefore,  $N$  does not deviate.

By condition (ii) and the fact that  $x(S_2)^* = x(m_2)$ ,  $S_2$  wins all the constituencies in region  $R_2$ . Therefore,  $S_2$  cannot win more seats by deviating. Similar arguments can be made if  $|R_2| \geq |R_1|$ . ■

When the voter distribution is homogeneous, the national party locates at the policy position of the median voter of the constituency median of the larger region. The regional parties locate at their respective region-wide medians.

Conditions (i) and (ii) in Proposition 8 require all medians in Region 1 to be located significantly apart from the medians in Region 2. The example below shows that conditions (i) and (ii) are necessary.

**EXAMPLE 14** Let  $R_1 = \{R_1^1, \dots, R_1^5\}$  and  $R_2 = \{R_2^1, \dots, R_2^7\}$  with the following policy positions of the medians :

(i) Region 1:  $x(m_1^1) = 0, x(m_1^2) = 0.1, x(m_1^3) = x(m_1) = 0.2, x(m_1^4) = 0.35, x(m_1^5) = 0.42$ .

(ii) Region 2:  $x(m_2^1) = 0.45, x(m_2^2) = 0.48, x(m_2^3) = 0.7, x(m_2^4) = x(m_2) = 0.8, x(m_2^5) = 0.85, x(m_2^6) = 0.9$  and  $x(m_2^7) = 1$ . These are illustrated in Figure 4.7.

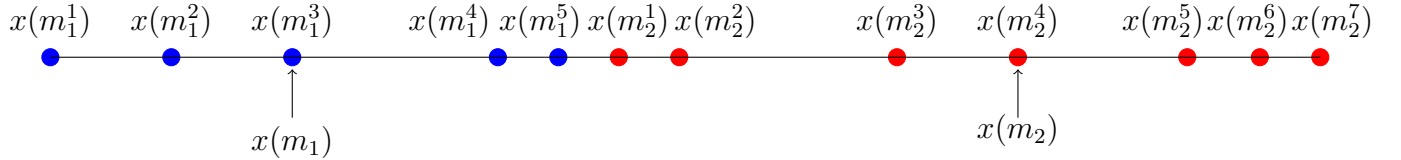


Figure 4.7: Illustration for Example 14

The distributional assumptions of Proposition 8 are not satisfied since

$$(i) \quad x(m_1^5) = 0.42 > \frac{3x(m_1) + x(m_2)}{4} = \frac{3 \times 0.2 + 0.8}{4} = 0.35.$$

$$(ii) \quad x(m_2^1) = 0.1 < \frac{x(m_1) + 3x(m_2)}{4} = \frac{0.2 + 3 \times 0.8}{4} = 0.65.$$

We show that the strategy profile  $x^*$  given in Proposition 8 is not an equilibrium. Let  $x(N)^* = x(S_2)^* = x(m_2)$  and  $x(S_1)^* = x(m_1)$ . The constituencies won by the parties are  $V_1 = 5, V_N = V_2 = 3.5$ .

We show that the national party can deviate beneficially. Consider a deviation by  $N$  to  $x(N)' = x(m_2^2) = 0.48$ . It gets votes from a majority of voters in constituencies  $R_1^4, R_1^5, R_2^1$  and  $R_2^2$ . This makes  $N$  strictly better-off. Therefore  $N$  deviates.

We show that no other equilibrium exists. Suppose  $x(N)^* = x(m_2^3)$ ,  $x(S_1)^* = x(m_1^5)$  and  $x(S_2)^* = x(m_2^3)$ . We have  $V_1 = 5, V_N = 3$  and  $V_2 = 4$ . Consider the following deviation by  $N$ :  $x(N)' = x(m_1^4)$ . Then  $V_N' = 4$  which makes  $N$  better-off. Similar arguments can be used to show that in  $x(N)^* \notin (x(m_1), x(m_2))$ . By Proposition 7 and the arguments above, no other equilibrium can exist.

Hence, additional conditions (i) and (ii) in the statement of Proposition 8 are essential in characterizing the equilibrium.

### 4.3.2 HETEROGENEOUS REGIONS

In this subsection we describe the equilibrium when the voter distribution is heterogeneous. Let  $\bar{A}_1, \dots, \bar{A}_K$  be the maximal isolation sets for  $(x_1, \dots, x_n)$ .

