Some Problems in Differential and Subdifferential Calculus of Matrices

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To

my family

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Contents

Introduction

A central problem in many subjects like matrix analysis, perturbation theory, numerical analysis and physics is to study the effect of small changes in a matrix A on a function $f(A)$. Among much studied functions on the space of matrices are trace, determinant, permanent, eigenvalues, norms. These are real or complex valued functions. In addition, there are some interesting functions that are matrix valued. For example, the (matrix) absolute value, tensor power, antisymmetric tensor power, symmetric tensor power.

When a function is differentiable, one of the ways to study the above problem is by using the derivative of f at A, denoted by $\mathrm{D}f(A)$. In order to obtain first order perturbation bounds, it is helpful to have information about $\|\mathrm{D}f(A)\|$. In general, finding the exact value of the norm of any operator is not an easy task. It might be easier and adequate to find good estimates on $\|\mathrm{D}f(A)\|$. Higher order perturbation bounds can be obtained using the norms of the higher order derivatives.

Some interesting functions like norms are not differentiable at some points. But they possess the useful property of being convex. In such a case, the notion of subderivative is used in place of the derivative.

This thesis consists of two parts. In one of them, we study (higher order) derivatives of the maps that take a matrix to its kth tensor power, kth antisymmetric tensor power and kth symmetric tensor power. We obtain explicit formulas for these derivatives and compute their norms. We also obtain expressions for the map that takes a matrix to its permanent. In the other part, we study the subdifferentials of norm functions and use them to investigate Birkhoff-James orthogonality in the space of matrices. These results are then applied to obtain some distance formulas. Such formulas have been of interest to many mathematicians.

Let $\mathbb{M}(n)$ denote the space of $n \times n$ complex matrices. Let $A(i|j)$ denote the $(n-1)\times(n-1)$ submatrix obtained from A by deleting its ith row and jth column. Let $\det : \mathbb{M}(n) \to \mathbb{C}$ be the map that takes a matrix A to its determinant. This map is differentiable and the famous Jacobi formula gives its derivative as

$$\mathrm{D}\det(A)(X) = \operatorname{tr}(\operatorname{adj}(A)X), \tag{0.1}$$

where the symbol $\operatorname{adj}(A)$, called the *adjugate* (or the *classical adjoint*) of A, stands for the transpose of the matrix whose (i,j)-entry is $(-1)^{i+j}\det A(i|j)$. The map $\mathrm{D}\det(A)$ is a linear operator from $\mathbb{M}(n)$ to $\mathbb{C}$. Its norm is defined as

$$\|\mathrm{D}\det(A)\| = \sup_{\|X\|=1} \|\mathrm{D}\det(A)(X)\|.$$

Let $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A) \geq 0$ denote the singular values of A. Given any k-tuple $(x_1,\ldots,x_k)$, let $p_m(x_1,\ldots,x_k)$ denote the mth elementary symmetric polynomial in $x_1,\ldots,x_k$. From (0.1), one can derive the relation

$$\|\mathrm{D}\det(A)\| = p_{n-1}(s_1(A),\ldots,s_n(A)). \tag{0.2}$$

Let $\wedge^k : \mathbb{M}(n) \to \mathbb{M}\left(\binom{n}{k}\right)$ be the map that takes a matrix A to its kth antisymmetric tensor power, $\wedge^k(A)$. In the special case $k = n$, this is the $\det$ map. Bhatia and Friedland [12] obtained a striking formula for the norm of $\mathrm{D}\wedge^k(A)$. They showed that

$$\|\mathrm{D}\wedge^k(A)\| = p_{k-1}(s_1(A),\ldots,s_k(A)). \tag{0.3}$$

Likewise we can consider the map $\vee^k : \mathbb{M}(n) \to \mathbb{M}\left(\binom{n+k-1}{k}\right)$ that takes a matrix A to its kth symmetric tensor power, $\vee^k(A)$. Bhatia [8] proved that

$$\|\mathrm{D}\vee^k(A)\| = k\|A\|^{k-1}. \tag{0.4}$$

In [14], Bhatia and Jain extended Jacobi's formula (0.1) to higher order derivatives of the function $\det$. Following this, Jain [31] derived an expression for higher order derivatives of $\wedge^k$. In this thesis, we obtain another expression for higher order derivatives of $\wedge^k$. Then we compute higher order derivatives of the maps $\otimes^k$ and $\vee^k$. We obtain an analogue of Jacobi's formula (0.1) for the map per that takes a matrix to its permanent. Expressions for higher order derivatives of the per map are also obtained. Two of the main results of this part of the thesis are the formulas

$$\|\mathrm{D}^m \wedge^k (A)\| = m!\, p_{k-m}(s_1(A), \ldots, s_k(A)) \tag{0.5}$$

and

$$\|\mathrm{D}^m \vee^k (A)\| = \frac{k!}{(k-m)!}\, \|A\|^{k-m} \tag{0.6}$$

for $1 \leq m \leq k$. The expressions (0.3) and (0.4) are particular cases of these for $m = 1$. The expression (0.5) for the norm of $\mathrm{D}^m \wedge^k (A)$ was first found in Jain [31]. We give a different proof for this. The main interest in this alternative approach is that we prove and use an analogue of a famous theorem of Russo and Dye about the norm of a positive linear map between C^*-algebras. We establish a multilinear version of this. We show that a positive m-linear map between C^*-algebras attains its norm at the m-tuple $(I,\, I, \ldots, I)$. This result is of independent interest.

Another function of matrices that is of obvious interest is a norm function $\|A\|$. This function may or may not be differentiable at a point A. But it is a convex function and so it is possible to compute its subdifferential $\partial\|\cdot\|$. This can then be applied to study the problem of finding best approximations to a matrix A from a given subspace $\mathbb{W}$ of $\mathbb{M}(n)$. Such problems are of importance in approximation theory and have intrigued many authors in the past few years (see [36, 42, 49, 50]). A particular case of this problem is when $\mathbb{W}$ is the subspace spanned by a matrix B. One specific question here is when is the zero matrix a best approximation to A from this subspace, that is, when does the following hold:

$$\min_{\lambda \in \mathbb{C}} \|A - \lambda B\| = \|A\|? \tag{0.7}$$

In other words, under what conditions do we have

$$\|A + \lambda B\| \geq \|A\| \text{ for all } \lambda \in \mathbb{C}? \tag{0.8}$$

A matrix A is said to be Birkhoff-James orthogonal to B if (0.8) holds. A substantial part of this thesis is devoted to the study of finding necessary and sufficient conditions for this to be the case. Let $f : \mathbb{C} \to \mathbb{R}_+$ be the map defined as $f(\lambda) = \|A + \lambda B\|$. To say that A is Birkhoff-James orthogonal to B is equivalent to saying that f attains its minimum at zero. If f were differentiable at A, then a necessary and sufficient condition for this would have been $\mathrm{D}f(0) = 0$. But our norm may not be differentiable at A. In this case, we can invoke the corresponding condition for the subdifferential: f attains its minimum at 0 if and only if 0 is in the subdifferential set $\partial f(0)$.

One of the most important norms in matrix analysis is the operator norm (also known as the spectral norm or the induced matrix 2-norm). Bhatia and Šemrl [15] obtained a very tractable condition for A to be Birkhoff-James orthogonal to B with respect to this norm. They showed that

$$\|A + \lambda B\| \geq \|A\| \text{ for all } \lambda \in \mathbb{C}$$

if and only if there exists a unit vector x such that

$$\|Ax\| = \|A\|$$

and

$$\langle Ax, Bx \rangle = 0.$$

The importance of this result is that it connects the more complicated Birkhoff-James orthogonality in the space $\mathbb{M}(n)$ to the standard orthogonality in the space $\mathbb{C}^n$. Different techniques for proving the Bhatia-Šemrl theorem have been studied by Kečkič [33] and Li and Schneider [35]. We introduce a new method to study such problems. Further, we use this method to investigate this problem for a special class of norms, namely Ky Fan k-norms.

If A and B are linear operators on an infinite dimensional Hilbert space, then a characterization for A to be Birkhoff-James orthogonal to B has been obtained in [15]. Using this, we give a necessary and sufficient condition for Birkhoff-James orthogonality in a C^*-algebra and more generally, in a Hilbert C^*-module $\mathcal{E}$ over a C^*-algebra $\mathcal{A}$. We show that for $e_1, e_2 \in \mathcal{E}$,

$$\|e_1 + \lambda e_2\| \geq \|e_1\| \text{ for all } \lambda \in \mathbb{C}$$

if and only if there exists a state φ on $\mathcal{A}$ such that

$$\varphi(\langle e_1, e_1\rangle) = \|e_1\|^2 \text{ and } \varphi(\langle e_1, e_2\rangle) = 0.$$

The distance of a matrix A to a subspace $\mathbb{W}$ of $\mathbb{M}(n)$ is defined as

$$\operatorname{dist}(A, \mathbb{W}) = \min_{W \in \mathbb{W}} \|A - W\|.$$

A useful consequence of the Bhatia-Šemrl theorem is a formula for $\operatorname{dist}(A, \mathbb{C}I)$:

$$\operatorname{dist}(A, \mathbb{C}I)^2 = \max_{\|x\|=1} \left(\|Ax\|^2 - |\langle x, Ax\rangle|^2\right).$$

The expression $\|Ax\|^2 - |\langle x, Ax\rangle|^2$ is known as variance of A with respect to the state x.

We study this problem in a much more general setting. Instead of the very special $\mathbb{C}I$, we consider any subalgebra $\mathcal{B}$ of $\mathbb{M}(n)$ and obtain a formula for $\operatorname{dist}(A, \mathcal{B})$. Let $\mathcal{C}_{\mathcal{B}}$ denote the orthogonal projection of $\mathbb{M}(n)$ onto $\mathcal{B}$. Then we show that

$$\operatorname{dist}(A, \mathcal{B})^2 = \max\{\operatorname{tr}(A^*AP - \mathcal{C}_{\mathcal{B}}(AP)^*\mathcal{C}_{\mathcal{B}}(AP)\mathcal{C}_{\mathcal{B}}(P)^{-1}) : P \geq 0, \operatorname{tr} P = 1\}, \tag{0.9}$$

where $\mathcal{C}_{\mathcal{B}}(P)^{-1}$ denotes the Moore-Penrose inverse of $\mathcal{C}_{\mathcal{B}}(P)$. As our work was in progress we came across a paper by Rieffel [41] where he raised the question of obtaining such a distance formula in any unital C^*-algebra. The formula (0.9) answers this question when the C^*-algebra is finite dimensional.

This thesis is organized as follows. In Chapter 1, we collect some basic facts about differential and subdifferential calculus. The most important convex functions in our study are norm functions on the space of matrices. We first describe the subdifferentials of some general classes of matrix norms, namely unitarily invariant norms and induced norms. These have been computed by Watson [45]. The most important norms for our study are the operator norm, the trace norm and the Ky Fan k-norms. An expression for the subdifferential of the Ky Fan k-norms has been given by Watson [47]. We obtain new formulas for this that can be used more easily in our problem. In Chapter 2, we introduce our method for approaching the problem of Birkhoff-James orthogonality of matrices. We then use this method to obtain similar results for the class of Ky Fan k-norms. In Section 2.2, we use a modification of the Bhatia-Šemrl theorem to obtain characterizations of Birkhoff-James orthogonality in $\mathcal{L}(\mathcal{H}, \mathcal{K})$, the space of bounded linear operators between infinite dimensional Hilbert spaces $\mathcal{H}, \mathcal{K}$. The space $\mathcal{L}(\mathcal{H}, \mathcal{K})$ is a Hilbert C^*-module over the C^*-algebra $\mathcal{L}(\mathcal{H})$. In Section 2.3, we obtain a necessary and sufficient condition for Birkhoff-James orthogonality in any Hilbert C^*-module. In Chapter 3, we investigate the notion of orthogonality to a subspace of $\mathbb{M}(n)$. We use the methods of subdifferential calculus to provide a necessary and sufficient condition for a matrix A to be orthogonal to a subspace $\mathbb{W}$ of $\mathbb{M}(n)$. This characterization is then used to obtain the expression (0.9) for the distance of A to any C^*-subalgebra of $\mathbb{M}(n)$. In Chapter 4, we first obtain some formulas for the higher order derivatives of the maps that take a matrix to its tensor power, antisymmetric tensor power, and symmetric tensor power. Then we go on to compute the norms of these derivatives. In Section 4.4, we obtain some expressions for the derivatives of the permanent function.

Most of the work in this thesis has already been published, some of it with coauthors. Chapter 2 is from [18], with Tirthankar Bhattacharyya. The work in Chapter 3 has been published in [28]. A part of Chapter 4 has appeared in a survey article [27]. The rest of the work in Chapter 4 is from [13], with Rajendra Bhatia and Tanvi Jain. I thank my coauthors for their permissions to include the joint works in my thesis.

Chapter 1

Differential and subdifferential calculus

In this chapter, we first recall some elementary facts about calculus on normed spaces. (For more details, see [9, Chapter X], [25, Chapter VIII].) Some important functions in our study are norm functions on the space of matrices. There are many norms of interest in matrix analysis, some of which are not differentiable, but all of them are convex. We study subdifferentials of these norms, and use them in latter chapters to characterize Birkhoff-James orthogonality. For convenience, we first collect some facts about the directional derivatives of a convex function, and then use them to define its subdifferential. More details on subdifferential calculus can be found in [30, Chapter D], [48, Chapter 2]. After stating the basic rules, we list some known results from [45, 47] on special norms of interest in matrix analysis. In Section 1.5, we draw special attention to the subdifferentials of the Ky Fan k-norms, for which we obtain some new expressions.

We first fix some notations. Let $\mathbb{X}$ be a real or complex Banach space and let $\mathbb{X}^*$ denote the real or complex dual of $\mathbb{X}$, respectively. We note that every complex Banach space $\mathbb{X}$ is also a real Banach space. There is a one to one correspondence between $\mathbb{X}^*$ and the real dual of $\mathbb{X}$ (see [23, Lemma 6.3]). We have $\varphi \in \mathbb{X}^*$ if and only if $\operatorname{Re}\varphi$ is a bounded real linear functional of $\mathbb{X}$. Similarly, if $\mathcal{H}$ is a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, then it is also a real Hilbert space with inner product $\mathrm{Re}\langle\cdot,\cdot\rangle$. All inner products on complex Hilbert spaces are assumed to be conjugate linear in the first component and linear in the second.

In this chapter, facts on differential calculus are stated without proofs, but for the convenience of the reader, most of the results on subdifferential calculus are given with proofs. Watson [45, 47] has computed the subdifferentials of norms on $\mathbb{M}(m, n; \mathbb{R})$, the space of $m \times n$ real matrices. Our interest is in the analogues of these results for the space of square complex matrices. Let $\mathbb{M}(n)$ denote the space of $n \times n$ complex matrices. We state the results on the subdifferentials of norms on $\mathbb{M}(n)$, by treating it as a real space. The proofs of these are small modifications of the real case.

1.1 The Fréchet derivative

Let $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{Z}$ be real or complex Banach spaces. Let $\mathbb{U}$ be an open subset of $\mathbb{X}$. Let $a \in \mathbb{U}$. Let $\mathbb{U}'$ be an open neighbourhood of 0 such that $a + h \in \mathbb{U}$ for all $h \in \mathbb{U}'$.

Definition 1.1.1. *Let $f : \mathbb{U} \to \mathbb{Y}$ be a continuous map. Then the map f is called Fréchet differentiable at a if there exists a bounded real linear operator $\mathrm{D}f(a) : \mathbb{X} \to \mathbb{Y}$ such that*

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - \mathrm{D}f(a)(h)\|}{\|h\|} = 0. \tag{1.1.1}$$

For details, see [25, Chapter VIII], [9, Chapter X]. The map f is said to be *Gâteaux differentiable* at a in the direction $x \in \mathbb{X}$ if $\lim_{t\to 0} \frac{f(a+tx)-f(a)}{t}$ exists. The Gâteaux derivative at a in the direction x, denoted by $\mathrm{D}_x f(a)$ or $f'(a, x)$, is given by

$$f'(a, x) = \lim_{t\to 0} \frac{f(a+tx) - f(a)}{t} = \frac{d}{dt}\bigg|_{t=0} f(a+tx).$$

The map f is said to be Gâteaux differentiable at a if $f'(a, x)$ exists for all $x \in \mathbb{X}$. If f is Fréchet differentiable at a, then the action of the map $\mathrm{D}f(a)$ at any $x \in \mathbb{X}$ is given by

$$\mathrm{D}f(a)(x) = f'(a, x). \tag{1.1.2}$$

We shall say f is differentiable to mean f is Fréchet differentiable at every point of $\mathbb{U}$. Some important rules of differentiation are stated below.

