Proximinality Properties of Subspaces and Intersection Properties of Balls in Banach Spaces

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Dedicated

to

My Parents

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Contents

CHAPTER 1

Introduction

In this chapter, we explain the background and the main theme of this thesis and provide a chapter-wise summary of its principal results. We introduce some notations and preliminaries that will be used in the subsequent chapters.

Study of proximinality related properties and ball intersection related properties of Banach spaces have been an active area of research in the field of geometry of Banach spaces. In this thesis, we mainly study these two classes of Banach space theoretic properties.

We consider only Banach spaces over the real field $\mathbb{R}$ and all subspaces we consider are assumed to be closed.

1.1 Preliminaries

For a Banach space X and a subspace Y, one of the basic problems in the field of approximation theory is the existence of a best approximation from Y for an element x of X. If this happens for every $x \in X$, then Y is said to be a proximinal subspace of X.

Definition 1.1.1. Let K be a non-empty closed subset of a Banach space X. For $x \in X$, the distance of x from K, denoted by $d(x, K)$, is given by $d(x, K) = \inf\{\|x - k\| : k \in K\}$.

The set-valued mapping $P_K : X \to 2^K$ defined by $P_K(x) = \{k \in K : d(x, K) = \|x - k\|\}$ is called the metric projection onto K. An element of $P_K(x)$ is called a best approximation from K to x. The set K is said to be *proximinal* in X if $P_K(x) \neq \emptyset$ for all $x \in X$.

Some of the natural examples of proximinal subspaces are reflexive subspaces and $\ker(f)$, where $f \in X^*$ is such that $\|f\| = f(x)$ for some $x \in X$ with $\|x\| = 1$ (the so-called norm attaining functional). The earliest results concerning characterization of proximinal subspaces of finite co-dimension (that is $\dim(X/Y) < \infty$) are mainly due to Garkavi (see [17, 18] for details). For instance, in [18], he characterized finite co-dimensional proximinal subspaces of $C(K)$, the space of all continuous functions on a compact Hausdorff space K, equipped with the supremum norm.

Theorem 1.1.2 ([18])**.** *Let K be a compact Hausdorff space and let Y be a finite co-dimensional subspace of $C(K)$. Then Y is proximinal in $C(K)$ if and only if the annihilator $Y^\perp$ satisfies the following three conditions*:

(a) $\mathrm{supp}(\mu^+) \bigcap \mathrm{supp}(\mu^-) = \emptyset$ *for each* $\mu \in Y^\perp \setminus \{0\}$,

(b) μ *is absolutely continuous with respect to* ν *on* $\mathrm{supp}(\nu)$ *for every pair* $\mu, \nu \in Y^\perp \setminus \{0\}$,

(c) $\mathrm{supp}(\nu) \setminus \mathrm{supp}(\mu)$ *is closed for each pair* $\mu, \nu \in Y^\perp \setminus \{0\}$.

The following result by Garkavi gives a necessary condition for factor reflexive subspaces to be proximinal. We recall that a subspace Y of a Banach space X is said to be a *factor reflexive subspace* if the quotient space X/Y is reflexive.

Proposition 1.1.3 ([46, Chapter III, Lemma 1.1])**.** *If Y is a factor reflexive proximinal subspace of a Banach space X, then every $f \in Y^\perp$ is a norm attaining functional on X.*

But in general the converse of the Proposition 1.1.3 need not be true though it is true when Y is of co-dimension one. Precisely, for a Banach space X and $f \in X^*$, $\ker(f)$ is proximinal in X if and only if f is a norm attaining functional on X.

In [19], Godefroy and Indumathi introduced a stronger version of proximinality called strong proximinality.

Definition 1.1.4. A proximinal subspace Y of a Banach space X is said to be *strongly proximinal* in X if for every $x \in X$ and every $\varepsilon > 0$, there exists a $\delta > 0$ such that $P_Y(x, \delta) \subseteq P_Y(x) + \varepsilon B_X$, where $P_Y(x, \delta) = \{y \in Y : \|x - y\| < d(x, Y) + \delta\}$ and B_X is the closed unit ball of X.

For a proximinal subspace Y of a Banach space X, one can easily observe that the above definition is equivalent to the following: for every element $x \in X$ and for every sequence (y_n) in Y with $\|x - y_n\| \to d(x, Y)$, $d(y_n, P_Y(x)) \to 0$.

Clearly, any finite dimensional subspace of a Banach space is strongly proximinal. In [38], Narayana proved that every infinite dimensional Banach space can be embedded isometrically as a non-strongly proximinal hyperplane in another Banach space.

In [16], Franchetti and Payá introduced a non-smooth extension of Fréchet differentiability, namely strong subdifferentiability, which in turn characterizes strongly proximinal hyperplanes.

Definition 1.1.5. The norm of a Banach space X is *strongly subdifferentiable* (in short SSD) at $x \in X$ if the one sided limit

$$d^+(x)(y) := \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

exists uniformly for $y \in B_X$. In this case, we call x an SSD point of X. If this happens for all unit vectors in X, then we say that the norm of X is SSD.

In [16], Franchetti and Payá observed that the norm of any finite dimensional Banach space X is SSD. They also proved that the norms of sequence spaces c_0 and ℓ_p $(1 < p < \infty)$ are SSD. In the case of ℓ_1, by [16, Theorem 1.2] and [15, Theorem 7], it follows that SSD-points of ℓ_1 are sequences with finite support. Combining [16, Theorem 1.2] and [15, Theorem 5], we have the following characterization of SSD-points of ℓ_∞.

Theorem 1.1.6. *Let $x \in \ell_\infty$. Then x is an* SSD*-point of ℓ_∞ if and only if*

$$\sup\{|x(n)| : |x(n)| \neq \|x\|\} < \|x\|.$$

The following result by Godefroy and Indumathi connects SSD-points with strongly proximinal subspaces of co-dimension one.

Theorem 1.1.7 ([19])**.** *Let X be a Banach space. Then for an $f \in X^*$, $\ker(f)$ is a strongly proximinal subspace of X if and only if f is an* SSD*-point of X^*.*

In the case of finite co-dimensional strongly proximinal subspaces, we have the following:

Theorem 1.1.8 ([19])**.** *Let Y be a finite co-dimensional subspace of a Banach space X. If Y is strongly proximinal in X, then $Y^\perp$ is contained in the set of all* SSD*-points of X^*.*

Later in [20], using the notion of strong subdifferentiability of convex functionals, Godefroy, Indumathi and Lust-Piquard gave a sufficient condition for the proximinality of finite co-dimensional subspaces in a Banach space. In the same article, they also proved that the converse of Theorem 1.1.8 holds for finite co-dimensional subspaces of the space of compact operators on the Hilbert space ℓ_2.

The following notion of quasi-polyhedral point, introduced in [2] by Amir and Deutsch, is stronger than the notion of an SSD-point.

Definition 1.1.9. A vector x in a Banach space X is called a *quasi-polyhedral* (in short QP) point of X if there exists a $\delta > 0$ such that $J_{X^*}(z) \subseteq J_{X^*}(x)$ if $\|z - x\| < \delta$ and $\|z\| = \|x\|$, where $J_{X^*}(x) = \{f \in B_{X^*} : f(x) = \|x\|\}$.

For a finite dimensional Banach space X, by [2, Theorem 2.19], each unit vector in X is a QP-point if and only if the closed unit ball of X has only finitely many extreme points. Lemma 3.3 of [19] shows that a QP-point is an SSD-point. But the converse need not be true. For example, consider $\mathbb{R}^n$ with usual norm. Then all unit vectors in $\mathbb{R}^n$ are SSD-points, but all unit vectors in $\mathbb{R}^n$ cannot be QP-points as the unit ball in $\mathbb{R}^n$ has infinitely many extreme points.

The following results by Dutta and Narayana show that, for a compact Hausdorff space K, the notions of SSD-point and QP-point coincide in $C(K)$ and in $C(K)^*$.

Theorem 1.1.10 ([13, Theorem 2.1]). *Let $f \in C(K)$ be such that $\|f\| = 1$. Then the following are equivalent*:

(i) $\{k \in K : |f(k)| = 1\}$ *is a clopen set.*

(ii) f *is an* SSD-*point.*

(iii) f *is a* QP-*point.*

Theorem 1.1.11 ([14, Theorem 2.1]). *Let $\mu \in C(K)^*$ be such that $\|\mu\| = 1$. Then the following are equivalent*:

(i) μ *is finitely supported.*

(ii) μ *is an* SSD-*point.*

(iii) μ *is a* QP-*point.*

In [1], Alfsen and Effros introduced the notion of an M-ideal in a Banach space. This well-studied concept in M-structure theory is stronger than proximinality (in fact stronger than strong proximinality).

Definition 1.1.12. Let X be a Banach space.

(a) A linear projection P on X is called an *M-projection* if

$$\|x\| = \max\{\|Px\|, \|x - Px\|\} \text{ for all } x \in X.$$

A linear projection P on X is called an *L-projection* if

$$\|x\| = \|Px\| + \|x - Px\| \text{ for all } x \in X.$$

(b) A subspace $Y \subseteq X$ is called an *M-summand* if it is the range of an M-projection. A subspace $Y \subseteq X$ is called an *L-summand* if it is the range of an L-projection.

(c) A subspace $Y \subseteq X$ is called an *M-ideal* if $Y^\perp$ is an *L-summand* in X^*. If a Banach space X, under the canonical embedding, is an M-ideal in X^{**}, then we say that X is an *M-embedded* space.

The following result will play an important role in Section 4.4 of Chapter 4.

Proposition 1.1.13 ([22, Page 66])**.** *Let Y be a subspace of a Banach space X. Then*

(a) *Y is an M-ideal in X if and only if Y is an M-ideal in* $\operatorname{span}\{Y, x\}$ *for all $x \in X$.*

(b) *Y is an M-summand in X if and only if Y is an M-summand in* $\operatorname{span}\{Y, x\}$ *for all $x \in X$.*

We now recall some results on M-ideals and M-summands which will be used in the subsequent chapters.

Proposition 1.1.14 ([22])**.** *For a Banach space X, we have the following*:

(a) *Every weak*-closed M-ideal in X^* is an M-summand.*

(b) *Every M-summand in X^* is weak*-closed and is of the form $Y^\perp$ for some L-summand Y in X.*

(c) *If X is an M-embedded space, then every subspace of X is also an M-embedded space.*

In [32], Lima introduced a weaker notion of M-ideal called semi M-ideal which is also stronger than being strongly proximinal.

Definition 1.1.15. A subspace Y of a Banach space X is called a *semi M-ideal* in X if there is a (nonlinear) projection P from X^* onto $Y^\perp$ such that

$$P(\lambda x^* + Py^*) = \lambda Px^* + Py^*,$$
$$\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$$

for all $x^*, y^* \in X^*$ and $\lambda \in \mathbb{R}$. Such a projection is called a semi L-projection and its range a semi L-summand.

The following result characterizes semi M-ideals in Banach spaces.

Theorem 1.1.16 ([32, Theorem 6.14])**.** *Let Y be a subspace of a Banach space X. Then Y is a semi L-summand in X if and only if $Y^\perp$ is a semi M-ideal in X^*.*

We now recall an example of a semi M-ideal which is not an M-ideal.

Example 1.1.17 ([22, Chapter I, Remark 2.3(a)]). Let μ be a positive measure and let $\dim(L_1(\mu)) > 2$. Then $Y = \{f \in L_1(\mu) : \int f\, d\mu = 0\}$ is a semi M-ideal in $L_1(\mu)$, but Y is not an M-ideal in $L_1(\mu)$.

The following result due to Lima gives a sufficient condition for a Banach space to be an M-embedded space.

Theorem 1.1.18 ([33, Corollary 3.4]). *Let X be a Banach space. If X is a semi M-ideal in X^{**}, then X is an M-embedded space.*

In [21], Godefroy, Kalton and Saphar introduced a weaker notion of M-ideal called an 'ideal'.

Definition 1.1.19. A subspace Y of a Banach space X is said to be an *ideal* in X if $Y^{\perp}$ is the kernel of a norm one projection on X^*.

Clearly, range of any norm one projection is an ideal. Also, every Banach space, under the canonical embedding, is an ideal in its bidual. For, if X is a Banach space, then the well-known projection $P : X^{***} \to X^{***}$ defined by $P(\Lambda) = \Lambda|_X$ is a projection of norm one with kernel $X^{\perp}$. It is well-known that c_0 is an example of an ideal in ℓ_∞ that is not the range of a projection of norm one.

The following theorem due to Lima characterizes ideals in Banach spaces.

Theorem 1.1.20 ([34, Theorem 1]). *For a subspace Y of a Banach space X, the following are equivalent:*

(i) *Y is an ideal in X.*

(ii) *$Y^{\perp\perp}$ is the range of a norm one projection in X^{**}.*

(iii) *If F is a finite dimensional subspace of X and $\varepsilon > 0$, then there exists an operator $T : F \to Y$ such that:*

(a) $T(x) = x$ *for* $x \in F \bigcap Y$,

(b) $\|T\| \leq (1 + \varepsilon)$.

Some of the important Banach space theoretic properties which are closely related to proximinality properties are intersection properties of balls in Banach space. In [1], Alfsen and Effros characterized M-ideals in terms of an intersection property of balls, namely the n-ball property ($n \in \mathbb{N}$).

For a Banach space X, we denote by $B_X(x, r)$ (or $B(x, r)$, if there is no scope for confusion) the closed ball of radius $r > 0$ around $x \in X$.

Definition 1.1.21 ([22])**.** Let $n \in \mathbb{N}$. A subspace Y of a Banach space X is said to have the *n-ball property* if, given n closed balls $\{B(a_i, r_i)\}_{i=1}^n$ in X such that $\bigcap_{i=1}^n B(a_i, r_i) \neq \emptyset$ and $Y \bigcap B(a_i, r_i) \neq \emptyset$ for all i, then $Y \bigcap \left(\bigcap_{i=1}^n B(a_i, r_i + \varepsilon)\right) \neq \emptyset$ for every $\varepsilon > 0$.

We now recall the following characterization of M-ideals.

Theorem 1.1.22 ([22, Chapter I, Theorem 2.2])**.** *Let Y be a subspace of a Banach space X. Then the following are equivalent*:

(i) *Y is an M-ideal in X.*

(ii) *Y has the n-ball property for all $n \in \mathbb{N}$.*

(iii) *Y has the 3-ball property.*

We also recall the following characterization of a semi M-ideal.

Theorem 1.1.23 ([32, Theorem 6.10])**.** *Let Y be a subspace of a Banach space X. Then Y is a semi M-ideal in X if and only if Y has the 2-ball property in X.*

In [50], Yost introduced another intersection property of balls, namely the $1\frac{1}{2}$-ball property, which is weaker than the n-ball property and is stronger than proximinality (in fact, stronger than strong proximinality).

Definition 1.1.24. A subspace Y of a Banach space X is said to have the (*strong*) $1\frac{1}{2}$-*ball property* if the conditions $x \in X$, $y \in Y$, $Y \cap B(x,r) \neq \emptyset$ and $\|x-y\| \leq r+s$ $(r,s>0)$ imply that $Y \cap B(x, r+\varepsilon) \cap B(y, s+\varepsilon) \neq \emptyset$ for all $(\varepsilon \geq 0)$ $\varepsilon > 0$.

We can easily see that the (strong) $1\frac{1}{2}$-ball property is equivalent to requiring the (strong) 2-ball property subject to the restriction that one of the centers lies in Y. Proposition 3.3 of [13] proves that a subspace having the $1\frac{1}{2}$-ball property is strongly proximinal. Hence through the intersection properties of balls, we can see that all geometric properties considered above, other than that of an ideal, lead to strong proximinality. We also have the following characterization of the strong $1\frac{1}{2}$-ball property due to Payá and Yost.

Theorem 1.1.25 ([39, Proposition 3(ii)])**.** *A subspace Y of a Banach space X has the strong $1\frac{1}{2}$-ball property in X if and only if for every $x \in X$, there exists an element $y \in P_Y(x)$ such that $\|x\| = \|x-y\| + \|y\|$.*

Recently in [5], Bandyopadhyay, Lin and Rao introduced a stronger version of proximinality called ball proximinality.

Definition 1.1.26. A subspace Y of a Banach space X is said to be *ball proximinal* in X if the closed unit ball B_Y of Y is proximinal in X.

In [5, Proposition 2.4], it is proved that ball proximinal subspaces of Banach spaces are proximinal. But Theorem 1 of [45] shows that the converse need not be true.

In [25], Indumathi and Lalithambigai characterized subspaces having the strong $1\frac{1}{2}$-ball property using ball proximinality.

Theorem 1.1.27 ([25])**.** *Let Y be a subspace of a Banach space X. Then Y has the strong $1\frac{1}{2}$-ball property in X if and only if Y is ball proximinal and has the $1\frac{1}{2}$-ball property in X.*

Another type of intersection property of balls studied by Grothendieck in 1950's and Lindenstrauss in 1960's is the $n.2$ intersection property, using which they obtained many significant results regarding projections in Banach spaces and operator version of Hahn-Banach theorem.

Definition 1.1.28 ([36]). Let $n \in \mathbb{N}$. A Banach space X has the $n.2$ intersection property ($n.2.I.P$) if for every family of n balls in X such that any two of them intersect, there is a point common to all the n balls.

The following result gives a sufficient condition for a Banach space to have $n.2.I.P$.

Lemma 1.1.29 ([36, Lemma 4.2]). *Let X be a Banach space such that every pair-wise intersecting family $\{B(x_i, r_i)\}_{i=1}^n$ of n balls in X satisfies $\bigcap_{i=1}^n B(x_i, r_i + \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. Then X has the $n.2.I.P$.*

Using this intersection property, Grothendieck and Lindenstrauss characterized the so-called L_1-predual spaces and $\mathcal{P}_1$-spaces.

Definition 1.1.30 ([29]).

(a) A Banach space X such that X^* is isometrically isomorphic to $L_1(\mu)$ for some positive measure μ is called an L_1-*predual space*.

(b) A Banach space X is said to be a $\mathcal{P}_1$-*space* if for every Banach space Z containing X, there is a linear projection P from Z onto X with $\|P\| \leq 1$.

(c) A subspace Y of a Banach space X is said to be 1-*complemented* in X if there exists a linear projection P of norm one on X with range Y.

It is well-known that for any compact Hausdorff space K, $C(K)$ is an L_1-predual space. We now recall the following example from [9] which will be used in the next chapter.

Example 1.1.31. Let K be a compact Hausdorff space and let $k_1, \ldots, k_n \in K$. If $\mu_1, \ldots, \mu_n$ are regular Borel measures on K with $\|\mu_i\| \leq 1$ and $|\mu_i|(\{k_1, \ldots, k_n\}) = 0$ for $i = 1, \ldots, n$; then $A = \{f \in C(K) : f(k_i) = \int f \, d\mu_i, \ i = 1, \ldots, n\}$ is an L_1-predual space.

The following theorem by Nachbin, Kelley, Goodner and Hasumi characterizes $\mathcal{P}_1$-spaces. We recall that a compact Hausdorff space K is *extremally disconnected* if the closure of every open subset of K is open (see [29, Section 7] for details).

Theorem 1.1.32 ([29, Section 11, Theorem 6]). *A Banach space X is a $\mathcal{P}_1$-space if and only if X is isometrically isomorphic to $C(K)$ for some extremally disconnected space K.*

The following theorem connects the $n.2.I.P$, L_1-predual spaces and $\mathcal{P}_1$-spaces.

Theorem 1.1.33 ([29, Section 21, Theorem 6]). *Let X be a Banach space. Then the following are equivalent:*

(i) *X is an L_1-predual space.*

(ii) *X^{**} is a $\mathcal{P}_1$-space.*

(iii) *X has the $n.2.I.P$ for all $n \in \mathbb{N}$.*

(iv) *X has the $4.2.I.P$.*

The following result gives a characterization of L_1-predual spaces in terms of ideals.

Theorem 1.1.34 ([41, Proposition 1]). *For any Banach space X, the following are equivalent:*

(i) *If Z is a Banach space such that X is isometric to a subspace of Z, then X is an ideal in Z.*

(ii) *X is isometric to an ideal in $C(K)$ for some compact Hausdorff space K.*

(iii) *X is an L_1-predual space.*

Apart from the n-ball property and the $n.2.I.P$, one of the ball intersection properties studied in the literature is the relative ball intersection property. These properties helps to study the class of Banach spaces that admit weighted Chebyshev centers for finite sets. In [47], Veselý called such class of Banach spaces as the class (GC) and developed a theory for such spaces.

Definition 1.1.35. Let X be a Banach space. Let $a_1, \ldots, a_n \in X$ and $\eta_1, \ldots, \eta_n > 0$. Minimizers of the function $\phi : X \to \mathbb{R}$ defined by $\phi(x) = \max_{1 \leq i \leq n} \eta_i \|x - a_i\|$ are called weighted Chebyshev centers with the weight $\eta = (\eta_1, \ldots, \eta_n)$. Classical Chebyshev centers are the weighted Chebyshev centers with the weight $\eta = (1, \ldots, 1)$.

Theorem 1.1.36 ([47, Theorem 2.7]). *For a Banach space X and $a_1, \ldots, a_n \in X$, the following are equivalent:*

(i) *If $r_1, \ldots, r_n > 0$ and $\bigcap\limits_{i=1}^{n} B_{X^{**}}(a_i, r_i) \neq \emptyset$, then $\bigcap\limits_{i=1}^{n} B_X(a_i, r_i) \neq \emptyset$.*

(ii) *$a_1, \ldots, a_n$ admits weighted Chebyshev centers for all weights $\eta = (\eta_1, ..., \eta_n)$, where $\eta_i > 0$ for all $i = 1, \ldots, n$.*

Definition 1.1.37 ([47, Definition 2.8]). We shall denote by (GC) the class of all Banach spaces X such that for every positive integer n and every $a_1, \ldots, a_n \in X$, one of the equivalent conditions (i), (ii) of Theorem 1.1.36 is satisfied.

The following proposition gives an intrinsic characterization of the class (GC).

Proposition 1.1.38 ([6, Proposition 2.9]). *Let X be a Banach space. Then $X \in$ (GC) if and only if for all $n \in \mathbb{N}$, $a_1, \ldots, a_n \in X$ and $r_1, \ldots, r_n > 0$, $\bigcap_{i=1}^{n} B(a_i, r_i + \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$ implies $\bigcap_{i=1}^{n} B(a_i, r_i) \neq \emptyset$.*

Later in [6], Bandyopadhyay and Rao generalized the concept (GC). In fact, this generalization comes from the subspace condition (i) of Theorem 1.1.36.

Definition 1.1.39 ([6, Definition 2.1]). Let X be a Banach space. We say that a subspace $Y \subseteq X$ is a *central subspace* of X if every finite family of closed balls with centers in Y that intersect in X, also intersect in Y.

Clearly, $X \in$ (GC) if and only if X is a central subspace of X^{**}. It follows from [6, Proposition 2.2 (a)] that Y is a central subspace of a Banach space X if and only if for any finite set $\{y_i\}_{i=1}^{n} \subset Y$ and $x \in X$, there exists an element $y \in Y$ such that $\|y - y_i\| \leq \|x - y_i\|$ for $1 \leq i \leq n$. Also, it is easy to see that if Y is a central subspace of a Banach space X, then finite subsets of Y that have Chebyshev centers in X have Chebyshev centers (relative to Y) in Y.

The following result characterizes L_1-predual spaces in terms of central subspaces.

Theorem 1.1.40 ([6, Theorem 3.3])**.** *A Banach space X is an L_1-predual space if and only if whenever X is a subspace of a dual space, it is a central subspace there.*

An infinite version of central subspaces called almost constrained subspaces was investigated in [3] and [4].

Definition 1.1.41. A subspace Y of a Banach space X is said to be an *almost constrained* (in short AC) subspace of X if any family of closed balls centered at points of Y that intersect in X, also intersect in Y.

One can easily see that 1-complemented subspaces are AC-subspaces and hence they are also central subspaces. We also observe that Y is an AC-subspace of a Banach space X if and only if for any family $\{y_\alpha\}_{\alpha\in I} \subseteq Y$ and $x \in X$, there exists an element $y \in Y$ such that $\|y - y_\alpha\| \leq \|x - y_\alpha\|$ for $\alpha \in I$.

The following result shows that the notion of being an AC-subspace is closely related to the notion of being a 1-complemented subspace.

Theorem 1.1.42 ([3, Proposition 2.2])**.** *For a subspace Y of a Banach space, the following are equivalent:*

(i) *Y is an* AC-*subspace of X.*

(ii) *Y is* 1-*complemented in* $\operatorname{span}\{Y, x\}$ *for every $x \in X$.*

We also recall the following theorem of Bandyopadhyay and Dutta that characterizes an AC-subspace of finite co-dimension in the space $C(K)$.

Theorem 1.1.43 ([4, Theorem 1.1])**.** *Let K be a compact Hausdorff space and Y be a subspace of co-dimension n in $C(K)$. Then the following are equivalent*:

(i) *Y is an* AC-*subspace of $C(K)$.*

(ii) *Y is* 1-*complemented in $C(K)$.*

(iii) *There exist measures* $\mu_1, \ldots, \mu_n$ *on* K *and distinct isolated points* $k_1, \ldots, k_n$ *of* K *such that*:

(a) $Y = \bigcap_{i=1}^{n} \ker(\mu_i)$,

(b) $\|\mu_i\| \leq 2|\mu_i(\{k_i\})|$, $i = 1, \ldots, n$.

We now recall the definition of the injective tensor product of two Banach spaces which will be needed in the subsequent chapters.

Definition 1.1.44 ([11])**.** Let X and Y be two Banach spaces and let $X \bigotimes Y$ be the algebraic tensor product of X and Y. Let $u \in X \bigotimes Y$. Define $\lambda(u)$ by

$$\lambda(u) = \sup\{|(x^* \otimes y^*)(u)| : x^* \in X^*, y^* \in Y^*, \|x^*\| \leq 1 \text{ and } \|y^*\| \leq 1\}.$$

Then λ is a norm on $X \bigotimes Y$. The injective tensor product of X and Y, denoted by $X \overset{\vee}{\bigotimes} Y$, is the completion of the normed linear space $X \bigotimes Y$, equipped with the norm λ.

Example 1.1.45 ([11, Chapter VIII, Example 6])**.** Let K be a compact Hausdorff space and let X be a Banach space. Then the space $C(K) \overset{\vee}{\bigotimes} X$ is isometrically isomorphic to the Banach space $C(K, X)$ of continuous functions $f : K \to X$, equipped with the norm $\|f\| = \sup\{\|f(k)\| : k \in K\}$.

In [48], Veselý defined a new direct sum called polyhedral direct sum of Banach spaces. This direct sum helps us to produce more examples of Banach spaces that belong to the class (GC), as the membership of the class (GC) is stable under the polyhedral direct sum.

For $n \in \mathbb{N}$, the set $[0, \infty)^n$ will be denoted by $\mathbb{R}^n_+$.

Definition 1.1.46 ([48])**.** A function $\pi : \mathbb{R}^n_+ \to \mathbb{R}_+$ is a norm on $\mathbb{R}^n_+$ if it is subadditive, positively homogeneous and $\pi(t) = 0 \Leftrightarrow t = 0$.

A norm π on $\mathbb{R}^n_+$ is called *polyhedral* if it is of the form $\pi(t) = \max_{1 \leq j \leq m} g_j(t)$, where $g_1, \ldots, g_m \in (\mathbb{R}^n)^*$. In this case, we say that the family $\{g_1, \ldots, g_m\}$ generates π. Now Lemma 1.5 of [48] shows that if $\{g_1, \ldots, g_m\}$ is a minimal family generating π, then $g_j(i) \geq 0$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

We say that a Banach space X is the *polyhedral direct sum* of Banach spaces $X_1, ..., X_n$ if $X = X_1 \oplus \ldots \oplus X_n$ and the norm on X is of the form

$$\|x\|_\pi = \pi(\|x(1)\|, \ldots, \|x(n)\|), \quad x = (x(1), \ldots, x(n)),$$

where π is a polyhedral non-decreasing norm on $\mathbb{R}^n_+$ with respect to the co-ordinate wise ordering on $\mathbb{R}^n_+$. In this case, we write

$$X = (X_1 \oplus \ldots \oplus X_n)_\pi.$$

1.2 Notations and Conventions

In this section, we introduce some notations and conventions which will be used in the subsequent chapters.