**PROPOSITION 9** We consider two cases.

- (i)  $\bar{A}_k(x_1, \dots, x_n) \cap \{x(m_1), x(m_2)\} \neq \emptyset$  for all  $k \in \{1, \dots, K\}$ . Note that this can only hold if there are at most two maximal isolation sets. Let these be denoted by  $A_1(x_1, \dots, x_n)$  and  $A_2(x_1, \dots, x_n)$ . Let  $\bar{A}_k = [x(m_2^{t_p}), x(m_1^{s_q})]$  for  $k \in \{1, 2\}$ . Suppose the following conditions hold:

- (a) If  $x(m_1) \in \bar{A}_k$  and  $x(m_2) \notin \bar{A}_k$ . Then

$$x(m_2^{t_p+1}) > \frac{x(m_1) + x(m_2)}{2} \text{ and } x(m_1^{s_{|R_1|}}) < \frac{3x(m_1) + x(m_2)}{4}.$$

- (b) If  $x(m_2) \in \bar{A}_k$  and  $x(m_1) \notin \bar{A}_k$ . Then

$$x(m_1^{s_q-1}) < \frac{x(m_1) + x(m_2)}{2} \text{ and } x(m_2^{t_{|R_1|}}) > \frac{x(m_1) + 3x(m_2)}{4}.$$

- (c) If  $x(m_2) \in \bar{A}_k$  and  $x(m_1) \in \bar{A}_{k'}$ . Then either (a) or (b) holds.

The following strategy profile is a sincere voting equilibrium:

$x(N)^* = x(m_j)$  where  $x(m_j) \in \bar{A}_k$  for some  $k \in \{1, 2\}$  and  $|R_j| \geq |R_{j'}|$  for all  $j' \in \{j \mid x(m_j) \in \bar{A}_{k'} \text{ for some } k'\}$ ,  $x(S_j)^* = x(m_j)$  for all  $j \in \{1, 2\}$ .

- (ii) Suppose exists  $k \in \{1, \dots, K\}$  such that  $\bar{A}_k(x_1, \dots, x_n) \neq \emptyset$  and  $\bar{A}_k(x_1, \dots, x_n) \cap \{x(m_1), x(m_2)\} = \emptyset$ . There are two subcases.

(A) Suppose the following conditions hold:

- (a)  $|R_1| \geq 2(s_q - 1)$   
(b)  $|R_1| - |R_2| \geq 2(s_q - t_p) - 1$ .

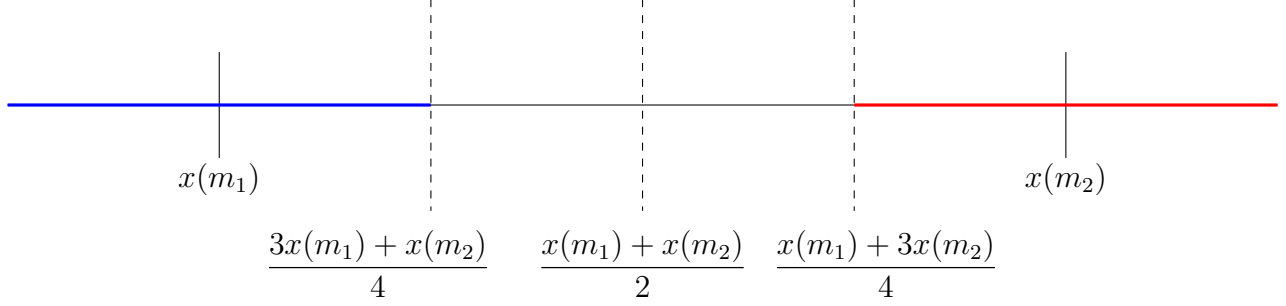


Figure 4.8: Illustration for Proposition 9

Then the following strategy is a sincere voting equilibrium:

$$x(N)^* \in \{x(m_1^{s_q}), x(m_2^{t_p})\}, x(S_1)^* = x(m_1^{s_q-1}) \text{ and } x(S_2)^* = x(m_1^{t_p+1}).$$

(B) Suppose the following conditions hold.