Proposition 1.1.2. *Let $f, g : \mathbb{U} \to \mathbb{Y}$ be two continuous mappings and let α be any scalar. If f and g are differentiable at $a \in \mathbb{U}$, then $f + g$ and αf are also differentiable at a, and the derivatives obey the rules*

$$\mathrm{D}(f+g)(a) = \mathrm{D}f(a) + \mathrm{D}g(a)$$

and

$$\mathrm{D}(\alpha f)(a) = \alpha \mathrm{D}f(a).$$

Proposition 1.1.3 (Chain Rule)**.** *Let $f : \mathbb{U} \to \mathbb{Y}$ be a continuous map. Let $a \in \mathbb{U}$ and let $b = f(a)$. Let $\mathbb{V}$ be an open subset of $\mathbb{Y}$ containing b. Let $g : \mathbb{V} \to \mathbb{Z}$ be a continuous map. If f is differentiable at a and g is differentiable at b, then the mapping $h = g \circ f$ is differentiable at a, and*

$$\mathrm{D}h(a) = \mathrm{D}g(b) \circ \mathrm{D}f(a).$$

Proposition 1.1.4. *Let f be a homeomorphism of an open subset $\mathbb{U}$ of $\mathbb{X}$ onto an open subset $\mathbb{V}$ of $\mathbb{Y}$ and let g be the inverse homeomorphism. If f is differentiable at $a \in \mathbb{U}$, and $\mathrm{D}f(a) : \mathbb{X} \to \mathbb{Y}$ is a real linear homeomorphism, then g is differentiable at $b = f(a)$ and $\mathrm{D}g(b) = \mathrm{D}f(a)^{-1}$.*

Some examples of differentiable functions are given below.

Example 1.1.5. The derivative of a bounded linear map $f : \mathbb{X} \to \mathbb{Y}$ exists at every point a of $\mathbb{X}$ and $\mathrm{D}f(a) = f$.

Example 1.1.6. Let $f : \mathbb{X} \times \mathbb{Y} \to \mathbb{Z}$ be a bounded bilinear mapping. Then f is differentiable at every point of $X \times \mathbb{Y}$ and its derivative at (a, b) is given by

$$\mathrm{D}f((a,b))((x,y)) = f((a,y)) + f((x,b)) \text{ for all } (x,y) \in \mathbb{X} \times \mathbb{Y}.$$

Example 1.1.7. Let $\mathbb{X}$ be a complex Hilbert space. Then the mapping $f(a) = \|a\|$ is differentiable at every point $a \neq 0$ and the derivative is given by

$$\mathrm{D}f(a)(x) = \frac{\operatorname{Re}\langle a, x\rangle}{\|a\|} \text{ for all } x \in \mathbb{X}.$$

We now give some interesting examples from matrix theory.

Example 1.1.8. Let $f : \mathbb{M}(n) \to \mathbb{M}(n)$ be the map defined by $f(A) = A^k$, where k is any natural number. Then $\mathrm{D}f(A)(X)$ is the coefficient of t in $(A + tX)^k$. The non-commutative binomial theorem says that

$$(A + X)^k = \sum_{\substack{j_1,\ldots,j_p \geq 0 \\ j_1+\cdots+j_p=k}} A^{j_1} X^{j_2} A^{j_3} \cdots X^{j_p}. \tag{1.1.3}$$

Using this, we get

$$\mathrm{D}f(A)(X) = \sum_{\substack{i,j \geq 0 \\ i+j=k-1}} A^i X A^j. \tag{1.1.4}$$

Example 1.1.9. Let $f : \mathbb{M}(n) \to \mathbb{M}(n)$ be the map defined by

$$f(A) = A^* A.$$

Then for each $A \in \mathbb{M}(n)$, we have

$$\mathrm{D}f(A)(X) = A^* X + X^* A \text{ for all } X \in \mathbb{M}(n).$$

Example 1.1.10. Let $f : \mathbb{M}(n) \to \mathbb{M}(n^k)$ be the map defined by $f(A) = \otimes^k(A)$, the kth tensor power of A. Since tensor power is a multilinear map, we have that $\mathrm{D}f(A)(X)$ is the coefficient of t in $\otimes^k(A + tX)$. We have the following expression similar to the binomial theorem (1.1.3):

$$\otimes^k(A + X) = \sum_{\substack{j_1,\ldots,j_p \geq 0 \\ j_1+\cdots+j_p=k}} (\otimes^{j_1} A) \otimes (\otimes^{j_2} X) \otimes (\otimes^{j_3} A) \otimes \cdots \otimes (\otimes^{j_p} X).$$

Using this, we obtain

$$\mathrm{D} \otimes^k (A)(X) = X \otimes A \otimes \ldots \otimes A + A \otimes X \otimes \ldots \otimes A + \ldots + A \otimes A \otimes \ldots \otimes X.$$

Let $\mathbb{GL}(n)$ denote the subset of invertible matrices in $\mathbb{M}(n)$.

Example 1.1.11. Let $f : \mathbb{GL}(n) \to \mathbb{GL}(n)$ be the map defined by

$$f(A) = A^{-1}.$$

Then f is differentiable at each $A \in \mathbb{GL}(n)$ and

$$\mathrm{D}f(A)(X) = -A^{-1}XA^{-1} \text{ for all } X \in \mathbb{M}(n).$$

For $A \in \mathbb{M}(n)$, let $s_1(A) \geq \cdots \geq s_n(A) \geq 0$ denote the singular values of A.

Example 1.1.12. Let $1 \leq p < \infty$. Let $f : \mathbb{M}(n) \to \mathbb{R}$ be defined as

$$f(A) = \|A\|_p^p = \sum_{i=1}^{n} s_i(A)^p. \tag{1.1.5}$$

The norm $\|A\|_p$ is known as the *Schatten p-norm* of A. Abatzoglou [1] and Aiken, Erdos and Goldstein [2] showed that for $1 < p < \infty$, f is Fréchet differentiable everywhere except at zero. Let $A = U|A|$ be a polar decomposition of A, where U is unitary and $|A| = (A^*A)^{1/2}$. Then

$$\mathrm{D}f(A)(X) = p \operatorname{Re} \operatorname{tr} |A|^{p-1}U^*X \text{ for all } X \in \mathbb{M}(n). \tag{1.1.6}$$

For $n \geq 2$ and $p = 1$, this map is nowhere Fréchet differentiable. However, it is Gâteaux differentiable at A if and only if A is invertible. In this case, we have for each $X \in \mathbb{M}(n)$,

$$f'(A, X) = \operatorname{Re} \operatorname{tr} U^*X. \tag{1.1.7}$$

Let $\mathcal{L}^R(\mathbb{X}; \mathbb{Y})$ denote the Banach space of all bounded real linear operators from $\mathbb{X}$ into $\mathbb{Y}$.

Suppose $f : \mathbb{X} \to \mathbb{Y}$ is a continuously differentiable mapping. Then the map $\mathrm{D}f$ is a continuous map of $\mathbb{X}$ into $\mathcal{L}^R(\mathbb{X}; \mathbb{Y})$. If $\mathrm{D}f$ is differentiable at $a \in \mathbb{X}$, then f is said to be twice differentiable at a, and the *second derivative* of f at a, denoted by $\mathrm{D}^2 f(a)$, is the derivative of $\mathrm{D}f$ at a. This is an element of $\mathcal{L}^R(\mathbb{X}; (\mathcal{L}^R(\mathbb{X}; \mathbb{Y}))$ which is identified with $\mathcal{L}_2^R(\mathbb{X}; \mathbb{Y})$, the space of bounded real bilinear mappings of $\mathbb{X} \times \mathbb{X}$ into $\mathbb{Y}$. Similarly, for any m, if $\mathrm{D}^{m-1} f$ is differentiable at $a \in \mathbb{X}$, then f is said to be m-times differentiable at a. The m^{th} derivative of f at a, denoted by $\mathrm{D}^m f(a)$, is an element of $\mathcal{L}_m^R(\mathbb{X}; \mathbb{Y})$, the space of bounded real multilinear mappings of $\mathbb{X} \times \cdots \times \mathbb{X}$ into $\mathbb{Y}$. If f is m-times differentiable at a, then for $x^1, \ldots, x^m \in \mathbb{X}$,

$$\mathrm{D}^m f(a)(x^1, \ldots, x^m) = \left.\frac{\partial^m}{\partial t_1 \cdots \partial t_m}\right|_{t_1 = \cdots = t_m = 0} f(a + t_1 x^1 + \cdots + t_m x^m), \tag{1.1.8}$$

and the multilinear mapping $\mathrm{D}^m f(a)$ is symmetric.

We give some examples of infinitely differentiable functions and of their higher order derivatives.

Example 1.1.13. A bounded linear map $f : \mathbb{X} \to \mathbb{Y}$ is infinitely differentiable and for each $m \geq 2$, we have $\mathrm{D}^m f(a) = 0$.

Example 1.1.14. Any bounded bilinear mapping $f : \mathbb{X} \times \mathbb{Y} \to \mathbb{Z}$ is infinitely differentiable. Its second derivative is given by

$$\mathrm{D}^2 f((a, b))((x_1, y_1), (x_2, y_2)) = f(x_1, y_2) + f(x_2, y_1)$$

and the derivatives of order greater than or equal to 3 are 0.

Example 1.1.15. Let $f(A) = A^k$ (see Example 1.1.8). Then for $1 \leq m \leq k$,

$$\mathrm{D}^m f(A)(X^1, \ldots, X^m) = \sum_{\sigma \in S_m} \sum_{\substack{j_1, \ldots, j_{m+1} \geq 0 \\ j_1 + \cdots + j_{m+1} = k - m}} A^{j_1} X^{\sigma(1)} A^{j_2} X^{\sigma(2)} \cdots A^{j_m} X^{\sigma(m)} A^{j_{m+1}},$$

where S_m is the set of all permutations on $\{1, 2, \ldots, \text{m}\}$. For $m > k$, we have $\mathrm{D}^m f(A) = 0$.

Example 1.1.16. Let $f(A) = A^*A$ (see Example 1.1.9). Then

$$\mathrm{D}^2 f(A)(X^1, X^2) = X^{1^*}X^2 + X^{2^*}X^1.$$

For $m > 2$, $\mathrm{D}^m f(A) = 0$.

Example 1.1.17. Let $f : \mathbb{GL}(n) \to \mathbb{GL}(n)$ be the map defined by $f(A) = A^{-1}$ (see Example 1.1.11). Then for each m,

$$\mathrm{D}^m f(A)(X^1, \ldots, X^m) = (-1)^m \sum_{\sigma \in S_m} A^{-1}X^{\sigma(1)}A^{-1}X^{\sigma(2)} \cdots A^{-1}X^{\sigma(m)}A^{-1}.$$

We shall study higher order derivatives of some other matrix functions in Chapter 4. These include determinant, permanent, tensor power, antisymmetric tensor power and symmetric tensor power of matrices.

Let $T : \mathbb{X} \to \mathbb{Y}$ be a bounded linear operator. The *operator norm* of T is defined as

$$\|T\| = \sup_{\|x\|=1} \|Tx\|. \tag{1.1.9}$$

Two of the most important theorems in differential calculus are the mean value theorem and Taylor's theorem.

Theorem 1.1.18 (Mean Value Theorem)**.** *Let $\mathbb{U}$ be an open convex subset of $\mathbb{X}$ and let $f : \mathbb{U} \to \mathbb{Y}$ be a differentiable map. Let $a, b \in \mathbb{U}$ and let L be the line segment joining them. Then*

$$\|f(b) - f(a)\| \leq \|b - a\| \sup_{u \in L} \|\mathrm{D}f(u)\|. \tag{1.1.10}$$

Theorem 1.1.19 (Taylor's Theorem)**.** *Let $f : \mathbb{U} \to \mathbb{Y}$ be a $(q+1)$-times differentiable function, then for all $a \in \mathbb{X}$ and for small $x \in \mathbb{X}$,*

$$f(a+x) = \sum_{m=1}^{q} \frac{1}{m!}\mathrm{D}^m f(a)(x, \ldots, x) + O(\|x\|^{q+1}). \tag{1.1.11}$$

1.2 The subdifferential of a convex function

Let $\mathbb{U}$ be a non-empty, convex and open subset of $\mathbb{X}$, and let $f : \mathbb{U} \to \mathbb{R}$ be a convex function. Let $a \in \mathbb{U}$ and $x \in \mathbb{X}$ be fixed. Let $t \in \mathbb{R}$ be such that $t \neq 0$ and $a + tx \in \mathbb{U}$. Let

$$q(t) = \frac{f(a + tx) - f(a)}{t}. \tag{1.2.1}$$

Theorem 1.2.1. *The map $t \mapsto q(t)$ is non-decreasing for $t > 0$ and for $t < 0$.*

Proof. We first show that if $0 < s \leq t$, then $q(s) \leq q(t)$. For such s, t, we have

$$s = (1 - \alpha)t \text{ for some } 0 \leq \alpha \leq 1. \tag{1.2.2}$$

Therefore

$$a + sx = a + (1 - \alpha)tx = \alpha a + (1 - \alpha)(a + tx).$$

Since f is convex, we have

$$f(a + sx) \leq \alpha f(a) + (1 - \alpha)f(a + tx).$$

Subtracting $f(a)$ from both sides and then dividing by t, we obtain

$$\frac{f(a + sx) - f(a)}{t} \leq (1 - \alpha)\frac{f(a + tx) - f(a)}{t}.$$

Using (1.2.2), we get

$$q(s) \leq q(t). \tag{1.2.3}$$

Now let $s \leq t < 0$. Then $0 < -t \leq -s$. So $-t = (1 - \alpha)(-s)$, that is, $t = (1 - \alpha)s$ for some $0 \leq \alpha \leq 1$. As before, we have

$$f(a + tx) \leq \alpha f(a) + (1 - \alpha)f(a + sx).$$

Subtracting $f(a)$ from both sides and then dividing by t, we obtain

$$\frac{f(a+tx)-f(a)}{t} \geq (1-\alpha)\frac{f(a+sx)-f(a)}{s}.$$

This gives (1.2.3) in this case too. □

Theorem 1.2.2. *The map $t \mapsto q(t)$ is bounded near zero.*

Proof. Consider the set $I_{a,x} = \{t \in \mathbb{R} | a + tx \in \mathbb{U}\}$. Since $\mathbb{U}$ is open and convex, $I_{a,x}$ is an open interval in $\mathbb{R}$. Define $\varphi_{a,x} : I_{a,x} \to \mathbb{R}$ as

$$\varphi_{a,x}(s) = f(a+sx).$$

Then for $t \neq 0$, we have

$$\frac{f(a+tx)-f(a)}{t} = \frac{\varphi_{a,x}(t)-\varphi_{a,x}(0)}{t}.$$

The map $t \mapsto q(t)$ is bounded near zero if and only if the map $t \mapsto \frac{\varphi_{a,x}(t)-\varphi_{a,x}(0)}{t}$ is bounded near zero. So it is enough to prove the theorem for $\mathbb{X} = \mathbb{R}$.

We show that f is locally Lipschitz continuous. Since $\mathbb{U}$ is an open convex subset of $\mathbb{R}$, there exist $c, d \in \mathbb{R} \cup \{-\infty, \infty\}$ such that $\mathbb{U} = (c, d)$. Now $a \in (c, d)$. So there exists $\delta > 0$ such that $(a - 2\delta, a + 2\delta) \subset (c, d)$. Let $y, y' \in [a - \delta, a + \delta]$. Without loss of generality, assume that $y \leq y'$. Choose t_1, t_2 such that $c < t_2 < a - \delta$ and $a + \delta < t_1 < d$. Since f is convex, it follows that

$$\frac{f(a-\delta)-f(t_2)}{a-\delta-t_2} \leq \frac{f(y')-f(y)}{y'-y} \leq \frac{f(t_1)-f(a+\delta)}{t_1-a-\delta}.$$

Let $L = \max\left\{\left|\frac{f(a-\delta)-f(t_2)}{a-\delta-t_2}\right|, \left|\frac{f(t_1)-f(a+\delta)}{t_1-a-\delta}\right|\right\}$. Hence

$$|f(y)-f(y')| \leq L|y-y'| \text{ for all } y, y' \in [a-\delta, a+\delta]. \tag{1.2.4}$$

□

Theorem 1.2.1 and Theorem 1.2.2 together tell us that $q(t)$ has a limit as t decreases to 0 or increases to 0.

Definition 1.2.3. *The right and left directional derivatives of f at a in the direction x are defined as*

$$f'_+(a,x) = \lim_{t\downarrow 0} \frac{f(a+tx)-f(a)}{t} = \inf\{q(t) : t > 0\} \tag{1.2.5}$$

and

$$f'_-(a,x) = \lim_{t\uparrow 0} \frac{f(a+tx)-f(a)}{t} = \sup\{q(t) : t < 0\}. \tag{1.2.6}$$

These are sometimes also denoted by $\mathrm{D}_+f(a)(x)$ and $\mathrm{D}_-f(a)(x)$, respectively. If f is Gâteaux differentiable, then $f'_+(a,x) = f'_-(a,x)$. In convex analysis, $f'_+(a,x)$ is sometimes also called as the directional derivative of f at a in the direction x, and is denoted by $f'(a,x)$. We will not use this notation.