In this thesis, we restrict ourselves to real scalars and all subspaces we consider are assumed to be closed. We consider every Banach space X, under the canonical embedding, as a subspace of X^{**} and do not write the embedding explicitly. Also, if a Banach space Y is isometric to a subspace of a Banach space X, then without loss of generality, we will consider Y as a subspace of X.

For a Banach space X, we denote by $B_X(x, r)$ (or $B(x, r)$, if there is no scope for confusion) the closed ball of radius $r > 0$ around $x \in X$. The closed unit ball and the unit sphere of X will be denoted by B_X and S_X respectively.

For a Banach space X, we denote by $NA(X)$ the set of all norm attaining functionals on X and by $NA_1(X)$ the set of all norm attaining functionals on X whose norm is one. Precisely,

$$NA(X) = \{f \in X^* : \text{there exists an element } x \in B_X \text{ such that } f(x) = \|f\|\} \text{ and}$$
$$NA_1(X) = NA(X) \bigcap S_{X^*}.$$

For a complete positive σ-finite measure space (Ω, Σ, μ), we denote by $L_p(\mu, X)$ the Banach space of all Bochner p-integrable (essentially bounded for $p = \infty$) functions on Ω with values in X, endowed with the usual p-norm (see [11] for details).

For a compact Hausdorff space K and a Banach space X, we denote by $C(K,X)$ the space of all X-valued continuous functions defined on K, endowed with the supremum norm. $C(K,\mathbb{R})$ will be denoted by $C(K)$.

For a completely regular space T, let βT denote the Stone-Čech compactification of T.

Definition 1.2.1. For an arbitrary collection $\{X_i : i \in I\}$ of Banach spaces,

(a) ℓ_p-sum ($1 \le p < \infty$) of X_i ($i \in I$) is defined by

$$\bigoplus_{i\in I}{}_p X_i = \left\{x \in \textstyle\prod_{i\in I} X_i : \|x\| = (\sum_{i\in I}\|x(i)\|^p)^{1/p} < \infty\right\}.$$

When $X_i = \mathbb{R}$ for all $i \in I$, $\bigoplus_{i\in I}{}_p X_i$ will be denoted by $\ell_p(I)$.

(b) ℓ_∞-sum of X_i ($i \in I$) is defined by

$$\bigoplus_{i\in I}{}_\infty X_i = \left\{x \in \textstyle\prod_{i\in I} X_i : \|x\| = \sup_{i\in I}\|x(i)\| < \infty\right\}.$$

When $X_i = \mathbb{R}$ for all $i \in I$, $\bigoplus_{i\in I}{}_\infty X_i$ will be denoted by $\ell_\infty(I)$.

For a finite family of Banach spaces $\{X_1, \dots, X_k\}$, ℓ_∞-sum of X_i ($1 \le i \le k$) will be denoted by $(X_1 \bigoplus \dots \bigoplus X_k)_{\ell_\infty^k}$.

(c) c_0-sum of X_i ($i \in I$) is defined by

$$\bigoplus_{i\in I}{}_{c_0} X_i = \left\{x \in \textstyle\prod_{i\in I} X_i : \{i \in I : \|x(i)\| > \varepsilon\} \text{ is finite for all } \varepsilon > 0\right\}$$

with the supremum norm on it.

When $X_i = \mathbb{R}$ for all $i \in I$, $\bigoplus_{i\in I}{}_{c_0} X_i$ will be denoted by $c_0(I)$.

When there is no confusion, we omit writing the index set for the direct sums.

Our notations are otherwise standard. Any unexplained terminology can be found in [11, 22, 29].

1.3 Chapter-wise Summary

In this section, we give a chapter-wise summary of the thesis.

In **Chapter 2**, we mainly consider various notions of proximinality in Banach spaces. In [40], Pollul raised the following question on transitivity of proximinality.

(P1) Which Banach spaces X have the following property: for any subspaces Y and Z of X with $Z \subseteq Y$, if $\dim(X/Y) = \dim(Y/Z) = 1$, Z is proximinal in Y and Y is proximinal in X, then Z is proximinal in X?

Later in [23], Indumathi asked a more general question.

(P2) Which Banach spaces X have the following property: for any subspaces Y and Z of X with $Z \subseteq Y$, if $\dim(X/Z) = n < \infty$, Z is proximinal in Y and Y is proximinal in X, then Z is proximinal in X?

A Banach space X with the property described in (P2) is called a $P(n)$ space and X is said to be a *Pollul space* if it is a $P(n)$ space for every $n \geq 2$. i.e., proximinality is transitive for finite co-dimensional subspaces. Clearly, reflexive spaces are Pollul spaces. c_0 and $K(\ell_2)$ (the space of all compact operators on ℓ_2) are non-trivial examples of Pollul spaces. It is also proved in [23] that the infinite dimensional $C(K)$ and $L_1(\mu)$ spaces are not $P(2)$ spaces and hence are not Pollul spaces. Motivated by this, we ask the transitivity problem for various degrees of proximinality (viz. proximinality, strong proximinality, the $1\frac{1}{2}$-ball property, semi M-ideal etc). Precisely, we ask the following:

(Q) Let (P) be one of these proximinality properties and let Y and Z be subspaces of X with $Z \subseteq Y \subseteq X$ such that Z has the property (P) in Y and Y has the property (P) in X , then is it necessary that Z has the property (P) in X?

We had already observed that the answer is not affirmative if the property (P) is proximinality. In [13], Dutta and Narayana proved that the transitivity of strong proximinality fails in ℓ_1. In [14], they also proved that the strong proximinality is a transitive relation for finite co-dimensional subspaces of $C(K)$. From [39, Example 6], it follows that the $1\frac{1}{2}$-ball property fails to be transitive in $\mathbb{R}^3$, equipped with ℓ_1-norm. Due to the fact that in a finite dimensional space the $1\frac{1}{2}$-ball property implies the strong $1\frac{1}{2}$-ball property, the same example also ensures that the strong $1\frac{1}{2}$-ball property fails to be transitive. But in [39], by using intersection properties of balls, Payá and Yost proved that the notions of being a semi M-ideal and being an M-ideal are transitive. In this chapter, we discuss the above mentioned transitivity problem and proximinality related problems. For instance, we prove that if (P) is strong proximinality and if Y is an M-ideal in X, then the question (Q) has an affirmative answer. In order to prove this, we prove that for a finite co-dimensional subspace Y of a Banach space X, Y is strongly proximinal in X if and only if $Y^{\perp\perp}$ is strongly proximinal in X^{**}.

In [19], Godefroy and Indumathi proved that if Y is a finite co-dimensional strongly proximinal subspace of X, then $Y^{\perp}$ is contained in the set of SSD-points of X^*. It remains an open problem whether the converse of this is true. In this chapter, we show that the converse is true in L_1-predual spaces. In order to prove this, we first show that for a positive measure μ, the notions of SSD-point and QP-point coincide in $L_1(\mu)$.

We also study the following problem: if Y is a subspace of a Banach space X and $f \in S_{Y^*}$ is an SSD-point of Y^*, then can we say that all the norm preserving Hahn-Banach extensions of f are SSD-points of X^*? We show that the answer is negative in general and is affirmative if the subspace Y is an M-ideal.

In [25, Corollary 2.5], it is stated that M-ideals are ball proximinal. In this chapter, we disprove this by giving an example. We also give a class of Banach spaces in which M-ideals are ball proximinal.

In **Chapter 3**, we discuss various notions of proximinality in vector-valued function spaces and direct sums. In approximation theory, one of the important problems is the following: Suppose that a subspace Y of a Banach space X has one of the proximinality properties in X. Does it follow that $L_1(\mu, Y)$ has the same proximinality property in $L_1(\mu, X)$? Since, for a measurable set E with $\mu(E) > 0$, the map $P : L_1(\mu, X) \to L_1(\mu, X)$ defined by $P(f) = f\chi_E$ is a non-trivial L-projection, by [22, Theorem 1.8], $L_1(\mu, X)$ cannot have any M-ideal provided that the dimension of $L_1(\mu, X)$ is greater than 2. Therefore if the dimension of $L_1(\mu, X)$ is greater than 2, then $L_1(\mu, Y)$ can never be an M-ideal in $L_1(\mu, X)$. Then under the assumption that Y is an M-ideal, one can ask about the strongest proximinality property that $L_1(\mu, Y)$ possesses. Proposition 3.1.14 gives a partial answer to this question.

In [37], Mendoza proved that $L_1(\mu, Y)$ is not proximinal in $L_1(\mu, X)$ even if Y is proximinal in X, but for a separable proximinal subspace Y of X, $L_1(\mu, Y)$ is proximinal in $L_1(\mu, X)$. For a non-atomic σ-finite countably generated measure space, an analogous result for the strong $1\frac{1}{2}$-ball property is proved in [44]. In this chapter, we prove these two results for every non-separable subspace Y of X satisfying a general condition: "each separable subspace of Y is contained in a separable subspace of Y that has the appropriate proximinality property in X ". We also give a class of Banach spaces and their subspaces where this general condition holds. But if the proximinality property under consideration is strong proximinality or the $1\frac{1}{2}$-ball property, an analogous result for the above problem is not known.

Now moving to the discrete version of the above problem, one can ask about the stability of these proximinality properties under ℓ_p-sums ($1 \leq p \leq \infty$) and c_0-sums of Banach spaces. In the discrete version, we ask the following question: Suppose that (P) is one of the proximinality properties and that for each $i \in I$, Y_i is a subspace of X_i having property (P) in X_i. Does this imply that, for $1 \leq p \leq \infty$, $\bigoplus_p Y_i$ and $\bigoplus_{c_0} Y_i$ have the same property in $\bigoplus_p X_i$ and $\bigoplus_{c_0} X_i$ respectively? The answer is affirmative if the property under consideration is proximinality. In [44], it is proved that the $1\frac{1}{2}$-ball property and the strong $1\frac{1}{2}$-ball property are stable under ℓ_∞-sums and c_0-sums. It is also proved in [44] that the $1\frac{1}{2}$-ball property is stable under ℓ_1-sums. But the stability of the strong $1\frac{1}{2}$-ball property under ℓ_1-sums is still not known.

It is proved in [30] that the strong proximinality is stable under countable c_0-sums and finite ℓ_∞-sums of Banach spaces. In this chapter, we prove that the strong proximinality is also stable under finite ℓ_1-sums. In fact, we prove that proximinality and strong proximinality (under an additional assumption) are stable under the polyhedral direct sums of

Banach spaces. Moreover, we characterize SSD-points of ℓ_1-sums of dual spaces.

In **Chapter 4**, we study the intersection properties of balls in Banach spaces. Different types of ball intersection properties were studied in the literature, namely the n-ball property and the $n.2.I.P.$ In this chapter, we study relative intersection properties of balls in Banach spaces. The main aim of this study is to investigate the class of Banach spaces which admit weighted Chebyshev centers for finite sets. Our motivation for this work comes from the work [47] of Veselý where he studied a new class of Banach spaces, namely the class (GC) which is defined in terms of the existence of weighted Chebyshev centers. In the same article, he also characterized such spaces using intersection properties of balls.

In [6], Bandyopadhyay and Rao considered some general results about the class (GC) using the concept of central subspaces. In fact, they characterized the class (GC) and produced several examples of Banach spaces which belong to the class (GC). In this chapter, we introduce and study a new notion of almost central subspaces which is weaker than that of central subspaces. Using this concept, we obtain some new results about the class (GC) and also about some of the other types of intersection properties of balls studied in the literature. In particular, we characterize L_1-predual spaces in terms of the almost central subspaces and give some examples of Banach spaces which belong to the class (GC).

The problem of characterizing 1-complemented subspaces of Banach spaces is of great importance in the theory of Banach spaces. It is well-known that subspaces of Hilbert spaces are 1-complemented. A classical result of Kakutani states that if every subspace of a Banach space X (with $\dim(X) \geq 3$) is 1-complemented in X, then X is a Hilbert space. In 1940, Phillips proved that c_0 is not 1-complemented in ℓ_∞. In [8], Baronti characterized finite co-dimensional 1-complemented subspaces of ℓ_∞ and in [7], Baronti and Papini characterized finite co-dimensional 1-complemented subspaces of c_0. In this chapter, we extend these results to $c_0(\Gamma)$ and $\ell_\infty(\Gamma)$ for any infinite discrete set Γ. Using this, we prove that $\ell_\infty(\Gamma)$ cannot have a finite co-dimensional 1-complemented subspace containing $c_0(\Gamma)$. We also give a simple proof for the implication (iii) $\implies$ (ii) of Theorem 1.1.43 when K is an extremally disconnected space.

In this chapter, we also derive several sufficient conditions for a semi M-ideal to be an M-ideal in terms of these intersection properties of balls. Some sufficient conditions for a central subspace to be an almost constrained subspace are also obtained. Moreover, we prove the stability of some of the ball intersection properties in quotient spaces, direct sums, vector-valued continuous function spaces and injective tensor product spaces. We also prove that the following question raised in [6] by Bandyopadhyay and Rao has an affirmative answer: for a family $\{X_\alpha : \alpha \in I\}$ of Banach spaces, is $\bigoplus_{c_0} X_\alpha$ a central subspace of $\bigoplus_\infty X_\alpha$?

Chapter 2

Transitivity of Proximinality Properties in Banach Spaces

In this chapter, we discuss the transitivity of various degrees of proximinality in Banach spaces. When transitivity does not hold, we investigate these properties under some additional assumptions on the intermediate space. For instance, we prove that if $Z \subseteq Y \subseteq X$, where Z is a finite co-dimensional subspace of X which is strongly proximinal in Y and Y is an M-ideal in X, then Z is strongly proximinal in X. In order to prove this, we show that for a finite co-dimensional proximinal subspace Y of a Banach space X, Y is strongly proximinal in X if and only if $Y^{\perp\perp}$ is strongly proximinal in X^{**}. Using this, we prove that in an abstract L_1-space, the notions of SSD-point and QP-point coincide. We also prove that if Y is an M-ideal in a Banach space X and $f \in S_{Y^*}$ is an SSD-point of Y^*, then any norm preserving Hahn-Banach extension of f to X is an SSD-point of X^*. Moreover, we give an example to show that M-ideals need not be ball proximinal and also we give a class of Banach spaces in which M-ideals are ball proximinal.

Most of the results in this chapter are from [28].

2.1 Transitivity of Strong Proximinality

In [42], Rao observed that the subspace $V = \{x \in c_0 : x(2n) = n\,x(2n-1), n \geq 1\}$ is proximinal in c_0, but not in ℓ_∞. Since c_0 is an M-ideal in ℓ_∞, this example shows that proximinality need not be transitive even when one of the subspace has the stronger property of being an M-ideal. But in [24], Indumathi proved that every finite co-dimensional proximinal subspace of c_0 continues to be proximinal in ℓ_∞.

We now give an example to show that the strong proximinality need not be transitive even with the finite co-dimensionality assumption on subspaces.

Example 2.1.1. There exist two subspaces Z and Y of finite co-dimension in ℓ_1 such that Z is strongly proximinal in Y and Y is strongly proximinal in ℓ_1, but Z is not strongly proximinal in ℓ_1.

Proof. Let $f = (0, 1, 1, \ldots)$ and $g = (1, -\frac{1}{2}, -\frac{1}{3}, \ldots)$. Then, by Theorem 1.1.6, f and g are SSD-points of ℓ_∞. Since $\ell_\infty = C(\beta\mathbb{N})$, by Theorem 1.1.10, we can see that the notions of SSD-point and QP-point coincide in ℓ_∞. Thus f and g are QP-points of ℓ_∞. Let $Z = \ker(f) \cap \ker(g)$ and $Y = \ker(f)$. Since f is a QP-point of ℓ_∞, Y is strongly proximinal in ℓ_1. Also, since g attains its norm on Y and g is a QP-point of ℓ_∞, by the proof of [13, Proposition 4.2], $g|_Y$ is a QP-point of Y^*. Hence $Z = \ker(g|_Y)$ is strongly proximinal in Y. Since $\sup\{|(f+g)(n)| : |(f+g)(n)| \neq 1\} = 1$, by Theorem 1.1.6, $f + g \in Z^\perp$ is not an SSD-point of ℓ_∞. Hence, by Theorem 1.1.8, Z is not strongly proximinal in ℓ_1. □

Our next result shows that transitivity holds under stronger assumptions on the intermediate space.

We call $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ a monotone map if $\varphi(\alpha_1, \beta) \geq \varphi(\alpha_2, \beta)$ whenever $\alpha_1 \geq \alpha_2$.

Proposition 2.1.2. *Let Y and Z be subspaces of a Banach space X such that $X = Y \oplus Z$ and let $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a monotone map such that for $x \in X$, $\|x\| = \varphi(\|y\|, \|z\|)$, where $x = y + z$ with $y \in Y$ and $z \in Z$. Suppose that for every sequence (α_n) in $\mathbb{R}_+$*

and for $\alpha, \beta \in \mathbb{R}_+$, $\varphi(\alpha_n, \beta) \to \varphi(\alpha, \beta)$ implies $\alpha_n \to \alpha$. If W is a proximinal (strongly proximinal) subspace of Y, then it is proximinal (strongly proximinal) in X.

Proof. Suppose W is proximinal in Y. Then for $x \in X$, we get

$$d(x, W) = \varphi(d(y, W), \|z\|), \quad \text{where } x = y + z \text{ with } y \in Y \text{ and } z \in Z.$$

For, let $w_0 \in P_W(y)$. Then

$$\begin{aligned} d(x, W) &= \inf\{\|x - w\| : w \in W\} \\ &= \inf\{\varphi(\|y - w\|, \|z\|) : w \in W\} \\ &\leq \varphi(\|y - w_0\|, \|z\|) \\ &= \varphi(d(y, W), \|z\|). \end{aligned}$$

On the other hand, it is clear from

$$\varphi(d(y, W), \|z\|) \leq \varphi(\|y - w\|, \|z\|) \leq \|x - w\| \text{ for all } w \in W$$

that $\varphi(d(y, W), \|z\|) \leq d(x, W)$. Thus $d(x, W) = \varphi(d(y, W), \|z\|)$. Now it follows that $w_0 \in P_W(x)$. Thus $P_W(y) \subseteq P_W(x)$. Hence W is proximinal in X.

Note that the convergence assumption on φ has not used yet.

Now let W be strongly proximinal in X and let (w_n) be a sequence in W such that $\|x - w_n\| \to d(x, W)$. Then, by the assumption on φ, $\|y - w_n\| \to d(y, W)$ and hence, by the strong proximinality of W in Y, $d(w_n, P_W(y)) \to 0$. Since $P_W(y) \subseteq P_W(x)$, $d(w_n, P_W(x)) \to 0$ and hence the strong proximinality of W in X follows. □

As an immediate consequence of Proposition 2.1.2, we have the following:

Corollary 2.1.3. *If Y is an L-summand in a Banach space X, then any proximinal (strongly proximinal) subspace of Y is proximinal (strongly proximinal) in X. Moreover, if Z is an M-summand in X, then any proximinal subspace of Z is proximinal in X.*

Proof. Let Z be a subspace of X such that $X = Y \bigoplus_1 Z$. Define $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by $\varphi(s, t) = s + t$ for $s, t \in \mathbb{R}_+$. Then φ is a monotone map such that $\|x\| = \varphi(\|y\|, \|z\|)$, where $x = y + z$ with $y \in Y$ and $z \in Z$. Since for every sequence (α_n) in $\mathbb{R}_+$ and for $\alpha, \beta \in \mathbb{R}_+$, $\varphi(\alpha_n, \beta) = \alpha_n + \beta \to \varphi(\alpha, \beta) = \alpha + \beta$ implies $\alpha_n \to \alpha$, the conclusion follows from Proposition 2.1.2. $\square$

We now recall a result from [24] to prove that the notion of strong proximinality pass through M-summands.

For a Banach space X, let $\mathcal{C}(X)$ denote the class of non-empty, bounded and closed subsets of X. Then the Hausdorff metric on $\mathcal{C}(X)$ is given by

$$h(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{z \in B} d(z, A)\right\} \quad \text{for } A, B \in \mathcal{C}(X).$$

Lemma 2.1.4 ([24, Fact 3.2])**.** *Let X be a Banach space and Y be a proximinal subspace of X. Let $x \in X \setminus Y$ and $\alpha > d(x, Y)$. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $z \in B(x, \delta)$ and for any $\beta > 0$ satisfying $|\beta - \alpha| < \delta$, we have*

$$h\left(B(x, \alpha) \bigcap Y,\, B(z, \beta) \bigcap Y\right) < \varepsilon,$$

where h is the Hausdorff metric on $\mathcal{C}(X)$.

Proposition 2.1.5. *Let X be a Banach space and let Y be an M-summand in X. If W is a strongly proximinal subspace of Y, then W is strongly proximinal in X.*

Proof. Suppose W is a strongly proximinal subspace of Y. Then, by Corollary 2.1.3, it follows that W is proximinal in X.

Now let $x \in X$ and $\varepsilon > 0$. Let Z be a subspace of X such that $X = Y \bigoplus_\infty Z$ and let $x = y + z$ with $y \in Y$ and $z \in Z$. An argument similar to the one used in the proof of Proposition 2.1.2 gives

$$d(x, W) = \max\{d(y, W), \|z\|\} \quad \text{and} \quad P_W(y) \subseteq P_W(x).$$

Case 1. $\|z\| > d(y, W)$.

In this case, since $d(x, W) = \|z\|$, we have

$$P_W(x) = B(y, \|z\|) \cap W \quad \text{and} \quad P_W(x, \eta) = B(y, \|z\| + \eta) \cap W \quad \text{for all } \eta > 0.$$

Since $\|z\| > d(y,W)$, by Lemma 2.1.4, there exists a $\delta > 0$ such that for $u \in Y$ with $\|u-y\| < 2\delta$ and for $\beta > 0$ with $|\beta - \|z\|| < 2\delta$, we get

$$h(B(y,\|z\|)\bigcap W,\ B(u,\beta)\bigcap W) < \varepsilon, \tag{2.1.1}$$

where h is the Hausdorff metric on $\mathcal{C}(Y)$.
Now put $u = y$ and $\beta = \|z\| + \delta$ in (2.1.1). Then we get

$$h\left(B(y,\|z\|)\bigcap W, B(y,\|z\|+\delta)\bigcap W\right) < \varepsilon.$$

Thus $B(y,\|z\|+\delta)\bigcap W \subseteq (B(y,\|z\|)\bigcap W) + \varepsilon B_X$ and hence $P_W(x,\delta) \subseteq P_W(x) + \varepsilon B_X$.
Case 2. $\|z\| \leq d(y,W)$.

In this case, since $d(x,W) = d(y,W)$, we have

$$P_W(x) = P_W(y) \text{ and } P_W(x,\eta) = P_W(y,\eta) \text{ for all } \eta > 0.$$

Since W is strongly proximinal in Y, there exists a $\delta > 0$ such that $P_W(y,\delta) \subseteq P_W(y)+\varepsilon B_Y$. Thus $P_W(x,\delta) \subseteq P_W(x) + \varepsilon B_X$.

Hence W is strongly proximinal in X. □

We now recall some notations from [19] which will be used in the remaining part of this section.
Let X be a Banach space and let $\{f_1,\ldots,f_n\}$ be a set of linearly independent functionals in X^*. Define $M_1, M_1^*, J_X(f_1)$ and $J_{X^{**}}(f_1)$ as:

$$\begin{aligned} M_1 &= \|f_1\|, & J_X(f_1) &= \{x \in S_X : f_1(x) = \|f_1\|\}, \\ M_1^* &= \|f_1\|, & J_{X^{**}}(f_1) &= \{x^{**} \in S_{X^{**}} : x^{**}(f_1) = \|f_1\|\}. \end{aligned}$$

Also, define $M_2, M_2^*, J_X(f_1,f_2)$ and $J_{X^{**}}(f_1,f_2)$ as:

$$\begin{aligned} M_2 &= \sup\{f_2(x) : x \in J_X(f_1)\}, & J_X(f_1,f_2) &= \{x \in J_X(f_1) : f_2(x) = M_2\}, \\ M_2^* &= \sup\{x^{**}(f_2) : x^{**} \in J_{X^{**}}(f_1)\}, & J_{X^{**}}(f_1,f_2) &= \{x^{**} \in J_{X^{**}}(f_1) : x^{**}(f_2) = M_2^*\}. \end{aligned}$$

Now, inductively obtain M_i, M_i^*, $J_X(f_1,\ldots,f_i)$ and $J_{X^{**}}(f_1,\ldots,f_i)$ for $1 \leq i \leq n$.

For $\varepsilon > 0$, define $J_X(f_1, \varepsilon)$ by

$$J_X(f_1, \varepsilon) = \{x \in B_X : f_1(x) > \|f_1\| - \varepsilon\}.$$

Having defined $J_X(f_1, \ldots, f_j, \varepsilon)$ for $1 \leq j \leq i < n$ inductively, define $J_X(f_1, \ldots, f_{i+1}, \varepsilon)$ by

$$J_X(f_1, \ldots, f_{i+1}, \varepsilon) = \{x \in J_X(f_1, \ldots, f_i, \varepsilon) : f_{i+1}(x) > M_{i+1} - \varepsilon\}.$$

The following result by Godefroy and Indumathi characterizes strongly proximinal subspaces of finite co-dimension.

Theorem 2.1.6. [19] *Let Y be a finite co-dimensional proximinal subspace of X. Then Y is strongly proximinal in X if and only if for any basis $\{f_1, \ldots, f_n\}$ of $Y^\perp$,*

$$\lim_{\varepsilon \to 0} [\sup\{d(x, J_X(f_1, \ldots, f_i)) : x \in J_X(f_1, \ldots, f_i, \varepsilon)\}] = 0$$

for $1 \leq i \leq n$.

In other words, a necessary and sufficient condition for the strong proximinality of a finite co-dimensional subspace Y of a Banach space X is: if $\{f_1, \ldots, f_n\}$ is a basis of $Y^\perp$ and $i \in \{1, \ldots, n\}$, then for every $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that $d(x, J_X(f_1, \ldots, f_i)) < \varepsilon$ whenever $x \in J_X(f_1, \ldots, f_i, \delta_\varepsilon)$.

We now recall some relations between the notations defined above.

Remark 2.1.7 ([19, Remark 1.2]). Let X be a Banach space and let $f \in S_{X^*}$ be an SSD-point of X^*. Then

(a) $J_{X^{**}}(f) = \overline{J_X(f)}^{w^*}$,

(b) $d(x, J_{X^{**}}(f)) = d(x, J_X(f))$.

Proposition 2.1.8. *Let X be a Banach space and let Y be a finite co-dimensional strongly proximinal subspace of X. Let $\{f_1, \ldots, f_n\} \subseteq S_{Y^\perp}$ be a basis of $Y^\perp$ and let M_i, M_i^*, $J_X(f_1, \ldots, f_i)$ and $J_{X^{**}}(f_1, \ldots, f_i)$ be defined as before. Then, for $1 \leq i \leq n$,*

(a) $M_i = M_i^*$,

(b) $J_{X^{**}}(f_1, \dots, f_i) = \overline{J_X(f_1, \dots, f_i)}^{w^*}$.