- (a)  $s_q - t_p \geq \frac{|R_j|}{2} + 1$
- (b)  $s_q \geq \frac{|R_1| + |R_2|}{2} + 1$ .
- (c) There does not exist  $k \in R_2^k$  such that  $x(m_2^k) \notin (x(m_1), \frac{x(m_1) + x(m_2)}{2})$ .

Then the following strategy is a sincere voting equilibrium:

$$x(N)^* = x(m_j) \text{ for some } j \in \{1, 2\} \text{ and } x(S_{j'})^* = x(m_{j'}) \text{ for } j' \in \{1, 2\}.$$

*Proof:* By Proposition 7, the national party will choose a policy position  $x(N)^* \in [x(m_1), x(m_2)]$ . The regional parties will stay on the side of the region-wide median from the national party's policy. The national party can win at most  $\frac{|R_j|}{2}$  from each region  $R_j, j \in \{1, 2\}$ . Therefore, by choosing a policy position in the maximal isolation set, the national party can isolate maximum number of constituencies from both regions. The national party can isolate at most  $\frac{|R_j|-1}{2}$  constituencies from each region. Moreover, if  $x(m_j) \in \bar{A}_k$ , then it can win at most  $\frac{|R_j|}{2}$  constituencies from  $R_j$  and  $\frac{|R_{j'}|-1}{2}$  constituencies from  $R_{j'}$  by isolating these constituencies. Therefore, in any equilibrium  $x(N)^* \in \bar{A}_k$  for some  $k \in \{1, \dots, K\}$ . By Proposition 7 and the arguments above we have  $x(S_1)^* \leq x(N)^*$  and  $x(S_2)^* \geq x(N)^*$ . We consider three cases separately.

- (i) Consider a maximal isolation set  $\bar{A}_k(x_1, \dots, x_n)$  such that  $x(m_j) \in \bar{A}_k(x_1, \dots, x_n)$  and  $|R_j| \geq |R_{j'}|$  for all  $j \in \{j \mid x(m_j) \in \bar{A}_{k'} \text{ for some } k'\}$ . Suppose  $x(m_1) \in \bar{A}_k$  for some  $k \in \{1, \dots, K\}$  and  $|R_1| \geq |R_2|$ . We show that  $x(S_1)^* = x(N)^* = x(m_1)$  and  $x(S_2)^* = x(m_2)$  is a sincere voting equilibrium.

Suppose  $S_1$  deviates to  $x(S_1)' \neq x(m_1)$ . Then  $S_1$  wins less than half the number of constituencies in  $R_1$  i.e.  $\Pi_1(x(S_1)') < \Pi_1(x^*) = \frac{|R_1|}{2}$ . Therefore,  $S_1$  cannot deviate beneficially.

By condition (a),  $x(m_2^{t_p+1}) > \frac{x(m_1)+x(m_2)}{2}$ . Therefore  $S_2$  is winning a majority of constituencies. Moreover, all constituencies that the national party is not able to isolate i.e.  $|R_2| - t_p$  are also obtained by  $S_2$ . Therefore, after a deviation to  $x(S_2)'$ ,  $S_2$  can get at most  $|R_2| - t_p$  constituencies. The payoff difference is  $\Pi_2(x(S_2)') - \Pi_2(x^*) \leq 0$ . Therefore,  $S_2$  cannot deviate beneficially.

Finally, we show that the national party cannot deviate profitably. By condition (a),  $x(m_1^{s_{|R_1|}}) < \frac{3(x(m_1)+x(m_2))}{4}$  and  $x(m_2^{t_p+1}) > \frac{x(m_1)+x(m_2)}{2}$ . Therefore, if  $N$  deviates to  $x(N)' > x(N)^*$ , it does not win more constituencies from  $R_2$  for all  $x(N)' \leq \frac{x(m_1)+x(m_2)}{2}$ . The payoff difference is  $\Pi_N(x(N)') - \Pi_N(x^*) = \frac{|R_1|}{2} + t_p - \frac{|R_1|}{2} - t_p \leq 0$ .

If  $N$  deviates to  $x(N)' \geq \frac{x(m_1)+x(m_2)}{2}$  it wins at most  $\frac{|R_2|}{2}$  constituencies from  $R_2$  but loses all from  $R_1$ . Since  $|R_1| \geq |R_2|$  the payoff difference is  $\Pi_N(x(N)') - \Pi_N(x^*) = \frac{|R_2|}{2} - \frac{|R_1|}{2} - t_p \leq 0$ . Therefore,  $N$  cannot deviate beneficially and the given strategy profile  $x^*$  is an equilibrium. Similar arguments can be made for cases (b) and (c).