Let $f : \mathbb{X} \to \mathbb{R}$ be a continuous convex function. It is easy to see that if f is differentiable at $a \in \mathbb{X}$, then

$$\mathrm{D}f(a)(y-a) \leq f(y) - f(a) \text{ for all } y \in \mathbb{X}.$$

So it is quite natural to consider the elements φ_R of the real dual of $\mathbb{X}$ which satisfy the condition

$$\varphi_R(y-a) \leq f(y) - f(a) \text{ for all } y \in \mathbb{X},$$

even when f is not differentiable at a. We know that if $\mathbb{X}$ is a complex Banach space, then there exists a unique $\varphi \in \mathbb{X}^*$ such that $\varphi_R = \operatorname{Re}\varphi$.

Definition 1.2.4. *A subgradient of f at $a \in \mathbb{X}$ is an element φ of $\mathbb{X}^*$ such that*

$$f(y) - f(a) \geq \operatorname{Re}\varphi(y-a) \quad \textit{for all } y \in \mathbb{X}. \tag{1.2.7}$$

The *subdifferential* is the set of bounded linear functionals $\varphi \in \mathbb{X}^*$ satisfying (1.2.7) and is denoted by $\partial f(a)$. It is a non-empty weak* compact convex subset of $\mathbb{X}^*$. For more details, see [30, Chapter D], [48, Chapter 2]. The following proposition is a direct consequence of the definition of the subdifferential. It is one of the most useful tools that we require in Chapter 2 and Chapter 3.

Proposition 1.2.5. *A continuous convex function $f : \mathbb{X} \to \mathbb{R}$ attains its minimum value at a if and only if $0 \in \partial f(a)$.*

Proof. It can be readily seen from (1.2.7) that $f(y) \geq f(a)$ for all $y \in \mathbb{X}$ if and only if 0 is a subgradient of f at a. $\square$

Definition 1.2.6. *Let $\mathbb{W}$ be a non-empty subset of $\mathbb{X}$. The support function of $\mathbb{W}$ is the function $\sigma_{\mathbb{W}} : \mathbb{X}^* \to \mathbb{R} \cup \{+\infty\}$ defined as*

$$\sigma_{\mathbb{W}}(\varphi) = \sup\{\operatorname{Re}\varphi(w) : w \in \mathbb{W}\}.$$

Let $\mathbb{B}$ be a non-empty subset of $\mathbb{X}^*$, then the support function $\sigma_{\mathbb{B}} : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$\sigma_{\mathbb{B}}(x) = \sup\{\operatorname{Re}\varphi(x) : \varphi \in \mathbb{B}\}.$$

Recall that a function $p : \mathbb{X} \to \mathbb{R}$ is said to be *sublinear* if it satisfies

(i) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in \mathbb{X}$,

(ii) $p(ax) = a\, p(x)$ for $a > 0, x \in \mathbb{X}$.

A support function is sublinear. Furthermore, the support function of $\mathbb{W}$ is finite everywhere if and only if $\mathbb{W}$ is weakly bounded. If $\mathbb{W}_1$ and $\mathbb{W}_2$ are two non-empty subsets of $\mathbb{X}$, then

$$\sigma_{\mathbb{W}_1+\mathbb{W}_2} = \sigma_{\mathbb{W}_1} + \sigma_{\mathbb{W}_2},$$

and

$$\sigma_{\mathbb{W}_1\cup\mathbb{W}_2} = \max\{\sigma_{\mathbb{W}_1}, \sigma_{\mathbb{W}_2}\}.$$

If $W_1 \subset W_2$, then $\sigma_{\mathbb{W}_1}(\varphi) \leq \sigma_{\mathbb{W}_2}(\varphi)$ for all $\varphi \in \mathbb{X}^*$. An important property of the support functions is given by the following proposition.

Proposition 1.2.7. *Let $T : \mathbb{X} \to \mathbb{Y}$ be a bounded linear map. Then for any non-empty subset $\mathbb{B}$ of $\mathbb{Y}^*$,*

$$\sigma_{T^*(\mathbb{B})}(x) = \sigma_{\mathbb{B}}(T(x)) \text{ for all } x \in \mathbb{X}. \tag{1.2.8}$$

Proof. By definition of the adjoint of the map T,

$$\operatorname{Re} T^*(\varphi)(x) = \operatorname{Re} \varphi(T(x)).$$

Taking $\sup$ over $\varphi \in \mathbb{B}$ on both the sides, we get (1.2.8). □

A function $\sigma : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ is said to be *closed* if, for each $x \in \mathbb{X}$,

$$\liminf_{x' \to x} \sigma(x') \geq \sigma(x).$$

There is a close connection between closed convex sets and closed sublinear functions.

Proposition 1.2.8. *Let $\mathbb{B}$ be a non-empty closed convex subset of $\mathbb{X}^*$ and let σ be a closed sublinear function on $\mathbb{X}$. Then the following are equivalent.*

(i) σ is the support function of $\mathbb{B}$,

(ii) $\mathbb{B} = \{\varphi \in \mathbb{X}^* : \operatorname{Re} \varphi(x) \leq \sigma(x) \text{ for all } x \in \mathbb{X}\}$.

The next theorem gives an alternate definition of the subdifferential of a convex function. We will use this later in sections 1.4 and 1.5 to compute subdifferentials of some matrix norms.

Theorem 1.2.9. *Let $f : \mathbb{X} \to \mathbb{R}$ be a continuous convex function. Then*

$$\partial f(a) = \{\varphi \in \mathbb{X}^* : \operatorname{Re}\ \varphi(x) \leq f'_+(a, x) \text{ for all } x \in \mathbb{X}\}. \tag{1.2.9}$$

Moreover, for each $x \in \mathbb{X}$,

$$f'_+(a, x) = \max\{\operatorname{Re} \varphi(x) : \varphi \in \partial f(a)\}.$$

Proof. First let $\varphi \in \mathbb{X}^*$ be such that

$$\operatorname{Re}\varphi(x) \leq f'_+(a,x) \text{ for all } x \in \mathbb{X}. \tag{1.2.10}$$

By (1.2.5), we obtain

$$\operatorname{Re}\varphi(x) \leq \frac{f(a+tx)-f(a)}{t} \text{ for all } t>0,\ x \in \mathbb{X}. \tag{1.2.11}$$

Let $y = a+tx$. As t varies over $(0,\infty)$ and x varies over $\mathbb{X}$, $y = a+tx$ varies over whole of $\mathbb{X}$. Using this in (1.2.11), we get

$$\operatorname{Re}\varphi(y-a) \leq f(y)-f(a) \text{ for all } y \in \mathbb{X}.$$

Now suppose (1.2.7) holds. For $t>0$, let $x = \frac{y-a}{t}$. As t varies over $(0,\infty)$ and y varies over $\mathbb{X}$, so does x. So we get (1.2.11). By using (1.2.5), we obtain (1.2.10).

We now show that for each $x \in \mathbb{X}$, there exists $\varphi_x \in \partial f(a)$ such that $\operatorname{Re}\varphi_x(x) = f'_+(a,x)$. Consider the one dimensional affine set $\mathbb{W}_x = a + \mathbb{R}x$ and the continuous affine function $h_x : \mathbb{W}_x \to \mathbb{R}$ defined as $h_x(a+tx) = f(a) + tf'_+(a,x)$. Then $h_x \leq f$ on $\mathbb{W}_x$. By a consequence of the Hahn-Banach theorem, there exists a continuous affine extension $\hat{h}_x : \mathbb{X} \to \mathbb{R}$ of h_x such that $\hat{h}_x \leq f$. Since $\hat{h}_x(a) = h_x(a) = f(a)$, we obtain $\hat{h}_x(y) = f(a) + \operatorname{Re}\varphi_x(y-a)$ for some $\varphi_x \in \mathbb{X}^*$. Thus $\operatorname{Re}\varphi_x(x) = \hat{h}_x(a+x) - f(a) = h_x(a+x) - f(a) = f'_+(a,x)$. $\square$

From the above theorem, we obtain

$$f'_+(a,x) = \sigma_{\partial f(a)}(x). \tag{1.2.12}$$

Corollary 1.2.10. *Let $f : \mathbb{X} \to \mathbb{R}$ be a continuous convex function. Then f is Gâteaux differentiable at $a \in \mathbb{X}$ if and only if its only subgradient at a is its Gâteaux derivative $f'(a,\cdot)$.*

Proof. By Theorem 1.2.9, we have that $\varphi \in \partial f(a)$ if and only if

$$f'_-(a,x) \leq \varphi(x) \leq f'_+(a,x) \text{ for all } x \in \mathbb{X}.$$

If f is Gâteaux differentiable at a, then $f'_-(a,x) = f'_+(a,x) = f'(a,x)$ for all $x \in \mathbb{X}$. So $\partial f(a) = \{f'(a,\cdot)\}$. If f is not Gâteaux differentiable at a, then there exists $x \in \mathbb{X}$ such that $f'_-(a,x) \neq f'_+(a,x)$. By Theorem 1.2.9, there exist $\varphi_1, \varphi_2 \in \partial f(a)$ such that $\varphi_1(x) = f'_+(a,x)$ and $\varphi_2(-x) = f'_+(a,-x) = -f'_-(a,x)$. Then $\varphi_1(x) - \varphi_2(x) = f'_+(a,x) - f'_-(a,x) \neq 0$ and hence $\partial f(a)$ is not a singleton. □

We now provide some rules of subdifferential calculus, which will be helpful in our analysis in Chapter 2 and Chapter 3.

Proposition 1.2.11. *Let $f_1, f_2 : \mathbb{X} \to \mathbb{R}$ be two continuous convex functions and let t_1, t_2 be positive numbers. Then*

$$\partial(t_1 f_1 + t_2 f_2)(a) = t_1 \partial f_1(a) + t_2 \partial f_2(a) \textit{ for all } a \in \mathbb{X}. \tag{1.2.13}$$

Proof. Note that $t_1 \partial f_1(a) + t_2 \partial f_2(a)$ is a compact convex set whose support function is

$$t_1 f'_{1+}(a,\cdot) + t_2 f'_{2+}(a,\cdot). \tag{1.2.14}$$

On the other hand, the support function of $\partial(t_1 f_1 + t_2 f_2)(a)$ is $(t_1 f_1 + t_2 f_2)'_+(a,\cdot)$, which is the same as (1.2.14). □

Proposition 1.2.12. *Let $S : \mathbb{X} \to \mathbb{Y}$ be a bounded linear map and let $L : \mathbb{X} \to \mathbb{Y}$ be the continuous affine map defined by $L(x) = S(x) + y_0$, for some $y_0 \in \mathbb{Y}$. Let $g : \mathbb{Y} \to \mathbb{R}$ be a continuous convex function. Then*

$$\partial(g \circ L)(a) = S^* \partial g(L(a)) \textit{ for all } a \in \mathbb{X}. \tag{1.2.15}$$

Proof. By Theorem 1.2.9,

$$\sigma_{\partial(g\circ L)(a)}(x) = (g \circ L)'_+(a,x).$$

To compute $(g \circ L)'_+(a, x)$, we note that

$$\begin{aligned} & \frac{(g \circ L)(a+tx) - (g \circ L)(a)}{t} \\ = \quad & \frac{g(L(a+tx)) - g(L(a))}{t} \\ = \quad & \frac{g(S(a+tx)+y_0) - g(L(a))}{t} \\ = \quad & \frac{g(S(a)+y_0+tS(x)) - g(L(a))}{t} \\ = \quad & \frac{g(L(a)+tS(x)) - g(L(a))}{t}. \end{aligned}$$

Taking infimum over $\{t : t > 0\}$, we get

$$(g \circ L)'_+(a, x) = g'_+(L(a), S(x)),$$

that is,

$$\sigma_{\partial(g \circ L)(a)}(x) = \sigma_{\partial g(L(a))}(S(x)).$$

By Proposition 1.2.7, this is equal to $\sigma_{S^*\partial g(L(a))}(x)$. Hence $\partial(g \circ L)(a) = S^*\partial g(L(a))$. $\square$

The following theorem provides a formula for the subdifferential of the supremum of convex functions. A proof of this can be found in [48, p. 97]. We shall use this theorem to obtain an expression for the subdifferential of Ky Fan k-norms in Section 1.5.

Proposition 1.2.13. *Let J be a compact set in a metric space. Let $\{f_j\}_{j \in J}$ be a collection of continuous convex functions from $\mathbb{X}$ to $\mathbb{R}$ such that the maps $j \to f_j(x)$ are upper semicontinuous for each $x \in \mathbb{X}$. Let $f : \mathbb{X} \to \mathbb{R}$ be defined as $f(x) = \sup\{f_j(x) : j \in J\}$, and let $J(x) = \{j \in J : f_j(x) = f(x)\}$. Assume that $f(x) < \infty$ for all $x \in \mathbb{X}$. Then for each $a \in \mathbb{X}$,*

$$\partial f(a) = \overline{\mathrm{conv}} \left\{ \cup_{j \in J(a)} \partial f_j(a) \right\}. \tag{1.2.16}$$

We now give some examples of convex functions and their subdifferentials.

Example 1.2.14. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = |x|.$$

This function is differentiable at all $a \neq 0$ and $\mathrm{D}f(a) = \operatorname{sgn}(a)$. At zero, it is not differentiable. Note that for $v \in \mathbb{R}$,

$$f(y) = |y| \geq f(0) + v.y = v.y$$

holds for all $y \in \mathbb{R}$ if and only if $|v| \leq 1$. Hence

$$\partial f(0) = [-1, 1]. \tag{1.2.17}$$

We know that f attains its minimum at 0. This is also the assertion of Proposition 1.2.5, since $0 \in \partial f(0)$.

Example 1.2.15. Let $f : \mathbb{R} \to \mathbb{R}$ be the map defined as

$$f(x) = \max\left\{0, \frac{x^2 - 1}{2}\right\}.$$

Then f is differentiable everywhere except at $x = -1, 1$. We have

$$\partial f(1) = [0, 1] \text{ and } \partial f(-1) = [-1, 0].$$

It is easy to see that f attains the minimum value 0 at all points of the interval $[-1, 1]$. This is also evident from Proposition 1.2.5.

Example 1.2.16. Let $f : \mathbb{X} \to \mathbb{R}$ be defined as

$$f(a) = \|a\|.$$

Then for $a \neq 0$,

$$\partial f(a) = \{\varphi \in \mathbb{X}^* : \operatorname{Re} \varphi(a) = \|a\|, \|\varphi\| \leq 1\}, \tag{1.2.18}$$

and

$$\partial f(0) = \{\varphi \in \mathbb{X}^* : \|\varphi\| \leq 1\}. \tag{1.2.19}$$

Proof. We prove the result for $a \neq 0$. The proof for the case $a = 0$ is similar. Let $\varphi \in \partial f(a)$. Then by definition of the subdifferential of f (1.2.7), we have

$$\|y\| - \|a\| \geq \operatorname{Re} \varphi(y - a) \text{ for all } y \in \mathbb{X}. \tag{1.2.20}$$

By putting $y = 0$ and $y = 2a$, we obtain

$$\|a\| = \operatorname{Re} \varphi(a). \tag{1.2.21}$$

By (1.2.20) and (1.2.21), we get that for any $y \in \mathbb{X}$ with $\|y\| \leq 1$,

$$\operatorname{Re} \varphi(y) \leq 1.$$

Therefore $\|\varphi\| \leq 1$. Now suppose that $\varphi \in \mathbb{X}^*$ is such that $\operatorname{Re} \varphi(a) = \|a\|$ and $\|\varphi\| \leq 1$. Then

$$\operatorname{Re} \varphi(y - a) = \operatorname{Re} \varphi(y) - \operatorname{Re} \varphi(a) \leq \|y\| - \|a\|.$$

Hence $\varphi \in \partial f(a)$. $\square$

Example 1.2.17. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined as $f(a) = \|a\|_\infty$. Then for $a \neq 0$,

$$\partial f(a) = \operatorname{conv}\{\pm e_i : |a_i| = \|a\|_\infty\}. \tag{1.2.22}$$

This can be obtained either from Proposition 1.2.13, or from Example 1.2.16.