Proof. (a) Clearly, $M_1 = M_1^*$ and $M_k \leq M_k^*$ for $k = 1, \dots, n$. Let $i \in \{1, \dots, n\}$. Now suppose that $M_j = M_j^*$ for $1 \leq j \leq i$. Since $J_{X^{**}}(f_1, \dots, f_i)$ is w^*-compact, f_{i+1} attains its supremum over $J_{X^{**}}(f_1, \dots, f_i)$ at some element $x_0^{**} \in J_{X^{**}}(f_1, \dots, f_i)$. Let (x_α) be a net in B_X such that $x_\alpha \to x_0^{**}$ in weak*-sense. Since $x_0^{**} \in J_{X^{**}}(f_1, \dots, f_i)$, $x_0^{**}(f_j) = M_j^* = M_j$ for $1 \leq j \leq i$. Hence for $1 \leq j \leq i$, $f_j(x_\alpha) \to M_j$. Since Y is a strongly proximinal subspace of X, by Theorem 2.1.6, it follows that $d(x_\alpha, J_X(f_1, \dots, f_i)) \to 0$. Now let (z_α) be a net in $J_X(f_1, \dots, f_i)$ such that $\|x_\alpha - z_\alpha\| \to 0$. Then $z_\alpha \to x_0^{**}$ in weak*-sense. Since $f_{i+1}(z_\alpha) \to x_0^{**}(f_{i+1}) = M_{i+1}^*$, we get $M_{i+1}^* = \lim_\alpha f_{i+1}(z_\alpha) \leq M_{i+1}$. Now the result follows by induction.

(b) Since f_1 is an SSD-point, we have $\overline{J_X(f_1)}^{w^*} = J_{X^{**}}(f_1)$. For $i = 2$, it is easy to see that $\overline{J_X(f_1, f_2)}^{w^*} \subseteq J_{X^{**}}(f_1, f_2)$. Now let $x^{**} \in J_{X^{**}}(f_1, f_2)$ and let (x_α) be a net in B_X such that $x_\alpha \to x^{**}$ in weak*-sense. Since $f_1(x_\alpha) \to x^{**}(f_1) = M_1$, $d(x_\alpha, J_X(f_1)) \to 0$. Choose a net (y_α) in $J_X(f_1)$ such that $\|x_\alpha - y_\alpha\| \to 0$. Hence $y_\alpha \to x^{**}$ in weak*-sense. Since $f_2(y_\alpha) \to x^{**}(f_2) = M_2$, $d(y_\alpha, J_X(f_1, f_2)) \to 0$. Hence there exists a net (z_α) in $J_X(f_1, f_2)$ such that $\|y_\alpha - z_\alpha\| \to 0$. Thus $z_\alpha \to x^{**}$ in weak*-sense. i.e., $\overline{J_X(f_1, f_2)}^{w^*} = J_{X^{**}}(f_1, f_2)$.

By a similar argument, we can prove (b) for $i > 2$. □

Lemma 2.1.9. *Let Y be a finite co-dimensional strongly proximinal subspace of a Banach space X and let $\{f_1, \dots, f_n\} \subset S_{Y^\perp}$ be a basis of $Y^\perp$. Then, for $x \in B_X$,*

$$d(x, J_X(f_1, \dots, f_i)) = d(x, J_{X^{**}}(f_1, \dots, f_i)).$$

Proof. If $n = 1$, then the conclusion follows from Remark 2.1.7(b).

Since no new ideas are required for $n > 2$, we prove the lemma only for $n = 2$.

Hence we have to show that for $x \in B_X$, $d(x, J_X(f_1, f_2)) = d(x, J_{X^{**}}(f_1, f_2))$.

Let $d = d(x, J_{X^{**}}(f_1, f_2))$. Since $J_{X^{**}}(f_1, f_2)$ is weak*-compact, it is proximinal. Choose $\phi \in J_{X^{**}}(f_1, f_2)$ such that $\|x - \phi\| = d$. Since Y is strongly proximinal in X, given $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that $d(x, J_X(f_1, f_2)) < \varepsilon$ whenever $x \in J_X(f_1, f_2, \delta_\varepsilon)$.

Now choose $\varepsilon > 0$ arbitrarily. Let $E = \text{span}\{x, \phi\} \subseteq X^{**}$ and $F = \text{span}\{f_1, f_2\} \subseteq X^*$. Let ε' be such that $0 < \varepsilon' < \min\{\delta_{\varepsilon/2^2}, \frac{\varepsilon}{2(d+1)}\}$. Now, by the principle of local reflexivity (see [35]), there exists a bounded linear map $T : E \to X$ such that:

(1) $Tx = x$,

(2) $(1 - \varepsilon') \leq \|T(z^{**})\| \leq (1 + \varepsilon')$ for $z^{**} \in S_E$,

(3) $f_i(T(z^{**})) = z^{**}(f_i)$ for $z^{**} \in E$ and $i = 1, 2$.

Let $x_1 = \frac{T(\phi)}{\|T(\phi)\|}$. Now

$$\begin{aligned}
\|x - x_1\| &\leq \|x - T\phi\| + \|T\phi - \frac{T\phi}{\|T\phi\|}\| \\
&= \|T(x - \phi)\| + |1 - \|T\phi\|| \\
&\leq (1 + \varepsilon')d + \varepsilon' \\
&= d + \varepsilon'(1 + d) < d + \frac{\varepsilon}{2}
\end{aligned}$$

and for $i = 1, 2$, we have

$$\begin{aligned}
f_i(x_1) &= f_i\left(\frac{T\phi}{\|T\phi\|}\right) \\
&\geq \frac{M_i}{1 + \varepsilon'} \\
&= M_i - \frac{M_i\varepsilon'}{1 + \varepsilon'} \\
&= M_i - \frac{f_i(T(\phi))\varepsilon'}{1 + \varepsilon'} \\
&> M_i - \varepsilon' \\
&> M_i - \delta_{\varepsilon/2^2}.
\end{aligned}$$

Thus $x_1 \in J_X(f_1, f_2, \delta_{\varepsilon/2^2})$ and hence we have

$$d(x_1, J_{X^{**}}(f_1, f_2)) \leq d(x_1, J_X(f_1, f_2)) < \varepsilon/2^2.$$

Let $\phi_1 \in J_{X^{**}}(f_1, f_2)$ be such that $\|x_1 - \phi_1\| < \varepsilon/2^2$. Then, again by principle of local reflexivity, there exists an element $x_2 \in B_X$ such that $\|x_1 - x_2\| < \varepsilon/2^2$ and $f_i(x_2) > M_i - \delta_{\varepsilon/2^3}$.

Proceeding inductively, we obtain a sequence (x_n) in B_X such that $\|x_n - x_{n-1}\| < \varepsilon/2^n$ and $f_i(x_n) > M_i - \delta_{\varepsilon/2^{n+1}}$ for all $n \in \mathbb{N}$ and for $i = 1, 2$. Without loss of generality, we assume that $\delta_{\varepsilon/2^n} \to 0$.

Clearly, (x_n) is a Cauchy sequence and hence there exists an element $z \in B_X$ such that $z = \lim_{n\to\infty} x_n$. Now $f_i(z) = M_i$ for $i = 1, 2$ and hence $z \in J_X(f_1, f_2)$. Also, for all $n \in \mathbb{N}$, we have $\|x - x_n\| \leq d + \varepsilon/2 + \ldots + \varepsilon/2^n$. Now letting $n \to \infty$, we get $\|x - z\| \leq d + \varepsilon$. Since $\varepsilon > 0$ is arbitrary and $z \in J_X(f_1, f_2)$, we have $d(x, J_X(f_1, f_2)) \leq d = d(x, J_{X^{**}}(f_1, f_2))$. Since $J_X(f_1, f_2) \subseteq J_{X^{**}}(f_1, f_2)$, we have $d(x, J_X(f_1, f_2)) \geq d(x, J_{X^{**}}(f_1, f_2))$ and hence the result follows. $\square$

We now recall some results regarding SSD-points to motivate our next result.

In [16, Theorem 1.2], Franchetti and Payá proved that for a Banach space X, an element $u \in S_X$ is an SSD-point of X if and only if u strongly exposes the set $J_{X^*}(u)$, in the sense that the distance $d(f_n, J_{X^*}(u))$ tends to zero for any sequence (f_n) in B_{X^*} such that $f_n(u) \to 1$. But we can easily observe that this is equivalent to the following: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(f, J_{X^*}(u)) < \varepsilon$ whenever $f \in B_{X^*}$ and $f(u) > 1 - \delta$. Hence we get:

Lemma 2.1.10. *Let X be a Banach space and $u \in S_X$. Then the following are equivalent:*

(i) *u is an* SSD-*point of X.*

(ii) *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$d(f, J_{X^*}(u)) < \epsilon \ \text{ whenever } f \in B_{X^*} \text{ and } f(u) > 1 - \delta.$$

We also have the following characterization of an SSD-point of a dual space.

Lemma 2.1.11 ([19, Lemma 1.1])**.** *Let X be a Banach space and $f \in S_{X^*}$. Then the following are equivalent:*

(i) *f is an* SSD-*point of X^*.*

(ii) *$f \in NA_1(X)$ and for every $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$d(x, J_X(f)) < \epsilon \ \textit{ whenever } \ x \in B_X \textit{ and } f(x) > 1 - \delta.$$

The following result also characterizes an SSD-point of a dual space.

Corollary 2.1.12. *Let X be a Banach space and $f \in S_{X^*}$. Then f is an* SSD*-point of X^* if and only if f is an* SSD*-point of X^{***}.*

Proof. Suppose f is an SSD-point of X^*. Let $\varepsilon > 0$. Since f is an SSD-point of X^*, $f \in NA_1(X)$. Thus $f \in NA_1(X^{**})$. Now, by Lemma 2.1.10, there exists a $\delta > 0$ such that $d(x^{**}, J_{X^{**}}(f)) < \varepsilon$ whenever $x^{**} \in B_{X^{**}}$ and $x^{**}(f) > 1 - \delta$. Then, by Lemma 2.1.11, it follows that f is an SSD-point of X^{***}.

Conversely, suppose that f is an SSD-point of X^{***}. Let $\varepsilon > 0$. Then, by Lemma 2.1.11, there exists a $\delta > 0$ such that $d(x^{**}, J_{X^{**}}(f)) < \varepsilon$ whenever $x^{**} \in B_{X^{**}}$ and $x^{**}(f) > 1 - \delta$. Hence, by Lemma 2.1.10, f is an SSD-point of X^*. □

Since for a Banach space X and for an element $f \in S_{X^*}$, f is an SSD-point of X^* if and only if $\ker(f)$ is strongly proximinal in X, the following result is immediate from Corollary 2.1.12.

Proposition 2.1.13. *Let Y be subspace of co-dimension 1 in a Banach space X. Then Y is strongly proximinal in X if and only if $Y^{\perp\perp}$ is strongly proximinal in X^{**}.*

Proof. Let $Y = \ker(f)$ for some $f \in S_{X^*}$. Then

$$\begin{aligned} Y \text{ is strongly proximinal in } X &\iff f \text{ is an SSD-point of } X^* \\ &\iff f \text{ is an SSD-point of } X^{***} \\ &\iff Y^{\perp\perp} = \ker(f) \text{ is strongly proximinal in } X^{**}. \quad \square \end{aligned}$$

Our next result generalizes Proposition 2.1.13 for finite co-dimensional case.

Proposition 2.1.14. *Let Y be a finite co-dimensional proximinal subspace of a Banach space X. Then Y is strongly proximinal in X if and only if $Y^{\perp\perp}$ is strongly proximinal in X^{**}.*

Proof. Suppose that Y is strongly proximinal in X. Let $\{f_1, \ldots, f_n\} \subset S_{Y^{\perp\perp\perp}}$ be a basis of $Y^{\perp\perp\perp}$. As $Y^{\perp}$ is finite dimensional, $Y^{\perp\perp\perp} = Y^{\perp}$. Thus $\{f_1, \ldots, f_n\}$ is also a basis of $Y^{\perp}$.

Now let $i \in \{1, \ldots, n\}$ and let $\varepsilon > 0$. Since Y is strongly proximinal in X, there exists a $\delta > 0$ such that $d(x, J_X(f_1, \ldots, f_i)) < \varepsilon$ whenever $x \in J_X(f_1, \ldots, f_i, \delta)$. Then for $x^{**} \in J_{X^{**}}(f_1, \ldots, f_i, \delta)$, we have $x^{**}(f_j) > M_j - \delta$ for $1 \leq j \leq i$. Let (x_α) be a net in B_X such that $x_\alpha \to x^{**}$ in weak*-sense. Now, without loss of generality, we assume that $f_j(x_\alpha) > M_j - \delta$ for all α and for $1 \leq j \leq i$. Hence there exists an element $z_\alpha \in J_X(f_1, \ldots, f_i)$ such that $\|x_\alpha - z_\alpha\| < \varepsilon$. Passing to a subnet of (z_α), if necessary, we may assume that $z_\alpha \to \phi$ in weak*-sense for some $\phi \in J_{X^{**}}(f_1, \ldots, f_i)$. Thus $(x_\alpha - z_\alpha) \to (x^{**} - \phi)$ in the weak*-sense. Then $\|x^{**} - \phi\| \leq \underline{\lim}_\alpha \|x_\alpha - z_\alpha\| \leq \varepsilon$. Therefore $d(x^{**}, J_{X^{**}}(f_1, \ldots, f_i)) \leq \|x^{**} - \phi\| < \varepsilon$. Hence, by Theorem 2.1.6, $Y^{\perp\perp}$ is strongly proximinal in X^{**}.

Conversely, suppose that $Y^{\perp\perp}$ is strongly proximinal in X^{**}. Let $\{f_1, \ldots, f_n\} \subset S_{Y^{\perp}}$ be a basis of $Y^{\perp}$ and let $\varepsilon > 0$. Since $Y^{\perp\perp\perp} = Y^{\perp}$, $\{f_1, \ldots, f_n\}$ is also a basis of $Y^{\perp\perp\perp}$. Let $i \in \{1, \ldots, n\}$. It is easy to observe that $J_X(f_1, \ldots, f_i, \delta) \subseteq J_{X^{**}}(f_1, \ldots, f_i, \delta)$. Since $Y^{\perp\perp}$ is strongly proximinal in X^{**}, there exists a $\delta > 0$ such that $d(x^{**}, J_{X^{**}}(f_1, \ldots, f_i)) < \varepsilon$ whenever $x^{**} \in J_{X^{**}}(f_1, \ldots, f_i, \delta)$. Then for $x \in J_X(f_1, \ldots, f_i, \delta)$, by Lemma 2.1.9, we have $d(x, J_X(f_1, \ldots, f_i)) = d(x, J_{X^{**}}(f_1, \ldots, f_i)) < \varepsilon$ and this completes the proof. $\square$

We are now ready to prove the main theorem of this section.

Theorem 2.1.15. *Let X be a Banach space and let Z be a finite co-dimensional proximinal subspace of X. Let Y be an M-ideal in X and let $Z \subseteq Y \subseteq X$. If Z is strongly proximinal in Y, then Z is strongly proximinal in X.*

Proof. Let Z be strongly proximinal in Y. Then, by Proposition 2.1.14, it follows that $Z^{\perp\perp}$ is strongly proximinal in $Y^{\perp\perp}$. Since $Y^{\perp\perp}$ is an M-summand in X^{**}, by Proposition 2.1.5, $Z^{\perp\perp}$ is strongly proximinal in X^{**}. Then, again by Proposition 2.1.14, Z is strongly proximinal in X. $\square$

Example 2.1.1 shows that the strong proximinality, in general, need not be transitive. We do not know whether we can replace the M-ideal assumption in Theorem 2.1.15 by the semi M-ideal assumption.

Question 2.1.16. Let Y be a semi M-ideal in X and Z be a strongly proximinal subspace of Y such that Z is of finite co-dimension in X. Is Z also strongly proximinal in X ?

Example 2.5.2 asserts that the transitivity of strong proximinality fails if we assume only that Y is an ideal.

Remark 2.1.17. We do not know whether the finite co-dimensionality assumption on Y in Theorem 2.1.15 is necessary. The answer is not known even if the strong proximinality in Theorem 2.1.15 is replaced by proximinality.

2.2 SSD-points and Strong Proximinality

For an SSD-point f of X^*, there always exists a norm preserving Hahn-Banach extension of f to X^{**} which is an SSD-point of X^{***}, namely the canonical image of f in X^{***}. But it is not known whether any norm preserving Hahn-Banach extension of f to X^{**} is again an SSD-point. Coming to a more general set up, since X is an ideal in X^{**}, it is natural to ask the following:

Question 2.2.1. Let Y be an ideal in X and let f be an SSD-point of Y^*. If $\widetilde{f}$ is a norm preserving Hahn-Banach extension of f to X, then is $\widetilde{f}$ an SSD-point of X^*?

We now give an example to show that the above question does not have an affirmative answer.

Example 2.2.2. There exist a strongly proximinal subspace Y of ℓ_1 and an SSD-point of Y^* such that one of its norm preserving Hahn-Banach extension is not an SSD-point of ℓ_∞.

Proof. Let $f = (0, 1, 1, \ldots)$ and $g = (1, -\frac{1}{2}, -\frac{1}{3}, \ldots)$. Then f and g are QP-points of ℓ_∞. Let $Z = \ker(f) \cap \ker(g)$ and $Y = \ker(f)$. Since f is a QP-point of ℓ_∞, Y is strongly proximinal in ℓ_1. Also, since g attains its norm on Y and g is a QP-point of ℓ_∞, by the proof of [13, Proposition 4.2], $g|_Y$ is a QP-point of Y^* and hence is an SSD-point of Y^*. We observe that $f+g$ is a norm preserving Hahn-Banach extension of $g|_Y$. But, by Theorem 1.1.6, $f+g \in Z^\perp$ is not an SSD-point of ℓ_∞. $\square$

But the answer to Question 2.2.1 is affirmative under some extra assumptions on Y. Our next theorem is a particular case of [16, Proposition 2.1], but for the sake of completeness we outline the proof below.

Theorem 2.2.3. *Let Y be a semi L-summand in a Banach space X and let $y \in Y$ be an* SSD *point of Y. Then y is also an* SSD*-point of X.*

Proof. Let $P : X \to X$ be a semi L-projection with range Y, then

$$d^+(y)(x) = d^+(y)(Px) + \|x - Px\|.$$

Now the conclusion follows from

$$\frac{\|y+tx\|-1}{t} - d^+(y)(x) = \|Px\| \left(\frac{\|y+t\|Px\|\frac{Px}{\|Px\|}\|-1}{t\|Px\|} - d^+(y)(\tfrac{Px}{\|Px\|}) \right). \qquad \square$$

Since, for an M-ideal Y in X, $X^* = Y^* \bigoplus_1 Y^\perp$, the following corollary is immediate from Theorem 2.2.3.

Corollary 2.2.4. *If Y is an M-ideal in a Banach space X and $f \in S_{Y^*}$ is an* SSD*-point of Y^*, then the unique a norm preserving Hahn-Banach extension of f to X is an* SSD*-point of X^*.*

We now prove the following lemma which will be used in the next proposition.

Lemma 2.2.5. *Let Y be subspace of a Banach space X and let $y \in S_Y$. If y is a* QP*-point of X, then y is a* QP*-point of Y.*

Proof. Let y be a QP-point of X. Then there exists a $\delta > 0$ such that $J_{X^*}(z) \subseteq J_{X^*}(y)$ for all $z \in S_X$ with $\|z - y\| < \delta$.

Now let $u \in S_Y$ be such that $\|u - y\| < \delta$. Then $J_{Y^*}(u) \subseteq J_{Y^*}(y)$. For, let $f \in J_{Y^*}(u)$. Let $\widetilde{f}$ be a norm preserving Hahn-Banach extension of f to X. Then it follows that $\widetilde{f} \in J_{X^*}(u) \subseteq J_{X^*}(y)$. Thus $f(y) = \widetilde{f}(y) = 1$ and hence $f \in J_{Y^*}(y)$. Therefore y is a QP-point of Y. $\square$

Our next result gives a class of Banach spaces where the notions of SSD-point and QP-point coincide.

Proposition 2.2.6. *For a positive measure μ, an* SSD*-point of $L_1(\mu)$ is also a* QP*-point.*

Proof. Let $f \in L_1(\mu)$ be an SSD-point. Since $L_1(\mu)$ is an L-summand in its bidual, by Theorem 2.2.3, f is an SSD-point of $L_1(\mu)^{**}$. But $L_1(\mu)^{**}$ is isometric to $C(K)^*$ for some compact Hausdorff space K. Then, by Theorem 1.1.11, f is a QP-point of $L_1(\mu)^{**}$. Hence, by Lemma 2.2.5, f is a QP-point of $L_1(\mu)$. $\square$

The following theorem characterizes strongly proximinal subspaces of $C(K)$.

Theorem 2.2.7 ([14, Corollary 2.3])**.** *Let K be a compact Hausdorff space and Y be a finite co-dimensional subspace of $C(K)$. Then the following are equivalent*:

(i) *Y is strongly proximinal in $C(K)$.*

(ii) $Y^{\perp} \subseteq \{f : f$ *is an* SSD*-point of* $C(K)^*\} = \{f : f$ *is a* QP*-point of* $C(K)^*\}$.

Our next result is the generalization of Theorem 2.2.7 to an L_1-predual space.

Proposition 2.2.8. *Let X be an L_1-predual space and $Y \subseteq X$ be a finite co-dimensional proximinal subspace of X. Then the following are equivalent*:

(i) *Y is strongly proximinal in X.*

(ii) $Y^{\perp} \subseteq \{f \in X^* : f$ *is an* SSD*-point of* $X^*\} = \{f \in X^* : f$ *is a* QP*-point of* $X^*\}$.

Proof. The implication (i) $\implies$ (ii) follows from Theorem 1.1.8. To prove (ii) $\implies$ (i), suppose that $Y^{\perp} \subseteq \{f \in X^* : f$ is an SSD-point of $X^*\}$. Since Y is of finite co-dimension in X, we can see that $Y^{\perp\perp\perp} \subseteq \{\varphi \in X^{***} : \varphi$ is an SSD-point of $X^{***}\}$. Since X is an L_1-predual space, by Theorem 1.1.33, it follows that $X^{**} = C(K)$ for some compact Hausdorff space K. Then, by Theorem 2.2.7, $Y^{\perp\perp}$ is strongly proximinal in X^{**}. Hence, by Proposition 2.1.14, Y is strongly proximinal in X. $\square$

If Y is a strongly proximinal subspace of finite co-dimension in a Banach space X, then, by Theorem 1.1.8, Y is the intersection of finitely many strongly proximinal hyperplanes. We now prove the converse of this result here.

Corollary 2.2.9. *Let X be an L_1-predual space. Then the intersection of finitely many strongly proximinal subspaces of finite co-dimension in X is strongly proximinal in X.*

Proof. Let X be an L_1-predual space and let $\{Y_i\}_{i=1}^m$ be a finite family of strongly proximinal subspaces of finite co-dimension in X. Let $Y = \bigcap_{i=1}^m Y_i$. For $1 \leq i \leq m$, let $f_{i,1}, \ldots, f_{i,n_i}$ be SSD-points of X^* such that $Y_i = \bigcap_{k=1}^{n_i} \ker(f_{i,k})$. Thus $Y = \bigcap_{i,k} \ker(f_{i,k})$ and hence $Y^\perp = \operatorname{span}\{f_{i,k} : 1 \leq i \leq m, 1 \leq k \leq n_i\} \subseteq \{f \in X^* : f \text{ is an SSD-point of } X^*\}$. Hence, by Proposition 2.2.8, Y is strongly proximinal in X. $\square$

2.3 Transitivity of Ball Intersection Properties

In this section, we discuss a few ball intersection properties and their transitivity which are closely related to the notion of proximinality in Banach spaces.

We first recall the following result from [44].

Proposition 2.3.1 ([44, Proposition 2.4])**.** *Let Y be an M-ideal in a Banach space X and let Z be a subspace of Y. If Z has the $1\frac{1}{2}$-ball property in Y, then Z has the $1\frac{1}{2}$-ball property in X.*

We now prove a similar result for the (strong) n-ball property under a stronger assumption on Y.

Lemma 2.3.2. *Let Y be an M-summand in a Banach space X and let Z be a subspace of Y. For $n \in \mathbb{N}$, if Z has the (strong) n-ball property in Y, then Z has the (strong) n-ball property in X.*

Proof. Let $\varepsilon > 0$ and let $\{B(x_i, r_i)\}_{1 \le i \le n}$ be a family of n balls in X such that

$$B(x_i, r_i) \cap Z \neq \emptyset \text{ for all } i = 1, \dots, n \text{ and } \bigcap_{i=1}^{n} B(x_i, r_i) \neq \emptyset.$$

Let $x \in \bigcap_{i=1}^{n} B(x_i, r_i)$ and let $P : X \to X$ be an M-projection with range Y. Then

$$P(x) \in \bigcap_{i=1}^{n} B(P(x_i), r_i) \text{ and } B(P(x_i), r_i) \cap Z \neq \emptyset \text{ for all } i = 1, \dots, n.$$

Since Z has the n-ball property in Y, there is an element $z \in Z \bigcap (\bigcap_{i=1}^{n} B(P(x_i), r_i + \varepsilon))$. Hence $\|z - x_i\| \le \max\{\|z - P(x_i)\|, \|x_i - P(x_i)\|\} \le r_i + \varepsilon$ for $1 \le i \le n$.

Now the strong n-ball property of Z in X follows by taking $\varepsilon = 0$ in the above proof. □

Lemma 2.3.3. *Let Y be a subspace of a Banach space X. Then Y is a semi M-ideal in X if and only if $Y^{\perp\perp}$ is a semi M-ideal in X^{**}.*

Proof. Suppose Y is a semi M-ideal in X. i.e., $Y^{\perp}$ is a semi L-summand in X^*. Then, by Theorem 1.1.16, it follows that $Y^{\perp\perp}$ is a semi M-ideal in X^{**}.

Conversely, suppose that $Y^{\perp\perp}$ is a semi M-ideal in X^{**}. Let $\varepsilon > 0$. Let $B(x_1, r_1)$ and $B(x_2, r_2)$ be balls in X such that $B(x_i, r_i) \cap Y \neq \emptyset$ for $i = 1, 2$ and $B(x_1, r_1) \cap B(x_2, r_2) \neq \emptyset$. Let $x \in B(x_1, r_1) \cap B(x_2, r_2)$ and let $y_i \in B(x_i, r_i) \cap Y$ for $i = 1, 2$. Since $Y^{\perp\perp}$ is a semi M-ideal in X^{**} and is a weak*-closed subspace of X^{**}, $Y^{\perp\perp}$ has the strong 2-ball property in X^{**}. Hence there exists an element $x^{**} \in Y^{\perp\perp}$ such that $\|x^{**} - x_i\| \le r_i$ for $i = 1, 2$. Let $E = \operatorname{span}\{x_1, x_2, y_1, y_2, x, x^{**}\}$ and $r = \max\{r_1, r_2\}$. Then, by an extended version of principle of local reflexivity (see [10, Theorem 3.2]), there exists an operator $T_\epsilon \colon E \to X$ such that:

(1) $T_\epsilon(z) = z$ if $z \in E \cap X$,

(2) $T_\epsilon(E \cap Y^{\perp\perp}) \subseteq Y$,

(3) $\|T_\epsilon\| \le 1 + \dfrac{\epsilon}{r}$.

Now take $z = T_\varepsilon(x^{**})$. Then $z \in Y$ and $\|z - x_i\| \le r_i + \varepsilon$ for $i = 1, 2$. Hence Y is a semi M-ideal in X. □

Our next result gives a sufficient condition for a semi M-ideal Y in a Banach space X to be a semi M-ideal in X^{**}.

Corollary 2.3.4. *Let Y be a semi M-ideal in a Banach space X. Then Y is a semi M-ideal in X^{**} if and only if Y is an M-embedded space.*

Proof. Suppose Y is a semi M-ideal in X^{**}. Then Y is a semi M-ideal in its bidual $Y^{**} = Y^{\perp\perp}$ and hence, by Theorem 1.1.18, Y is an M-ideal in Y^{**}.
Conversely, suppose that Y is an M-embedded space. Since Y is a semi M-ideal in X, by Lemma 2.3.3, $Y^{\perp\perp}$ is a semi M-ideal in X^{**}. Then, by using the transitivity property of semi M-ideals, it follows that Y is a semi M-ideal in X^{**}. $\square$

Our next result proves the transitivity of semi M-ideals under an M-ideal assumption on the intermediate space. Even though the transitivity property of semi M-ideal is proved in [39], we give an alternate proof of this when the intermediate space is an M-ideal.