(ii) Suppose there exists  $k \in \{1, \dots, K\}$  such that  $\bar{A}_k(x_1, \dots, x_n) \neq \emptyset$  and  $\bar{A}_k(x_1, \dots, x_n) \cap \{x(m_1), x(m_2)\} = \emptyset$ . We consider the two cases separately.

(A) We first show that both  $S_1$  and  $S_2$  cannot deviate beneficially. Since there exists some  $k \in \{1, 2\}$  such that  $\bar{A}_k(x_1, \dots, x_n) \cap \{x(m_1), x(m_2)\} = \emptyset$  we have  $x(m_1^{s_q}) > x(m_1)$  and  $x(m_2^{t_p}) < x(m_2)$ . Therefore,  $V_1 = s_q - 1 \geq \frac{|R_1|+1}{2}$  and  $V_2 = |R_2| - t_p + 1 \geq \frac{|R_2|+1}{2}$ . If  $S_1$  deviates to  $x(S_1)' > x(N)^*$ , then it wins less than  $\frac{|R_1|}{2}$  constituencies. This makes it worse-off. If it deviates to  $x(S_1)' \leq x(N)^*$  it gets at most  $V_1' = V_1$ . Therefore,  $S_1$  does not deviate. Similar arguments can be made to show that  $S_2$  does not deviate.

Suppose  $N$  deviates to  $x(N)' \leq x(N)^*$ . It wins at most  $s_q - 1 + t_p$  constituencies. The payoff difference is  $\Pi_N(x(N)') - \Pi_N(x^*) = s_q - 1 + t_p - |R_1| + s_q - 1 - t_p$ . By condition (a),  $|R_1| \geq 2(s_q - 1)$ . Therefore  $\Pi_N(x(N)') - \Pi_N(x^*) \leq 0$ .

Suppose  $N$  deviates to  $x(N)' \geq x(N)^*$ . Then it wins at most  $|R_2| - t_p + s_q$  constituencies. The payoff difference is  $\Pi_N(x(N)') - \Pi_N(x^*) = |R_2| - t_p + s_q - |R_1| + s_q - 1 - t_p$ . By condition (b),  $|R_1| - |R_2| \geq 2(s_q - t_p) - 1$ . Therefore  $\Pi_N(x(N)') - \Pi_N(x^*) \leq 0$ . Therefore,  $N$  does not deviate.

(B) Assume w.l.o.g.  $x(N)^* = x(m_1) = x(S_1)^*$  and  $x(S_2)^* = x(m_2)$ . We first show that both  $S_1$  and  $S_2$  cannot deviate beneficially. If  $S_1$  deviates to  $x(S_1)' \neq x(S_1)^*$ , then it wins less than  $\frac{|R_1|}{2}$  constituencies. This makes  $S_1$  worse-off. Therefore,  $S_1$  does not deviate. By condition (c)  $S_2$  wins all constituencies  $R_2^k$  such that  $x(m_2^k) > x(m_1)$ . If  $S_2$  deviates to  $x(S_2)' \neq x(S_2)^*$ , then its payoff difference is  $\Pi_2(x(S_2)') - \Pi_2(x^*) = V_2' - V_2 \leq 0$ . Therefore,  $S_2$  does not deviate.

There are three possible deviations by  $N$ . Suppose  $N$  deviates to  $x(N)' \in \bar{A}_k(x_1, \dots, x_n)$ . It wins at most  $|R_1| - s_q + 1 + t_p$  constituencies. Note that  $N$  gets at