By (1.2.19),

$$\partial f(0) = \{v \in \mathbb{R}^n : \sum_{i=1}^{n} |v_i| \leq 1\}. \tag{1.2.23}$$

Example 1.2.18. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined as $f(a) = \|a\|_1$. If a is a vector none of whose components are zero, then f is differentiable and

$$\partial f(a) = \{\mathrm{D}f(a)\} = \left\{ \begin{pmatrix} \operatorname{sgn}(a_1) \\ \vdots \\ \operatorname{sgn}(a_n) \end{pmatrix} \right\}.$$

Now suppose s components of a are zero. Let the components of a be ordered as $|a_1| \geq \cdots \geq |a_{n-s}| > |a_{n-s+1}| = \cdots = |a_n| = 0$. Then for $a \neq 0$,

$$\partial f(a) = \{v \in \mathbb{R}^n : v_i = \operatorname{sgn}(a_i), i = 1, \ldots, n-s;\ |v_i| \leq 1, v_i \operatorname{sgn}(a_i) \geq 0, i = n-s+1, \ldots, n\}$$

and

$$\partial f(0) = \{v \in \mathbb{R}^n : |v_i| \leq 1 \text{ for all } 1 \leq i \leq n\}.$$

Example 1.2.19. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined as

$$f(a) = \|a\|_{(k)},$$

the sum of k largest elements of $|a_1|, \ldots, |a_n|$. Let the components of a be ordered so that $|a_1| \geq \cdots \geq |a_n|$. Let the multiplicity of $|a_k|$ be $r + t$, where $r \geq 0$ and $t \geq 1$, such that $|a_{k-t+1}| = \cdots = |a_{k+r}|$. If $r = 0$, that is, $|a_k| > |a_{k+1}|$ (where a_{n+1} is assigned value zero), then f is differentiable and

$$\partial f(a) = \{\mathrm{D}f(a)\} = \left\{ \begin{pmatrix} \operatorname{sgn}(a_1) \\ \vdots \\ \operatorname{sgn}(a_k) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}.$$

If $r > 0$, then Watson [46] showed that for $a \neq 0$,

$$\begin{aligned}\partial f(a) \;=\; & \{v \in \mathbb{R}^n : \|v\|_\infty \leq 1; v_i = \operatorname{sgn}(a_i), i = 1, \ldots, k-t; \\ & v_i \operatorname{sgn}(a_i) \geq 0, i = k-t+1, \ldots, k+r; \\ & \sum_{i=k-t+1}^{k+r} |v_i| = t \text{ if } a_k \neq 0, \text{ otherwise } \sum_{i=k-t+1}^{k+r} |v_i| \leq t; \\ & v_i = 0, i = k+r+1, \ldots, n\}. \end{aligned} \tag{1.2.24}$$

For $a = 0$, we have

$$\partial f(0) = \{v \in \mathbb{R}^n : |v_i| \leq 1 \text{ for all } 1 \leq i \leq n, \sum_{i=1}^{n} |v_i| \leq k\}.$$

1.3 Norms on $\mathbb{M}(n)$

Let $\mathcal{H}$ be an n-dimensional complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{L}(\mathcal{H})$ denote the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. Fix an orthonormal basis for $\mathcal{H}$. Then $\mathcal{H}$ can be identified with $\mathbb{C}^n$ and $\mathcal{L}(\mathcal{H})$ can be identified with $\mathbb{M}(n)$. Let $A \in \mathbb{M}(n)$ with singular values $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A) \geq 0$. Then the operator norm (1.1.9) can be expressed as

$$\|A\| = \sup_{\|x\|=\|y\|=1} |\langle y, Ax \rangle|.$$

We also have

$$\|A\| = s_1(A) = \|A^* A\|^{1/2}. \tag{1.3.1}$$

There are other useful norms relevant to our study. The Schatten p-norms $\|.\|_p$, $1 \leq p < \infty$, are defined in (1.1.5). The norm $\|A\|_\infty$ is defined as

$$\|A\|_\infty = s_1(A) = \|A\|.$$

The Schatten p-norms for $p = 1$ and $p = 2$, are called the *trace norm* and the *Hilbert-Schmidt* norm of A, respectively. These can also be expressed as

$$\|A\|_1 = \operatorname{tr}(|A|) \tag{1.3.2}$$

and

$$\|A\|_2 = \operatorname{tr}(A^*A)^{1/2}. \tag{1.3.3}$$

If A has entries a_{ij}, then

$$\|A\|_2 = \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2}.$$

The dual of a matrix norm $\|\cdot\|$, denoted by $\|\cdot\|^*$, is defined as

$$\|A\|^* = \max_{\|X\|=1} |\operatorname{tr}(A^*X)| = \max_{\|X\|=1} \operatorname{Re}\operatorname{tr}(A^*X). \tag{1.3.4}$$

For $1 < p < \infty$, the dual of $\|\cdot\|_p$ is $\|\cdot\|_q$, where $\frac{1}{p} + \frac{1}{q} = 1$. The trace norm is the dual of the operator norm and vice versa. Another useful class of norms is the *Ky Fan k-norms*, $1 \leq k \leq n$. They are denoted by $\|A\|_{(k)}$, and are defined as

$$\|A\|_{(k)} = s_1(A) + s_2(A) + \cdots + s_k(A). \tag{1.3.5}$$

The particular cases $k = 1$ and $k = n$ correspond to the operator norm and the trace norm, respectively.

All the norms listed above belong to a more general class of norms, namely the unitarily invariant norms. A norm $|||\cdot|||$ is said to be unitarily invariant if

$$|||UAV||| = |||A|||$$

for any unitary matrices U, V.

Definition 1.3.1. *A symmetric gauge function* $\Phi : \mathbb{R}^n \to \mathbb{R}$ *is a function with the following properties.*

(i) Φ *is a norm on* $\mathbb{R}^n$,

(ii) $\Phi(\Pi x) = \Phi(x)$ *for all* $x \in \mathbb{R}^n$ *and for all permutation matrices* Π,

(iii) $\Phi(\epsilon_1 x_1, \ldots, \epsilon_n x_n) = \Phi(x_1, \ldots, x_n)$, *if* $\epsilon_j = \pm 1$,

(iv) $\Phi(1, 0, \ldots, 0) = 1$.

Let $s(A)$ denote the n-tuple $(s_1(A), \ldots, s_n(A))$. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $\operatorname{diag}(x)$ denote the diagonal matrix with entries $x_1, \ldots, x_n$ on its diagonal. Then given a symmetric gauge function Φ on $\mathbb{R}^n$, the function given by $|||A|||_\Phi = \Phi(s(A))$ defines a unitarily invariant norm on $\mathbb{M}(n)$. Conversely, given any unitarily invariant norm $|||\cdot|||$ on $\mathbb{M}(n)$, the function on $\mathbb{R}^n$ defined by $\Phi_{|||\cdot|||}(x) = |||\operatorname{diag}(x)|||$, is a symmetric gauge function.

Every norm on $\mathbb{C}^n$ induces a norm on $\mathbb{M}(n)$. Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$. Then the norm on $\mathbb{M}(n)$ *induced* by $\|\cdot\|$ is the norm defined as

$$\|A\|' = \max_{\|x\|=1} \|Ax\|. \tag{1.3.6}$$

In particular, when the vector norm is taken to be the Euclidean norm, then the induced norm is the operator norm.

1.4 Subdifferentials of matrix norms

Let $\|\cdot\|$ be any norm on the space $\mathbb{M}(n)$. The space $\mathbb{M}(n)$ is a complex Hilbert space with inner product $\langle A, B\rangle = \operatorname{tr}(A^*B)$. By Example 1.2.16, we have that for $A \neq 0$,

$$\partial\|A\| = \{G \in \mathbb{M}(n) : \|A\| = \operatorname{Re}\operatorname{tr}(G^*A), \|G\|^* \leq 1\}. \tag{1.4.1}$$

We study the subdifferentials of various norms described in Section 1.3. In this section, we formulate analogues for $\mathbb{M}(n)$ of some known results on $\mathbb{M}(n;\mathbb{R})$ from [45] and [47]. Similar results can be stated for $\mathbb{M}(m,n)$, the space of $m \times n$ complex matrices. Since we require these results for square matrices in our analysis in subsequent chapters, we restrict ourselves to $\mathbb{M}(n)$.

Unitarily invariant norms

Let $|||\cdot|||$ be a unitarily invariant norm and let Φ be the corresponding symmetric gauge function.

Theorem 1.4.1. *Let $g : \mathbb{M}(n) \to \mathbb{R}$ be defined as $g(A) = |||A|||$ and let $X \in \mathbb{M}(n)$. Then there exist left and right singular vectors, $u_i, v_i (1 \le i \le n)$ of A corresponding to $s_i(A)$ such that*

$$g'_+(A, X) = \max_{z \in \partial\Phi(s(A))} \sum_{i=1}^{n} z_i \operatorname{Re}\langle u_i, X v_i\rangle. \tag{1.4.2}$$

From the proof of Theorem 1 in [47], we obtain the following.

Proposition 1.4.2. *Let A be any matrix in $\mathbb{M}(n)$ with singular values $s_1(A) \ge \cdots \ge s_n(A) \ge 0$. Let $S = \begin{bmatrix} s_1(A) & & \\ & \ddots & \\ & & s_n(A) \end{bmatrix}$. Consider the set*

$$\begin{aligned} \mathbb{W}(A) \;=\; & \{UDV^* : A = USV^* \text{ is a singular value decomposition of } A, D = \operatorname{diag}(d) \\ & \text{with } d \in \partial\Phi(s(A))\}, \end{aligned}$$

that is, $B \in \mathbb{W}(A)$ if and only if for any singular value decomposition USV^* of A, the matrix U^*BV is a diagonal matrix D with the property that the vector d formed by the diagonal elements of D is in $\partial\Phi(s(A))$. Then $\mathbb{W}(A)$ is a convex set.

By Theorem 1 in [47], we get the following.

Theorem 1.4.3. *For any $A \in \mathbb{M}(n)$,*

$$\begin{aligned} \partial|||A||| \;=\; & \{UDV^* : A = USV^* \text{ is a singular value decomposition of } A, D = \operatorname{diag}(d) \text{ with} \\ & d \in \partial\Phi(s(A))\}. \end{aligned} \tag{1.4.3}$$

As corollaries, Watson [45] obtained characterizations of the subdifferential of the operator norm and the trace norm. Let $A = USV^*$ be any singular value decomposition of A. Let $\|\cdot\|$ be the operator norm on $\mathbb{M}(n)$.

Corollary 1.4.4. *Let the multiplicity of $s_1(A)$ be t. Let the matrices U, V be partitioned as $U = [U_1 : U_2]$ and $V = [V_1 : V_2]$, where $U_1, V_1 \in \mathbb{M}(n,t)$ and $U_2, V_2 \in \mathbb{M}(n, n-t)$. Then $G \in \partial\|A\|$ if and only there exists a positive semidefinite matrix $T \in \mathbb{M}(t)$ with $\operatorname{tr} T = 1$ such that $G = U_1 T V_1^*$.*

Corollary 1.4.5. *Let the number of zero singular values of A be ℓ. Let the matrices U, V be partitioned as $U = [U_1 : U_2]$ and $V = [V_1 : V_2]$, where $U_1, V_1 \in \mathbb{M}(n, n-\ell)$ and $U_2, V_2 \in \mathbb{M}(n, \ell)$. Then $G \in \partial\|A\|_1$ if and only if there exists $T \in \mathbb{M}(\ell)$ with $s_1(T) \leq 1$ such that $G = U_1 V_1^* + U_2 T V_2^*$.*

Induced norms

Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$ and let $\|\cdot\|'$ be the norm on $\mathbb{M}(n)$ induced by $\|\cdot\|$. Let $\mathbb{K}(A)$ be the set

$$\mathbb{K}(A) = \{(v, w) : u, v, w \in \mathbb{C}^n \text{ with } \|u\| = \|v\| = 1, Av = \|A\|' u, w \in \partial\|u\|\}. \tag{1.4.4}$$

Let v be a right singular vector of A, corresponding to its maximum singular value $\|A\|'$ and let u be the corresponding left singular vector. Then the set $\mathbb{K}(A)$ contains the tuples (v, w) where $w \in \partial\|u\|$. Note that $\mathbb{K}(0) = \{(v, w) : v, w \in \mathbb{C}^n \text{ with } \|w\|^* = \|v\| = 1\}$.

We state the next two theorems along with their proofs. We will use similar ideas in the proofs of Theorem 1.5.3 and Theorem 1.5.6.

Theorem 1.4.6. *Let $g : \mathbb{M}(n) \to \mathbb{R}$ be the map defined as $g(A) = \|A\|'$. Then*

$$g'_+(A, X) = \max_{(v,w) \in \mathbb{K}(A)} \operatorname{Re}\langle w, Xv\rangle.$$

Proof. For any $(v, w) \in \mathbb{K}(A)$, we have

$$\|A + tX\|' \geq \|(A + tX)v\|.$$

By the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\|A+tX\|' &\geq \mathrm{Re}\langle w,(A+tX)v\rangle\\ &= \|A\|'\,\mathrm{Re}\langle w,u\rangle + t\,\mathrm{Re}\langle w,Xv\rangle.\end{aligned}$$

Since $w \in \partial\|u\|$, we get from Example 1.2.16 that $\mathrm{Re}\langle w,u\rangle = \|u\| = 1$. Thus we obtain

$$\|A+tX\|' \geq \|A\|' + t\,\mathrm{Re}\langle w,Xv\rangle.$$

This implies that for $t>0$,

$$\frac{\|A+tX\|' - \|A\|'}{t} \geq \max_{(v,w)\in\mathbb{K}(A)} \mathrm{Re}\langle w,Xv\rangle. \tag{1.4.5}$$

Similarly for any $(v(t),w(t)) \in \mathbb{K}(A+tX)$, we have

$$\begin{aligned}\|A\|' &\geq \|Av(t)\|\\ &\geq \mathrm{Re}\langle w(t),Av(t)\rangle\\ &= \|A+tX\|'\,\mathrm{Re}\langle w(t),u(t)\rangle - t\,\mathrm{Re}\langle w(t),Xv(t)\rangle\\ &= \|A+tX\|' - t\,\mathrm{Re}\langle w(t),Xv(t)\rangle.\end{aligned}$$

Thus we get that for $t>0$,

$$\frac{\|A+tX\|' - \|A\|'}{t} \leq \mathrm{Re}\langle w(t),Xv(t)\rangle. \tag{1.4.6}$$

Let $\{t_n\}$ be a sequence of positive real numbers such that $t_n \to 0$ as $n\to\infty$. Since the unit ball in $\mathbb{C}^n$ is compact, there exists a subsequence $\{t_{n_m}\}$ of $\{t_n\}$ and vectors u',v',w' such that

$$v(t_{n_m}) \to v',\, u(t_{n_m}) \to u',\, w(t_{n_m}) \to w' \text{ as } m\to\infty.$$

So

$$\|u'\| = \|v'\| = \|w'\|^* = 1,$$

$$Av' = \|A\|'u',$$

and

$$\mathrm{Re}\langle w', u'\rangle = \|u'\| = 1.$$

So $w' \in \partial\|u'\|$. Thus $(v', w') \in \mathbb{K}(A)$. By (1.4.6), we get that

$$\lim_{t\to 0^+} \frac{\|A + tX\|' - \|A\|'}{t} \le \mathrm{Re}\langle w', Xv'\rangle.$$

Combining this with (1.4.5), we obtain the required result. □

Theorem 1.4.7. *We have*

$$\partial\|A\|' = \mathrm{conv}\{wv^* : (v, w) \in \mathbb{K}(A)\}. \tag{1.4.7}$$

Proof. First let $G \in \mathrm{conv}\{wv^* : (v, w) \in \mathbb{K}(A)\}$. Then $G = \sum_i \alpha_i w_i v_i^*$, where $0 \le \alpha_i \le 1$, $\sum_i \alpha_i = 1$, $(v_i, w_i) \in \mathbb{K}(A)$. We get

$$\begin{aligned} \mathrm{Re}\,\mathrm{tr}(G^*A) &= \sum_i \alpha_i \,\mathrm{Re}\langle w_i, Av_i\rangle \\ &= \|A\|' \sum_i \alpha_i \,\mathrm{Re}\langle w_i, u_i\rangle \\ &= \|A\|' \end{aligned}$$

and

$$\begin{aligned} \|G\|'^* &= \max_{\|X\|'\le 1} \mathrm{Re}\,\mathrm{tr}(G^*X) \\ &\le \sum_i \alpha_i \max_{\|X\|'\le 1} \mathrm{Re}\langle w_i, Xv_i\rangle \\ &\le 1. \end{aligned}$$

By (1.4.1), we get that $G \in \partial\|A\|'$.

Now let $G \in \partial\|A\|'$. Suppose $G \notin \operatorname{conv}\{wv^* : (v,w) \in \mathbb{K}(A)\}$. By the Separating Hyperplane Theorem, there exists $X \in \mathbb{M}(n)$ such that

$$\operatorname{Re}\operatorname{tr}(X^* wv^*) < \operatorname{Re}\operatorname{tr}(X^* G) \text{ for all } (v,w) \in \mathbb{K}(A).$$

This gives

$$\max_{(v,w)\in\mathbb{K}(A)} \operatorname{Re}\langle w, Xv\rangle < \max_{G\in\partial\|A\|'} \operatorname{Re}\operatorname{tr}(X^* G). \tag{1.4.8}$$

By using Theorem 1.2.9, the right hand side of (1.4.8) is the right directional derivative of $\|\cdot\|'$ at A in the direction X. By Theorem 1.4.6, this must be equal to the left hand side of (1.4.8). This gives a contradiction. Hence $G \in \operatorname{conv}\{wv^* : (v,w) \in \mathbb{K}(A)\}$. $\square$

The most frequently used induced norm is the one induced by the Euclidean norm. By the above theorem, we obtain a description for its subdifferential. In the following corollaries, $\|u\|$ denotes the Euclidean norm of the vector u.