Theorem 2.3.5. *Let Y be an M-ideal in X and let Z be a subspace of Y. If Z is a semi M-ideal in Y, then Z is a semi M-ideal in X.*

Proof. Since $Z \subseteq Y \subseteq X$, we have $Z^{\perp\perp} \subseteq Y^{\perp\perp} \subseteq X^{**}$. Thus, by Lemma 2.3.3, $Z^{\perp\perp}$ is a semi M-ideal in $Y^{\perp\perp}$ and hence, by Lemma 2.3.2, $Z^{\perp\perp}$ is a semi M-ideal in X^{**}. Then, by Lemma 2.3.3, Z is a semi M-ideal in X. $\square$

2.4 M-ideals and Ball Proximinality

Corollary 2.5 of [25] claims that M-ideals are ball proximinal. In this section, we disprove this claim by giving an example.

Example 2.4.1. Let X be the disc algebra (i.e., the Banach space of continuous functions on the closed unit disc which are analytic in the open unit disc, equipped with the supremum norm) and let $Y = \{f \in X : f(1) = 0\}$. It is known that Y is an M-ideal in X and hence Y has the $1\frac{1}{2}$-ball property in X. It is proved in [49] that Y does not have the strong $1\frac{1}{2}$-ball property. Since, by Theorem 1.1.27, every ball proximinal subspace with the

$1\frac{1}{2}$-ball property has the strong $1\frac{1}{2}$-ball property, it follows that Y is not ball proximinal in X.

We now give a class of Banach spaces in which M-ideals are ball proximinal.

Theorem 2.4.2. *Let X be a Banach space. If X has the $3.2.I.P.$, then every M-ideal in X satisfies the strong 3-ball property. In particular, M-ideals in an L_1-predual space have the strong 3-ball property.*

Proof. Let Y be an M-ideal in X and let $\{B(x_i, r_i)\}_{i=1}^3$ be a family of 3 closed balls in X such that $B(x_i, r_i) \cap Y \neq \emptyset$ for all $i = 1, 2, 3$ and $\bigcap_{i=1}^3 B(x_i, r_i) \neq \emptyset$. Let $\varepsilon > 0$. Since Y is an M-ideal in X, there exists an element $y_0 \in Y$ such that $y_0 \in \bigcap_{i=1}^3 B(x_i, r_i + \varepsilon)$. Then $\{B(x_i, r_i)\}_{i=1}^3 \cup \{B(y_0, \varepsilon)\}$ is a mutually intersecting family of closed balls in X. Now, for every $i \in \{1, 2, 3\}$, $\{B(x_j, r_j)\}_{j \neq i} \cup \{B(y_0, \varepsilon)\}$ is a mutually intersecting family of 3 balls in X. Since X has $3.2.I.P.$, these three balls have non-empty intersection. Since Y is an M-ideal, for every $i \in \{1, 2, 3\}$ and for every $\delta > 0$, there exists a point $y_i = y_i(\delta)$ satisfying

$$\|y_i - x_j\| \leq r_j + \delta, \quad j \neq i \text{ and } \|y_i - y_0\| \leq \varepsilon + \delta \text{ for all } i = 1, 2, 3.$$

We now follow the technique used in the proof of [36, Lemma 4.2] in the rest of the proof.

Let $y = \frac{1}{3}\sum_{i=1}^3 y_i$, then $\|y - y_0\| \leq \varepsilon + \delta$ and $\|y - x_j\| \leq r_j + \delta + \frac{2}{3}\varepsilon$.

Now for $\delta \leq \varepsilon/6$, we have

$$\|y - y_0\| \leq 2\varepsilon \text{ and } \|y - x_j\| \leq r_j + \tfrac{5}{6}\varepsilon \text{ for } j = 1, 2, 3.$$

From the above inequalities, it follows that there exists a sequence (z_m) in Y (with $z_0 = y_0$) such that

$$\|z_{m+1} - z_m\| \leq 2(\tfrac{5}{6})^m \varepsilon$$

and

$$\|z_{m+1} - x_j\| \leq r_j + (\tfrac{5}{6})^m \varepsilon \text{ for } j = 1, 2, 3.$$

Hence (z_m) is a Cauchy sequence in Y. Let $z = \lim_{m\to\infty} z_m$. Then $z \in \bigcap_{j=1}^3 B(x_j, r_j) \cap Y$ and this concludes the proof of the theorem. $\square$

Corollary 2.4.3. *Let X be a Banach space. If X has the $3.2.I.P.$, then every M-ideal in X is ball proximinal. In particular, M-ideals in an L_1-predual space are ball proximinal.*

Proof. Suppose X has the $3.2.I.P.$ and let Y be an M-ideal in X. Then, by Theorem 2.4.2, Y has the strong 3-ball property. Since, by Theorem 1.1.27, subspaces with strong $1\frac{1}{2}$-ball property are ball proximinal, we can see that Y is also ball proximinal in X. □

2.5 Some Examples

Our first example shows that the strong proximinality assumption on a subspace is not sufficient to guarantee that any proximinal subspace of it is also proximinal in the bigger space.

Example 2.5.1. There exist two subspaces Z and Y of finite co-dimension in $C[0,1]$ such that Z is proximinal in Y and Y is strongly proximinal in $C[0,1]$, but Z is not proximinal in $C[0,1]$.

Proof. Let $k \in [0,1] \setminus \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Now define $\mu, \nu \in C[0,1]^*$ as

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{\frac{1}{n}} \qquad \text{and} \qquad \nu = \frac{1}{2}(\delta_0 - \delta_k).$$

Then $\|\mu\| = \|\nu\| = 1$. Now take $Z = \ker(\mu) \cap \ker(\nu)$ and $Y = \ker(\nu)$. Since $\mathrm{supp}(\nu)$ is finite, by Theorem 1.1.11, $\ker(\nu)$ is strongly proximinal in $C[0,1]$. Since $1 \in \ker(\nu)$ and $\mu(1) = 1$, $\mu|_{\ker(\nu)}$ is a norm attaining functional on $\ker(\nu)$. Hence it follows that $\ker(\mu) \cap \ker(\nu) = \ker(\mu|_{\ker(\nu)})$ is a proximinal subspace of $\ker(\nu)$. But ν is not absolutely continuous with respect to μ on $\mathrm{supp}(\mu)$. Hence, by Theorem 1.1.2, $\ker(\mu) \cap \ker(\nu)$ is not proximinal in $C[0,1]$. □

Our next example is a variant of Example 2.5.1. In fact, it shows that the notion of strong proximinality may not pass through ideals.

Example 2.5.2. There exist two subspaces Z and Y of finite co-dimension in $C[0,1]$ such that Z is strongly proximinal in Y and Y is an ideal in $C[0,1]$, but Z is not proximinal in $C[0,1]$.

Proof. Let μ, ν and k be as in the proof of Example 2.5.1. Take $Z = \ker(\mu) \cap \ker(\nu)$ and $Y = \ker(\mu)$. Choose a continuous function $g\colon [0,1] \to [-1,1]$ such that $g(\frac{1}{n}) = g(0) = 1$ for $n \geq 2$ and $g(1) = g(k) = -1$. Then $g \in \ker(\mu)$ and $\nu(g) = 1$. Since $\nu|_{\ker(\mu)}$ attains its norm over $\ker(\mu)$, $\ker(\mu) \cap \ker(\nu) = \ker(\nu|_{\ker(\mu)})$ is proximinal in $\ker(\mu)$. Let $\lambda = -\sum_{n=2}^{\infty} \frac{1}{2^n} \delta_{\frac{1}{n}}$. Then $\ker(\mu) = \ker(\lambda - \delta_1)$ and $\|\lambda\| \leq 1$ and hence, by Example 1.1.31, $\ker(\mu)$ is an L_1-predual space. Then, by Theorem 1.1.34, $\ker(\mu)$ is an ideal in $C[0,1]$. Since ν is not absolutely continuous with respect to μ on $\operatorname{supp}(\mu)$, by Theorem 1.1.2, it follows that $\ker(\mu) \cap \ker(\nu)$ is not proximinal in $C[0,1]$. $\square$

Our next example shows that the property of being a semi M-ideal may not pass through ideals.

Example 2.5.3. There exist a Banach space X and a semi M-ideal Y in X such that Y is not a semi M-ideal in X^{**}.

Proof. Take $X = \ell_1$. Then, for the constant sequence $1 \in \ell_\infty$, by Example 1.1.17, it follows that $Y = \ker(1)$ is a semi M-ideal in ℓ_1. But $\ker(1)$ is not a semi M-ideal in $(\ell_\infty)^*$. For, if $\ker(1)$ were a semi M-ideal in $(\ell_\infty)^*$, then, by Corollary 2.3.4, $\ker(1)$ is an M-embedded space. From [22, Chapter III, Corollary 3.3.C and Theorem 3.4], it follows that a non-reflexive M-embedded space contains a subspace isomorphic to c_0. Since ℓ_1 cannot contain an isomorphic copy of c_0, $\ker(1)$ is reflexive. But this is a contradiction as ℓ_1 cannot have a reflexive subspace of co-dimension 1. Hence $\ker(1)$ is not a semi M-ideal in $(\ell_\infty)^*$. $\square$

CHAPTER 3

Proximinality Properties in Vector-valued Function Spaces

In this chapter, for a closed subspace Y of a Banach space X, we define a separably determined property for Y in X. If the property (P) is either proximinality or the strong $1\frac{1}{2}$-ball property and if (P) is separably determined for Y in X, then we prove that $L_1(\mu, Y)$ has the same property (P) in $L_1(\mu, X)$. For an M-embedded space X, we give a class of elements in $L_1(\mu, X^{**})$ having a best approximation from $L_1(\mu, X)$. We also prove that some of these proximinality properties are stable under polyhedral direct sums of Banach spaces. As a corollary, we prove that strong proximinality is stable under finite ℓ_1-sums. Moreover, we characterize SSD-points of ℓ_1-sums of dual spaces.

Most of the results in this chapter are from [27].

3.1 Separably Determined Properties

We begin this section by defining a separably determined property which plays a major role in this section.

Definition 3.1.1. Let Y be a non-separable subspace of a non-separable Banach space X and let (P) be a property in X. We call (P) a separably determined property for Y in X if for every separable subspace Z of Y, there exists a separable subspace Z' of X such that $Z \subseteq Z' \subseteq Y$ and Z' has the property (P) in X.

For some of the proximinality properties (P), our next result shows that if (P) is separably determined for Y in X, then Y has the property (P) in X.

Theorem 3.1.2. *Let Y be a non-separable subspace of a non-separable Banach space X and let* (P) *be one of the following properties*:

(a) *Proximinality.*

(b) *Ball proximinality.*

(c) *Strong proximinality.*

(d) *The $1\frac{1}{2}$-ball property.*

(e) *The strong $1\frac{1}{2}$-ball property.*

If (P) *is separably determined for Y in X, then Y has the property* (P) *in X.*

Proof. (a) Let $x \in X$. Choose a sequence (y_n) in Y such that $d(x, Y) = \lim_{n\to\infty} \|x - y_n\|$. Now let $Z = \overline{\text{span}}\{y_n\}_{n\geq 1}$. Then there exists a separable subspace Z' of X such that $Z \subseteq Z' \subseteq Y$ and Z' is proximinal in X. Thus there exists an element $z' \in Z'$ such that $d(x, Z') = \|x - z'\|$. Now we have

$$\|x - z'\| = d(x, Z') \leq \lim_{n\to\infty} \|x - y_n\| = d(x, Y) \leq d(x, Z') = \|x - z'\|.$$

Therefore $d(x, Y) = \|x - z'\|$ and hence Y is proximinal in X.

(b) Let $x \in X$. Suppose (y_n) is a sequence in B_Y such that $d(x, B_Y) = \lim_{n\to\infty} \|x - y_n\|$. Now let $Z = \overline{\text{span}}\{y_n\}_{n\geq 1}$. Then there exists a separable subspace Z' of X such that

$Z \subseteq Z' \subseteq Y$ and Z' is ball proximinal in X. Thus there exists an element $z' \in B_{Z'}$ such that $d(x, B_{Z'}) = \|x - z'\|$. Now we have

$$\|x - z'\| = d(x, B_{Z'}) \leq \lim_{n \to \infty} \|x - y_n\| = d(x, B_Y) \leq d(x, B_{Z'}) = \|x - z'\|.$$

Thus $d(x, B_Y) = \|x - z'\|$ and hence Y is ball proximinal in X.

(c) Suppose Y is not strongly proximinal in X. Then there exists an $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, there exists an element $y_n \in P_Y(x, \frac{1}{n})$ such that $d(y_n, P_Y(x)) > \varepsilon$. Now let $Z = \overline{\text{span}}\{y_n\}$. Then, by assumption, there exists a separable subspace Z' of X such that $Z \subseteq Z' \subseteq Y$ and Z' is strongly proximinal in X. Therefore there exists a $\delta > 0$ such that $P_{Z'}(x, \delta) \subseteq P_{Z'}(x) + \varepsilon B_Y$. Since $y_n \in P_Y(x, \frac{1}{n})$, it follows that $d(x, Y) = d(x, Z') = \lim_{n \to \infty} \|x - y_n\|$. Therefore $P_{Z'}(x) \subseteq P_Y(x)$ and hence $d(y_n, P_{Z'}(x)) \geq d(y_n, P_Y(x)) > \varepsilon$ for all n. Now since $\|x - y_n\|$ converges to $d(x, Z')$, we have $\|x - y_n\| < d(x, Z') + \delta$ for sufficiently large n. Hence for such n, $d(y_n, P_{Z'}(x)) \leq \varepsilon$. This contradiction proves (c).

(d) Let $x \in X$, $y \in Y$, $B(x, r) \cap Y \neq \emptyset$ and $\|x - y\| \leq r + s$ $(r,\ s > 0)$. Let $y_0 \in B(x, r) \cap Y$ and $Z = \text{span}\{y, y_0\}$. Then, by assumption, there exists a separable subspace Z' of X such that $Z \subseteq Z' \subseteq Y$ and Z' has the $1\frac{1}{2}$-ball property in X. Thus $B(x, r+\varepsilon) \cap B(y, s+\varepsilon) \cap Z' \neq \emptyset$ for all $\varepsilon > 0$. Therefore $B(x, r+\varepsilon) \cap B(y, s+\varepsilon) \cap Y \neq \emptyset$ for all $\varepsilon > 0$ and hence Y has the $1\frac{1}{2}$-ball property in X.

(e) Let $x \in X$, $y \in Y$, $B(x, r) \cap Y \neq \emptyset$ and $\|x - y\| \leq r+s$ $(r,\ s > 0)$. Let $y_0 \in B(x, r) \cap Y$ and $Z = \text{span}\{y, y_0\}$. Then, by assumption, there exists a separable subspace Z' of X such that $Z \subseteq Z' \subseteq Y$ and Z' has the strong $1\frac{1}{2}$-ball property in X. Thus $B(x, r) \cap B(y, s) \cap Z' \neq \emptyset$. Therefore $B(x, r) \cap B(y, s) \cap Y \neq \emptyset$ and hence Y has the strong $1\frac{1}{2}$-ball property in X. $\square$

We now give examples of Banach spaces X and their subspaces Y such that proximinality is separably determined for Y in X. Since subspaces of reflexive spaces are proximinal, we use reflexive spaces to produce such examples.

Lemma 3.1.3. *Let $\{X_i : i \in \mathbb{N}\}$ be a countable collection of reflexive spaces and let $X = \bigoplus_{c_0} X_i$. Then, for every separable subspace Y of X, there exists a separable proximinal subspace Z of X such that $Y \subseteq Z \subseteq X$.*

Proof. Since Y is separable, there exists a countable set $\{y_n\} \subseteq Y$ such that $Y = \overline{\text{span}}\{y_n\}$. Let $Z_i = \overline{\text{span}}\{y_n(i) : n = 1, 2, \ldots\}$ and $Z = \bigoplus_{c_0} Z_i$. Clearly, $Y \subseteq Z \subseteq X$. Since each Z_i is a separable proximinal subspace of X_i, Z is a separable proximinal subspace of X. $\square$

Theorem 3.1.4. *Let $\{X_i : i \in I\}$ be a family of reflexive spaces and let $X = \bigoplus_{c_0} X_i$. If Y is a proximinal factor reflexive subspace of X, then proximinality is separably determined for Y in X.*

Proof. Since Y is a proximinal factor reflexive subspace of X, by Proposition 1.1.3, every $f \in Y^\perp$ is norm attaining. Hence there exists an element $x \in S_X$ such that $f(x) = 1 = \|f\|$. Since $f \in X^* = \bigoplus_1 X_i^*$, we have $\sum_{i\in I} f(i)(x(i)) = \sum_{i\in I} \|f(i)\|$. Hence $f(i)(x(i)) = \|f(i)\|$ for all $i \in I$.

Now suppose $f(i) \neq 0$ for infinitely many i. Then, for these infinitely many i, we have

$$1 = \frac{f(i)}{\|f(i)\|}(x(i)) = \left|\frac{f(i)}{\|f(i)\|}(x(i))\right| \leq \|x(i)\|,$$

which contradicts the fact that $x \in \bigoplus_{c_0} X_i$. Hence $f(i) = 0$ for all but finitely many i. Hence we can find a finite subset A of I such that $f(i) = 0$ for all $f \in Y^\perp$ and $i \notin A$. For, if there is no finite subset A of I such that $f(i) = 0$ for all $f \in Y^\perp$ and $i \notin A$, then we can construct a Cauchy sequence in $Y^\perp$ which does not converge to a point in $Y^\perp$, which is a contradiction. Thus, by the canonical identification, we can see that $Y^\perp \subseteq \bigoplus_{i\in A}{}_1 X_i^*$. Hence we get

$$Y = \left(Y \cap \bigoplus_{i\in A}{}_\infty X_i\right) \bigoplus_\infty \left(\bigoplus_{i\notin A}{}_{c_0} X_i\right).$$

For, let $y \in Y$. Since $X = \left(\bigoplus_{i\in A}{}_\infty X_i\right) \bigoplus_\infty \left(\bigoplus_{i\notin A}{}_{c_0} X_i\right)$, there exist $y_1 \in \left(\bigoplus_{i\in A}{}_\infty X_i\right)$ and $y_2 \in \left(\bigoplus_{i\notin A}{}_{c_0} X_i\right)$ such that $y = y_1 + y_2$ and $\|y\| = \max\{\|y_1\|, \|y_2\|\}$. Now for $f \in Y^\perp$, since $f(y) = f(y_2) = 0$, we get $f(y_1) = 0$. Thus $f(y_1) = 0$ for all $f \in Y^{\perp}$. Then $y_1 \in Y$ and hence $Y \subseteq \left(Y \cap \bigoplus_{i \in A} {}_{\infty} X_i \right) \bigoplus_{\infty} \left(\bigoplus_{i \notin A} {}_{c_0} X_i \right)$. Since $f(i) = 0$ for all $i \notin A$ and $f \in Y^{\perp}$, we can see that $f(z) = 0$ for all $z \in \left(\bigoplus_{i \notin A} {}_{c_0} X_i \right)$ and $f \in Y^{\perp}$. Hence $Y = \left(Y \cap \bigoplus_{i \in A} {}_{\infty} X_i \right) \bigoplus_{\infty} \left(\bigoplus_{i \notin A} {}_{c_0} X_i \right)$.

Let Z be a separable subspace of Y and let $\{z_n\}_{n \in \mathbb{N}} \subseteq Z$ be such that $Z = \overline{\text{span}}\{z_n\}_{n \in \mathbb{N}}$. Then, for every $n \in \mathbb{N}$, there exist $v_n \in Y \cap \bigoplus_{i \in A} {}_{\infty} X_i$ and $w_n \in \bigoplus_{i \notin A} {}_{c_0} X_i$ such that $z_n = v_n + w_n$ and $\|z_n\| = \max\{\|v_n\|, \|w_n\|\}$. Now let $V = \overline{\text{span}}\{v_n\}_{n \in \mathbb{N}}$ and also let $W = \overline{\text{span}}\{w_n\}_{n \in \mathbb{N}}$. Clearly, $V \subseteq Y \cap \bigoplus_{i \in A} {}_{\infty} X_i$. Since A is finite, $\bigoplus_{i \in A} {}_{\infty} X_i$ is a reflexive space and hence V is a separable proximinal subspace of $\bigoplus_{i \in A} {}_{\infty} X_i$. For $n \in \mathbb{N}$, let $A_n = \{i \in I \setminus A : w_n(i) \neq 0\}$. Then $A_0 = \bigcup_{n=1}^{\infty} A_n$ is a countable subset of $I \setminus A$. Now, by the canonical identification, we can see that $W \subseteq \bigoplus_{i \in A_0} {}_{\infty} X_i$. Then, by Lemma 3.1.3, there exists a separable proximinal subspace W' of $\bigoplus_{i \in A_0} {}_{\infty} X_i$ such that $W \subseteq W' \subseteq \bigoplus_{i \in A_0} {}_{\infty} X_i$. Since $\bigoplus_{i \in A_0} {}_{\infty} X_i$ is an M-summand in $\bigoplus_{i \notin A} {}_{\infty} X_i$, by Corollary 2.1.3, W' is proximinal in $\bigoplus_{i \notin A} {}_{c_0} X_i$. Now let $Z' = V \bigoplus_{\infty} W'$. Then Z' is a separable proximinal subspace of X such that $Z \subseteq Z' \subseteq Y$ and hence the theorem follows. $\square$

Corollary 3.1.5. *Let Y be a finite co-dimensional proximinal subspace of $c_0(I)$, where I is a non-empty discrete set. Then proximinality is separably determined for Y in $c_0(I)$.*

Our next result gives the ℓ_1-sum version of Lemma 3.1.3.

Lemma 3.1.6. *Let $\{X_i : i \in \mathbb{N}\}$ be a countable collection of reflexive spaces and let $X = \bigoplus_1 X_i$. Then, for every separable subspace Y of X, there exists a separable proximinal subspace Z of X such that $Y \subseteq Z \subseteq X$.*

Proof. Since Y is separable, there exists a countable set $\{y_n\} \subseteq Y$ such that $Y = \overline{\text{span}}\{y_n\}$. Let $Z_i = \overline{\text{span}}\{y_n(i) : n = 1, 2, \ldots\}$ and $Z = \bigoplus_1 Z_i$. Clearly, $Y \subseteq Z \subseteq X$. Since subspaces of reflexive spaces are proximinal, each Z_i is a separable proximinal subspace of X_i. Also, since countable ℓ_1-sums of separable spaces are separable and ℓ_1-sums of proximinal subspaces are proximinal, Z is a separable proximinal subspace of X. $\square$

The following result gives the ℓ_1-sum version of Theorem 3.1.4.

Theorem 3.1.7. *Let $\{X_i : i \in I\}$ be any family of reflexive spaces and let $X = \bigoplus_1 X_i$. If Y is a subspace of X such that there exists a finite subset A of I with $f(i) = 0$ for all $f \in Y^\perp$ and $i \notin A$, then proximinality is separably determined for Y in X.*

Proof. Since $f(i) = 0$ for all $f \in Y^\perp$ and $i \notin A$, by the canonical identification, we have $Y^\perp \subseteq \bigoplus_{i\in A}{}_\infty X_i^*$. Hence, as observed in the proof of Theorem 3.1.4, we get

$$Y = \left(Y \cap \bigoplus_{i\in A}{}_1 X_i\right) \bigoplus{}_1 \left(\bigoplus_{i\notin A}{}_1 X_i\right).$$

Let Z be a separable subspace of Y and let $\{z_n\}_{n\in\mathbb{N}} \subseteq Z$ be such that $Z = \overline{\text{span}}\{z_n\}_{n\in\mathbb{N}}$. Then, for every $n \in \mathbb{N}$, there exist $v_n \in Y \cap \bigoplus_{i\in A}{}_1 X_i$ and $w_n \in \bigoplus_{i\notin A}{}_1 X_i$ such that $z_n = v_n + w_n$ and $\|z_n\| = \|v_n\| + \|w_n\|$. Now let $V = \overline{\text{span}}\{v_n\}_{n\in\mathbb{N}}$ and let $W = \overline{\text{span}}\{w_n\}_{n\in\mathbb{N}}$. Clearly, $V \subseteq Y \cap \bigoplus_{i\in A}{}_1 X_i$. Since A is finite, V is a separable proximinal subspace of $\bigoplus_{i\in A}{}_1 X_i$. Since W is a separable subspace of $\bigoplus_{i\notin A}{}_1 X_i$, by a similar argument used in the proof of Theorem 3.1.4, there exists a countable subset A_0 of $I \setminus A$ such that $W \subseteq \bigoplus_{i\in A_0}{}_1 X_i$. Then, by Lemma 3.1.6, there exists a separable proximinal subspace W' of $\bigoplus_{i\in A_0}{}_1 X_i$ such that $W \subseteq W' \subseteq \bigoplus_{i\in A_0}{}_1 X_i$. Since $\bigoplus_{i\in A_0}{}_1 X_i$ is an L-summand in $\bigoplus_{i\notin A}{}_1 X_i$, by Corollary 2.1.3, W' is proximinal in $\bigoplus_{i\notin A}{}_1 X_i$. Now let $Z' = V \bigoplus_1 W'$. Then Z' is a separable proximinal subspace of X such that $Z \subseteq Z' \subseteq Y$ and hence the theorem follows. $\square$

For $1 \leq p \leq \infty$ and for a subspace Y of a Banach space X, Example 3.1 of [37] shows that the proximinality of Y in X need not imply the proximinality of $L_p(\lambda, Y)$ in $L_p(\lambda, X)$, where λ is the Lebesgue measure on $[0,1]$. However, for a complete positive σ-finite measure μ, the following theorem gives a sufficient condition for $L_p(\mu, Y)$ to be proximinal in $L_p(\mu, X)$.

Theorem 3.1.8 ([37, Theorem 3.4]). *Let (Ω, Σ, μ) be a complete positive σ-finite measure space and let $1 \leq p \leq \infty$. If Y is a separable proximinal subspace of a Banach space X, then $L_p(\mu, Y)$ is proximinal in $L_p(\mu, X)$.*

Moreover, Corollary 3.5 of [37] proves that if every separable subspace of Y is proximinal in X, then $L_p(\mu, Y)$ is proximinal in $L_p(\mu, X)$ for $1 \leq p \leq \infty$. Our next result generalizes this fact. Even though it is noted in [37, Remark 3.6], we give a proof of it for the sake of completeness.

Theorem 3.1.9. *Let (Ω, Σ, μ) be a complete positive σ-finite measure space. Let X be a Banach space and Y be a subspace of X such that proximinality is separably determined for Y in X. Then, for $1 \leq p \leq \infty$, $L_p(\mu, Y)$ is proximinal in $L_p(\mu, X)$.*

Proof. Let $f \in L_p(\mu, X)$. Now suppose that (f_n) is a sequence in $L_p(\mu, Y)$ satisfying $d(f, L_p(\mu, Y)) = \lim_{n\to\infty} \|f - f_n\|$. Since $f_n's$ are μ-essentially separably valued, without loss of generality, we can assume that range(f_n) is separable for all $n \in \mathbb{N}$. Now for $n \in \mathbb{N}$, let $Z_n = \overline{\text{range}(f_n)}$ and let $Z = \overline{\text{span}}\{\cup_{n=1}^{\infty} Z_n\}$. Since Z is a separable subspace of Y, there exists a separable proximinal subspace Z' of X such that $Z \subseteq Z' \subseteq Y$. Then, by Theorem 3.1.8, $L_p(\mu, Z')$ is proximinal in $L_p(\mu, X)$. Hence there exists an element $g \in L_p(\mu, Z')$ such that $\|f - g\| = d(f, L_p(\mu, Z'))$. Then

$$d(f, L_p(\mu, Y)) \leq \|f - g\| = d(f, L_p(\mu, Z')) \leq \lim_{n\to\infty} \|f - f_n\| = d(f, L_p(\mu, Y)).$$

Therefore $L_p(\mu, Y)$ is proximinal in $L_p(\mu, X)$. $\square$

In the case of the strong $1\frac{1}{2}$-ball property, we recall the following:

Theorem 3.1.10 ([44, Theorem 3.1]). *Let (Ω, Σ, μ) be a non-atomic σ-finite countably generated measure space. Let Y be a separable subspace of a Banach space X. If Y has the strong $1\frac{1}{2}$-ball property in X, then $L_1(\mu, Y)$ has the strong $1\frac{1}{2}$-ball property in $L_1(\mu, X)$.*

Our next theorem generalizes this result.