least  $\frac{|R_1|}{2}$  in equilibrium. Therefore, the payoff difference is  $\Pi_N(x(N)') - \Pi_N(x^*) = |R_1| - s_q + 1 + t_p - \frac{|R_1|}{2}$ . By condition (a),  $\Pi_N(x(N)') - \Pi_N(x^*) = \frac{|R_1|}{2} + 1 - s_q + t_p \leq 0$ . Suppose  $N$  deviates to a policy  $x(N)' = x(m_2)$ . It wins at most  $\frac{|R_2|}{2} + |R_1| - s_q + 1$  constituencies. The payoff difference is  $\Pi_N(x(N)') - \Pi_N(x^*) = \frac{|R_2|}{2} + |R_1| - s_q + 1 - \frac{|R_1|}{2}$ . By condition (b),  $\Pi_N(x(N)') - \Pi_N(x^*) \leq 0$ . For any other deviation by  $N$  to  $x(N)' \neq x(N)^*$ ,  $N$  wins at most  $|R_1| - s_q + 1 + t_p$  constituencies. Similar arguments as the ones used earlier can be made to show that  $\Pi_N(x(N)') - \Pi_N(x^*) \leq 0$ . Therefore,  $N$  does not deviate. ■

Proposition 9 describes the equilibrium when regions are heterogeneous. There are two cases to consider.

Case 1: Suppose all the maximal isolation sets contain the policy position of a region-wide median. Note that this is only possible when there are at most two maximal isolation sets, each containing the policy position of a region-wide median. The national party locates at the policy position of the region-wide median of the region with the greater number of constituencies if the latter is in a maximal isolation set. The regional parties are located at their respective region-wide medians.

Case 2: Suppose there exists a maximal isolation set which lies in the interior of the interval containing the two region-wide medians. Then there are two types of possible equilibria. In the first type, the national party locates in the maximal isolation in the interior of the interval containing the two region-wide medians. The regional parties locate as close to the national party's position on the same side of their respective region-wide medians. In the second type, the national party locates at the policy position of a region-wide median which isolates a significant number of constituencies from the other region. The regional parties choose the policy positions of their respective region-wide medians.

In both cases described above, certain additional conditions (as in Proposition 8) need to be imposed. If these conditions are not met, then our proposed equilibrium does not exist. It is quite possible that no pure strategy equilibrium exists. However, we have been unable to verify non-existence. The following example illustrates these difficulties.

**EXAMPLE 15** Let  $R_1 = \{R_1^1, \dots, R_1^5\}$  and  $R_2 = \{R_2^1, \dots, R_2^7\}$ . The median positions are at equal distance from each other with  $x(m_1^1) = 0$ ,  $x(m_2^1) = 0.091$ ,  $x(m_2^2) = 0.182$ ,  $x(m_1^2) = 0.273$ ,  $x(m_1^3) = 0.364$ ,  $x(m_2^3) = 0.455$ ,  $x(m_1^4) = 0.546$ ,  $x(m_2^4) = 0.637$ ,  $x(m_2^5) = 0.728$ ,  $x(m_2^6) = 0.819$ ,  $x(m_1^5) = 0.91$  and  $x(m_2^7) = 1$ . This is shown in the figure below.

As shown in Example 13 there are three maximal isolation sets: (i)  $[x(m_2^2), x(m_1^3)]$  (ii)  $[x(m_2^3), x(m_1^4)]$  (iii)  $[x(m_2^4), x(m_1^5)]$ .

We show that the following strategy profile  $x^*$  is not an equilibrium:  $x(N)^* = x(S_2)^* = x(m_2)$  and  $x(S_1)^* = x(m_1)$ . We have  $V_1 = 3$ ,  $V_N = 5.5$  and  $V_2 = 3.5$ .



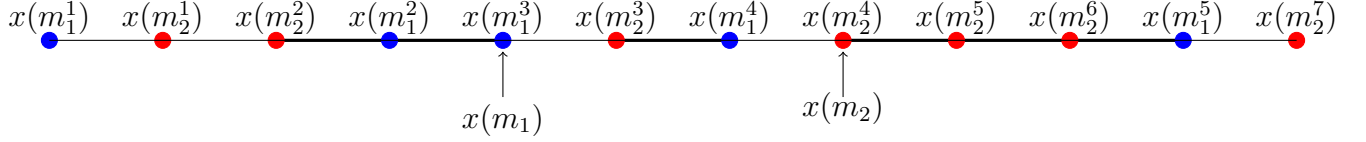


Figure 4.9: Illustration for Example 15

We show that  $S_1$  can deviate beneficially. Consider deviation  $x(S_1)' = x(m_1^4)$ . The payoff difference is  $\Pi_1(V') - \Pi_1(V) = 4 - 3 = 1 > 0$ . Therefore,  $S_1$  is better-off after the deviation.