Corollary 1.4.8. *We have*

$$\partial\|A\| = \operatorname{conv}\{uv^* : \|u\| = \|v\| = 1, Av = \|A\|u\}. \tag{1.4.9}$$

We have noted in Corollary 1.4.4 that $\partial\|A\| = \{U_1TV_1^* : T \in \mathbb{M}(t), T$ positive semidefinite, $\operatorname{tr} T = 1\}$, where t is the multiplicity of $s_1(A)$. If $T \in \mathbb{M}(t)$ is such that T is positive semidefinite and $\operatorname{tr} T = 1$, then the eigenvalues of T, $\lambda_i(T), 1 \leq i \leq t$, satisfy $\lambda_i(T) \geq 0$ and $\sum_{i=1}^t \lambda_i(T) = 1$. By taking spectral decomposition of T, we get $U_1TV_1^* = \sum_{i=1}^t \lambda_i(T) u_i v_i^*$, where $Av_i = \|A\|u_i, 1 \leq i \leq t$. On the other hand, any matrix of the form uv^*, where $\|u\| = \|v\| = 1, Av = \|A\|u$, can be written as $U_1 \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} V_1^*$, where U_1, V_1 satisfy the conditions in Corollary 1.4.4. Thus $uv^* \in \{U_1TV_1^* : T \in \mathbb{M}(t), T$ positive semidefinite, $\operatorname{tr} T = 1\}$. Since this set is convex, we have $\operatorname{conv}\{uv^* : \|u\| = \|v\| = 1, Av = \|A\|u\} \subseteq \{U_1TV_1^* : T \in$

$\mathbb{M}(t), T$ positive semidefinite, $\operatorname{tr} T = 1\}$.

Corollary 1.4.9. *If $A \neq 0$ is positive semidefinite, then*

$$\partial\|A\| = \operatorname{conv}\{uu^* : \|u\| = 1, Au = \|A\|u\}. \tag{1.4.10}$$

1.5 Subdifferentials of the Ky Fan k-norms

Let $1 \leq k \leq n$. Let the the multiplicity of $s_k(A)$ be $r + t$, where $r \geq 0$ and $t \geq 1$, such that

$$s_{k-t+1}(A) = \cdots = s_{k+r}(A).$$

By using the result in Example 1.2.19 and Theorem 1.4.3, Watson [47] gave the following characterization of the subdifferential of the map that takes a matrix A to its Ky Fan k-norm $\|A\|_{(k)}$.

Theorem 1.5.1. *Let $A = USV^*$ be a singular value decomposition of A and let the matrices U, V be partitioned as $U = [U_1 : U_2 : U_3]$ and $V = [V_1 : V_2 : V_3]$ where $U_1, V_1 \in \mathbb{M}(n, k - t); U_2, V_2 \in \mathbb{M}(n, r + t); U_3, V_3 \in \mathbb{M}(n, n - k - r)$. If $s_k(A) > 0$, then $G \in \partial\|A\|_{(k)}$ if and only if there exists $T \in \mathbb{M}(r + t)$ with $s_1(T) \leq 1, T$ positive semidefinite and $\sum_{j=1}^{r+t} s_j(T) = t$ such that $G = U_1V_1^* + U_2TV_2^*$. If $s_k(A) = 0$, then $G \in \partial\|A\|_{(k)}$ if and only if there exists $T \in \mathbb{M}(r + t)$ with $s_1(T) \leq 1$ and $\sum_{j=1}^{r+t} s_j(T) \leq t$ such that $G = U_1V_1^* + U_2TV_2^*$.*

We now obtain new expressions for the set $\partial\|A\|_{(k)}$. One of these expressions will be more useful for us to obtain a characterization of Birkhoff-James orthogonality in $\mathbb{M}(n)$, with respect to $\|\cdot\|_{(k)}$. The Ky Fan k-norm of a matrix A is also given by

$$\|A\|_{(k)} = \max_{\substack{U,V \in \mathbb{M}(n,k) \\ U^*U = V^*V = I_k}} \operatorname{Re} \operatorname{tr} U^*AV = \max_{\substack{U,V \in \mathbb{M}(n,k) \\ U^*U = V^*V = I_k}} |\operatorname{tr} U^*AV|. \tag{1.5.1}$$

(See [37, p. 791].) If A is positive semidefinite, then

$$\|A\|_{(k)} = \max_{\substack{U \in \mathbb{M}(n,k) \\ U^*U = I_k}} \operatorname{tr} U^*AU. \tag{1.5.2}$$

Theorem 1.5.2. *For $A \in \mathbb{M}(n)$,*

$$\partial\|A\|_{(k)} = \operatorname{conv}\{UV^* : U, V \in \mathbb{M}(n,k),\ U^*U = V^*V = I_k, \|A\|_{(k)} = \operatorname{Re}\operatorname{tr} U^*AV\}. \tag{1.5.3}$$

If A is positive semidefinite, then

$$\partial\|A\|_{(k)} = \operatorname{conv}\{UU^* : U \in \mathbb{M}(n,k),\ U^*U = I_k, \|A\|_{(k)} = \operatorname{tr} U^*AU\}. \tag{1.5.4}$$

Proof. For any $U, V \in \mathbb{M}(n,k)$ with $U^*U = V^*V = I_k$, let $h_{(U,V)} : \mathbb{M}(n) \to \mathbb{R}$ be defined as

$$h_{(U,V)}(A) = \operatorname{Re}\operatorname{tr} U^*AV = \operatorname{Re}\langle UV^*, A\rangle.$$

Since each $h_{(U,V)}$ is linear, it is differentiable and $\mathrm{D}h_{(U,V)}(A) = h_{(U,V)}$. Now $\|A\|_{(k)} = \max\{h_{(U,V)}(A) : U^*U = V^*V = I_k\}$. Let J be the set given by

$$J = \{(U,V) \in \mathbb{M}(n,k) \times \mathbb{M}(n,k) : U^*U = V^*V = I_k\}. \tag{1.5.5}$$

Then J is compact. For each $A \in \mathbb{M}(n)$, the map $(U,V) \to h_{(U,V)}(A)$ is continuous. Therefore by Proposition 1.2.13, we get that

$$\partial\|A\|_{(k)} = \operatorname{conv}\{h_{(U,V)} : U, V \in \mathbb{M}(n,k),\ U^*U = V^*V = I_k, \|A\|_{(k)} = \operatorname{Re}\operatorname{tr} U^*AV\}.$$

So we get

$$\partial\|A\|_{(k)} = \operatorname{conv}\{UV^* : U, V \in \mathbb{M}(n,k),\ U^*U = V^*V = I_k, \|A\|_{(k)} = \operatorname{Re}\operatorname{tr} U^*AV\}.$$

When A is positive semidefinite, (1.5.4) follows from (1.5.2). □

Let $g : \mathbb{M}(n) \to \mathbb{R}$ be the function defined as $g(A) = \|A\|_{(k)}$. We obtain another expression for the subdifferential of the Ky Fan k-norms. To do so, we first calculate $g'_+(A, \cdot)$.

Theorem 1.5.3. *For $X \in \mathbb{M}(n)$,*

$$g'_+(A, X) = \max_{\substack{u_1,\dots,u_k \text{ o.n.} \\ v_1,\dots,v_k \text{ o.n.} \\ Av_i = s_i(A)u_i}} \sum_{i=1}^{k} \mathrm{Re}\langle u_i, Xv_i \rangle. \tag{1.5.6}$$

Proof. From (1.5.1), we have

$$\|A\|_{(k)} = \max_{\substack{u_1,\dots,u_k \text{ o.n.} \\ v_1,\dots,v_k \text{ o.n.}}} \sum_{i=1}^{k} \mathrm{Re}\langle u_i, Xv_i \rangle. \tag{1.5.7}$$

For any sets of k orthonormal vectors $u_1, \dots, u_k$ and $v_1, \dots, v_k$ satisfying $Av_i = s_i(A)u_i, 1 \leq i \leq k$, we have

$$\begin{aligned}
\|A + tX\|_{(k)} &\geq \sum_{i=1}^{k} \mathrm{Re}\langle u_i, (A + tX)v_i \rangle \\
&= \sum_{i=1}^{k} s_i(A) + t \sum_{i=1}^{k} \mathrm{Re}\langle u_i, Xv_i \rangle \\
&= \|A\|_{(k)} + t \sum_{i=1}^{k} \mathrm{Re}\langle u_i, Xv_i \rangle.
\end{aligned}$$

This gives for $t > 0$,

$$\frac{\|A + tX\|_{(k)} - \|A\|_{(k)}}{t} \geq \max_{\substack{u_1,\dots,u_k \text{ o.n.} \\ v_1,\dots,v_k \text{ o.n.} \\ Av_i = s_i(A)u_i}} \sum_{i=1}^{k} \mathrm{Re}\langle u_i, Xv_i \rangle. \tag{1.5.8}$$

Now for any sets of k orthonormal vectors $u_1(t), \dots, u_k(t)$ and $v_1(t), \dots, v_k(t)$ satisfying

$$(A + tX)v_i(t) = s_i(A + tX)u_i(t), \quad 1 \leq i \leq k, \tag{1.5.9}$$

we have

$$\|A\|_{(k)} \geq \sum_{i=1}^{k} \mathrm{Re}\langle u_i(t), Av_i(t) \rangle$$

$$
\begin{aligned}
&= \sum_{i=1}^{k} s_i(A+tX) - t\sum_{i=1}^{k} \mathrm{Re}\langle u_i(t), Xv_i(t)\rangle \\
&= \|A+tX\|_{(k)} - t\sum_{i=1}^{k} \mathrm{Re}\langle u_i(t), Xv_i(t)\rangle.
\end{aligned}
$$

So for each $t > 0$ we obtain

$$\frac{\|A+tX\|_{(k)} - \|A\|_{(k)}}{t} \leq \sum_{i=1}^{k} \mathrm{Re}\langle u_i(t), Xv_i(t)\rangle. \tag{1.5.10}$$

Consider a sequence $\{t_n\}$ of positive real numbers converging to zero as $n \to \infty$. Since the unit ball in $\mathbb{C}^n$ is compact, there exists a subsequence $\{t_{n_m}\}$ of $\{t_n\}$ such that for each $1 \leq i \leq k$, there exist u_i' and v_i' such that $\{u_i(t_{n_m})\}$ and $\{v_i(t_{n_m})\}$ converge to u_i' and v_i', respectively, as $m \to \infty$. Then the sets of vectors $u_1', \ldots, u_k'$ and $v_1', \ldots, v_k'$ are orthonormal. By continuity of singular values, we also know that

$$s_i(A + t_{n_m}B) \to s_i(A) \text{ as } m \to \infty. \tag{1.5.11}$$

Hence we obtain $Av_i' = s_i(A)u_i'$ for all $1 \leq i \leq k$. By (1.5.10), we get that

$$g_+'(A,X) = \lim_{m\to\infty} \frac{\|A+t_{n_m}X\|_{(k)} - \|A\|_{(k)}}{t_{n_m}} \leq \max_{\substack{u_1,\ldots,u_k \text{ o.n.} \\ v_1,\ldots,v_k \text{ o.n.} \\ Av_i = s_i(A)u_i}} \sum_{i=1}^{k} \mathrm{Re}\langle u_i, Xv_i\rangle. \tag{1.5.12}$$

Combining this with (1.5.8), we obtain the required result. $\square$

Remark 1.5.4. The above proof works equally well if the maximum in (1.5.6) is taken over the sets of orthonormal vectors $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ such that for each $1 \leq i \leq k$, u_i and v_i are left and right singular vectors of A, respectively, corresponding to the ith singular value $s_i(A)$ of A. We note here that for each $t > 0$, if along with (1.5.9), we also have

$$(A+tX)^*u_i(t) = s_i(A+tX)v_i(t),$$

then by passing onto a subsequence $\{t_{n_m}\}$ as in the above proof, and taking limit as $m \to \infty$, we obtain

$$A^* u_i' = s_i(A) v_i'.$$

So for each $X \in \mathbb{M}(n)$, we get

$$g_+'(A, X) = \max_{\substack{u_1,\ldots,u_k \text{ o.n.} \\ v_1,\ldots,v_k \text{ o.n.} \\ Av_i = s_i(A)u_i \\ A^*u_i = s_i(A)v_i}} \sum_{i=1}^{k} \mathrm{Re}\langle u_i, Xv_i \rangle. \tag{1.5.13}$$

Corollary 1.5.5. *Let A be positive semidefinite. Let $\lambda_1(A) \geq \cdots \geq \lambda_n(A) \geq 0$ be the eigenvalues of A, with $\lambda_k(A) > 0$. Then*

$$g_+'(A, X) = \max_{\substack{u_1,\ldots,u_k \text{ o.n.} \\ Au_i = \lambda_i(A)u_i}} \sum_{i=1}^{k} \mathrm{Re}\langle u_i, Xu_i \rangle. \tag{1.5.14}$$

Proof. We know that that if $Av = \lambda u$ and $Au = \lambda v$, where $\lambda > 0$, then $u = v$. Using this, the required result follows from (1.5.13). □

Theorem 1.5.6. *We have*

$$\begin{aligned}
\partial\|A\|_{(k)} &= \mathrm{conv}\{\sum_{i=1}^{k} u_i v_i^* : u_1,\ldots,u_k, v_1,\ldots,v_k \in \mathbb{C}^n, u_1,\ldots,u_k \text{ o.n.}, \mathrm{v}_1,\ldots,\mathrm{v}_\mathrm{k} \text{ o.n.}, \\
&\qquad Av_i = s_i(A)u_i \textit{ for all } 1 \leq i \leq k\} && (1.5.15) \\
&= \mathrm{conv}\{\sum_{i=1}^{k} u_i v_i^* : u_1,\ldots,u_k, v_1,\ldots,v_k \in \mathbb{C}^n, u_1,\ldots,u_k \text{ o.n.}, \mathrm{v}_1,\ldots,\mathrm{v}_\mathrm{k} \text{ o.n.}, \\
&\qquad Av_i = s_i(A)u_i, A^*u_i = s_i(A)v_i \textit{ for all } 1 \leq i \leq k\}. && (1.5.16)
\end{aligned}$$

Proof. Denote the set on the right hand side of (1.5.15) by $\mathbb{H}(A)$. Let $G \in \mathbb{H}(A)$. Then

$$G = \sum_{i=1}^{k} u_i v_i^*,$$

where $u_1,\ldots,u_k$ and $v_1,\ldots,v_k$ are orthonormal sets of vectors such that $Av_i = s_i(A)u_i$ for all

$1 \le i \le k$. So

$$\begin{aligned} \operatorname{Re}\operatorname{tr}(G^*A) &= \sum_{i=1}^{k} \operatorname{Re}\langle u_i, Av_i\rangle \\ &= \sum_{i=1}^{k} s_i(A) \\ &= \|A\|_{(k)}, \end{aligned}$$

and

$$\begin{aligned} \operatorname{Re}\operatorname{tr}(G^*X) &= \sum_{i=1}^{k} \operatorname{Re}\langle u_i, Xv_i\rangle \\ &\le \|X\|_{(k)}. \end{aligned}$$

Thus

$$\|G\|^* \le 1.$$

So we get by (1.4.1) that $\mathbb{H}(A) \subseteq \partial\|A\|_{(k)}$, and therefore $\operatorname{conv}\mathbb{H}(A) \subseteq \partial\|A\|_{(k)}$.

Now let $G \in \partial\|A\|_{(k)}$. Suppose $G \notin \operatorname{conv}\mathbb{H}(A)$. The set $\mathbb{H}(A)$ is compact, and so is its convex hull. By the Separating Hyperplane Theorem, there exists $X \in \mathbb{M}(n)$ such that for all sets of k orthonormal vectors $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ satisfying $Av_i = s_i(A)u_i$ for $1 \le i \le k$, we have

$$\operatorname{Re}\operatorname{tr}\left(X^*\left(\sum_{i=1}^{k} u_i v_i^* - G\right)\right) < 0.$$

This implies

$$\max_{\substack{u_1,\ldots,u_k \text{ o.n.} \\ v_1,\ldots,v_k \text{ o.n.} \\ Av_i = s_i(A)u_i}} \sum_{i=1}^{k} \operatorname{Re}\langle u_i, Xv_i\rangle < \max_{G\in\partial\|A\|_{(k)}} \operatorname{Re}\operatorname{tr}(X^*G).$$

By Theorem 1.2.9, the right hand side is $g'_+(A, X)$. By (1.5.6), this should be equal to the left hand side. This gives a contradiction. Thus we obtain (1.5.15).