Theorem 3.1.11. *Let (Ω, Σ, μ) be a non-atomic σ-finite countably generated measure space. Let X be a Banach space and let Y be a subspace of X such that the strong $1\frac{1}{2}$-ball property is separably determined for Y in X. Then $L_1(\mu, Y)$ has the strong $1\frac{1}{2}$-ball property in $L_1(\mu, X)$.*

Proof. Let $f \in L_1(\mu, X)$. Now suppose that (f_n) is a sequence in $L_1(\mu, Y)$ satisfying $d(f, L_1(\mu, Y)) = \lim_{n\to\infty} \|f - f_n\|$. Since $f_n's$ are μ-essentially separably valued, without loss of generality, we can assume that range(f_n) is separable for all $n \in \mathbb{N}$. Now let $Z_n = \overline{\text{range}(f_n)}$ and let $Z = \overline{\text{span}}\{\cup_{n=1}^{\infty} Z_n\}$. Since Z is separable subspace of Y, there exists a separable subspace Z' of X such that $Z \subseteq Z' \subseteq Y$ and Z' has the strong $1\frac{1}{2}$-ball property in X. Then, by Theorem 3.1.10, $L_1(\mu, Z')$ has the strong $1\frac{1}{2}$-ball property in $L_1(\mu, X)$. Hence, by Theorem 1.1.25, there exists an element $g \in P_{L_1(\mu,Z')}(f)$ such that $\|f\| = \|f - g\| + \|g\|$. Since $L_1(\mu, Z') \subseteq L_1(\mu, Y)$,

$$d(f, L_1(\mu, Y)) \leq \|f - g\| = d(f, L_1(\mu, Z')) \leq \lim_{n\to\infty} \|f - f_n\| = d(f, L_1(\mu, Y)).$$

Hence $g \in P_{L_1(\mu,Y)}(f)$ and the result follows. □

Theorem 3.1.10 shows that if Y is a separable subspace of a Banach space X having the strong $1\frac{1}{2}$-ball property in X, then $L_1(\mu, Y)$ has the strong $1\frac{1}{2}$-ball property in $L_1(\mu, X)$. But even for a separable M-ideal Y in X, we do not know whether $L_1(\mu, Y)$ has the strong $1\frac{1}{2}$-ball property in $L_1(\mu, X)$. Now since M-embedded spaces are 'weakly compactly generated', we can find a class of elements in $L_1(\mu, X^{**})$ having a best approximation from $L_1(\mu, X)$. Our next result proves this.

We recall that a Banach space is called *weakly compactly generated* if it is the closed linear span of some weakly compact set.

The following theorem gives examples of weakly compactly generated spaces.

Theorem 3.1.12 ([22, Chapter III, Theorem 4.6])**.** *M-embedded spaces are weakly compactly generated.*

We also recall the following important property of weakly compactly generated spaces.

Theorem 3.1.13 ([12, Chapter 5, Section 2, Theorem 3])**.** *Let X be a weakly compactly generated Banach space and let Z be a separable subspace of X. Then there exists a separable 1-complemented subspace Y of X containing Z.*

We already noted in Example 2.4.1 that M-ideals need not be ball proximinal. Therefore we add ball proximinality as an additional assumption in our next result.

Proposition 3.1.14. *Let X be an M-embedded space and let X be ball proximinal in X^{**}. Let (Ω, Σ, μ) be a non-atomic σ-finite countably generated measure space. Let $f \in L_1(\mu, X^{**})$ be such that* $\text{range}(f) \subseteq Z^{\perp\perp}$*, where Z is a separable subspace of X. Then there exists an element $f_0 \in P_{L_1(\mu,X)}(f)$ such that $\|f\| = \|f - f_0\| + \|f_0\|$.*

Proof. Let $f \in L_1(\mu, X^{**})$ and Z be a separable subspace of X such that $\text{range}(f) \subseteq Z^{\perp\perp}$. Since X is an M-embedded space, by Theorem 3.1.12 and Theorem 3.1.13, there exists a separable subspace Y of X such that $Z \subseteq Y \subseteq X$ and a projection $P : X \to X$ such that $\|P\| = 1$ and $\text{range}(P) = Y$. Then Y is ball proximinal in $Y^{\perp\perp} = Y^{**}$. For, let $x^{**} \in Y^{\perp\perp}$. Since X is ball proximinal in X^{**}, there exists an element $x \in B_X$ such that $d(x^{**}, B_X) = \|x^{**} - x\|$. Then $P(x) \in B_Y$ and

$$d(x^{**}, B_Y) \geq d(x^{**}, B_X) \geq \|x^{**} - x\| \geq \|P^{**}(x^{**} - x)\| = \|x^{**} - P(x)\| \geq d(x^{**}, B_Y).$$

Hence Y is ball proximinal in $Y^{\perp\perp} = Y^{**}$. Moreover, by Proposition 1.1.14(c), we know that subspace of an M-embedded space is an M-embedded space. Thus Y is an M-embedded space. Since, by Theorem 1.1.27, a ball proximinal subspace having the $1\frac{1}{2}$-ball property has the strong $1\frac{1}{2}$-ball property, we can see that Y has the strong $1\frac{1}{2}$-ball property in $Y^{\perp\perp}$. Then, by Theorem 3.1.10, $L_1(\mu, Y)$ has the strong $1\frac{1}{2}$-ball property in $L_1(\mu, Y^{**})$. Since $Z^{\perp\perp} \subseteq Y^{\perp\perp} = Y^{**}$, it follows that $f \in L_1(\mu, Y^{**})$. Hence, by Theorem 1.1.25, there exists an element $f_0 \in P_{L_1(\mu,Y)}(f)$ such that $\|f\| = \|f - f_0\| + \|f_0\|$. Now let $g \in L_1(\mu, X)$. Then, for $t \in \Omega$, we have

$$\|f(t) - g(t)\| \geq \|P^{**}(f(t) - g(t))\| = \|f(t) - P(g(t))\|.$$

Hence $\|f - g\| \geq \|f - P \circ g\| \geq \|f - f_0\|$. Therefore $f_0 \in P_{L_1(\mu,X)}(f)$ and the result follows. □

For a probability measure μ, our next theorem gives a necessary condition for $L_p(\mu, Y)$ to be strongly proximinal in $L_p(\mu, X)$.

Theorem 3.1.15. *Let Y be a subspace of a Banach space X. Let (Ω, Σ, μ) be a probability space and let $1 \leq p \leq \infty$. If $L_p(\mu, Y)$ is strongly proximinal in $L_p(\mu, X)$, then Y is strongly proximinal in X.*

Proof. Suppose $L_p(\mu, Y)$ is strongly proximinal in $L_p(\mu, X)$. Let $x \in X$ and $\varepsilon > 0$. Define $f \in L_p(\mu, X)$ as $f = x\chi_\Omega$. Then there exists a $\delta > 0$ such that

$$P_{L_p(\mu,Y)}(f, \delta) \subseteq P_{L_p(\mu,Y)}(f) + \varepsilon B_{L_p(\mu,X)}.$$

Now let $y \in P_Y(x, \delta)$. Define $g \in L_p(\mu, Y)$ as $g = y\chi_\Omega$. Then $\|x - y\| = \|f - g\|$.

Case 1. $1 \leq p < \infty$.

For $h \in L_p(\mu, Y)$,

$$\|f - h\|^p = \int_\Omega \|f(\omega) - h(\omega)\|^p \, d\mu \geq \int_\Omega d(f(\omega), Y)^p \, d\mu = d(x, Y)^p.$$

Hence $d(x, Y) \leq d(f, L_p(\mu, Y))$. Therefore $g \in P_{L_p(\mu,Y)}(f, \delta)$. Then, by assumption, there exists an element $g' \in P_{L_p(\mu,Y)}(f)$ such that $\|g - g'\| \leq \varepsilon$. Now put $y_0 = \int_\Omega g' \, d\mu$. Then, for all $h \in L_p(\mu, Y)$,

$$\|x - y_0\| = \left\| \int_\Omega f \, d\mu - \int_\Omega g' \, d\mu \right\| \leq \|f - g'\| = d(f, L_p(\mu, Y)) \leq \|f - h\|.$$

Now for $u \in Y$, define $h' \in L_p(\mu, Y)$ as $h' = u\chi_\Omega$. Then $\|x - y_0\| \leq \|f - h'\| = \|x - u\|$. Hence $y_0 \in P_Y(x)$. Since $\|y - y_0\| \leq \|g - g'\| \leq \varepsilon$, $y \in P_Y(x) + \varepsilon B_X$. This completes the proof for $1 \leq p < \infty$.

Case 2. $p = \infty$.

For $h \in L_\infty(\mu, Y)$, since $\|f(\omega) - h(\omega)\| \leq \|f - h\|$ for almost all $\omega \in \Omega$, $d(x, Y) \leq \|f - h\|$. Hence $d(x, Y) \leq d(f, L_\infty(\mu, Y))$. Therefore $g \in P_{L_\infty(\mu,Y)}(f, \delta)$. Then, by assumption, there exists an element $g' \in P_{L_\infty(\mu,Y)}(f)$ such that $\|g - g'\| \leq \varepsilon$. Hence there exists a measure zero set E such that $\|f(\omega) - g'(\omega)\| \leq d(f, L_\infty(\mu, Y))$ and $\|g(\omega) - g'(\omega)\| \leq \varepsilon$ for all $\omega \notin E$. Fix an $\omega_0 \notin E$ and define $y_0 \in Y$ as $y_0 = g'(\omega_0)$. Now for $u \in Y$, define $h' \in L_p(\mu, Y)$ as $h' = u\chi_\Omega$. Then $\|x - y_0\| \leq \|f - h'\| = \|x - u\|$. Hence $y_0 \in P_Y(x)$. Since $\|y - y_0\| \leq \|g(\omega_0) - g'(\omega_0)\| \leq \varepsilon$, $y \in P_Y(x) + \varepsilon B_X$. This completes the proof for $p = \infty$. $\square$

But the converse of Theorem 3.1.15 is still not known.

Question 3.1.16. Let (Ω, Σ, μ) be a probability space and let Y be a strongly proximinal subspace of a Banach space X. Let $1 \leq p \leq \infty$. Is $L_p(\mu, Y)$ strongly proximinal in $L_p(\mu, X)$?

3.2 Stability of Proximinality Properties Under Direct Sums

We begin this section with two lemmas which describe the distance function in ℓ_p-sums and ℓ_∞-sums of Banach spaces.

Lemma 3.2.1. *Let $\{X_i : i \in I\}$ be a family of Banach spaces and let Y_i be a proximinal subspace of X_i. Let $1 \leq p < \infty$. Let $X = \bigoplus_p X_i$ and $Y = \bigoplus_p Y_i$. Then, for an element $x = (x(i)) \in X$, $d(x, Y) = \left(\sum_{i \in I} d(x(i), Y_i)^p\right)^{1/p}$.*

Proof. Since each Y_i is proximinal in X_i, there exists an element $y_i' \in Y_i$ such that $d(x(i), Y_i) = \|x(i) - y_i'\|$. Define $y' \in \prod_{i \in I} Y_i$ as $y'(i) = y_i'$. Since $\|x(i) - y'(i)\| \leq \|x(i)\|$ for all $i \in I$, we have $y' \in \bigoplus_p Y_i$. Then $d(x, Y)^p \leq \sum_{i \in I} \|x(i) - y'(i)\|^p = \sum_{i \in I} d(x(i), Y_i)^p$. Now for any $y \in Y$, $\sum_{i \in I} d(x(i), Y_i)^p \leq \sum_{i \in I} \|x(i) - y(i)\|^p \leq \|x - y\|^p$. Hence the lemma follows. $\square$

The following result gives the ℓ_∞-sum version and c_0-sum version of the above lemma.

Lemma 3.2.2 ([31]). *Let $\{X_i : i \in I\}$ be a family of Banach spaces and let Y_i be a proximinal subspace of X_i. Let $X = \bigoplus_\infty X_i$ ($X = \bigoplus_{c_0} X_i$) and $Y = \bigoplus_\infty Y_i$ ($Y = \bigoplus_{c_0} Y_i$). Then, for $x \in X$, $d(x, Y) = \sup_{i\in I} d(x(i), Y_i)$.*

As an immediate consequence of Lemma 3.2.1 and Lemma 3.2.2, we have the following known result.

Theorem 3.2.3. *Let $\{X_i : i \in I\}$ be a family of Banach spaces and let Y_i be a subspace of X_i. Let $1 \leq p \leq \infty$. Then the following are equivalent:*

(i) *Y_i is proximinal in X_i for all $i \in I$.*

(ii) *$\bigoplus_p Y_i$ is proximinal in $\bigoplus_p X_i$.*

(iii) *$\bigoplus_{c_0} Y_i$ is proximinal in $\bigoplus_{c_0} X_i$.*

Proof. (i) $\implies$ (ii): Suppose Y_i is proximinal in X_i for all $i \in I$. For $1 \leq p \leq \infty$, let $x \in \bigoplus_p X_i$. Then there exists an element $y_i \in Y_i$ such that $d(x(i), Y_i) = \|x(i) - y_i\|$ for all $i \in I$. Define $y \in \prod_{i\in I} Y_i$ as $y(i) = y_i$ for all $i \in I$. Since $\|x(i) - y(i)\| \leq \|x(i)\|$ for all $i \in I$, we can see that the element $y \in \bigoplus_p Y_i$. Then, by Lemma 3.2.1 and Lemma 3.2.2, we get $d(x, \bigoplus_p Y_i) = \|x - y\|$. Hence $\bigoplus_p Y_i$ is proximinal in $\bigoplus_p X_i$.
(ii) $\implies$ (i): Let $\bigoplus_p Y_i$ be proximinal in $\bigoplus_p X_i$. Fix an $i \in I$. Let $x_i \in X_i$. Now define an element $x \in \bigoplus_p X_i$ by

$$x(j) = \begin{cases} x_i & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists an element $y \in \bigoplus_p Y_i$ such that $d(x, \bigoplus_p Y_i) = \|x - y\|$. Since

$$\|x_i - y(i)\| \leq \|x - y\| = d(x, \bigoplus{}_p Y_i) = d(x_i, Y_i),$$

we get that Y_i is proximinal in X_i.
(i) $\implies$ (iii): Suppose Y_i is proximinal in X_i for all $i \in I$. Let $x \in \bigoplus_{c_0} X_i$. Then there exists an element $y_i \in Y_i$ such that $d(x(i), Y_i) = \|x(i) - y_i\|$ for all $i \in I$. Define $y \in \prod_{i\in I} Y_i$

as $y(i) = y_i$ for all $i \in I$. Since $\|y(i)\| \leq \|x(i) - y(i)\| + \|x(i)\| \leq 2\|x(i)\|$ for all $i \in I$, we can see that the element $y \in \bigoplus_{c_0} Y_i$. Then, by Lemma 3.2.2, we get $d(x, \bigoplus_{c_0} Y_i) = \|x - y\|$. Hence $\bigoplus_{c_0} Y_i$ is proximinal in $\bigoplus_{c_0} X_i$.

(iii) $\Longrightarrow$ (i): Suppose $\bigoplus_{c_0} Y_i$ is proximinal in $\bigoplus_{c_0} X_i$. Fix an $i \in I$. Let $x_i \in X_i$. Now define an element $x \in \bigoplus_{c_0} X_i$ by

$$x(j) = \begin{cases} x_i & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists an element $y \in \bigoplus_{c_0} Y_i$ such that $d(x, \bigoplus_{c_0} Y_i) = \|x - y\|$. Since

$$\|x_i - y(i)\| \leq \|x - y\| = d(x, \bigoplus{}_{c_0} Y_i) = d(x_i, Y_i),$$

we get that Y_i is proximinal in X_i. □

We now prove the stability of some proximinality properties under polyhedral direct sums.

For $i = 1, 2, \ldots, n$, $e_i \in \mathbb{R}^n$ is defined by $e_i(j) = 0$ if $i \neq j$ and $e_i(j) = 1$ if $i = j$.

We first prove the stability of proximinality under polyhedral direct sums.

Theorem 3.2.4. *Let X be a polyhedral direct sum of Banach spaces X_i $(1 \leq i \leq n)$ and let Y_i be a subspace of X_i $(1 \leq i \leq n)$. Let π be the corresponding polyhedral norm and suppose $\pi(e_i) \neq 0$ for all i. Then the polyhedral direct sum Y of Y_i $(1 \leq i \leq n)$ is proximinal in X if and only if each Y_i is proximinal in X_i $(1 \leq i \leq n)$.*

Proof. Suppose that each Y_i is proximinal in X_i $(1 \leq i \leq n)$ and let $x \in X$. Then there exists an element $y_i \in Y_i$ such that $\|x(i) - y_i\| = d(x(i), Y_i)$ $(1 \leq i \leq n)$. Now define $y \in Y$ as $y(i) = y_i$ $(1 \leq i \leq n)$. Then, for $z \in Y$, we have $\|x(i) - y(i)\| \leq \|x(i) - z(i)\|$. Since π is non-decreasing, $\|x - y\|_\pi \leq \|x - z\|_\pi$ for all $z \in Y$. Hence Y is proximinal in X.

Conversely, suppose Y is proximinal in X and let $x_i \in X_i$.

Define $x \in X$ by

$$x(j) = \begin{cases} x_i & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists an element $y \in Y$ such that $\|x - y\|_\pi = d(x, Y)$. Now, let $z_i \in Y_i$. Define $z \in Y$ by

$$z(j) = \begin{cases} z_i & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\|x(i) - y(i)\|\pi(e_i) \le \|x - y\|_\pi \le \|x - z\|_\pi = \|x(i) - z_i\|\pi(e_i).$$

Hence Y_i is proximinal in X_i. □

Our next lemma characterizes the distance function in polyhedral direct sums of Banach spaces.

Lemma 3.2.5. *Let X be a polyhedral direct sum of Banach spaces X_i $(1 \le i \le n)$ and let π be the corresponding polyhedral norm. Let Y_i be a proximinal subspace of X_i $(1 \le i \le n)$ and let Y be the polyhedral direct sum of Y_i $(1 \le i \le n)$. Then, for an element $x \in X$,*

$$d(x, Y) = \pi(d(x(1), Y_1), \ldots, d(x(n), Y_n)).$$

Proof. Let $y_i' \in Y_i$ be such that $\|x(i) - y_i'\| = d(x(i), Y_i)$ for all $i = 1, \ldots, n$. Then

$$\begin{aligned} d(x, Y) &\le \pi(\|x(1) - y_1'\|, \ldots, \|x(n) - y_n'\|) \\ &= \pi(d(x(1), Y_1), \ldots, d(x(n), Y_n)) \\ &\le \pi(\|x(1) - y_1\|, \ldots, \|x(n) - y_n\|) \text{ for all } y_i \in Y_i \\ &= \|x - y\|_\pi \text{ for all } y \in Y. \end{aligned}$$

i.e., $d(x, Y) \le \pi\left(d(x(1), Y_1), \ldots, d(x(n), Y_n)\right) \le d(x, Y)$, which proves the lemma. □

Lemma 3.2.6. *Let X be a polyhedral direct sum of Banach spaces X_i $(1 \le i \le n)$ and let π be the corresponding polyhedral norm. Let Y_i be a subspace of X_i $(1 \le i \le n)$ and let Y be the polyhedral direct sum of Y_i $(1 \le i \le n)$. Then, for $x \in X$, we get*

$$P_{Y_1}(x(1)) \times \ldots \times P_{Y_n}(x(n)) \subseteq P_Y(x)$$

and equality holds if $g_j(e_i) > 0$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$, where $\{g_1, \ldots, g_m\}$ is a minimal family generating π.

Proof. Let $y_i \in P_{Y_i}(x(i))$ for all $i = 1, \ldots, n$ and let $z \in Y$. Define $y \in Y$ as $y(i) = y_i$ $(1 \le i \le n)$. Then, for any $z \in Y$, $\|x - y\|_\pi \le \pi(\|x(1) - z(1)\|, \ldots, \|x(n) - z(n)\|) = \|x - z\|_\pi$. Hence $y \in P_Y(x)$.

Now suppose that $g_j(e_i) > 0$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Let $y \in P_Y(x)$. Suppose that there exists an element $j \in \{1, \ldots, n\}$ such that $y(j) \notin P_{Y_j}(x(j))$. Without loss of generality, we can assume that $j = 1$. Now let $r_1 = \min_{1 \le j \le m} g_j(e_1)$ and let $\delta > 0$ be such that $\|y(1) - x(1)\| > d(x(1), Y_1) + \delta$. Then

$$\begin{aligned}
\|x - y\|_\pi &= \pi\left(\|x(1) - y(1)\|, \ldots, \|x(n) - y(n)\|\right) \\
&\ge \pi\left(d(x(1), Y_1) + \delta, d(x(2), Y_2), \ldots, d(x(n), Y_n)\right) \\
&= \max_j g_j\left(d(x(1), Y_1) + \delta, d(x(2), Y_2), \ldots, d(x(n), Y_n)\right) \\
&\ge \max_j g_j\left(d(x(1), Y_1), d(x(2), Y_2), \ldots, d(x(n), Y_n)\right) + \delta r_1 \\
&= d(x, Y) + \delta r_1,
\end{aligned}$$

which is a contradiction. □

Our next theorem shows that with an additional assumption on the polyhedral norm, strong proximinality is stable under polyhedral direct sums.

Theorem 3.2.7. *Let X be a polyhedral direct sum of Banach spaces X_i $(1 \le i \le n)$ and let Y_i be a subspace of X_i $(1 \le i \le n)$. Let π be the corresponding polyhedral norm with $g_j(e_i) > 0$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$, where $\{g_1, \ldots, g_m\}$ is a minimal family generating π. Then the polyhedral direct sum Y of Y_i $(1 \le i \le n)$ is strongly proximinal in X if and only if each Y_i is strongly proximinal in X_i $(1 \le i \le n)$.*

Proof. Suppose Y is strongly proximinal in X. Now fix an $i \in \{1, \ldots, n\}$. Then, by Theorem 3.2.4, Y_i is proximinal in X_i.

Now let $x_i \in X_i$ and $\varepsilon > 0$. Define $x \in X$ by

$$x(j) = \begin{cases} x_i & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists a $\delta > 0$ such that $P_Y(x,\delta) \subseteq P_Y(x) + r_0\varepsilon B_Y$, where $r_0 = \min_{1\le i\le n}\pi(e_i)$. Let $r = \max_{1\le i\le n}\pi(e_i)$ and $y_i \in P_Y(x(i), \frac{\delta}{r})$. Define $y \in Y$ by

$$y(j) = \begin{cases} y_i & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Now $\|x-y\|_\pi = \|x(i)-y(i)\|\pi(e_i) < d(x(i), Y_i)\pi(e_i) + \frac{\delta}{r}\pi(e_i) \le d(x,Y)+\delta$. Hence $y \in P_Y(x,\delta)$. Then there exists an element $\widetilde{z} \in P_Y(x)$ such that $\|\widetilde{z}-y\| \le r_0\varepsilon$. Hence $\widetilde{z}(i) \in P_{Y_i}(x(i))$ and $\|\widetilde{z}(i)-y(i)\|\pi(e_i) \le \|y-\widetilde{z}\| \le r_0\varepsilon < \pi(e_i)\varepsilon$. Then we have $P_{Y_i}(x_i, \frac{\delta}{r}) \subseteq P_{Y_i}(x_i) + \varepsilon B_{Y_i}$ and hence Y_i is strongly proximinal in X_i.

Conversely, suppose that each Y_i is strongly proximinal in X_i ($1 \le i \le n$). Then, by Theorem 3.2.4, Y is proximinal in X. Now let $x \in X$ and let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that $P_{Y_i}(x(i),\delta) \subseteq P_{Y_i}(x(i)) + \frac{\varepsilon}{\pi(1)}B_{Y_i}$ for $1 \le i \le n$. Let $r_i = \min_{1\le j\le m} g_j(e_i)$ and let $r' = \min_{1\le i\le n} r_i$. Let $y \in P_Y(x,\delta r')$. Then $y(i) \in P_{Y_i}(x(i),\delta)$ for all $i = 1,\ldots,n$. If not, then there exists an element $j \in \{1,\ldots,n\}$ such that $y(j) \notin P_{Y_j}(x(j),\delta)$. Without loss of generality, we can assume that $j = 1$. Then

$$\begin{aligned}
\|x-y\|_\pi &= \pi(\|x(1)-y(1)\|,\ldots,\|x(n)-y(n)\|) \\
&\ge \pi(d(x(1),Y_1)+\delta, d(x(2),Y_2),\ldots,d(x(n),Y_n)) \\
&= \max_j g_j(d(x(1),Y_1)+\delta, d(x(2),Y_2),\ldots,d(x(n),Y_n)) \\
&\ge \max_j g_j(d(x(1),Y_1), d(x(2),Y_2),\ldots,d(x(n),Y_n)) + \delta r_1 \\
&\ge d(x,Y) + \delta r'.
\end{aligned}$$

The above contradiction proves that $y(1) \in P_{Y_1}(x(1),\delta)$ and hence $y(i) \in P_{Y_i}(x(i),\delta)$ for all i. Then, for every $i \in \{1,\ldots,n\}$, there exists an element $y_i' \in P_{Y_i}(x(i))$ such that $\|y(i)-y_i'\| \le \frac{\varepsilon}{\pi(1)}$. Define an element $y' \in Y$ as $y'(i) = y_i'$. Then $y' \in P_Y(x)$ and $\|y-y'\| = \pi(\|y(1)-y'(1)\|,\ldots,\|y(n)-y'(n)\|) \le \frac{\varepsilon}{\pi(1)}\pi(1) = \varepsilon$. Hence $y \in P_Y(x) + \varepsilon B_X$ and the converse follows. $\square$

In Theorem 3.2.7, if we take $X_i = \mathbb{R}$ ($1 \le i \le n$) and $\pi(t) = g(t)$, where $g \in \mathbb{R}^n$ is given by $(1,1,\ldots,1)$, then we have the following:

Corollary 3.2.8. *Strong proximinality is stable under finite ℓ_1-sums.*

Since for a Banach space X and for an $f \in S_{X^*}$, $\ker(f)$ is strongly proximinal in X if and only if f is an SSD-point of X^*, the problem of stability of strong subdifferentiability under infinite sums of Banach spaces is of great importance. In [16], Franchetti and Payá proved that the strong subdifferentiability of the norm is preserved under the formation of arbitrary c_0-sums and arbitrary ℓ_p-sums $(1 < p < \infty)$. In [16, Theorem 2.5], they also characterized SSD-points of arbitrary ℓ_∞-sums of Banach spaces. In our next theorem, we characterize SSD-points of ℓ_1-sums of dual spaces.