We show that the following strategy profile  $x^*$  is also not an equilibrium:  $x(N)^* = x(m_1^4)$ ,  $x(S_2)^* = x(m_2)$  and  $x(S_1)^* = x(m_1)$ . The seat shares are  $V_1 = 3$ ,  $V_N = 5$  and  $V_2 = 4$ .

Suppose  $N$  deviates to a policy position  $x(N)' = x(m_1)$ . After the deviation  $N$  will win 2.5 constituencies (in expectation) from  $R_1$  and 3 constituencies from  $R_2$ . Therefore,  $V_N' = 5.5 > V_N = 5$ . This makes  $N$  better. Hence  $x^*$  is not an equilibrium.

We give an example to illustrate an equilibrium.

**EXAMPLE 16** Let  $R_1 = \{R_1^1, \dots, R_1^5\}$  and  $R_2 = \{R_2^1, \dots, R_2^7\}$ . The location of the constituency medians are shown in Figure 4.10.

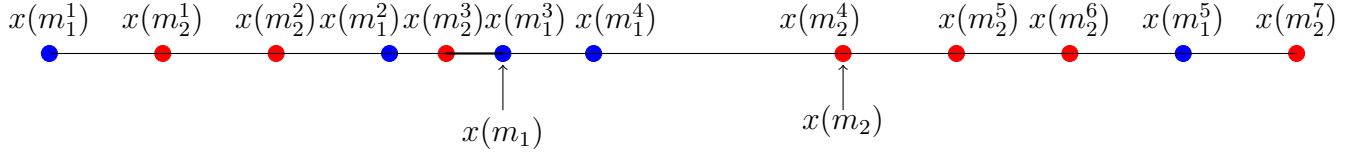


Figure 4.10: Illustration for Example 16

The median positions are as shown in the figure above. There is one maximal isolation set: (i)  $[x(m_2^3), x(m_1^3)]$ .

We show that the following strategy profile  $x^*$  is an equilibrium:  $x(N)^* = x(S_1)^* = x(m_1)$  and  $x(S_2)^* = x(m_2)$ . The expected seat shares are  $V_1 = 2.5$ ,  $V_N = 5.5$  and  $V_2 = 4$ .

Suppose  $S_1$  cannot to  $x(S_1)' \neq x(S_1)^*$ . Then it wins at most 3 constituencies. Therefore  $V_1' - V_1 = 2 - 2.5 = -0.5$ . Hence,  $S_1$  does not deviate.

Suppose deviates to any policy  $x(S_2)' \neq x(S_2)^*$ . Then it wins at most 3 constituencies. We have  $V_2' - V_2 = 3 - 4 = -1$ . Therefore  $S_2$  does not deviate.

Finally, we show that  $N$  cannot deviate beneficially. Clearly, any deviation  $x(N)' \in (x(m_1), x(m_2))$  is not profitable. If  $N$  deviates to  $x(N)' \leq x(m_1)$  it wins at most 5 constituencies. Therefore, the payoff difference is  $V_N' - V_N = 5 - 5.5 = -0.5$ . If it deviates to

$x(N)' \geq x(m_2)$  it wins at most 4.5 constituencies. We have  $V'_N - V_N = 4 - 5.5 = -1.5$ . Therefore,  $N$  does not deviate.

## 4.4 DISCUSSION

The national party can win a greater number of constituencies from both the regions when the voter distribution is heterogeneous. Suppose the heterogeneity is maximum i.e. there exists a maximal isolation set which isolates  $\frac{|R_j|-1}{2}$  constituencies from both the regions. In an equilibrium of the nature described in Proposition 9 the national party can win at most  $\max_{j \in \{1,2\}} \frac{|R_j|}{2}$  from one region and  $\frac{|R_{j'}-1|}{2}$  from the other region i.e. almost half of both the regions. However, when regions are homogeneous it can win at most  $\max_{j \in R} \frac{|R_j|}{2}$  or half the larger region.