The expression (1.5.16) can be proved similarly by using (1.5.13), instead of (1.5.6).

□

Corollary 1.5.7. *Let A be a positive semidefinite matrix, with eigenvalues $\lambda_1(A) \ge \cdots \lambda_n(A) \ge$*

0 *such that* $\lambda_k(A) > 0$. *Then*

$$\partial\|A\|_{(k)} = \mathrm{conv}\{\sum_{i=1}^{k} u_i u_i^* : u_1, \dots, u_k \in \mathbb{C}^n, u_1, \dots, u_k \text{ o.n.}, \mathrm{Au_i} = \lambda_{\mathrm{i}}(\mathrm{A})\mathrm{u_i}\, \textit{for all}\ 1 \leq \mathrm{i} \leq \mathrm{k}\}. \tag{1.5.17}$$

Chapter 2

Characterization of Birkhoff-James orthogonality

Let $\mathbb{X}$ be a complex Banach space. An element x of $\mathbb{X}$ is said to be *Birkhoff-James orthogonal* to another element y of $\mathbb{X}$ if

$$\|x + \lambda y\| \geq \|x\| \text{ for all complex numbers } \lambda. \tag{2.0.1}$$

Birkhoff-James orthogonality is equivalent to the usual orthogonality when $\mathbb{X}$ is a Hilbert space. Henceforth, orthogonality will mean Birkhoff-James orthogonality.

We study orthogonality in the space $\mathbb{M}(n)$ equipped with various norms. The starting point of this is the work of Bhatia and Šemrl [15] where they obtained an attractive characterization in case of the operator norm. This was followed by Li and Schneider [35] in which they studied the problem for other induced norms on $\mathbb{M}(n)$. We introduce a new approach to this problem based on subdifferential calculus. Using this method, we first obtain new proofs of the results in [15],[35]. Then we use this technique to obtain new results for the Ky Fan k-norms.

The results can be extended to a more general setting of a C^*-algebra and of a Hilbert C^*-module. We obtain a characterization of orthogonality in Hilbert C^*-modules in terms of the states of the underlying C^*-algebra. We first deal with the special module $\mathcal{L}(\mathcal{H}, \mathcal{K})$, the space of bounded linear operators from a Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{K}$, and then obtain a necessary and sufficient condition for orthogonality in a general Hilbert C^*-module.

2.1 Orthogonality in $\mathbb{M}(n)$

The Bhatia-Šemrl theorem

An interesting characterization of orthogonality in the space $\mathbb{M}(n)$ with the operator norm $\|\cdot\|$, was found by Bhatia and Šemrl in [15].

Theorem 2.1.1. *Let $A, B \in \mathbb{M}(n)$. Then $\|A+\lambda B\| \geq \|A\|$ for all $\lambda \in \mathbb{C}$ if and only if there is a unit vector x such that $\|Ax\| = \|A\|$ and $\langle Ax, Bx\rangle = 0$.*

Consider the function $f(\lambda) = \|A+\lambda B\|$. The crux of our approach lies in the observation that A is orthogonal to B is equivalent to saying f attains its minimum at $\lambda = 0$. Therefore, the problem should be amenable to the subdifferential calculus, introduced in Chapter 1.

We shall first look at the real version of the Bhatia-Šemrl theorem.

Theorem 2.1.2. *Let $A, B \in \mathbb{M}(n)$. Then $\|A+tB\| \geq \|A\|$ for all $t \in \mathbb{R}$ if and only if there exists a unit vector x such that $\|Ax\| = \|A\|$ and $\operatorname{Re}\langle Ax, Bx\rangle = 0$.*

Proof. If there exists a unit vector x such that $\|Ax\| = \|A\|$ and $\operatorname{Re}\langle Ax, Bx\rangle = 0$, then for $t \in \mathbb{R}$,

$$\begin{aligned}
\|A+tB\|^2 &\geq \|(A+tB)x\|^2 \\
&= \|Ax\|^2 + t^2\|Bx\|^2 + 2t\operatorname{Re}\langle Ax, Bx\rangle \\
&= \|Ax\|^2 + t^2\|Bx\|^2 \\
&\geq \|Ax\|^2 \\
&= \|A\|^2.
\end{aligned}$$

Conversely let

$$\|A+tB\| \geq \|A\| \text{ for all } t \in \mathbb{R}. \tag{2.1.1}$$

We first note that it is enough to show that if A is a positive semidefinite matrix and $B \in \mathbb{M}(n)$ such that (2.1.1) holds, then there exists a unit vector y such that

$$Ay = \|A\|y \text{ and } \operatorname{Re}\langle Ay, By\rangle = 0. \tag{2.1.2}$$

The general case may be reduced to this by using a singular value decomposition of A. Let $A = USV^*$ be a singular value decomposition of A. Then (2.1.1) implies

$$\|S + tU^*BV\| \geq \|S\| \text{ for all } t \in \mathbb{R}. \tag{2.1.3}$$

If there exists a unit vector y such that

$$Sy = \|S\|y \text{ and } \operatorname{Re}\langle Sy, U^*BVy\rangle = 0,$$

then for $x = Vy$, we have

$$\|Ax\| = \|A\| \text{ and } \operatorname{Re}\langle Ax, Bx\rangle = 0.$$

Thus we can assume that A is a positive semidefinite matrix in (2.1.1). Let $S : \mathbb{R} \to \mathbb{M}(n)$ be the linear map defined as

$$S(t) = tB$$

and let $L : \mathbb{R} \to \mathbb{M}(n)$ be the affine map

$$L(t) = A + S(t).$$

Let $g : \mathbb{M}(n) \to \mathbb{R}$ be the convex map given by

$$g(T) = \|T\|.$$

Then (2.1.1) can be rewritten as

$$(g \circ L)(t) \geq (g \circ L)(0).$$

By Proposition 1.2.5 we get

$$0 \in \partial(g \circ L)(0). \tag{2.1.4}$$

Using Proposition 1.2.12, we get that

$$0 \in S^* \partial \|A\|. \tag{2.1.5}$$

The map $S^* : \mathbb{M}(n) \to \mathbb{R}$ is given by

$$S^*(T) = \mathrm{Re}\,\mathrm{tr}(B^*T).$$

By Corollary 1.4.9, we obtain

$$S^* \partial \|A\| = \mathrm{conv}\{\mathrm{Re}\langle u, Bu\rangle : \|u\| = 1, Au = \|A\|u\}. \tag{2.1.6}$$

Let $\mathcal{M}$ denote the eigenspace of A corresponding to the maximum eigenvalue $\|A\|$. Let $P_{\mathcal{M}}$ be the orthogonal projection onto $\mathcal{M}$ and let $i_{\mathcal{M}}$ be its adjoint. Then $i_{\mathcal{M}}$ is the inclusion map of $\mathcal{M}$ into $\mathbb{C}^n$. Then the set $\{\langle u, Bu\rangle : \|u\| = 1, Au = \|A\|u\}$ is the numerical range of $P_{\mathcal{M}} B i_{\mathcal{M}}$. By the Hausdorff-Toeplitz theorem [29, p. 113], this is a convex set. Therefore the set $\{\mathrm{Re}\langle u, Bu\rangle : \|u\| = 1, Au = \|A\|u\}$ is convex. By (2.1.5) and (2.1.6) we get that there exists a unit vector y such that

$$Ay = \|A\|y \text{ and } \mathrm{Re}\langle y, By\rangle = 0.$$

These together imply that

$$\mathrm{Re}\langle Ay, By\rangle = 0. \tag{2.1.7}$$

□

Remark 2.1.3. Another necessary and sufficient condition for A to be orthogonal to B is given in [35, Theorem 3.1(c)]. Suppose that A is positive semidefinite and r is the multiplicity of the largest eigenvalue $\|A\|$. Let U be an $n \times r$ matrix such that the columns of U form an orthonormal basis for the eigenspace of A corresponding to $\|A\|$. Then a corresponding result of [35] says that A is orthogonal to B if and only if 0 belongs to the numerical range of U^*BAU. This is equivalent to saying that 0 belongs to *the numerical range of* B*, restricted to the eigenspace of* A *corresponding to the largest eigenvalue* $\|A\|$. An analogous condition for $\|A + tB\| \geq \|A\|$ for all $t \in \mathbb{R}$ is established through our proof. We have shown that if A is positive semidefinite and $\|A + tB\| \geq \|A\|$ for all $t \in \mathbb{R}$ then 0 belongs to the real part of the numerical range of B, restricted to the eigenspace of A corresponding to $\|A\|$. The result for the case when A is not positive semidefinite can be obtained from this by using a singular value decomposition of A.

Proof of the Bhatia-Šemrl theorem. If there exists a unit vector x such that $\|Ax\| = \|A\|$ and $\langle Ax, Bx\rangle = 0$, then by an argument similar to the one in the proof of Theorem 2.1.2 we get that

$$\|A + \lambda B\| \geq \|A\| \text{ for all } \lambda \in \mathbb{C}. \tag{2.1.8}$$

Now suppose (2.1.8) holds. This can also be written as

$$\|A + re^{i\theta}B\| \geq \|A\| \text{ for all } r, \theta \in \mathbb{R}.$$

Fix θ and let $B_\theta = e^{i\theta}B$. Then we have

$$\|A + rB_\theta\| \geq \|A\| \text{ for all } r \in \mathbb{R}.$$

We can assume A to be positive semidefinite as in the proof of Theorem 2.1.2. By (2.1.2), there exists a unit vector y_θ such that

$$Ay_\theta = \|A\|y_\theta \text{ and } \operatorname{Re}\, e^{i\theta}\langle Ay_\theta, By_\theta\rangle = 0. \tag{2.1.9}$$

Consider the set $\{\langle y, B^*Ay\rangle : \|y\| = 1,\ Ay = \|A\|y\}$. This is the numerical range of

$P_{\mathcal{M}}B^*Ai_{\mathcal{M}}$, where $\mathcal{M}$ is the eigenspace of A corresponding to its maximum eigenvalue $\|A\|$, $P_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M}$ and $i_{\mathcal{M}}$ is the adjoint of $P_{\mathcal{M}}$. By the Hausdorff-Toeplitz theorem, this is a compact convex set in $\mathbb{C}$. If 0 does not belong to this set, then by the Separating Hyperplane Theorem, there exists $\lambda_0 \in \mathbb{C}$ such that for all unit vectors y satisfying $Ay = \|A\|y$, we have

$$\operatorname{Re} \overline{\lambda_0}\langle y, B^*Ay\rangle > 0. \tag{2.1.10}$$

Let $\lambda_0 = |\lambda_0|e^{i\theta_0}$. Then by (2.1.10), we get

$$\operatorname{Re}\ e^{-i\theta_0}\langle y, B^*Ay\rangle > 0 \text{ for all } y \text{ such that } \|y\| = 1,\ Ay = \|A\|y.$$

This is a contradiction to (2.1.9). Thus we get that

$$0 \in \{\langle y, B^*Ay\rangle : \|y\| = 1,\ Ay = \|A\|y\}.$$

So there exists a unit vector y such that

$$Ay = \|A\|y \text{ and } \langle Ay, By\rangle = 0.$$

□

The Schatten p-norms

A characterization for orthogonality in $\mathbb{M}(n)$ with the Schatten p-norms has been given in [15] as well as [35]. We have seen in Example 1.1.12 that $\|\cdot\|_p$ is differentiable for $1 < p < \infty$ and the derivative is given by (1.1.6). For $p = 1$, it is Gâteaux differentiable at A if and only if A is invertible. In this case, the Gâteaux derivative is given by (1.1.7). So we can derive the following characterization given in [15, 35].

Theorem 2.1.4. *Let $1 < p < \infty$. If $A = U|A|$ is a polar decomposition of A then $\|A+\lambda B\|_p \geq \|A\|_p$ for all $\lambda \in \mathbb{C}$ if and only if $\operatorname{tr}|A|^{p-1}U^*B = 0$. If $p = 1$, then $\operatorname{tr} U^*B = 0$ implies $\|A + \lambda B\|_1 \geq \|A\|_1$ for all $\lambda \in \mathbb{C}$. The converse is true when A is invertible.*

The trace norm

Theorem 2.1.4 gives a characterization of orthogonality in the norm $\|\cdot\|_1$ in case A is invertible. The general case was obtained by Li and Schneider in [35]. We shall prove their theorem using our technique.

Theorem 2.1.5. *Let the number of zero singular values of A be ℓ. Let $A = USV^*$ be a singular value decomposition of A. Let*

$$B = U \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} V^*, \text{ where } B_{11} \in \mathbb{M}(n-\ell), B_{22} \in \mathbb{M}(\ell).$$

Then

$$\|A + \lambda B\|_1 \geq \|A\|_1 \text{ for all } \lambda \in \mathbb{C}$$

if and only if $|\operatorname{tr} B_{11}| \leq \|B_{22}\|$.

Proof. First note that $|\operatorname{tr} B_{11}| \leq \|B_{22}\|$ holds if and only if there exists $T \in \mathbb{M}(\ell)$ with $\|T\| \leq 1$ such that $\operatorname{tr} B_{11} + \operatorname{tr}(T^* B_{22}) = 0$. Now by arguments similar to the one used in our proof of the Bhatia-Šemrl theorem, it is enough to prove that $\|A + tB\|_1 \geq \|A\|_1$ for all $t \in \mathbb{R}$ if and only if there exists $T \in \mathbb{M}(\ell)$ with $s_1(T) \leq 1$ such that $\operatorname{Re}(\operatorname{tr} B_{11} + \operatorname{tr}(T^* B_{22})) = 0$. Let S, L, g be the maps

$$S(t) = tB,$$

$$L(t) = A + tB$$

and

$$g(T) = \|T\|_1.$$

Then

$$\|A + tB\|_1 \geq \|A\|_1 \text{ for all } t \in \mathbb{R}$$

if and only if

$$(g \circ L)(t) \geq (g \circ L)(0) \text{ for all } t \in \mathbb{R}.$$

A necessary and sufficient condition for this to hold is $0 \in \partial(g \circ L)(0)$. By Proposition 1.2.12, $\partial(g \circ L)(0) = S^*\partial\|A\|_1$. Let the matrices U, V be partitioned as $U = [U_1 : U_2]$ and $V = [V_1 : V_2]$, where $U_1, V_1 \in \mathbb{M}(n, n-\ell)$ and $U_2, V_2 \in \mathbb{M}(n, \ell)$. By Corollary 1.4.5, $0 \in S^*\partial\|A\|_1$ if and only if there exists $T \in \mathbb{M}(\ell)$ with $s_1(T) \leq 1$ such that $0 = \operatorname{Re}\operatorname{tr} B^*(U_1V_1^* + U_2TV_2^*)$. A calculation shows that this is equivalent to $\operatorname{Re}(\operatorname{tr} B_{11}^* + \operatorname{tr}(B_{22}^*T)) = 0$. □

Remark 2.1.6. With the hypothesis of the above theorem, the result can be reformulated as follows:

$$\|A + \lambda B\|_1 \geq \|A\|_1 \text{ for all } \lambda \in \mathbb{C}$$

if and only if there exists $T \in \mathbb{M}(\ell)$ with $s_1(T) \leq 1$ such that $\operatorname{tr} B_{11} + \operatorname{tr}(T^*B_{22}) = 0$.

Remark 2.1.7. Another necessary and sufficient condition given in [35] for orthogonality in $\|\cdot\|_1$ is that there exists a matrix $G \in \mathbb{M}(n)$ such that $\|G\|_\infty \leq 1$, $\operatorname{tr}(G^*A) = \|A\|$ and $\operatorname{tr}(G^*B) = 0$. Let S, L, g be the maps as defined above in the proof of Theorem 2.1.5. Then the condition $0 \in S^*\partial\|A\|_1$ gives the required result by using (1.4.1).

The Ky Fan k-norms

In this section, we generalize the Bhatia-Šemrl theorem and the results for orthogonality in the trace norm. The Bhatia-Šemrl theorem can also be stated as follows. If $A = U|A|$ is a polar decomposition of A, then A is orthogonal to B in $\|\cdot\|$ if and only if there exists a unit vector x such that $|A|x = \|A\|x$ and $\langle x, U^*Bx\rangle = 0$. We prove the following.

Theorem 2.1.8. *Let $A = U|A|$ be a polar decomposition of A. If there exist k orthonormal vectors $u_1, u_2, \ldots, u_k$ such that*

$$|A|\, u_i = s_i(A)u_i \textit{ for all } 1 \leq i \leq k \tag{2.1.11}$$

and

$$\sum_{i=1}^{k} \langle u_i, U^*Bu_i\rangle = 0, \tag{2.1.12}$$

then

$$\|A+\lambda B\|_{(k)} \geq \|A\|_{(k)} \text{ for all } \lambda \in \mathbb{C}. \tag{2.1.13}$$

If $s_k(A) > 0$, then the converse is also true.