Theorem 3.2.9. *Let $\{X_i : i \in I\}$ be a family of Banach spaces and let $X = \bigoplus_{c_0} X_i$. Then $f \in S_{X^*}$ is an* SSD*-point of X^* if and only if f has only finitely many non-zero components and for all $i \in I$ with $f(i) \neq 0$, $\frac{f(i)}{\|f(i)\|}$ is an* SSD*-point of X_i^*.*

Proof. It is well-known that $\left(\bigoplus_{c_0} X_i\right)^* = \bigoplus_1 X_i^*$. Now let $f \in S_{X^*}$ be an SSD-point of X^*. Since an SSD-point of X^* is norm attaining, there exists an element $x \in S_X$ such that $f(x) = 1 = \|f\|$. Hence $f(i)(x(i)) = \|f(i)\|$.
Suppose $f(i) \neq 0$ for infinitely many i. Then, for these infinitely many i, we have

$$1 = \frac{f(i)}{\|f(i)\|}(x(i)) = \left|\frac{f(i)}{\|f(i)\|}(x(i))\right| \leq \|x(i)\|$$

which contradicts the fact that $x \in \bigoplus_{c_0} X_i$.
Now let A be a finite subset of I such that $f(i) \neq 0$ for $i \in A$ and $f(i) = 0$ for $i \notin A$. Now for $g \in B_{X^*}$ and $t > 0$,

$$\frac{\|f + tg\| - 1}{t} = \sum_{i \in A} \frac{\|f(i) + tg(i)\| - \|f(i)\|}{t} + \sum_{i \notin A} \|g(i)\|.$$

Now letting $t \to 0^+$, we get

$$d^+(f)(g) = \sum_{i \in A} d^+(\tfrac{f(i)}{\|f(i)\|})(g(i)) + \sum_{i \notin A} \|g(i)\|.$$

Hence for $g \in B_{X^*}$ and $t > 0$, we have

$$\frac{\|f + tg\| - 1}{t} - d^+(f)(g) = \sum_{i \in A} \left(\frac{\left\| \frac{f(i)}{\|f(i)\|} + \frac{t}{\|f(i)\|} g(i) \right\| - 1}{(\frac{t}{f(i)})} - d^+(\tfrac{f(i)}{\|f(i)\|})(g(i)) \right). \quad (3.2.1)$$

Now the necessity follows from the fact that

$$0 \leq \frac{\left\| \frac{f(i)}{\|f(i)\|} + \frac{t}{\|f(i)\|} g(i) \right\| - 1}{\left(\frac{t}{f(i)}\right)} - d^+\left(\frac{f(i)}{\|f(i)\|}\right)(g(i)) \leq \frac{\|f+tg\|-1}{t} - d^+(f)(g)$$

for all $i \in A$.

Conversely, suppose that there exists a finite subset A of I such that $f(i) = 0$ for $i \notin A$ and $\frac{f(i)}{\|f(i)\|}$ is an SSD-point of X_i^* for each $i \in A$. We now observe as before that for every $g \in B_{X^*}$, (3.2.1) holds. Let $\varepsilon > 0$ and m be the cardinality of A. Since A is finite and $\frac{f(i)}{\|f(i)\|}$ is an SSD-point of X_i^* for each $i \in A$, there exists a $\delta > 0$ such that

$$\frac{\left\| \frac{f(i)}{\|f(i)\|} + \frac{t}{\|f(i)\|} g_i \right\| - 1}{\left(\frac{t}{f(i)}\right)} - d^+\left(\frac{f(i)}{\|f(i)\|}\right)(g_i) < \frac{\varepsilon}{m} \text{ for all } i \in A,\ g_i \in B_{X_i^*} \text{ and } 0 < t < \delta.$$

Then

$$0 \leq \frac{\|f + tg\| - 1}{t} - d^+(f)(g) < \varepsilon \text{ for all } g \in B_{X^*} \text{ and } 0 < t < \delta.$$

Hence f is an SSD-point of B_{X^*}. □

By taking $X_i = \mathbb{R}$ for all $i \in I$ in Theorem 3.2.9, we get:

Corollary 3.2.10. *Let I be a non-empty set. Then* SSD*-points of $\ell_1(I)$ are precisely the finitely supported points of $\ell_1(I)$.*

Proceeding as in the proof of Theorem 3.2.9, we get:

Theorem 3.2.11. *Let X_i $(1 \leq i \leq n)$ be Banach spaces and let $X = \bigoplus_1 X_i$. Then $x \in S_X$ is an* SSD*-point of X if and only if for all $i \in \{1, \ldots, n\}$ with $x(i) \neq 0$, $\frac{x(i)}{\|x(i)\|}$ is an* SSD*-point of X_i.*

CHAPTER 4

Intersection Properties of Balls in Banach Spaces

In this chapter, we introduce a weaker notion of central subspace called almost central subspace and study Banach spaces that belong to the class (GC). In particular, we prove that if Y is an almost central subspace of a Banach space X such that Y is in the class (GC), then Y is a central subspace of X^{**}. We also prove that a Banach space X is an L_1-predual space if and only if X is an almost central subspace of every Banach space that contains it. Using these intersection properties of balls, we obtain some sufficient conditions for a semi M-ideal to be an M-ideal. For instance, we prove that if Y is a semi M-ideal in X such that $Y^{\perp\perp}$ is an almost central subspace of X^{**}, then Y is an M-ideal in X. We also obtain some results on 1-complemented subspaces. Moreover, we prove the stability of some of the ball intersection properties in quotient spaces, direct sums, vector-valued continuous function spaces and injective tensor product spaces.

Most of the results in this chapter are from [26].

4.1 Almost Central Subspaces

We begin this section with the definition of an 'almost central subspace' of a Banach space which is a generalization of the concept called central subspace, defined in [6].

Definition 4.1.1. A subspace Y of a Banach space X is called an almost central subspace if for every finite set $\{y_1, ..., y_n\} \subseteq Y$, $x \in X$ and $\varepsilon > 0$, there exists an element $y_\varepsilon \in Y$ such that $\|y_\varepsilon - y_i\| \leq \|x - y_i\| + \varepsilon$ for $1 \leq i \leq n$.

Our next proposition summarizes some observations regarding almost central subspaces.

Proposition 4.1.2.

(a) *Central subspaces of Banach spaces are almost central.*

(b) *A subspace Y of a Banach space X is an almost central subspace of X if and only if for each finite family $\{B_Y(y_i, r_i)\}_{i=1}^n$ of closed balls in Y having non-empty intersection in X, the family $\{B_Y(y_i, r_i + \varepsilon)\}_{i=1}^n$ of closed balls in Y has non-empty intersection in Y for all $\varepsilon > 0$.*

(c) *A weak*-closed almost central subspace of a dual space is an* AC*-subspace .*

(d) *If Z is an almost central subspace of a Banach space Y and Y is an almost central subspace of a Banach space X, then Z is an almost central subspace of X.*

Proof. (a) Let Y be a central subspace of a Banach space X and let $x \in X$, $y_1, \ldots, y_n \in Y$ and $\varepsilon > 0$. Since Y is a central subspace of X, there exists an element $y \in Y$ such that $\|y - y_i\| \leq \|x - y_i\|$ for $1 \leq i \leq n$. Hence Y is an almost central subspace of X.

(b) Suppose Y is an almost central subspace of X. Let $\{B_Y(y_i, r_i)\}_{i=1}^n$ be a family of n balls in Y such that there exists an element $x \in X$ with $x \in \bigcap_{i=1}^n B_Y(y_i, r_i)$. Since Y is an almost central subspace of X, for every $\varepsilon > 0$, there exists an element $y_\varepsilon \in Y$ such that $\|y_\varepsilon - y_i\| \leq \|x - y_i\| + \varepsilon$ for all $i \in \{1, \ldots, n\}$. Hence $y_\varepsilon \in \bigcap_{i=1}^n B_Y(y_i, r_i + \varepsilon)$.

Conversely, suppose that for every family $\{B_Y(y_i, r_i)\}_{i=1}^n$ of closed balls in Y having non-empty intersection in X, $\bigcap_{i=1}^n B_Y(y_i, r_i + \varepsilon) \neq \emptyset$. Now let $y_1, \dots, y_n \in Y$, $x \in X$ and $\varepsilon > 0$. Consider the family $\{B_Y(y_i, \|x - y_i\|)\}_{i=1}^n$ of closed balls in Y. Clearly, $x \in \bigcap_{i=1}^n B_Y(y_i, \|x - y_i\|)$. Then, by assumption, there exists an element $y_\varepsilon \in Y$ such that $y_\varepsilon \in \bigcap_{i=1}^n B_Y(y_i, \|x - y_i\| + \varepsilon)$. Thus $\|y_\varepsilon - y_i\| \leq \|x - y_i\| + \varepsilon$ for $i = 1, \dots, n$. Hence Y is an almost central subspace of X.

(c) Let X be a Banach space and Y be a weak*-closed almost central subspace of X^*. Let $\{B_Y(y_\alpha, r_\alpha)\}_{\alpha \in I}$ be a family of balls in Y having non-empty intersection in X. Now consider the family $\{B_Y(y_\alpha, r_\alpha + \varepsilon)\}_{\alpha \in I, \varepsilon > 0}$. Since Y is an almost central subspace of X^*, any finite subfamily of $\{B_Y(y_\alpha, r_\alpha + \varepsilon)\}_{\alpha \in I, \varepsilon > 0}$ has non-empty intersection in Y. Since Y is a weak*-closed subspace of X^*, $B_Y(y_\alpha, r_\alpha + \varepsilon)$ is weak*-compact for all $\alpha \in I$ and $\varepsilon > 0$. Then there exists an element $y \in Y$ such that $y \in \bigcap_{\alpha \in I, \varepsilon > 0} B_Y(y_\alpha, r_\alpha + \varepsilon)$. Thus $y \in \bigcap_{\alpha \in I} B_Y(y_\alpha, r_\alpha)$ and hence Y is an AC-subspace of X^*.

(d) Let Z be an almost central subspace of a Banach space Y and let Y be an almost central subspace of a Banach space X. Let $\{B_Z(z_i, r_i)\}_{i=1}^n$ be a family of n balls in Z having non-empty intersection in X. Since Y is an almost central subspace of X, by (b), the family $\{B_Y(z_i, r_i + \varepsilon/2)\}_{i=1}^n$ has non-empty intersection in Y for all $\varepsilon > 0$. Since Z is an almost central subspace of Y, again by (b), the family $\{B_Z(z_i, r_i + \varepsilon)\}_{i=1}^n$ has non-empty intersection in Z for all $\varepsilon > 0$. Hence Z is an almost central subspace of X. $\square$

The following lemma gives examples of almost central subspaces.

Lemma 4.1.3. *Let X be a Banach space and Y be an ideal in X. Then Y is an almost central subspace of X.*

Proof. Let $\{y_1, ..., y_n\} \subseteq Y$, $x \in X$ and $\varepsilon > 0$. Choose an $\eta > 0$ such that $\eta\|x - y_i\| \leq \varepsilon$ for all $i \in \{1, \dots, n\}$. Define $F = \operatorname{span}\{y_1, ..., y_n, x\}$. Since Y is an ideal in X, by Theorem 1.1.20, there exists an operator $T_\eta : F \to Y$ such that

$$T_\eta(y) = y \text{ for } y \in F \cap Y \text{ and } \|T_\eta\| \leq 1 + \eta.$$

Now define $y_\eta = T_\eta(x)$. Then $y_\eta \in Y$ and for $1 \leq i \leq n$,

$$\|y_\eta - y_i\| = \|T_\eta(x) - T_\eta(y_i)\| \leq (1+\eta)\|x - y_i\| \leq \|x - y_i\| + \varepsilon.$$

Hence Y is an almost central subspace of X. □

Since every Banach space is an ideal in its bidual, the following result is immediate from Lemma 4.1.3.

Corollary 4.1.4. *Every Banach space is almost central in its bidual.*

Since every M-ideal is an ideal, by Lemma 4.1.3, M-ideals are almost central. We now give an example to show that a semi M-ideal may not be an almost central subspace.

Example 4.1.5. Let ℓ_1^3 denote the three dimensional space $\mathbb{R}^3$, endowed with the norm $\|x\| = |x(1)|+|x(2)|+|x(3)|$ for $x = (x(1), x(2), x(3)) \in \mathbb{R}^3$. Now consider the subspace G of ℓ_1^3 defined as $G = \{(x(1), x(2), -x(1) - x(2)) : x(1), x(2) \in \mathbb{R}\} \subseteq \ell_1^3$. Then Example 1.1.17 shows that G is a semi M-ideal in ℓ_1^3. But G is not a central subspace of ℓ_1^3. For, let $g_1 = (-1, -1, 2)$, $g_2 = (-1, 2, -1)$, $g_3 = (2, -1, -1)$ and let $x = (-1, -1, -1)$. Then g_1, g_2, $g_3 \in G$ and $x \in \ell_1^3$. Clearly, $\|g_i - x\| = 3$ for all $i = 1, 2, 3$. Suppose there is an element $\alpha \in G$ such that $\|\alpha - g_i\| \leq 3$ for all $i = 1, 2, 3$. Then

$$|\alpha(1) + 1| + |\alpha(2) + 1| + |\alpha(1) + \alpha(2) + 2| \leq 3. \tag{4.1.1}$$

$$|\alpha(1) + 1| + |\alpha(2) - 2| + |\alpha(1) + \alpha(2) - 1| \leq 3. \tag{4.1.2}$$

$$|\alpha(1) - 2| + |\alpha(2) + 1| + |\alpha(1) + \alpha(2) - 1| \leq 3. \tag{4.1.3}$$

But (4.1.1) shows that both $\alpha(1)$ and $\alpha(2)$ cannot be positive simultaneously. But the symmetric inequalities (4.1.2) and (4.1.3) rule out other possibilities. Thus G is not a central subspace of ℓ_1^3. Then, by a compactness argument, we can see that G is not an almost central subspace of ℓ_1^3.

In [47, Example 5.6], Veselý gave an example of a three-dimensional Banach space X such that $C([0,1], X)$ is not a central subspace of its bidual. Since every Banach space is an ideal in its bidual, the same example shows that an ideal (in particular, an almost central subspace) need not be a central subspace. We now give a sufficient condition for an almost central subspace to be a central subspace.

Theorem 4.1.6. *Let Y be an almost central subspace of a Banach space X such that $Y \in$ (GC). Then Y is a central subspace of X.*

Proof. Let $\{y_1, ..., y_n\} \subseteq Y$ and $x \in X$. Since $Y \in$ (GC), by Proposition 1.1.38, it is enough to show that $\bigcap_{i=1}^n B_Y(y_i, \|x - y_i\| + \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$.
Now let $\varepsilon > 0$. Also, let $\eta > 0$ be such that $\eta\|x - y_i\| \leq \varepsilon$ for all $i \in \{1, \ldots, n\}$. Since Y is an almost central subspace of X, there exists an element $y \in Y$ such that $\|y - y_i\| \leq (1 + \eta)\|x - y_i\|$ for all $i \in \{1, \ldots, n\}$. Hence $\|y - y_i\| \leq \|x - y_i\| + \varepsilon$ for all $i \in \{1, \ldots, n\}$ and the result follows. □

Our next result gives a sufficient condition for an almost central subspace to be an AC-subspace.

Proposition 4.1.7. *Let Y be an almost central subspace of a Banach space X such that Y is isometric to the range of a projection of norm one in some dual space. Then Y is an* AC-*subspace of X.*

Proof. Let Z be a Banach space and $P : Z^* \to Z^*$ be a projection of norm one such that Y is isometric to range(P). Let $\phi : Y \to$ range(P) be the corresponding onto isometry. Now let $\{B_Y(y_\alpha, r_\alpha)\}_{\alpha\in I}$ be any family of closed balls in Y and $x \in X$ be such that $\|x-y_\alpha\| \leq r_\alpha$ for all $\alpha \in I$. Consider the family $\{B_Y(y_\alpha, r_\alpha + \varepsilon)\}_{\alpha\in I,\varepsilon>0}$. Since Y is an almost central subspace of X, any finite collection of balls from this family has non-empty intersection in Y. Hence any finite collection of balls from the family $\{B_{Z^*}(\phi(y_\alpha), r_\alpha + \varepsilon)\}_{\alpha\in I,\varepsilon>0}$ has non-empty intersection in range(P). Now, by weak*-compactness, there exists an element $f \in Z^*$ such that $\|f-\phi(y_\alpha)\| \leq r_\alpha+\varepsilon$ for all $\alpha \in I$ and for all $\varepsilon > 0$. Hence $\|f-\phi(y_\alpha)\| \leq r_\alpha$ for all $\alpha \in I$. Now define $y = \phi^{-1}(P(f))$. Then, for all $\alpha \in I$, we have

$$\|y - y_\alpha\| = \|\phi^{-1}(P(f)) - \phi^{-1}(\phi(y_\alpha))\| = \|P(f) - \phi(y_\alpha)\| = \|P(f - \phi(y_\alpha))\| \leq r_\alpha.$$

Hence Y is an AC-subspace of X. □

We now give a class of Banach spaces where almost central subspaces are central.

Proposition 4.1.8. *Let X be an L_1-predual space and let Y be an almost central subspace of X. Then Y is an L_1-predual space. Moreover, Y is a central subspace of X.*

Proof. Let $\{B_Y(y_i, r_i)\}_{i=1}^n$ be any family of n balls in Y such that any two of them intersect in Y. Since X is an L_1-predual space, by Theorem 1.1.33, there exists an element $x \in X$ such that $\|x - y_i\| \leq r_i$ for all i. Also, since Y is an almost central subspace of X, we have $\bigcap_{i=1}^n B_Y(y_i, r_i + \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. Then, by Lemma 1.1.29 and Theorem 1.1.33, it follows that Y is an L_1-predual space. Now let $\{B_Y(y_i, r_i)\}_{i=1}^n$ be a family of n balls in Y that has non-empty intersection in X. It is well-known that two balls intersect if and only if the distance between the centers is less than or equal to the sum of the radii. Thus $\{B_Y(y_i, r_i)\}_{i=1}^n$ is a pairwise intersecting family in Y. Since Y is an L_1-predual space, by Theorem 1.1.33, it follows that $\{B_Y(y_i, r_i)\}_{i=1}^n$ intersect in Y. Hence Y is a central subspace of X. $\square$

Our next result gives a characterization of L_1-predual spaces in terms of almost central subspaces.

Theorem 4.1.9. *A Banach space X is an L_1-predual space if and only if X is an almost central subspace of every Banach space that contains it.*

Proof. Let X be an L_1-predual space and let Z be a Banach space such that $X \subseteq Z$. Then, by Theorem 1.1.40, X is a central subspace of Z^{**}. Thus X is a central subspace of Z and hence X is an almost central subspace of Z.

Conversely, suppose that X is an almost central subspace of every Banach space that contains it. In particular, X is an almost central subspace of $\ell_\infty(\Gamma)$ for some non-empty discrete space Γ. Since $\ell_\infty(\Gamma)$ is an L_1-predual space, by Proposition 4.1.8, it follows that X is an L_1-predual space. $\square$

Proposition 14 of [43] proves that if Y is an ideal in a Banach space X with $Y \in$ (GC), then Y is a central subspace of X^{**}. Since every ideal is an almost central subspace, our next proposition generalizes this result.

Proposition 4.1.10. *Let Y be an almost central subspace of a Banach space X. Then Y is a central subspace of X^{**} if and only if $Y \in$ (GC).*

Proof. Let Y be a central subspace of X^{**}. Since $Y \subseteq Y^{\perp\perp} \subseteq X^{**}$ and $Y^{\perp\perp} = Y^{**}$, Y is a central subspace of Y^{**}. Hence $Y \in$ (GC).

Conversely, suppose that $Y \in$ (GC). Since Y is an almost central subspace of X and X is an almost central subspace of X^{**}, by Proposition 4.1.2(d), Y is an almost central subspace of X^{**}. Hence, by Theorem 4.1.6, it follows that Y is a central subspace of X^{**}. $\square$

By a similar transitivity argument, we have the following corollary.

Corollary 4.1.11. *Let Y be a subspace of X such that $Y^{\perp\perp}$ is an almost central subspace of X^{**}. Then Y is an almost central subspace of X^{**}. In addition, if $Y \in$ (GC), then Y is a central subspace of X^{**}.*

Proof. Since Y is an almost central subspace of $Y^{**} = Y^{\perp\perp}$ and $Y^{\perp\perp}$ is an almost central subspace of X^{**}, by Proposition 4.1.2(d), Y is an almost central subspace of X^{**}. If $Y \in$ (GC), then, by Theorem 4.1.6, Y is a central subspace of X^{**}. $\square$

4.2 Stability Results

Coming to quotient spaces, one can easily observe that if Y is 1-complemented in a Banach space X, then for any subspace Z of Y, Y/Z is 1-complemented in X/Z. Motivated by this, we consider the following problem: Let Y be a subspace of a Banach space X having some property (P) in X. Then for a subspace Z of Y, when can we say that Y/Z has the property (P) in X/Z ? We study this problem when the property (P) under consideration is almost constrained, almost central, central and ideal.

For a subspace Y of a Banach space X and $x \in X$, we denote by $[x]$ the equivalence class in X/Y containing x.

Our next result solves the above problem for AC-subspaces.

Proposition 4.2.1. *Let Y be an* AC*-subspace of a Banach space X and let Z be a subspace of Y. Then Y/Z is an* AC*-subspace of X/Z.*

Proof. Let $\{B_{Y/Z}([y_i], r_i)\}_{i\in I}$ be a family of balls in Y/Z and also let $x \in X$ be such that $[x] \in \bigcap_{i\in I} B_{Y/Z}([y_i], r_i)$. Then for each $\varepsilon > 0$ and $i \in I$, there exists an element $z_{\varepsilon,i} \in Z$ such that

$$\|x - y_i + z_{\varepsilon,i}\| \leq \|[x] - [y_i]\| + \varepsilon \leq r_i + \varepsilon \text{ for all } i \in I \text{ and } \varepsilon > 0.$$

We now consider the family $\{B_Y(y_i - z_{\varepsilon,i}, r_i + \varepsilon)\}_{i\in I,\varepsilon>0}$ of closed balls in Y. Clearly, $x \in \bigcap_{i\in I,\varepsilon>0} B_Y(y_i - z_{\varepsilon,i}, r_i + \varepsilon)$. Since Y is an AC-subspace of X, there exists an element $y \in Y$ such that $y \in \bigcap_{i\in I,\varepsilon>0} B_Y(y_i - z_{\varepsilon,i}, r_i + \varepsilon)$. Then, for $i \in I$, we have

$$\|[y] - [y_i]\| \leq \|y - y_i + z_{\varepsilon,i}\| \leq r_i + \varepsilon \text{ for all } \varepsilon > 0.$$

Therefore $\|[y] - [y_i]\| \leq r_i$ for all $i \in I$ and hence Y/Z is an AC-subspace of X/Z. □

We now prove the stability of ideals in quotient spaces.

Proposition 4.2.2. *Let Y be an ideal in a Banach space X and let Z be a subspace of Y. Then Y/Z is an ideal in X/Z.*

Proof. Since Y is an ideal in X, by Theorem 1.1.20, $Y^{\perp\perp}$ is 1-complemented in X^{**}. Then $Y^{\perp\perp}/Z^{\perp\perp}$ is 1-complemented in $X^{**}/Z^{\perp\perp}$. But $X^{**}/Z^{\perp\perp}$ is isometric to $(X/Z)^{**}$ and this isometry takes $Y^{\perp\perp}/Z^{\perp\perp}$ onto $(Y/Z)^{\perp\perp}$. Hence $(Y/Z)^{\perp\perp}$ is 1-complemented in $(X/Z)^{**}$. Then, again by Theorem 1.1.20, Y/Z is an ideal in X/Z. □

Our next result proves the stability of almost central subspaces in quotient spaces.

Proposition 4.2.3. *Let Y be an almost central subspace of a Banach space X and let Z be a subspace of Y. Then Y/Z is an almost central subspace of X/Z.*

Proof. Let $[x] \in X/Z$, $\{[y_1], ..., [y_n]\} \subseteq Y/Z$ and $\varepsilon > 0$. Then, for $1 \leq i \leq n$, there exists an element $z_{\varepsilon,i} \in Z$ such that

$$\|x - y_i + z_{\varepsilon,i}\| \leq \|[x] - [y_i]\| + \varepsilon/2.$$

Since Y is an almost central subspace of X, there exists an element $y_\varepsilon \in Y$ such that

$$\|y_\varepsilon - y_i + z_{\varepsilon,i}\| \leq \|x - y_i + z_{\varepsilon,i}\| + \varepsilon/2 \text{ for } 1 \leq i \leq n.$$

Now, for $1 \leq i \leq n$, we have

$$\|[y_\varepsilon] - [y_i]\| \leq \|y_\varepsilon - y_i + z_{\varepsilon,i}\| \leq \|x - y_i + z_{\varepsilon,i}\| + \varepsilon/2 \leq \|[x] - [y_i]\| + \varepsilon.$$

Hence Y/Z is an almost central subspace of X/Z. □

Now, for Banach spaces X, Y, Z with $Z \subseteq Y \subseteq X$, our next set of results give some sufficient conditions for Y/Z to be a central subspace of X/Z.

Combining Proposition 4.2.3 and Theorem 4.1.6, we get:

Corollary 4.2.4. *Let Y be an almost central subspace of a Banach space X and let Z be a subspace of Y. If $Y/Z \in$ (GC), then Y/Z is a central subspace of X/Z.*

As a consequence of the above corollary, we have the following result.

Corollary 4.2.5. *Let Y be a subspace of a Banach space X and let Z be a subspace of Y such that $Y/Z \in$ (GC). If $Y^{\perp\perp}$ is an almost central subspace of X^{**}, then Y/Z is a central subspace of X/Z.*

Proof. By Corollary 4.1.11, Y is an almost central subspace of X^{**}. Hence Y is an almost central subspace of X. Since $Y/Z \in$ (GC), by Corollary 4.2.4, Y/Z is a central subspace of X/Z. □

Since every reflexive space is in the class (GC), the following corollary is easy to see.

Corollary 4.2.6. *Let Y be a subspace of a Banach space X such that $Y^{\perp\perp}$ is an almost central subspace of X^{**}. Then, for any factor reflexive subspace Z of Y, Y/Z is a central subspace of X/Z.*

We now prove the converse of Proposition 4.2.3 under some additional assumptions.

Proposition 4.2.7. *Let X be an L_1-predual space, Z be an M-ideal in X and Y be a subspace of X such that $Z \subseteq Y \subseteq X$. If Y/Z is almost central in X/Z, then Y is a central subspace of X.*

Proof. Let $x \in X$, $y_1 \dots, y_n \in Y$ and $\varepsilon > 0$. Then, by assumption, there exists an element $y_\varepsilon \in Y$ such that

$$\|[y_\varepsilon] - [y_i]\| \leq \|[x] - [y_i]\| + \varepsilon/2 \leq \|x - y_i\| + \varepsilon/4.$$

Let $z_{\varepsilon,i} \in Z$ be such that $\|y_\varepsilon - y_i - z_{\varepsilon,i}\| \leq \|x - y_i\| + \varepsilon/2$ for all $i \in \{1, \dots, n\}$. Now consider the finite family of balls $\{B_X(y_\varepsilon - y_i, \|x - y_i\| + \varepsilon/2)\}_{i=1}^n$ in X. Since this is a pairwise intersecting family of balls in X and X is an L_1-predual space, by Theorem 1.1.33, $\bigcap_{i=1}^n B_X(y_\varepsilon - y_i, \|x - y_i\| + \varepsilon/2) \neq \emptyset$. Also, since Z is an M-ideal in X, by Theorem 1.1.22, it follows that Z has the n-ball property in X. Then there exists an element $z_\varepsilon \in Z$ such that $\|z_\varepsilon - y_\varepsilon + y_i\| \leq \|x - y_i\| + \varepsilon$ for all $i \in \{1, \dots, n\}$. Therefore Y is an almost central subspace of X and hence, by Proposition 4.1.8, Y is a central subspace of X. □

The following corollary is the converse of Proposition 4.2.2 under some additional assumptions.

Corollary 4.2.8. *Let X be an L_1-predual space, Z be an M-ideal in X and Y be a subspace of X such that $Z \subseteq Y \subseteq X$. If Y/Z is an ideal in X/Z, then Y is an ideal in X.*

Proof. Since Y/Z is an ideal in X/Z, by Lemma 4.1.3, Y/Z is an almost central subspace of X/Z. Thus, by Proposition 4.2.7, Y is a central subspace of X. Then, by Proposition 4.1.8, Y is an L_1-predual space. Hence, by Theorem 1.1.34, Y is an ideal in X. □

Remark 4.2.9. It is easy to observe that for any family $\{X_\alpha : \alpha \in \Gamma\}$ of Banach spaces, if Y_α is an almost central subspace of X_α, then $\bigoplus_\infty Y_\alpha$ is an almost central subspace of $\bigoplus_\infty X_\alpha$.