Suppose  $|R_1| \geq |R_2|$ . Let  $V_N^{hom}$ ,  $V_N^{het}$  be the maximum number of constituencies that the national party can win in equilibrium when voter distribution is homogeneous and when the heterogeneity is maximum respectively. Then  $V_N^{het} - V_N^{hom} = \frac{|R_1|}{2} + \frac{|R_2|-1}{2} - \frac{|R_1|}{2} = \frac{|R_2|-1}{2} \geq 0$ . Hence,  $V_N^{het} - V_N^{hom} > 0$  if  $|R_2| \geq 2$ .

Similarly, the regional parties win a greater number of constituencies when the voter distribution is homogeneous than when it is heterogeneous. When the voter distribution is homogeneous, the regional party of the region with the smaller number of constituencies wins all the constituencies from its region. On the other hand, if the voter distribution has maximum heterogeneity, then the regional party can only win at most  $\frac{|R_j|-1}{2}$  constituencies in equilibrium.

## 4.5 CONCLUSION

The model can be easily generalized to multiple regions in a straightforward way. The nature of the competition between regional and national parties will remain the same. The regional parties will stay on the same side of the national party's policy as their respective region-wide medians. The national party will isolate all the constituencies across regions.

It would also be interesting to consider a multidimensional policy space with certain dimensions corresponding to regional issues. Although such a model would be more realistic, the existence of equilibrium will present significant challenges (See [Plott \(1967\)](#)). We hope to address these issues in future work.

# Bibliography

- AUSTEN-SMITH, D. (1984): “Two-Party Competition with Many Constituencies,” *Mathematical Social Sciences*, 177–198.
- (1987): “Parties, Districts and the Spatial Theory of Elections,” *Social Choice and Welfare*, 4, 9–23.
- AUSTEN-SMITH, D. AND J. S. BANKS (2000): *Positive Political Theory I: Collective Preference*, University of Michigan Press.
- (2005): *Positive Political Theory II: Strategy and Structure*, University of Michigan Press.
- BAILEY, M. AND D. W. BRADY (1998): “Heterogeneity and Representation: The Senate and Free Trade,” *American Journal of Political Science*, 524–544.
- BARBERÀ, S., H. SONNENSCHN, AND L. ZHOU (1991): “Voting by Committees,” *Econometrica: Journal of the Econometric Society*, 595–609.
- BERVOETS, S. AND V. MERLIN (2012): “Gerrymander-proof Representative Democracies,” *International Journal of Game Theory*, 41(3), 473–488.
- BESLEY, T. AND S. COATE (1997): “An Economic Model of Representative Democracy,” *The Quarterly Journal of Economics*, 112, 85–114.
- BHATTACHARYA, S. (2014): “Strategic Communication and Group Formation,” Working paper.
- BROWNE, E. C. AND M. N. FRANKLIN (1973): “Aspects of Coalition Payoffs in European Parliamentary Democracies,” *The American Political Science Review*, 453–469.
- CALLANDER, S. (2005): “Electoral Competition in Heterogenous Districts,” *Journal of Political Economy*, 113(5), 1116–1145.
- CASAMATTA, G. AND P. DE DONDER (2005): “On the Influence of Extreme Parties in Electoral Competition with Policy-motivated Candidates,” *Social Choice and Welfare*, 25, 1–29.

- CHAMBERS, C. P. (2008): “Consistent Representative Democracy,” *Games and Economic Behavior*, 62(2), 348–363.
- (2009): “An Axiomatic Theory of Political Representation,” *Journal of Economic Theory*, 144(1), 375–389.
- DHILLON, A. (2005): “Political Parties and Coalition Formation,” *Demange, G and M. Wooders, M. (Editors), Group formation in economics: Networks, Clubs and Coalitions*, Cambridge University Press, New York, 289–311.
- DUGGAN, J. AND M. FEY (2005): “Electoral Competition with Policy-Motivated Candidates,” *Games and Economic Behavior*, 51, 490–522.
- EYSTER, E. AND T. KITTSSTEINER (2007): “Party Platforms in Electoral Competition with Heterogenous Constituencies,” *Theoretical Economics*, 2, 41–70.
- FUNG, L. W. AND K. S. FU (1975): “An Axiomatic Approach to Rational Decision Making in a Fuzzy Environment,” *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, 227–256.
- GALLAGHER, M. AND P. MITCHELL (2005): *The Politics of Electoral Systems*, Cambridge University Press.
- GAMSON, W. A. (1961): “A Theory of Coalition Formation,” *American Sociological Review*, 373–382.
- GERBER, E. R. AND J. B. LEWIS (2004): “Beyond the Median: Voter Preferences, District Heterogeneity, and Political Representation,” *Journal of Political Economy*, 112, 1364–1383.
- GOODIN, R. E. AND C. LIST (2006): “A Conditional Defense of Plurality Rule: Generalizing May’s Theorem in a Restricted Informational Environment,” *American Journal of Political Science*, 50(4), 940–949.
- GOYAL, S. (2012): *Connections: An Introduction to the Economics of Networks*, Princeton University Press.
- HEATH, A., S. GLOUHAROVA, AND O. HEATH (2005): “India: Two Party Contests Within a Multiparty System,” *The Politics of Electoral Systems*, 137–156.
- JACKSON, M., L. MATHEVET, AND K. MATTES (2007): “Nomination Processes and Policy Outcomes,” *Quarterly Journal of Political Science*, 2, 67–92.
- JACKSON, M. O. (2008): *Social and Economic Networks*, Princeton University Press.