We use ideas similar to those in the proof of the Bhatia-Šemrl theorem 2.1.1. We first prove a lemma.

Lemma 2.1.9. *Let $X, Y \in \mathbb{M}(n)$ and let Y be positive semidefinite. Let $\lambda_1(Y) \geq \cdots \geq \lambda_n(Y) \geq 0$ be the eigenvalues of Y. For $1 \leq r \leq n$, let*

$$\mathcal{W}(X,Y) = \left\{ \sum_{i=1}^{r} \langle u_i, Xu_i \rangle : u_1, \ldots, u_r \in \mathbb{C}^n, u_1, \ldots, u_r \text{ o.n.}, Yu_i = \lambda_i(Y)u_i \text{ for all } 1 \leq i \leq r \right\}.$$

Then $\mathcal{W}(X,Y)$ is convex.

Proof. Let the number of distinct eigenvalues of Y be ℓ and let $\mathcal{H}_1, \ldots, \mathcal{H}_\ell$ be the respective eigenspaces. Let $m_1, \ldots, m_\ell$ be the dimensions of $\mathcal{H}_1, \ldots, \mathcal{H}_\ell$, respectively. Let $1 \leq \ell' \leq \ell$. Suppose $m_1 + \cdots + m_{\ell'-1} < r \leq m_1 + \cdots + m_{\ell'}$. Let $m = r - (m_1 + \cdots + m_{\ell'-1})$. Set

$$\mathcal{W}_j(X) = \left\{ \sum_{i=1}^{m_j} \langle u_i, Xu_i \rangle : u_1, \ldots, u_{m_j} \in \mathcal{H}_j, u_1, \ldots, u_{m_j} \text{ o.n.} \right\} \text{ for } 1 \leq j \leq \ell' - 1,$$

$$\mathcal{W}_{\ell'}(X) = \left\{ \sum_{i=1}^{m} \langle u_i, Xu_i \rangle : u_1, \ldots, u_m \in \mathcal{H}_{\ell'}, u_1, \ldots, u_m \text{ o.n.} \right\}.$$

Since $\mathcal{H}_1, \ldots, \mathcal{H}_\ell$ are mutually orthogonal, we have

$$\mathcal{W}(X,Y) = \sum_{j=1}^{\ell'} \mathcal{W}_j(X). \tag{2.1.14}$$

Note that $\mathcal{W}_j(X)$ is a singleton set for $1 \leq j \leq \ell' - 1$. Hence it is sufficient to show that $\mathcal{W}_{\ell'}(X)$ is convex. Let $P_{\ell'}$ be the orthogonal projection from $\mathbb{C}^n$ onto $\mathcal{H}_{\ell'}$, and let $i_{\ell'}$ denote its adjoint (which is the inclusion map of $\mathcal{H}_{\ell'}$ into $\mathbb{C}^n$). Then $\mathcal{W}_{\ell'}(X)$ is the m-numerical range of $P_{\ell'} X i_{\ell'}$, which is convex (see [29, p. 315]). $\square$

Theorem 2.1.10. *Let $A = U|A|$ be a polar decomposition of A. If there exist k orthonormal vectors $u_1, u_2, \ldots, u_k$ such that*

$$|A|\, u_i = s_i(A) u_i \text{ for all } 1 \le i \le k \tag{2.1.15}$$

and

$$\sum_{i=1}^{k} \mathrm{Re}\langle u_i, U^* B u_i\rangle = 0, \tag{2.1.16}$$

then

$$\|A + tB\|_{(k)} \ge \|A\|_{(k)} \text{ for all } t \in \mathbb{R}. \tag{2.1.17}$$

If $s_k(A) > 0$, then the converse is also true.

Proof. First suppose that there exist k orthonormal vectors $u_1, u_2, \ldots, u_k$, such that $|A|\, u_i = s_i(A)\, u_i$ for all $1 \le i \le k$ and $\sum_{i=1}^{k} \mathrm{Re}\langle u_i, U^* B u_i\rangle = 0$. We have

$$\|A + tB\|_{(k)} = \||A| + tU^* B\|_{(k)}$$

and by (1.5.1),

$$\||A| + tU^* B\|_{(k)} \ge \sum_{i=1}^{k} \mathrm{Re}\langle u_i, (|A| + tU^* B) u_i\rangle.$$

So

$$\begin{aligned} \|A + tB\|_{(k)} &\ge \sum_{i=1}^{k} \langle u_i, |A| u_i\rangle + \sum_{i=1}^{k} \mathrm{Re}\langle u_i, U^* B u_i\rangle \\ &= \sum_{i=1}^{k} s_i(A) \\ &= \|A\|_{(k)}. \end{aligned}$$

Now suppose that $s_k(A) > 0$ and

$$\|A + tB\|_{(k)} \ge \|A\|_{(k)} \text{ for all } t \in \mathbb{R}.$$

This can also be written as

$$\||A| + tU^*B\|_{(k)} \geq \||A|\|_{(k)} \text{ for all } t \in \mathbb{R}. \tag{2.1.18}$$

Let $S : \mathbb{R} \to \mathbb{M}(n)$ be the map given by $S(t) = tU^*B$, $L : \mathbb{R} \to \mathbb{M}(n)$ be the map defined as $L(t) = |A| + tU^*B$ and $g : \mathbb{M}(n) \to \mathbb{R}_+$ be the map defined by $g(X) = \|X\|_{(k)}$. So we have that $g \circ L$ attains its minimum at zero. By Proposition 1.2.5, we obtain that $0 \in \partial(g \circ L)(0)$. Using Proposition 1.2.12, we obtain

$$0 \in S^* \partial \|A\|_{(k)}. \tag{2.1.19}$$

By Corollary 1.5.7, this is equivalent to saying that

$$0 \in \operatorname{conv}(\operatorname{Re} \mathcal{W}(U^*B, |A|)).$$

By Lemma 2.1.9, $\operatorname{Re} \mathcal{W}(U^*B, |A|)$ is a convex set. So there exist k orthonormal vectors $u_1, \ldots, u_k$ such that

$$|A|u_i = s_i(A)u_i$$

and

$$\operatorname{Re} \sum_{i=1}^{k} \langle u_i, U^*Bu_i \rangle = 0.$$

$\square$

Proof of Theorem 2.1.8. Suppose that there exist k orthonormal vectors satisfying (2.1.11) and (2.1.12), then we can easily obtain (2.1.13), as done in the proof of Theorem 2.1.10.

Conversely, let $s_k(A) > 0$ and (2.1.13) hold. Arguing as in our proof of the Bhatia-Šemrl theorem 2.1.1, we get that for each fixed $\theta \in \mathbb{R}$,

$$\||A| + rU^*B_\theta\|_{(k)} \geq \||A|\|_{(k)} \text{ for all } r \in \mathbb{R}.$$

By Theorem 2.1.10, there exist k orthonormal vectors $u_1^{(\theta)}, \ldots, u_k^{(\theta)}$ such that

$$|A|u_i^{(\theta)} = s_i(A)u_i^{(\theta)} \text{ for all } 1 \leq i \leq k$$

and

$$\operatorname{Re} e^{i\theta} \sum_{j=1}^{k} \langle u_j^{(\theta)}, U^*Bu_j^{(\theta)} \rangle = 0. \tag{2.1.20}$$

By Lemma 2.1.9, the set $\mathcal{W}(U^*B, |A|)$ is convex in $\mathbb{C}$. It is also compact in $\mathbb{C}$. If $0 \notin \mathcal{W}(U^*B, |A|)$, then by the Separating Hyperplane Theorem, there exists a θ_0 such that

$$\operatorname{Re} e^{i\theta_0} \sum_{i=1}^{k} \langle u_i, U^*Bu_i \rangle > 0 \text{ for all } u_1, \ldots, u_k \text{ o.n., } |A|u_i = s_i(A)u_i \text{ for } 1 \leq i \leq k.$$

This is a contradiction to (2.1.20). Thus $0 \in \mathcal{W}(U^*B, |A|)$, and so there exist k orthonormal vectors $u_1, \ldots, u_k$ such that

$$|A|u_i = s_i(A)u_i \text{ for all } 1 \leq i \leq k$$

and

$$\sum_{i=1}^{k} \langle u_i, U^*Bu_i \rangle = 0.$$

□

We have seen in Remark 2.1.6 that if

$$B = U \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} V^*, \text{ where } B_{11} \in \mathbb{M}(n-\ell), B_{22} \in \mathbb{M}(\ell),$$

then $\|A + \lambda B\|_1 \geq \|A\|_1$ for all $\lambda \in \mathbb{C}$ if and only if there exists $T \in \mathbb{M}(\ell)$ with $s_1(T) \leq 1$ such that $\operatorname{tr} B_{11} + \operatorname{tr}(T^*B_{22}) = 0$. We obtain analogous results for orthogonality in the Ky Fan k-norms, which also follow as a special case of Theorem 4 in [47]. Let the multiplicity of $s_k(A)$

be $r+t$, where $r \geq 0$ and $t \geq 1$, such that

$$s_{k-t+1}(A) = \cdots = s_{k+r}(A).$$

Theorem 2.1.11. *Let $A = USV^*$ be a singular value decomposition of A. Let*

$$B = U \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} V^*, \text{ where } B_{11} \in \mathbb{M}(k-t), B_{22} \in \mathbb{M}(r+t), B_{33} \in \mathbb{M}(n-k-r).$$

(a) Let $s_k(A) > 0$. Then

$$\|A + \lambda B\|_{(k)} \geq \|A\|_{(k)} \text{ for all } \lambda \in \mathbb{C}$$

if and only if there exists positive semidefinite $T \in \mathbb{M}(r+t)$ with $s_1(T) \leq 1$ and $\sum_{j=1}^{r+t} s_j(T) = t$ such that

$$\operatorname{tr} B_{11} + \operatorname{tr}(T^* B_{22}) = 0.$$

(b) Let $s_k(A) = 0$. Then

$$\|A + \lambda B\|_{(k)} \geq \|A\|_{(k)} \text{ for all } \lambda \in \mathbb{C}$$

if and only if there exists $T \in \mathbb{M}(r+t)$ with $s_1(T) \leq 1$, and $\sum_{j=1}^{r+t} s_j(T) \leq t$ such that

$$\operatorname{tr} B_{11} + \operatorname{tr}(T^* B_{22}) = 0.$$

Proof. By arguments similar to the ones used in our proof of the Bhatia-Šemrl theorem, it is enough to prove that $\|A+tB\|_{(k)} \geq \|A\|_{(k)}$ for all $t \in \mathbb{R}$ if and only if $\operatorname{Re}\operatorname{tr}(B_{11}+T^*B_{22}) = 0$. Let S, L, g be the maps as defined in the proof of Theorem 2.1.10. Then $\|A+tB\|_{(k)} \geq \|A\|_{(k)}$ for all $t \in \mathbb{R}$ if and only if $0 \in S^* \partial \|A\|_{(k)}$. Let the matrices U, V be partitioned as $U = [U_1 : U_2 : U_3]$ and $V = [V_1 : V_2 : V_3]$, where $U_1, V_1 \in \mathbb{M}(n, k-t); U_2, V_2 \in$

$\mathbb{M}(n, r+t); U_3, V_3 \in \mathbb{M}(n, n-k-r)$. If $s_k(A) > 0$, then by Theorem 1.5.1, we get that $0 \in S^*\partial\|A\|_{(k)}$ if and only if there exists $T \in \mathbb{M}(r+t)$ with $s_1(T) \leq 1$, T positive semidefinite, and $\sum_{j=1}^{r+t} s_j(T) = t$ such that $0 = \operatorname{Re}\operatorname{tr} B^*(U_1V_1^* + U_2TV_2^*)$. Similarly, if $s_k(A) = 0$, $0 \in S^*\partial\|A\|_{(k)}$ if and only if there exists $T \in \mathbb{M}(r+t)$ with $s_1(T) \leq 1$ and $\sum_{j=1}^{r+t} s_j(T) \leq t$ such that $0 = \operatorname{Re}\operatorname{tr} B^*(U_1V_1^* + U_2TV_2^*)$. A calculation shows that $\operatorname{tr} B^*(U_1V_1^* + U_2TV_2^*) = \operatorname{tr} B_{11}^* + \operatorname{tr}(B_{22}^*T)$. This gives the required result.

□

Induced norms

Theorem 2.1.12. *Let* $\|\cdot\|'$ *be an induced norm as defined in* (1.3.6). *Let* $\mathbb{V}(A) = \{wv^* : \|w\|^* = 1, \|v\| = 1, \langle w, Av\rangle = \|A\|'\}$. *Then*

$$\|A + \lambda B\|' \geq \|A\|' \textit{ for all } \lambda \in \mathbb{C} \tag{2.1.21}$$

if and only there exist vectors $w_1, \ldots, w_\ell, v_1, \ldots, v_\ell$ *and numbers* $\alpha_1, \ldots, \alpha_\ell$ *such that* $w_1v_1^*, \ldots, w_\ell v_\ell^* \in \mathbb{V}(A)$, $0 \leq \alpha_j \leq 1$ *for all* $1 \leq j \leq \ell$ *,* $\sum_{j=1}^{\ell} \alpha_j = 1$ *such that*

$$\sum_{j=1}^{\ell} \alpha_j \langle w_j, Bv_j\rangle = 0. \tag{2.1.22}$$

Proof. Arguing as in our proof of the Bhatia-Šemrl theorem 2.1.1, it is sufficient to prove that $\|A + tB\|' \geq \|A\|'$ for all $t \in \mathbb{R}$ if and only if there exist $w_1, \ldots, w_\ell, v_1, \ldots, v_\ell$ with $w_1v_1^*, \ldots, w_\ell v_\ell^* \in \mathbb{V}(A)$, and numbers $\alpha_j (1 \leq j \leq \ell)$ with $0 \leq \alpha_j \leq 1$, $\sum_{j=1}^{\ell} \alpha_j = 1$ such that

$$\sum_{j=1}^{\ell} \alpha_j \operatorname{Re}\langle w_j, Bv_j\rangle = 0.$$

We obtain $\|A + tB\|' \geq \|A\|'$ for all $t \in \mathbb{R}$ if and only if $0 \in S^*\partial\|A\|'$, where $S(t) = tB$ for all $t \in \mathbb{R}$. Let $\mathbb{K}(A)$ be the set defined in (1.4.4). By Theorem 1.4.7, $0 \in S^*\partial\|A\|'$ is the same as saying that there exist matrices of the form $w_1v_1^*, \ldots, w_\ell v_\ell^*$, where $(v_j, w_j) \in \mathbb{K}(A)$ for all

$1 \leq j \leq \ell$, and numbers $\alpha_j (1 \leq j \leq \ell)$ with $0 \leq \alpha_j \leq 1$, $\sum_{j=1}^{\ell} \alpha_j = 1$ such that

$$\sum_{j=1}^{\ell} \alpha_j \operatorname{Re}\langle w_j, Bv_j \rangle = 0. \tag{2.1.23}$$

By Example 1.2.16, one can easily see that each $w_j v_j^* \in \mathbb{V}(A)$. Hence we get the required result. $\square$

The above theorem is Proposition 4.2 in [35] where the authors obtain this result with $\ell \leq 3$ in the complex case and $\ell \leq 2$ in the real case, using a theorem of Singer [42, p. 170].

2.2 Orthogonality in $\mathcal{L}(\mathcal{H},\mathcal{K})$

Bhatia and Šemrl made a remark in [15] on extending Theorem 2.1.1 to bounded linear operators on an infinite dimensional Hilbert space $\mathcal{H}$. We note this as a theorem.

Theorem 2.2.1. *Let $A, B \in \mathcal{L}(\mathcal{H})$. Then $\|A + \lambda B\| \geq \|A\|$ for all $\lambda \in \mathbb{C}$ if and only if there exists a sequence $\{x_n\}$ of unit vectors such that*

$$\lim_{n\to\infty} \|Ax_n\| = \|A\| \tag{2.2.1}$$

and

$$\lim_{n\to\infty} \langle Ax_n, Bx_n \rangle = 0. \tag{2.2.2}$$

Proof. See Remark 3.1 in [15]. $\square$

Let $\mathcal{K}$ be another infinite dimensional Hilbert space.