We now prove the stability of almost central subspaces in vector-valued continuous function spaces.

Let K be a compact Hausdorff space and X be a Banach space. Then, for $f \in C(K)$ and $x \in X$, an element $f \otimes x \in C(K, X)$ is defined as $(f \otimes x)(k) = f(k)x$ for $k \in K$.

Proposition 4.2.10. *Let Y be an almost central subspace of a Banach space X and K be a compact Hausdorff space. Then $C(K, Y)$ is an almost central subspace of $C(K, X)$.*

Proof. Let $f_1, \dots, f_n \in C(K, Y)$, $f \in C(K, X)$ and $\varepsilon > 0$. Then, by the proof of [36, Page 43, Corollary 2], for the finite family $\{f_1, \dots, f_n\}$, there exists a partition of unity $\{\varphi_j\}_{j=1}^m$ and a closed subspace B of $C(K, Y)$ spanned by the elements of the form $\sum_{j=1}^m \varphi_j \otimes y_j$ with $y_j \in Y$ such that $d(f_i, B) < \varepsilon/4$ for $1 \leq i \leq n$ and B is isometric to $(Y \bigoplus \dots \bigoplus Y)_{\ell_\infty^m}$. Similarly for f, there exists a partition of unity $\{\varphi_l'\}_{l=1}^k$ and a closed subspace B' of $C(K, X)$ spanned by the elements of the form $\sum_{l=1}^k \varphi_l' \otimes x_l$ with $x_l \in X$ such that $d(f, B') < \varepsilon/4$ and B' is isometric to $(X \bigoplus \dots \bigoplus X)_{\ell_\infty^k}$. Now let $\widetilde{f} \in B'$ be such that $\|f - \widetilde{f}\| < \varepsilon/4$ and $\widetilde{f}_i \in B$ be such that $\|f_i - \widetilde{f}_i\| < \varepsilon/4$ for $1 \leq i \leq n$.

Case 1. $m \leq k$.

Since B is isometric to $(Y \bigoplus \dots \bigoplus Y)_{\ell_\infty^m}$, B is an M-summand in $(Y \bigoplus \dots \bigoplus Y)_{\ell_\infty^k}$. Since M-summands are central subspaces, by Remark 4.2.9 and Remark 4.1.2(d) , it follows that B is an almost central subspace of B'. Then there exists an element $g \in B$ such that $\|g - \widetilde{f}_i\| \leq \|\widetilde{f} - \widetilde{f}_i\| + \varepsilon/4$ for $1 \leq i \leq n$. Hence we have

$$\begin{aligned}
\|g - f_i\| &\leq \|g - \widetilde{f}_i\| + \|\widetilde{f}_i - f_i\| \\
&\leq \|\widetilde{f} - \widetilde{f}_i\| + \varepsilon/4 + \varepsilon/4 \\
&\leq \|\widetilde{f} - f\| + \|f - f_i\| + \|f_i - \widetilde{f}_i\| + \varepsilon/2 \\
&\leq \|f - f_i\| + \varepsilon.
\end{aligned}$$

Case 2. $k < m$.

In this case, we can isometrically embed B' into $(X \bigoplus \dots \bigoplus X)_{\ell_\infty^m}$. Since B is isometric to $(Y \bigoplus \dots \bigoplus Y)_{\ell_\infty^m}$, by Remark 4.2.9, B is an almost central subspace of $(X \bigoplus \dots \bigoplus X)_{\ell_\infty^m}$. Then there exists an element $g \in B$ such that $\|g - \widetilde{f}_i\| \leq \|\widetilde{f} - \widetilde{f}_i\| + \varepsilon/4$ for $1 \leq i \leq n$.

Hence we have

$$\begin{aligned}\|g - f_i\| &\leq \|g - \widetilde{f}_i\| + \|\widetilde{f}_i - f_i\| \\ &\leq \|\widetilde{f} - f\| + \|f - f_i\| + \|f_i - \widetilde{f}_i\| + \varepsilon/2 \\ &\leq \|f - f_i\| + \varepsilon.\end{aligned}$$

Thus in all cases there exists an element $g \in B \subseteq C(K, Y)$ such that $\|g - f_i\| \leq \|f - f_i\| + \varepsilon$ for $1 \leq i \leq n$. Hence $C(K, Y)$ is an almost central subspace of $C(K, X)$. $\square$

For a central subspace Y of a Banach space X and for a compact Hausdorff space K, it is not known whether $C(K, Y)$ is a central subspace of $C(K, X)$. But if $C(K, Y) \in$ (GC) and Y is an almost central subspace of X, then, by Proposition 4.2.10 and Theorem 4.1.6, $C(K, Y)$ is a central subspace of $C(K, X)$. Now for a Banach space X, Theorem 3.6 of [48] gives a sufficient condition for $C(K, X)$ to be in the class (GC). Precisely, if X is a polyhedral Banach space such that $X \in$ (GC) and $\{f \in B_{X^*} : f(x) = 1\} \bigcap \text{ext}(B_{X^*})$ is finite for each $x \in S_X$, then $C(K, X) \in$ (GC) (by $\text{ext}(B_{X^*})$, we denote the set of all extreme points of B_{X^*} and a Banach space is called polyhedral if the closed unit ball of each of its finite dimensional subspace is a polytope). Since dual of a finite dimensional polyhedral space is polyhedral, this will imply that if X is a finite dimensional polyhedral space, then $C(K, X) \in$ (GC). This information together with Proposition 4.2.10 give the following corollary.

Corollary 4.2.11. *Let Y be an almost central subspace of a Banach space X and K be a compact Hausdorff space. If Y is a polyhedral Banach space such that $Y \in$ (GC) and $\{g \in B_{Y^*} : g(y) = 1\} \bigcap \text{ext}(B_{Y^*})$ is finite for each $y \in S_Y$, then $C(K, Y)$ is a central subspace of $C(K, X)$. In particular, if Y is a finite dimensional polyhedral central subspace of X, then $C(K, Y)$ is a central subspace of $C(K, X)$.*

We now discuss the stability problem in injective tensor product spaces. We first recall the following:

Lemma 4.2.12 ([41, Lemma 2])**.** *Let X and Z be Banach spaces and let Y be an ideal in Z. Then the injective tensor product $Y \overset{\vee}{\bigotimes} X$ is an ideal in $Z \overset{\vee}{\bigotimes} X$.*

We now discuss the stability of almost central subspaces under injective tensor product.

Proposition 4.2.13. *Let K be a compact Hausdorff space and let A be an almost central subspace of $C(K)$. Then, for any Banach space X, the injective tensor product $A\check{\bigotimes}X$ is almost central in $C(K,X)$.*

Proof. Since A is an almost central subspace of $C(K)$, by Proposition 4.1.8, it follows that A is an L_1-predual space. Then, by Theorem 1.1.34, A is an ideal in $C(K)$. Hence, by Lemma 4.2.12, $A\check{\bigotimes}X$ is an ideal in $C(K)\check{\bigotimes}X$. Since $C(K,X) = C(K)\check{\bigotimes}X$, by Lemma 4.1.3, $A\check{\bigotimes}X$ is almost central in $C(K,X)$. □

Theorem 4.2.14. *Let K be a compact Hausdorff space and let A be an almost central subspace of $C(K)$. If Y is an almost central subspace of a Banach space X, then the injective tensor product $A\check{\bigotimes}Y$ is an almost central subspace of $C(K)\check{\bigotimes}X$. In particular, $A\check{\bigotimes}Y$ is an almost central subspace of $A\check{\bigotimes}X$.*

Proof. By Proposition 4.2.13, $A\check{\bigotimes}Y$ is almost central in $C(K)\check{\bigotimes}Y = C(K,Y)$. Then, by Proposition 4.2.10 and Remark 4.1.2(d), it follows that $A\check{\bigotimes}Y$ is an almost central subspace of $C(K,X) = C(K)\check{\bigotimes}X$. Since $A\check{\bigotimes}Y \subseteq A\check{\bigotimes}X \subseteq C(K,X)$, $A\check{\bigotimes}Y$ is an almost central subspace of $A\check{\bigotimes}X$. □

Corollary 4.2.15. *Let Z be an L_1-predual space. Then, for any almost central subspace Y of a Banach space X, the injective tensor product $Z\check{\bigotimes}Y$ is an almost central subspace of $Z\check{\bigotimes}X$.*

Proof. Since Z is an L_1-predual space, by Theorem 1.1.33, Z^{**} is isometric to $C(K)$ for some compact Hausdorff space K. Then, by Theorem 4.1.9, Z is an almost central subspace of $C(K)$. Therefore, by Theorem 4.2.14, $Z\check{\bigotimes}Y$ is an almost central subspace of $Z\check{\bigotimes}X$. □

In [6], Bandyopadhyay and Rao raised the following question: for a family $\{X_\alpha : \alpha \in I\}$ of Banach spaces, is $\bigoplus_{c_0} X_\alpha$ a central subspace of $\bigoplus_\infty X_\alpha$? In [43], Rao proved that if $X_\alpha \in$ (GC) for all $\alpha \in I$, then this question has an affirmative answer. But our next result shows that the above question has an affirmative answer even without any additional assumption.

Proposition 4.2.16. *Let Γ be a non-empty set and X_α $(\alpha \in \Gamma)$ be Banach spaces. Then $\bigoplus_{c_0} X_\alpha$ is a central subspace of $\bigoplus_\infty X_\alpha$.*

Proof. Let $x \in \bigoplus_\infty X_\alpha$ and $y_1, ..., y_n \in \bigoplus_{c_0} X_\alpha$. Let $r = \min_{1\leq i\leq n} \|x - y_i\|$. Since $y_1, ..., y_n \in \bigoplus_{c_0} X_\alpha$, there exists a finite set A such that $\|y_i(\alpha)\| \leq r$ whenever $\alpha \notin A$. Define $z \in \bigoplus_{c_0} X_\alpha$ as

$$z(\alpha) = \begin{cases} x(\alpha) & \text{if } \alpha \in A, \\ 0 & \text{if } \alpha \notin A. \end{cases}$$

Now for $1 \leq i \leq n$,

if $\alpha \in A$, then $\|z(\alpha) - y_i(\alpha)\| = \|x(\alpha) - y_i(\alpha)\| \leq \|x - y_i\|$ and

if $\alpha \notin A$, then $\|z(\alpha) - y_i(\alpha)\| = \|y_i(\alpha)\| \leq r \leq \|x - y_i\|$.

Hence $\|z - y_i\| \leq \|x - y_i\|$ for all i. □

Corollary 4.2.17. *The class* (GC) *is stable under c_0-direct sum of Banach spaces.*

Proof. Let Γ be a non-empty set and let $X_\alpha \in$ (GC) for all $\alpha \in \Gamma$. Then $\bigoplus_{c_0} X_\alpha$ is a central subspace of $\bigoplus_{c_0} X_\alpha^{**}$. Since $\bigoplus_{c_0} X_\alpha^{**}$ is a central subspace of $\bigoplus_\infty X_\alpha^{**} = (\bigoplus_{c_0} X_\alpha)^{**}$, the result follows from the transitivity property of central subspaces. □

We now prove the stability of some ball intersection properties under polyhedral direct sums.

Our next theorem proves that the property of being a central subspace is stable under polyhedral direct sums.

Theorem 4.2.18. *Let X be a polyhedral direct sum of Banach spaces X_i $(1 \leq i \leq n)$ and Y_i be a subspace of X_i $(1 \leq i \leq n)$. Let π be the corresponding polyhedral norm and suppose $\pi(e_i) \neq 0$ for all i. Then the polyhedral direct sum Y of Y_i $(1 \leq i \leq n)$ is a central subspace of X if and only if Y_i is a central subspace of X_i for all i.*

Proof. Suppose Y is a central subspace of X. Fix an $m \in \{1, \ldots, n\}$. Let $x_m \in X_m$ and $y_{m,k} \in Y_m$ $(1 \leq k \leq p)$. Define $x \in X$ and $y_k \in Y$ $(1 \leq k \leq p)$ as

$$x(i) = \begin{cases} x_m & \text{if } m = i, \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad y_k(i) = \begin{cases} y_{m,k} & \text{if } m = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists an element $y \in Y$ such that $\|y - y_k\|_\pi \leq \|x - y_k\|_\pi$ for $1 \leq k \leq p$. Therefore, for $1 \leq k \leq p$, we have

$$\begin{aligned} \|y(m) - y_k(m)\|\pi(e_m) =& \pi(\|y(m) - y_k(m)\|e_m) \\ \leq& \pi(\|(y(1) - y_k(1)\|, \ldots, \|y(n) - y_k(n)\|) \\ \leq& \pi(\|(x(1) - y_k(1)\|, \ldots, \|x(n) - y_k(n)\|) \\ =& \|x(m) - y_k(m)\|\pi(e_m). \end{aligned}$$

Since $\pi(e_i) \neq 0$ for all i, we get $\|y(m) - y_{m,k}\| \leq \|x(m) - y_{m,k}\|$ for $1 \leq k \leq p$. Hence Y_m is a central subspace of X_m.

Conversely, suppose that Y_i is a central subspace of X_i for $1 \leq i \leq n$. Let $x \in X$ and $y_k \in Y$ $(1 \leq k \leq p)$. Then, for $1 \leq m \leq n$, there exists an element $y_m \in Y_m$ such that $\|y_m - y_k(m)\| \leq \|x(m) - y_k(m)\|$ for $1 \leq k \leq p$. Define $y \in Y$ as $y(i) = y_i$ $(1 \leq i \leq n)$. Now using the monotonicity of π, we get

$$\|y - y_k\|_\pi \leq \pi(\|(x - y_k)(1)\|, \ldots, \|(x - y_k)(n)\|) = \|x - y_k\|_\pi \text{ for } 1 \leq k \leq p.$$

Hence Y is a central subspace of X. □

Similarly we can prove the following:

Theorem 4.2.19. *Let X be a polyhedral direct sum of Banach spaces X_i $(1 \leq i \leq n)$ and Y_i be a subspace of X_i $(1 \leq i \leq n)$. Let π be the corresponding polyhedral norm and suppose $\pi(e_i) \neq 0$ for all i. Then the polyhedral sum Y of Y_i $(1 \leq i \leq n)$ is an* AC*-subspace of X if and only if Y_i is an* AC*-subspace of X_i for all i.*

Proof. Suppose Y is an AC-subspace of X. Fix an $m \in \{1, \dots, n\}$. Let $x_m \in X_m$ and $\{y_{m,\alpha}\}_{\alpha \in I} \subseteq Y_m$. Define $x \in X$ and $y_\alpha \in Y$ $(\alpha \in I)$ as

$$x(i) = \begin{cases} x_m & \text{if } m = i, \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad y_\alpha(i) = \begin{cases} y_{m,\alpha} & \text{if } m = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists an element $y \in Y$ such that $\|y - y_\alpha\|_\pi \leq \|x - y_\alpha\|_\pi$ for $\alpha \in I$. Therefore, for $\alpha \in I$, we have

$$\begin{aligned} \|y(m) - y_\alpha(m)\|\pi(e_m) =& \pi(\|y(m) - y_\alpha(m)\|e_m) \\ \leq& \pi(\|(y(1) - y_\alpha(1)\|, \dots, \|y(n) - y_\alpha(n)\|) \\ \leq& \pi(\|(x(1) - y_\alpha(1)\|, \dots, \|x(n) - y_\alpha(n)\|) \\ =& \|x(m) - y_\alpha(m)\|\pi(e_m). \end{aligned}$$

Since $\pi(e_i) \neq 0$ for all i, we get $\|y(m) - y_{m,\alpha}\| \leq \|x_m - y_{m,\alpha}\|$ for $\alpha \in I$. Hence Y_m is an AC-subspace of X_m.

Conversely, suppose that Y_i is an AC-subspace of X_i for all $i \in \{1, \dots, n\}$. Now let $x \in X$ and $\{y_\alpha\}_{\alpha \in I} \subseteq Y$. Then, for $m \in \{1, \dots, n\}$, there exists an element $y_m \in Y_m$ such that $\|y_m - y_\alpha(m)\| \leq \|x(m) - y_\alpha(m)\|$ for $\alpha \in I$. Define $y \in Y$ as $y(i) = y_i$ $(1 \leq i \leq n)$. Now using the monotonicity of π, we get

$$\|y - y_\alpha\|_\pi \leq \pi(\|(x - y_\alpha)(1)\|, \dots, \|(x - y_\alpha)(n)\|) = \|x - y_\alpha\|_\pi \text{ for } \alpha \in I.$$

Hence Y is an AC-subspace of X. $\square$

4.3 1-complemented Subspaces of $C(K)$

In this section, we first recall the notion of orthogonality in Banach spaces. We also recall the characterization of 1-complemented subspaces in terms of this orthogonality notion.

Definition 4.3.1. Let X be a Banach space and let $x, y \in X$. We say that x is orthogonal to y, denoted by $x \perp y$, if $\|x\| \leq \|x + \lambda y\|$ for every scalar λ. For subspaces M and N of X, if $x \perp y$ for all $x \in M$ and $y \in N$, then we write $M \perp N$.

For any non-empty set I and for any $i, j \in I$, $\delta_{i,j}$ is defined by

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

We now recall some results from [8] regarding orthogonality and 1-complemented subspaces.

Lemma 4.3.2. *Let Y be a subspace of a Banach space X. Then we have the following*:

(a) *Y is 1-complemented in X if and only if there exists a subspace Z of X such that $Y \perp Z$ and $X = Y \bigoplus Z$.*

(b) *If Y is of co-dimension n in X, then for any projection P from X onto Y there exist $f_1, \dots, f_n \in X^*$ and $z_1, \dots, z_n \in X$ such that*:

  (1) $Y = \bigcap_{i=1}^{n} \ker(f_i)$,

  (2) $P(x) = x - \sum_{i=1}^{n} f_i(x) z_i$,

  (3) $f_i(z_j) = \delta_{i,j}$ *for all* $i, j = 1, \dots, n$.

In [7], Baronti and Papini characterized finite co-dimensional 1-complemented subspaces of c_0.

Theorem 4.3.3 ([7, Theorem 6.3])**.** *Let Y be a subspace of co-dimension n in c_0. Then Y is 1-complemented in c_0 if and only if there exist n different indices $t_1, \dots, t_n$ and a basis $\{f_1, \dots, f_n\}$ of $Y^\perp$ such that*

$$\|f_i\| \leq 2|f_i(t_i)| \quad \textit{for} \ \ i = 1, \dots, n.$$

Our next theorem extends this result to the non-separable case, $c_0(\Gamma)$.

Theorem 4.3.4. *Let Γ be a non-empty discrete set and Y be a subspace of co-dimension n in $c_o(\Gamma)$. Then Y is 1-complemented in $c_o(\Gamma)$ if and only if there exist n different indices $t_1, \dots, t_n \in \Gamma$ and a basis $\{f_1, \dots, f_n\}$ of $Y^\perp$ such that*

$$\|f_i\| \leq 2|f_i(t_i)| \ \textit{for} \ i = 1, \dots, n.$$

Proof. Let Y be a 1-complemented subspace of $c_0(\Gamma)$ of co-dimension n. Let P be a norm one projection from $c_0(\Gamma)$ onto Y with kernel V. i.e., $c_0(\Gamma) = Y \oplus V$.

Let $\{v_1, \ldots, v_n\}$ be a basis of V and let $A_1 = \bigcup_{i=1}^{n}\{t \in \Gamma : v_i(t) \neq 0\}$. Then A_1 is countable. Since $v_i(t) = 0$ for $t \notin A_1$ and $i \in \{1, \ldots, n\}$, we can consider each v_i as an element of $c_0(A_1)$. Also, we can suppose that there exist n distinct indices $t_1, \ldots, t_n$ such that $v_i(t_j) = \delta_{ij}$ (in fact, there exist n distinct indices $t_1, \ldots, t_n$ such that $\det(v_i(t_j)) \neq 0$ and so we can choose n^2 scalars a_{ij} $(1 \leq i, j \leq n)$ such that the elements $\widetilde{v_i} = \sum_{i=1}^{n} a_{ij} z_j$ satisfy the condition $\widetilde{v_i}(t_j) = \delta_{ij}$ for $1 \leq i, j \leq n$).

Now, by Hahn-Banach theorem, choose $f_1, \ldots, f_n \in c_0(\Gamma)^* = \ell_1(\Gamma)$ such that $f_i|_Y = 0$ and $f_i(v_j) = \delta_{ij}$ for $1 \leq i, j \leq n$. Now let $A_2 = \bigcup_{i=1}^{n}\{t \in \Gamma : f_i(t) \neq 0\}$ and $A = A_1 \cup A_2$. Then, for $1 \leq i, j \leq n$, we have $v_i \in c_0(A)$, $f_i \in \ell_1(A)$, $v_i(t_j) = \delta_{ij}$ and $f_i(v_j) = \delta_{ij}$. Also, $P(x) = x - \sum_{i=1}^{n} f_i(x) v_i$. Let $A = \{s_1, s_2, \ldots\}$.
For $j, m \in \mathbb{N}$, define

$$x_j^m(s_k) = \begin{cases} \operatorname{sgn}\left(\delta_{jk} - \displaystyle\sum_{i=1}^{n} f_i(s_k) v_i(s_j)\right) & \text{if } k \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|x_j^m\| \leq 1$ for every $j,\ m \in \mathbb{N}$. Also, for $j,\ m \in \mathbb{N}$, we have

$$\begin{aligned} P(x_j^m)(s_k) =& x_j^m(s_k) - \sum_{i=1}^{n} \left(\textstyle\sum_{l=1}^{\infty} f_i(s_l) x_j^m(s_l)\right) v_i(s_k) \\ =& x_j^m(s_k) - \sum_{l=1}^{\infty} \left(\textstyle\sum_{i=1}^{n} f_i(s_l) v_i(s_k)\right) x_j^m(s_l) \\ =& \sum_{l=1}^{\infty} \delta_{lk} x_j^m(s_l) - \sum_{l=1}^{\infty} \left(\textstyle\sum_{i=1}^{n} f_i(s_l) v_i(s_k)\right) x_j^m(s_l) \\ =& \sum_{l=1}^{m} \left(\delta_{lk} - \textstyle\sum_{i=1}^{n} f_i(s_l) v_i(s_k)\right) x_j^m(s_l). \end{aligned}$$

$$\begin{aligned} 1 = \|P\| \geq& |P(x_j^m)(s_j)| \\ =& \sum_{l=1}^{m} \left(\delta_{lj} - \textstyle\sum_{i=1}^{n} f_i(s_l) v_i(s_j)\right) \operatorname{sgn}\left(\delta_{lj} - \textstyle\sum_{i=1}^{n} f_i(s_l) v_i(s_j)\right) \end{aligned}$$

$$= \sum_{l=1}^{m} |\delta_{lj} - \textstyle\sum_{i=1}^{n} f_i(s_l)v_i(s_j)|.$$

Hence for $j \in \mathbb{N}$, we have

$$1 \geq \sum_{l=1}^{\infty} |\delta_{lj} - \textstyle\sum_{i=1}^{n} f_i(s_l)v_i(s_j)|. \tag{4.3.1}$$

From (4.3.1) we have

$$\left|1 - \sum_{i=1}^{n} f_i(s_j)v_i(s_j)\right| + \sum_{l=1}^{\infty} |\textstyle\sum_{i=1}^{n} f_i(s_l)v_i(s_j)| - \left|\displaystyle\sum_{i=1}^{n} f_i(s_j)v_i(s_j)\right| \leq 1.$$

Hence

$$\sum_{l=1}^{\infty} |\textstyle\sum_{i=1}^{n} f_i(s_l)v_i(s_j)| \leq 2 \left|\displaystyle\sum_{i=1}^{n} f_i(s_j)v_i(s_j)\right| \quad \text{for all } j \in \mathbb{N}.$$

If $s_j = t_j$ $(1 \leq j \leq n)$, then $\sum_{l=1}^{\infty} |f_j(s_l)| \leq 2\,|f_j(t_j)|$. Hence $\|f_j\| \leq 2|f_j(t_j)|$ for $1 \leq j \leq n$.

Conversely, suppose that there exist $t_1, \ldots, t_n \in \Gamma$ and a basis $\{f_1, \ldots, f_n\}$ of $Y^{\perp}$ such that $\|f_i\| \leq 2|f_i(t_i)|$ for $1 \leq i \leq n$. Let $A = \bigcup_{i=1}^{n} \{t \in \Gamma : f_i(t) \neq 0\} = \{s_1, s_2, \ldots\}$.

Claim 1. If $y \in Y \backslash \{0\}$ and $\|y\| = |y(t_i)|$ for some $i \in \{1, \ldots, n\}$, then $|y(s_p)| = \|y\|$ whenever $f_i(s_p) \neq 0$.

For, suppose $|y(s_p)| < \|y\|$ and $f_i(s_p) \neq 0$. Since $f_i(t) = 0$ for $t \notin A$ and $i \in \{1, \ldots, n\}$, we can consider each f_i as an element of $\ell_1(A)$. Then we have

$$\begin{aligned}
|f(t_i)y(t_i)| &= |\sum_{\substack{k=1\\ s_k \neq t_i}}^{\infty} f_i(s_k)y(s_k)| \\
&\leq \sum_{\substack{k=1\\ s_k \neq t_i}}^{\infty} |f_i(s_k)|\,|y(s_k)| \\
&= |f_i(s_p)||y(s_p)| + \sum_{\substack{k=1\\ s_k \neq t_i, s_p}}^{\infty} |f_i(s_k)|\,|y(s_k)| \\
&< \|y\|\,(\|f_i\| - |f_i(t_i)|) \\
&\leq |y(t_i)|\,|f_i(t_i)|,
\end{aligned}$$

which is a contradiction.

Claim 2. If $y \in Y\backslash\{0\}$, then $\|y\| = |y(s_p)|$ for some $s_p \notin \{t_1, \ldots, t_n\}$.

For, let $y \in Y\backslash\{0\}$. Assume that $\|y\| = \max\{|y(t_1)|, \ldots, |y(t_n)|\} > |y(s_p)|$ for every $s_p \notin \{t_1, \ldots, t_n\}$. Without loss of generality, we assume that y attains its norm only at the components $t_1, \ldots, t_m$ $(m \leq n)$. i.e., $\|y\| = |y(t_j)|$ for every $j \in \{1, \ldots, m\}$. Then, by Claim 1, $f_j(h) = 0$ for $h \notin \{t_1, \ldots, t_m\}$ and $j = 1, \ldots, m$. Hence

$$f_j(y) = 0 \implies \sum_{i=1}^{m} f_j(t_i)y(t_i) = 0 \text{ for } 1 \leq j \leq m.$$

Since $f_1, \ldots, f_n$ are linearly independent, this system has only trivial solution. i.e., $y(t_j) = 0$ for $1 \leq j \leq m$. Hence $y = 0$. This contradiction proves Claim 2.

For $s \in \Gamma$, $e_s \in c_0(\Gamma)$ is defined by $e_s(t) = 1$ if $s = t$ and $e_s(t) = 0$ if $s \neq t$. Then it follows that $e_{t_i} \notin Y$. For, if $e_{t_i} \in Y$, then $f_i(t_i) = f_i(e_{t_i}) = 0$ and hence $\|f_i\| = 0$, which is a contradiction.