- JACKSON, M. O. AND B. MOSELLE (2002): “Coalition and Party Formation in a Legislative Voting Game,” *Journal of Economic Theory*, 103, 49–87.
- KASHER, A. AND A. RUBINSTEIN (1997): “On the Question “Who is a J?”: A Social Choice Approach,” *Logique et Analyse*, 160, 385–395.
- KATZ, R. S. (1998): “Malapportionment and Gerrymandering in Other Countries and Alternative Electoral Systems,” *Contributions in Political Science*, 384, 245–260.
- LASLIER, J.-F. (2005): “Party Objectives in the “Divide a Dollar” Electoral Competition,” in *Social Choice and Strategic Decisions*, Springer, 113–130.
- LEVY, G. (2002): “Endogenous Parties: Cooperative and Non-Cooperative Analysis,” *Mimeo*.
- (2004): “A Model of Political Parties,” *Journal of Economic Theory*, 115, 250–277.
- MACÉ, A. (2013): “Voting with Evaluations: When Should We Sum? What Should We Sum?” *Mimeo*.
- MASSICOTTE, L. (2005): “Canada: Sticking to First-Past-the-Post, for the Time Being,” *The Politics of Electoral Systems*, 99–118.
- NERMUTH, M. (1994): “Two-Stage Discrete Aggregation: The Ostrogorski Paradox and Related Phenomena,” *Social Choice and Welfare*, 9, 99–116.
- OSBORNE, M. AND A. SLIVINSKY (1996): “A Model of Political Competition with Citizen-Candidates,” *Quarterly Journal of Economics*, 1, 65–96.
- OSBORNE, M. J. AND R. TOURKY (2008): “Party Formation in Single-Issue Politics,” *Journal of European Economic Association*, 6(5), 974–1005.
- PERESS, M. (2010): “The Spatial Model with Non-Policy Factors: A Theory of Policy-Motivated Candidates,” *Social Choice and Welfare*, 34, 265–294.
- PEROTE-PEÑA, J. (2005): “Gerrymandering-proof Social Welfare Functions,” *Mimeo*.
- PLOTT, C. R. (1967): “A Notion of Equilibrium and its Possibility Under Majority Rule,” *The American Economic Review*, 787–806.
- (1973): “Path Independence, Rationality, and Social Choice,” *Econometrica*, 41, 1075–1091.
- RIKER, W. H. (1962): *The Theory of Political Coalitions*, vol. 578, Yale University Press New Haven.

- RIVIERE, A. (1999): "Citizen Candidacy, Party Formation, and Duverger's Law," *Working paper*.
- SAMUELS, D. AND R. SNYDER (2001): "The Value of a Vote: Malapportionment in Comparative Perspective," *British Journal of Political Science*, 31, 651–671.
- TEORELL, J. (1999): "A Deliberative Defence of Intra-Party Democracy," *Party politics*, 5, 363–382.
- WARWICK, P. V. AND J. N. DRUCKMAN (2001): "Portfolio Salience and the Proportionality of Payoffs in Coalition Governments," *British Journal of Political Science*, 31, 627–649.
- WITTMAN, D. (1983): "Candidate Motivation: A Synthesis of Alternative Theories," *The American political science review*, 142–157.