Now let $Z = \operatorname{span}\{e_{t_1}, \ldots, e_{t_n}\}$. Then $c_0(\Gamma) = Y \oplus Z$. Let $P\colon c_0(\Gamma) \to c_0(\Gamma)$ be defined by $P(y+z) = y$ for $y \in Y$ and $z \in Z$. Then P is a projection onto Y. For each $y \in Y \setminus \{0\}$, define $g_y\colon \Gamma \to \mathbb{R}$ by $g_y = \operatorname{sgn}(y(s_p))e_{s_p}$, where $s_p \notin \{t_1, \ldots, t_n\}$ is such that $\|y\| = |y(s_p)|$. Then $g_y \in \ell_1(\Gamma)$, $\|g_y\| = 1$, $g_y(y) = \|y\|$ and $g_y(z) = 0$ for all $z \in Z$ and hence

$$\|P(y+z)\| = \|y\| = g_y(y) = g_y(y+z) \leq \|y+z\| \text{ for } y \in Y \text{ and } z \in Z.$$

Therefore Y is 1-complemented in $c_0(\Gamma)$. □

The following result by Baronti characterizes finite co-dimensional 1-complemented subspaces of ℓ_∞.

Theorem 4.3.5 ([8, Theorem]). *A subspace Y of co-dimension n in ℓ_∞ is 1-complemented if and only if there exist n distinct elements $t_1, \ldots, t_n$ and n linearly independent functionals $f_1, \ldots, f_n$ in $(\ell_\infty)^*$ such that:*

(a) $f_i = h_i + g_i$ *with* $h_i \in l_1$, $g_i \in c_0^\perp$, $i = 1, \ldots, n$;

(b) $Y = \bigcap_{i=1}^{n} f_i^{-1}(0)$;

(c) $\|g_i\| \leq 2|h_i(t_i)| - \|h_i\|$, $i = 1, \ldots, n$.

Our next result extends Theorem 4.3.5 to $\ell_\infty(\Gamma)$, for any non-empty discrete set Γ.

Theorem 4.3.6. *Let Γ be any infinite discrete set and Y be a subspace of co-dimension n in $\ell_\infty(\Gamma)$. Then Y is 1-complemented in $\ell_\infty(\Gamma)$ if and only if there exist n distinct elements $t_1, \ldots, t_n$ in Γ and n linearly independent functionals $f_1, \ldots, f_n$ in $(\ell_\infty(\Gamma))^*$ such that:*

(a) $f_i = h_i + g_i$ *with* $h_i \in l_1(\Gamma)$, $g_i \in c_0(\Gamma)^\perp$, $i = 1, \ldots, n$;

(b) $Y = \bigcap_{i=1}^n f_i^{-1}(0)$;

(c) $\|g_i\| \leq 2|h_i(t_i)| - \|h_i\|$, $i = 1, \ldots, n$.

Proof. Let Y be 1-complemented in $\ell_\infty(\Gamma)$. Then there exist n linearly independent elements $z_1, \ldots, z_n$ of $\ell_\infty(\Gamma)$ such that $Y \perp \operatorname{span}\{z_1, \ldots, z_n\}$. We can suppose that there exist n distinct indices $t_1, \ldots, t_n$ such that $z_i(t_j) = \delta_{i,j}$ for $i, j = 1, \ldots, n$. Let P be a norm one projection from $\ell_\infty(\Gamma)$ onto Y with kernel $\operatorname{span}\{z_1, \ldots, z_n\}$. Then there exist n linearly independent functionals $f_1, \ldots, f_n \in (\ell_\infty(\Gamma))^*$ such that $P(x) = x - \sum_{i=1}^n f_i(x) z_i$ for $x \in \ell_\infty(\Gamma)$; $f_i(z_j) = \delta_{i,j}$ for $i, j = 1, \ldots, n$ and $f_i = h_i + g_i$ with $h_i \in \ell_1(\Gamma)$ and $g_i \in c_0(\Gamma)^\perp$ for $i = 1, \ldots, n$.

Let $\operatorname{supp}(h_i)$ denote the set of all non-zero co-ordinates of h_i. Since $h_i \in \ell_1(\Gamma)$, $\operatorname{supp}(h_i)$ is countable. Let $\operatorname{supp}(h_i) = \{i_1, i_2, \ldots\}$.

Claim 1. $t_i \in \operatorname{supp}(h_i)$ for $i = 1, \ldots, n$.

For, suppose there exists an element $i \in \{1, \ldots, n\}$ such that $t_i \notin \operatorname{supp}(h_i)$. Now for $k \in \mathbb{N}$, define $x_k \in \ell_\infty(\Gamma)$ as

$$x_k(p) = \begin{cases} -\operatorname{sgn}(h_i(p)) & \text{if } p = i_1, \ldots, i_k; \\ 1 & \text{if } p = t_i; \\ 0 & \text{otherwise.} \end{cases}$$

We now observe that $x_k \in c_0(\Gamma)$ and so $g_i(x_k) = 0$. Then we have

$$f_i(x_k) = h_i(x_k) = -\sum_{j=1}^k |h_i(i_j)| \quad \text{and}$$

$$1 \geq |P(x_k)(t_i)| = |1 - h_i(x)| \geq 1 + \sum_{j=1}^{k} |h_i(i_j)|.$$

Hence $\sum_{j=1}^{k} |h_i(i_j)| = 0$. Now letting $k \to \infty$, we get $\|h_i\| = 0$. Thus $h_i = 0$.

We note that $g_i \neq 0$. For, if $g_i = 0$, then $f_i = 0$, which is a contradiction. Since $g_i \neq 0$, for every $\varepsilon > 0$, there exists an element $x^\varepsilon \in \ell_\infty(\Gamma)$ such that $\|x^\varepsilon\| = 1$ and $-g_i(x^\varepsilon) > \|g_i\| - \varepsilon$. Now for $k \in \mathbb{N}$, define $x_k^\varepsilon \in \ell_\infty(\Gamma)$ as

$$x_k^\varepsilon(p) = \begin{cases} -\operatorname{sgn}(h_i(p)) & \text{if } p = i_1, \ldots, i_k; \\ 1 & \text{if } p = t_i; \\ x^\varepsilon(p) & \text{otherwise.} \end{cases}$$

We now observe that $x_k^\varepsilon - x^\varepsilon \in c_0(\Gamma)$ and so $g_i(x_k^\varepsilon) = g_i(x^\varepsilon)$. Then we have

$$1 \geq |P(x_k^\varepsilon)(t_i)| = |1 - g_i(x_k^\varepsilon)| = |1 - g_i(x^\varepsilon)|.$$

Thus $g_i(x^\varepsilon) \geq 0$ and hence $\|g_i\| - \varepsilon \leq -g_i(x^\varepsilon) \leq 0$. Now letting $\varepsilon \to 0$, we get $g_i = 0$. This contradiction shows that $t_i \in \operatorname{supp}(h_i)$ for all $i \in \{1, \ldots, n\}$.

Now fix $i \in \{1, \ldots, n\}$. Let $t_i = i_{k_0}$. Now for $k > k_0$, define $x_k \in \ell_\infty(\Gamma)$ as

$$x_k(p) = \begin{cases} -\operatorname{sgn}(h_i(p)) & \text{if } p = i_1, \ldots, i_k;\ p \neq t_i, \\ 1 & \text{if } p = t_i, \\ 0 & \text{otherwise.} \end{cases}$$

We now observe that $x_k \in c_0(\Gamma)$ and so $g_i(x_k) = 0$. Then we have

$$f_i(x_k) = h_i(x_k) = h_i(t_i) - \sum_{\substack{j=1 \\ j \neq k_0}}^{k} |h_i(i_j)| \quad \text{and}$$

$$1 \geq |P(x_k)(t_i)| = |1 - h_i(x_k)| \geq 1 - h_i(t_i) + \sum_{\substack{j=1 \\ j \neq k_0}}^{k} |h_i(i_j)|.$$

Thus $\sum_{\substack{j=1 \\ j \neq t_i}}^{k} |h_i(i_j)| \leq h_i(t_i)$ and hence $\|h_i\| \leq 2|h_i(t_i)|$.

We can suppose that $g_i \neq 0$ (in fact, if $g_i = 0$ we have the required conclusion). Then, for every $\varepsilon > 0$, there exists an element $x^\varepsilon \in \ell_\infty(\Gamma)$ such that $\|x^\varepsilon\| = 1$ and $-g_i(x^\varepsilon) > \|g_i\| - \varepsilon$. Now for $k > k_0$, define $x_k^\varepsilon \in \ell_\infty(\Gamma)$ as

$$x_k^\varepsilon(p) = \begin{cases} -\operatorname{sgn}(h_i(p)) & \text{if } p = i_1, \ldots, i_k;\ p \neq t_i, \\ 1 & \text{if } p = t_i, \\ x^\varepsilon(p) & \text{otherwise.} \end{cases}$$

Then we observe that $x_k^\varepsilon - x^\varepsilon \in c_0(\Gamma)$ and so $g_i(x_k^\varepsilon) = g_i(x^\varepsilon)$. Hence we have

$$1 \geq |P(x_k^\varepsilon)(t_i)| = |1 - (h_i(x_k^\varepsilon) + g_i(x_k^\varepsilon))|.$$

Thus $h_i(x_k^\varepsilon) \geq -g_i(x_k^\varepsilon) \geq \|g_i\| - \varepsilon$. Since $h_i(x_k^\varepsilon) \leq |h_i(t_i)| - \sum_{\substack{j=1 \\ j \neq k_0}}^{k} |h_i(i_j)| + \sum_{i=k+1}^{\infty} h_i(j) x^\varepsilon(j)$, we have

$$\|g_i\| \leq |h_i(t_i)| - \sum_{\substack{j=1 \\ j \neq k_0}}^{k} |h_i(i_j)| + \sum_{i=k+1}^{\infty} h_i(j) x^\varepsilon(j) + \varepsilon.$$

Now letting $k \to \infty$ and $\varepsilon \to 0$, we get (c).

Conversely, suppose that there exist n distinct elements $t_1, \ldots, t_n$ in Γ and n linearly independent functionals $f_1, \ldots, f_n$ in $(\ell_\infty(\Gamma))^*$ such that:

(a) $f_i = h_i + g_i$ with $h_i \in l_1(\Gamma)$, $g_i \in c_0(\Gamma)^\perp$, $i = 1, \ldots, n$;

(b) $Y = \bigcap_{i=1}^n f_i^{-1}(0)$;

(c) $\|g_i\| \leq 2|h_i(t_i)| - \|h_i\|,\ i = 1, \ldots, n$.

Claim 2. Let $y \in Y \setminus \{0\}$. If there exists an element $i \in \{1, \ldots, n\}$ such that $\|y\| = |y(t_i)|$, then we get:

(1) $g_i = 0$ if $h_i(y) = 0$,

(2) $\|y\| = |y(p)|$ if $h_i(p) \neq 0$,

(3) $y \perp c_0(\Gamma)$ if $h_i(y) \neq 0$.

To prove this claim, we first see that

$$
\begin{aligned}
|h_i(y)| &= \left|\sum_{s\in\Gamma} h_i(s)y(s)\right| \\
&\geq |h_i(t_i)y(t_i)| - \left|\sum_{s\neq t_i} h_i(s)y(s)\right| \\
&\geq |h_i(t_i)||y(t_i)| - \sum_{s\neq t_i} |h_i(s)||y(s)| \\
&= \|y\||h_i(t_i)| - \sum_{s\neq t_i} |h_i(s)||y(s)| \\
&\geq \|y\|(2|h_i(t_i)| - \|h_i\|) \\
&\geq \|y\|\|g_i\| \geq |g_i(y)| = |h_i(y)|.
\end{aligned}
$$

Hence all the above inequalities are equalities. Therefore

$$
|h_i(y)| = \|y\|\|g_i\| = \|y\|(2|h_i(t_i)| - \|h_i\|) = \|y\||h_i(t_i)| - \sum_{s\neq t_i} |h_i(s)||y(s)|. \tag{4.3.2}
$$

We now prove the Claim 2.

(1) Now if $h_i(y) = 0$, then, by (4.3.2), $\|g_i\| = 0$.

(2) Let $p \in \Gamma$ be such that $h_i(p) \neq 0$ and $|y(p)| < \|y\|$. Then, by (4.3.2), we get

$$
\begin{aligned}
\|y\|(\|h_i\| - |h_i(t_i)|) &= \sum_{s\neq t_i} |h_i(s)||y(s)| \\
&= |y(p)||h_i(p)| + \sum_{s\neq t_i,p} |h_i(s)||y(s)| \\
&< \|y\||h_i(p)| + \sum_{s\neq t_i,p} |h_i(s)||y(s)| \\
&\leq \|y\|(\|h_i\| - |h_i(t_i)|).
\end{aligned}
$$

This contradiction proves (2).

(3) Let $h_i(y) \neq 0$. Since $f_i(y) = 0$, $g_i(y) = -h_i(y)$. Let $F \in (\ell_\infty(\Gamma))^*$ be defined by $F = -\dfrac{\|y\|}{h_i(y)} g_i$. Since $g_i \in c_0(\Gamma)^\perp$, $F \in c_0(\Gamma)^\perp$. Also, since $F(y) = \|y\|$, from the first equation in (4.3.2), we get $\|F\| = 1$. Then for $w \in c_0(\Gamma)$, we have

$$
\|y\| = F(y) = F(y + tw) \leq \|y + tw\|.
$$

Hence $y \perp c_0(\Gamma)$. This completes the proof of Claim 2.

For $1 \leq i \leq n$, $e_{t_i} \in \ell_\infty(\Gamma)$ be defined as $e_{t_i}(s) = \delta_{s,t_i}$. Clearly, $e_{t_i} \notin Y$ for $1 \leq i \leq n$. Thus $\ell_\infty(\Gamma) = Y \bigoplus \operatorname{span}\{e_{t_1}, \ldots, e_{t_n}\}$.

Now let $y \in Y \setminus \{0\}$. We split the proof into two cases.

Case 1. Suppose $\|y\| > \max\{|y(t_1)|, \ldots, |y(t_n)|\} = \bar{t}$.

Let k be an integer larger than 2. Then there exists an element $t_0 \in \Gamma$ such that

$$|y(t_0)| > \|y\| - \frac{\|y\| - \bar{t}}{k} \equiv \|y\| - \bar{t_k}.$$

Now let $t, a_1, \ldots, a_n$ be real numbers. Then

$$\|y + t(a_1 e_{t_1} + \ldots + a_n e_{t_n})\| \geq |y(t_0)| > \|y\| - \bar{t_k}.$$

Then for $k \to \infty$, we have $\|y\| \leq \|y + t(a_1 e_{t_1} + \ldots + a_n e_{t_n})\|$. Hence $y \perp \operatorname{span}\{e_{t_1}, \ldots, e_{t_n}\}$.

Case 2. Suppose $\|y\| = \max\{|y(t_1)|, \ldots, |y(t_n)|\}$.

Let $t, a_1, \ldots, a_n$ be real numbers. We consider the following subcases.

Subcase 1. $\|y\| = |y(t)|$ for some $t \in \Gamma \setminus \{t_1, \ldots, t_n\}$.

In this case, we get

$$\|y + t(a_1 e_{t_1} + \ldots + a_n e_{t_n})\| \geq |y(t)| = \|y\|.$$

So $y \perp \operatorname{span}\{e_{t_1}, \ldots, e_{t_n}\}$.

Subcase 2. Suppose $\|y\| > |y(t)|$ for all $t \in \Gamma \setminus \{t_1, \ldots, t_n\}$.

In this case, without loss of generality, we can suppose that $t_1, \ldots, t_p$ are the only components at which y attains its norm.

If $h_i(y) \neq 0$ for some $i \in \{1, \ldots, p\}$, then, by Claim 2, we have $y \perp \operatorname{span}\{e_{t_1}, \ldots, e_{t_n}\}$.

Now suppose $h_i(y) = 0$ for any $i \in \{1, \ldots, p\}$. Then, by Claim 2, we have $g_i = 0$ for every $i \in \{1, \ldots, p\}$ and also $h_i(t) = 0$ for $t \notin \{t_1, \ldots, t_p\}$ and for $i = 1, \ldots, p$. So the linear system $h_i(y) = 0,\ i = 1, \ldots, p$ is a Cramer's system. Since $h_1, \ldots, h_p$ are linearly independent, this linear system has no non-trivial solution. Therefore $y_k = 0$ for $1 \leq k \leq p$. Hence $y = 0$. This contradiction proves that the last subcase is not possible. Therefore, in any case, we have $y \perp \operatorname{span}\{e_{t_1}, \ldots, e_{t_n}\}$ and hence Y is 1-complemented in $\ell_\infty(\Gamma)$. $\square$

The following result shows that the space $\ell_\infty(\Gamma)$ cannot have a finite co-dimensional 1-complemented subspace containing $c_0(\Gamma)$.

Corollary 4.3.7. *Let X be a Banach space such that $c_0(\Gamma) \subset X \subset \ell_\infty(\Gamma)$ for some infinite discrete space Γ. If X is a finite co-dimensional subspace of $\ell_\infty(\Gamma)$, then X cannot be a 1-complemented subspace of $\ell_\infty(\Gamma)$.*

Proof. Suppose X is 1-complemented in $\ell_\infty(\Gamma)$. Then, by Theorem 4.3.6, there exist n distinct elements $t_1, \ldots, t_n$ in Γ and n linearly independent functionals $f_1, \ldots, f_n$ in $(\ell_\infty(\Gamma))^*$ such that:

(a) $f_i = h_i + g_i$ with $h_i \in l_1(\Gamma)$, $g_i \in c_0(\Gamma)^\perp$, $i = 1, \ldots, n$;

(b) $X = \bigcap_{i=1}^n f_i^{-1}(0)$;

(c) $\|g_i\| \leq 2|h_i(t_i)| - \|h_i\|$, $i = 1, \ldots, n$.

Since $X^\perp \subseteq c_0(\Gamma)^\perp$, $f_i \in c_0(\Gamma)^\perp$ for all $i \in \{1, \ldots, n\}$. Hence $h_i = f_i - g_i \in c_0(\Gamma)^\perp$ for all $i \in \{1, \ldots, n\}$. Since $\ell_\infty(\Gamma)^* = \ell_1(\Gamma) \bigoplus_1 c_0(\Gamma)^\perp$, we get $h_i = 0$ for all $i \in \{1, \ldots, n\}$. Then, by (c), $g_i = 0$ and hence $f_i = 0$ for all $i \in \{1, \ldots, n\}$. This contradiction proves that X cannot be 1-complemented in $\ell_\infty(\Gamma)$. $\square$

Let K be a compact Hausdorff space and E be a closed subset of K. Also, let $\mathcal{B}(K)$ be the class of Borel subsets of K. Now, for $\mu \in C(E)^*$, we define $\widetilde{\mu} \in C(K)^*$ as

$$\widetilde{\mu}(B) = \begin{cases} \mu(B) & \text{if } B \in \mathcal{B}(K) \text{ and } B \subseteq E, \\ 0 & \text{if } B \in \mathcal{B}(K) \text{ and } B \bigcap E = \emptyset. \end{cases}$$

Lemma 4.3.8. *Let K be a compact Hausdorff space and E be a closed subset of K such that there exists a continuous map $\varphi : K \to E$ which is identity on E. For $1 \leq i \leq n$, let $\mu_i \in C(E)^*$. If $\bigcap_{i=1}^n \ker(\widetilde{\mu}_i)$ is a 1-complemented subspace of $C(K)$, then $\bigcap_{i=1}^n \ker(\mu_i)$ is a 1-complemented subspace of $C(E)$.*

Proof. Suppose $P : C(K) \to C(K)$ is a projection of norm one with range $\bigcap_{i=1}^{n} \ker(\widetilde{\mu_i})$. Now define $P' : C(E) \to C(E)$ by

$$P'(f) = P(f \circ \varphi)|_E \quad \text{for} \;\; f \in C(E).$$

Since

$$\int_E P'(f)\ d\mu_i = \int_E P(f \circ \varphi)|_E\ d\mu_i = \int_K P(f \circ \varphi)\ d\widetilde{\mu_i} = 0 \quad \text{for all} \;\; f \in C(E),$$

we get $P'(f) \in \bigcap_{i=1}^{n} \ker(\mu_i)$ and hence P' is well-defined. Clearly, P' is a linear map. Now let $f \in \bigcap_{i=1}^{n} \ker(\mu_i)$. Since φ is identity on E, we have

$$\int_K P(f \circ \varphi)\ d\widetilde{\mu_i} = \int_E f\ d\mu_i = 0 \quad \text{for } 1 \le i \le n.$$

Thus $f \circ \varphi \in \bigcap_{i=1}^{n} \ker(\widetilde{\mu_i})$ and $P(f \circ \varphi) = f \circ \varphi$. Therefore $P'(f) = f$ and hence P' is a projection onto $\bigcap_{i=1}^{n} \ker(\mu_i)$. Since $\|P'(f)\| = \|P(f \circ \varphi)|_E\| \le \|P(f \circ \varphi)\| \le \|f \circ \varphi\| = \|f\|$, $\|P'\| = 1$. Hence P' is the required projection. □

We now recall the following property of an extremally disconnected space which will be used in our next proposition.

Lemma 4.3.9 ([29, Section 7, Lemma 3 and Theorem 3])**.** *Let K be an extremally disconnected space. Then there exists a topological space T, a continuous map $r : T \to K$ and a homeomorphic embedding $s : K \to T$ such that $r \circ s$ is the identity map on K and T is the Stone-Čech compactification of its isolated points.*

In our next result, we observe a simple proof for the implication (iii) $\Longrightarrow$ (ii) of Theorem 1.1.43 when K is an extremally disconnected space.

Proposition 4.3.10. *Let K be an extremally disconnected space. If there exist measures $\mu_1, \ldots, \mu_n$ on K and distinct isolated points $k_1, \ldots, k_n$ of K such that $\|\mu_i\| \le 2|\mu_i(\{k_i\})|$, then $\bigcap_{i=1}^{n} \ker(\mu_i)$ is a 1-complemented subspace of $C(K)$.*

Proof. Let Γ be a dense subset of K. Since each k_i is an isolated point of K, $k_i \in \Gamma$ for all $i \in \{1, \ldots, n\}$. Now consider Γ with the discrete topology and its Stone-Čech compactification $\beta(\Gamma)$. Then, by Lemma 4.3.9, K is homeomorphically embedded into $\beta(\Gamma)$ and also there exists a continuous map $\varphi : \beta(\Gamma) \to K$ such that φ is identity on K. Now consider measures $\widetilde{\mu_i}$ on $\beta(\Gamma)$ such that $\widetilde{\mu_i}(D) = 0$ for any Borel set D disjoint from K and $\widetilde{\mu_i}(D) = \mu_i(D)$ for any Borel set $D \subseteq K$. Since $k_i \in \Gamma$, $2|\mu_i(\{k_i\})| \geq \|\mu_i\| = \|\widetilde{\mu}\|$. Since $C(\beta(\Gamma))$ is isometric to $\ell_\infty(\Gamma)$, by Theorem 4.3.6, $\bigcap_{i=1}^n \ker(\widetilde{\mu_i})$ is 1-complemented in $\ell_\infty(\Gamma)$. Then, by Lemma 4.3.8, $\bigcap_{i=1}^n \ker(\mu_i)$ is 1-complemented in $C(K)$. $\square$

In an L_1-predual space, we do not know whether every AC-subspace of finite co-dimension is the range of a norm one projection and/or is the intersection of AC-subspaces of co-dimension one.

4.4 Some Applications

Example 1.1.17 shows that a semi M-ideal need not be an M-ideal. In this section, we give some sufficient conditions for a semi M-ideal to be an M-ideal in terms of the relative intersection properties of balls.

The following result by Rao gives a sufficient condition for a semi M-ideal to be an M-ideal in terms of ideals.

Proposition 4.4.1 ([43, Proposition 23])**.** *Let Y be an ideal in a Banach space X. Then Y is a semi M-ideal in X if and only if Y is an M-ideal in X.*

Our next theorem gives a sufficient condition for a semi M-ideal to be an M-ideal in terms of almost central subspaces.

Theorem 4.4.2. *Let Y be a semi M-ideal in a Banach space X such that $Y^{\perp\perp}$ is an almost central subspace of X^{**}. Then Y is an M-ideal in X.*

Proof. Since Y is a semi M-ideal in X, by Lemma 2.3.3, $Y^{\perp\perp}$ is a semi M-ideal in X^{**}. Also, since $Y^{\perp\perp}$ is a weak*-closed almost central subspace of X^{**}, $Y^{\perp\perp}$ is an AC-subspace of X^{**}. Hence for any $x^{**} \notin Y^{\perp\perp}$, by Theorem 1.1.42, $Y^{\perp\perp}$ is 1-complemented in span$\{Y^{\perp\perp}, x^{**}\}$ and hence is an ideal in span$\{Y^{\perp\perp}, x^{**}\}$. Now, for every $x^{**} \notin Y^{\perp\perp}$, since $Y^{\perp\perp}$ is a semi M-ideal in span$\{Y^{\perp\perp}, x^{**}\}$, by Proposition 4.4.1, it follows that $Y^{\perp\perp}$

is an M-ideal in span$\{Y^{\perp\perp}, x^{**}\}$. Hence, by Proposition 1.1.13(a), $Y^{\perp\perp}$ is an M-ideal in X^{**}. Since $Y^{\perp\perp}$ is a weak*-closed M-ideal in X^{**}, by Proposition 1.1.14(a), $Y^{\perp\perp}$ is an M-summand in X^{**} . Hence, by Proposition 1.1.14(b), there exists an L-summand V in X^* such that $X^{**} = Y^{\perp\perp} \bigoplus_\infty V^\perp$. Then, by the duality between L- and M-projections, we get $X^* = Y^\perp \bigoplus_1 V$ and hence Y is an M-ideal in X. □

The following result gives a sufficient condition for an M-ideal to be an M-summand.

Theorem 4.4.3 ([22, Chapter I, Corollary 1.3])**.** *Let X be a Banach space and let Y be an M-ideal in X. If Y is* 1*-complemented in X, then Y is an M-summand in X.*

Our next result gives a sufficient condition for a semi M-ideal to be an M-summand and it also improves Theorem 4.4.3.

Theorem 4.4.4. *Let Y be an* AC*-subspace of a Banach space X. Then Y is a semi M-ideal in X if and only if Y is an M-summand in X.*

Proof. Suppose Y is a semi M-ideal in X and is an AC-subspace of X. Since Y is an AC-subspace of X, by Theorem 1.1.42, Y is 1-complemented in span$\{Y, x\}$ for all $x \in X$. Also, since Y is a semi M-ideal in X, Y is a semi M-ideal in span$\{Y, x\}$ for all $x \in X$. Thus, by Proposition 4.4.1, Y is an M-ideal in span$\{Y, x\}$ for all $x \in X$. Then, by Theorem 4.4.3, Y is an M-summand in span$\{Y, x\}$ for all $x \in X$. Hence, by Proposition 1.1.13(b), Y is an M-summand in X. □

Our next theorem gives another sufficient condition for a semi M-ideal to be an M-ideal. In fact, this result improves Proposition 4.4.1.

Theorem 4.4.5. *Let Y be a subspace of a Banach space X such that Y is an ideal in* span$\{Y, x\}$ *for all $x \in X$. Then Y is a semi M-ideal in X if and only if Y is an M-ideal in X.*

Proof. Suppose Y is a semi M-ideal in X and is an ideal in span$\{Y, x\}$ for all $x \in X$. Then, by Proposition 4.4.1, Y is an M-ideal in span$\{Y, x\}$ for all $x \in X$. Hence, by Proposition 1.1.13(a), Y is an M-ideal in X. □

Remark 4.4.6. For a subspace Y of a Banach space X, it is not known whether Y is an ideal in X even if Y is an ideal in $\operatorname{span}\{Y, x\}$ for all $x \in X$.

Publications

(1) C. R. Jayanarayanan and Tanmoy Paul, *Transitivity of various notions of proximinality in Banach spaces*, Communicated (2012)

(2) C. R. Jayanarayanan, *Proximinality properties in* $L_p(\mu, X)$ *and polyhedral direct sums of Banach spaces*, To appear in Numer. Funct. Anal. Optim.

(3) C. R. Jayanarayanan, *Intersection properties of balls in Banach spaces*, To appear in J. Funct. Spaces Appl.

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