Theorem 2.2.2. *Let $A, B \in \mathcal{L}(\mathcal{H},\mathcal{K})$. Then $\|A + \lambda B\| \geq \|A\|$ for all $\lambda \in \mathbb{C}$ if and only if there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that*

$$\lim_{n\to\infty} \|Ax_n\| = \|A\|$$

and

$$\lim_{n\to\infty} \langle Ax_n, Bx_n\rangle = 0.$$

Proof. First suppose that there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $\lim_{n\to\infty} \|Ax_n\| = \|A\|$ and $\lim_{n\to\infty} \langle Ax_n, Bx_n\rangle = 0$. Then for every $\lambda \in \mathbb{C}$,

$$\begin{aligned} \|A+\lambda B\|^2 &\geq \|(A+\lambda B)x_n\|^2 \\ &= \|Ax_n\|^2 + |\lambda|^2\|Bx_n\|^2 + 2\,\mathrm{Re}\ \lambda \langle Ax_n, Bx_n\rangle \\ &\geq \|Ax_n\|^2 + 2\,\mathrm{Re}\ \lambda \langle Ax_n, Bx_n\rangle . \end{aligned}$$

This holds for all n. Taking limit on both the sides as $n \to \infty$ we get

$$\|A+\lambda B\| \geq \|A\|.$$

Conversely, let $\|A+\lambda B\| \geq \|A\|$ for all $\lambda \in \mathbb{C}$. For any $T \in \mathcal{L}(\mathcal{H},\mathcal{K})$, we denote by $\tilde{T}$, the operator on $\mathcal{H} \oplus \mathcal{K}$ defined as

$$\tilde{T} = \begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}.$$

Note that $\|\tilde{T}\| = \|T\|$. Therefore we have $\|\tilde{A} + \lambda\tilde{B}\| \geq \|\tilde{A}\|$ for all $\lambda \in \mathbb{C}$. By Theorem 2.2.1, we get a sequence $\{h_n \oplus k_n\}$ of unit vectors in $\mathcal{H} \oplus \mathcal{K}$ such that

$$\|\tilde{A}(h_n \oplus k_n)\| \to \|\tilde{A}\| \text{ and } \left\langle \tilde{A}(h_n \oplus k_n), \tilde{B}(h_n \oplus k_n)\right\rangle \to 0 \text{ as } n \to \infty. \tag{2.2.3}$$

The first equation gives

$$\|Ah_n\| \to \|A\| \text{ as } n \to \infty. \tag{2.2.4}$$

This gives that $h_n \neq 0$ for all but finitely many n. So we assume $h_n \neq 0$ for all n. Now

$$\|A\| = \lim_{n\to\infty} \|Ah_n\| \leq \|A\| \liminf_{n\to\infty} \|h_n\|.$$

Therefore $\liminf_{n\to\infty} \|h_n\| \geq 1$. Since $\|h_n\| \leq 1$ for every n, we have $\limsup_{n\to\infty} \|h_n\| \leq 1$. So

$$\lim_{n\to\infty} \|h_n\| = 1.$$

Let $x_n = \frac{h_n}{\|h_n\|}$. Hence we get a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that

$$\|Ax_n\| \to \|A\| \text{ and } \langle Ax_n, Bx_n\rangle \to 0 \text{ as } n \to \infty.$$

$\square$

Corollary 2.2.3. *Let $A, B \in \mathcal{L}(\mathcal{H},\mathcal{K})$. Then $\|A+\lambda B\| \geq \|A\|$ for all $\lambda \in \mathbb{C}$ if and only if there exists a state φ on $\mathcal{L}(\mathcal{H})$ such that $\varphi(A^*A) = \|A\|^2$ and $\varphi(A^*B) = 0$. (Note that this state φ may not be of the form $\varphi(T) = \langle x, Tx\rangle$ for any x.)*

Proof. First suppose that there exists a state φ on $\mathcal{L}(\mathcal{H})$ such that $\varphi(A^*A) = \|A\|^2$ and $\varphi(A^*B) = 0$. For every $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} \|A+\lambda B\|^2 &\geq \varphi((A+\lambda B)^*(A+\lambda B)) \\ &= \varphi(A^*A) + \overline{\lambda}\varphi(B^*A) + \lambda\varphi(A^*B) + |\lambda|^2 \varphi(B^*B) \\ &\geq \|A\|^2. \end{aligned} \tag{2.2.5}$$

Conversely, let $\|A+\lambda B\| \geq \|A\|$ for all $\lambda \in \mathbb{C}$. Then by Theorem 2.2.2, there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $\|Ax_n\| \to \|A\|$ and $\langle Ax_n, Bx_n\rangle \to 0$ as $n \to \infty$. Define $\varphi_n : \mathcal{L}(\mathcal{H}) \to \mathbb{C}$ as

$$\varphi_n(T) = \langle x_n, Tx_n\rangle.$$

Then φ_n is a state on $\mathcal{L}(\mathcal{H})$. Note that $\varphi_n(A^*A) = \langle Ax_n, Ax_n\rangle \to \|A\|^2$ and $\varphi_n(A^*B) = \langle Ax_n, Bx_n\rangle \to 0$ as $n \to \infty$. Since the collection of all states on any C^*-algebra is weak* compact, $\{\varphi_n\}$ has a convergent subnet $\{\psi_\alpha\}$ converging to a state ψ in weak* topology. We have

$$\psi(A^*A) = \lim_{\alpha} \psi_\alpha(A^*A) = \|A\|^2$$

and

$$\psi(A^*B) = \lim_{\alpha} \psi_\alpha(A^*B) = 0.$$

□

Remark 2.2.4. Let $A, B \in \mathbb{M}(m,n)$. Then $\|A+\lambda B\| \geq \|A\|$ for all $\lambda \in \mathbb{C}$ if and only if there exists a unit vector x such that $\|Ax\| = \|A\|$ and $\langle Ax, Bx\rangle = 0$. This follows from Theorem 2.2.2. If $\|A+\lambda B\| \geq \|A\|$ for all $\lambda \in \mathbb{C}$, then from Theorem 2.2.2 we obtain a sequence $\{x_n\}$ of unit vectors such that $\|Ax_n\| \to \|A\|$ and $\langle Ax_n, Bx_n\rangle \to 0$ as $n \to \infty$. Since $\{x_n\}$ is a bounded sequence, it has a convergent subsequence converging to a vector x. This x serves as the required unit vector.

In the following three corollaries, we reformulate the result of the above remark 2.2.4 to show what it looks like in three specific situations. In these corollaries, all the Hilbert spaces considered are finite dimensional.

Corollary 2.2.5. *Let $A_j \in \mathcal{L}(\mathcal{H}, \mathcal{K}_j)$ for $j = 1, \ldots, d$. Consider the column operator* $\begin{pmatrix} A_1 \\ \vdots \\ A_d \end{pmatrix} : \mathcal{H} \to \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_d$ *which takes $x \in \mathcal{H}$ to* $\begin{pmatrix} A_1x \\ \vdots \\ A_dx \end{pmatrix}$. *Then*

$$\left\| \begin{pmatrix} A_1 + \lambda B_1 \\ \vdots \\ A_d + \lambda B_d \end{pmatrix} \right\| \geq \left\| \begin{pmatrix} A_1 \\ \vdots \\ A_d \end{pmatrix} \right\| \quad \textit{for all } \lambda \in \mathbb{C}$$

if and only if there exists a unit vector $x \in \mathcal{H}$ such that

$$\sum_{j=1}^{d} \|A_jx\|^2 = \| \sum_{j=1}^{d} A_j^* A_j \| \textit{ and } \sum_{j=1}^{d} \langle A_jx, B_jx\rangle = 0.$$

Corollary 2.2.6. *Let $A_j \in \mathcal{L}(\mathcal{H}_j, \mathcal{K})$ for $j = 1, \ldots, d$. Consider the row operator*

$(A_1,\dots,A_d) : \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_d \to \mathcal{K}$ *which takes* $\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}$ *to* $A_1x_1 + \cdots + A_dx_d$. *Then*

$$\|(A_1+\lambda B_1,\dots,A_d+\lambda B_d)\| \geq \|(A_1,\dots,A_d)\| \quad \textit{for all } \lambda \in \mathbb{C}$$

if and only if there exists a unit vector $\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_d$ *such that*

$$\|(A_1,\dots,A_d)\|^2 = \sum_{j=1}^{d} \|A_jx_j\|^2 + \sum_{\substack{i,j=1 \\ i\neq j}}^{d} \langle A_ix_i, A_jx_j\rangle \textit{ and } \sum_{i,j=1}^{d} \langle A_ix_i, B_jx_j\rangle = 0.$$

Corollary 2.2.7. *Let* $A_j \in \mathcal{L}(\mathcal{H}_j,\mathcal{K}_j)$ *for* $j = 1,\dots,d$. *Consider the "diagonal" operator*

$$\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_d \end{pmatrix} : \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_d \to \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_d.$$

Then

$$\left\| \begin{pmatrix} A_1+\lambda B_1 & & \\ & \ddots & \\ & & A_d+\lambda B_d \end{pmatrix} \right\| \geq \left\| \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_d \end{pmatrix} \right\| \quad \textit{for all } \lambda \in \mathbb{C}$$

if and only if there exists a unit vector $\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_d$ *such that*

$$\max_{1\leq k\leq d} \|A_k\|^2 = \sum_{j=1}^{d} \|A_jx_j\|^2 \textit{ and } \sum_{j=1}^{d} \langle A_jx_j, B_jx_j\rangle = 0.$$

As an interesting application of this, we have the following result.

Corollary 2.2.8. *Let $n_1, \ldots, n_k$ be a partition of a positive integer n, that is, $\sum_{j=1}^{k} n_j = n$. Let $A, B \in \mathbb{M}(m, n)$. Let $A = [A_1 : \ldots : A_k]$, where each $A_j \in \mathbb{M}(m, n_j)$. Define*

$$\|A\|_{\text{col}} = \max_{1 \leq j \leq k} \|A_j\|.$$

Suppose this maximum is attained at d indices, say $j_1, \ldots, j_d$. Then A is orthogonal to B in $\|\cdot\|_{\text{col}}$ if and only if $[A_{j_1} : \ldots : A_{j_d}]$ is orthogonal to $[B_{j_1} : \ldots : B_{j_d}]$ in $\|\cdot\|_{\text{col}}$.

Proof. If $[A_{j_1} : \ldots : A_{j_d}]$ is orthogonal to $[B_{j_1} : \ldots : B_{j_d}]$ in $\|\cdot\|_{\text{col}}$, then for all $\lambda \in \mathbb{C}$,

$$\begin{aligned}
\|A + \lambda B\|_{\text{col}} &= \max_{1 \leq j \leq k} \|A_j + \lambda B_j\| \\
&\geq \max_{1 \leq p \leq d} \|A_{j_p} + \lambda B_{j_p}\| \\
&= \|[A_{j_1} : \ldots : A_{j_d}] + \lambda [B_{j_1} : \ldots : B_{j_d}]\|_{\text{col}} \\
&\geq \|[A_{j_1} : \ldots : A_{j_d}]\|_{\text{col}} \\
&= \|A_{j_p}\| \quad \text{for all } 1 \leq p \leq d \\
&= \|A\|_{\text{col}}.
\end{aligned}$$

For the converse, first note that, by virtue of the norm on A being maximum of the norms of A_j, the matrix $[A_1 : \ldots : A_k]$ being orthogonal to the matrix $[B_1 : \ldots : B_k]$ in $\|\cdot\|_{\text{col}}$ is the same as saying that

$$\left\| \begin{pmatrix} A_1 + \lambda B_1 & & \\ & \ddots & \\ & & A_k + \lambda B_k \end{pmatrix} \right\| \geq \left\| \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix} \right\| \quad \text{for all } \lambda \in \mathbb{C}. \tag{2.2.6}$$

Assume, without loss of generality, that $j_p = p$ for all $1 \leq p \leq d$, that is,

$$\|A\|_{\text{col}} = \|A_1\| = \cdots = \|A_d\|.$$

Thus we have to prove that

$$\left\|\begin{pmatrix} A_1+\lambda B_1 & & \\ & \ddots & \\ & & A_d+\lambda B_d \end{pmatrix}\right\| \geq \left\|\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_d \end{pmatrix}\right\| \quad \text{for all } \lambda \in \mathbb{C}.$$

Now we use equation (2.2.6) and Corollary 2.2.7 to conclude that there exist $x_j \in \mathbb{C}^{n_j},\ j = 1, \ldots, k$, such that

$$\sum_{j=1}^{k} \|x_j\|^2 = 1, \quad \sum_{j=1}^{k} \|A_j x_j\|^2 = \|A_1\|^2 = \cdots = \|A_d\|^2 \text{ and } \sum_{j=1}^{k} \langle A_j x_j, B_j x_j\rangle = 0.$$

Now

$$\|A_1\|^2 = \sum_{j=1}^{k} \|A_j x_j\|^2 \leq \sum_{j=1}^{k} \|A_j\|^2 \|x_j\|^2 \leq \|A_1\|^2.$$

This gives

$$\sum_{j=1}^{k} \|A_j\|^2 \|x_j\|^2 = \|A_1\|^2 = \cdots = \|A_d\|^2.$$

Therefore we get $x_{d+1} = \cdots = x_k = 0$. So now we have $x_1, \ldots, x_d$ in $\mathbb{C}^{n_1}, \ldots, \mathbb{C}^{n_d}$, respectively, such that

$$\sum_{j=1}^{d} \|x_j\|^2 = 1, \quad \sum_{j=1}^{d} \|A_j x_j\|^2 = \|A_1\|^2 = \cdots = \|A_d\|^2 \text{ and } \sum_{j=1}^{d} \langle A_j x_j, B_j x_j\rangle = 0.$$

Again by using Corollary 2.2.7, we get the required result. $\square$

2.3 Orthogonality in Hilbert C^*-modules

Motivated by the results in the previous sections, we explore orthogonality in the setting of Hilbert C^*-modules. As our work was in progress, this problem was also studied by Arambašić and Rajić [3]. Some of the results in this section overlap with theirs, but the proofs we present are different and seem more natural. We start by an elementary Hilbert C^*-module, namely a C^*-algebra.

Theorem 2.3.1. *Let $\mathcal{A}$ be a C^*-algebra. Let $a, b \in \mathcal{A}$. Then $\|a + \lambda b\| \geq \|a\|$ for all $\lambda \in \mathbb{C}$ if and only if there exists a state φ on $\mathcal{A}$ such that*

$$\varphi(a^*a) = \|a\|^2 \text{ and } \varphi(a^*b) = 0.$$

Proof. If there exists a state on $\mathcal{A}$ such that $\varphi(a^*a) = \|a\|^2$ and $\varphi(a^*b) = 0$, then arguments similar to the one in (2.2.5) show that $\|a + \lambda b\| \geq \|a\|$ for all $\lambda \in \mathbb{C}$. For the converse, let $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be a faithful representation. Let $A = \pi(a)$ and $B = \pi(b)$. Then we have $\|A + \lambda B\| \geq \|A\|$ for all $\lambda \in \mathbb{C}$. By Corollary 2.2.3, there exists a state ψ on $\mathcal{L}(\mathcal{H})$ such that

$$\psi(A^*A) = \|A\|^2 \text{ and } \psi(A^*B) = 0. \tag{2.3.1}$$

Let $\varphi : \mathcal{A} \to \mathbb{C}$ be defined as $\varphi(a) = \psi(\pi(a))$. Then φ is a state on $\mathcal{A}$. Equation (2.3.1) implies that

$$\varphi(a^*a) = \|a\|^2 \text{ and } \varphi(a^*b) = 0.$$

$\square$

For any given state φ on $\mathcal{A}$ and $a \in \mathcal{A}$, let the *variance* of a with respect to φ, denoted by $\operatorname{var}_\varphi(a)$, be defined as

$$\operatorname{var}_\varphi(a) = \varphi(a^*a) - |\varphi(a)|^2. \tag{2.3.2}$$

Let $\operatorname{dist}(a, \mathbb{C}1) = \min\{\|a - \lambda 1\| : \lambda \in \mathbb{C}\}$ be the distance of a from $\mathbb{C}1$. In Audenaert [6, Theorem 9], it has been shown that for any $A \in \mathbb{M}(n)$,

$$\operatorname{dist}(A, \mathbb{C}I)^2 = \max\{\operatorname{tr} A^*AP - |\operatorname{tr} AP|^2 : P \text{ positive semidefinite}, \operatorname{tr} P = 1\}. \tag{2.3.3}$$

The following corollary is a generalization of (2.3.3) and was first obtained by Rieffel [41, Theorem 3.10]. We provide a proof of it using Theorem 2.3.1.

Corollary 2.3.2. *With the notations as above, we have for any $a \in \mathcal{A}$,*

$$\operatorname{dist}(a, \mathbb{C}1)^2 = \max\{\operatorname{var}_\varphi(a) : \varphi \in S(\mathcal{A})\}, \tag{2.3.4}$$

where $S(\mathcal{A})$ denotes the state space of $\mathcal{A}$.

Proof. First note that for any $\varphi \in S(\mathcal{A})$,

$$\varphi(a^*a) \leq \|a\|^2.$$

Therefore

$$\operatorname{var}_\varphi(a) = \varphi(a^*a) - |\varphi(a)|^2 \leq \|a\|^2.$$

Let $\lambda \in \mathbb{C}$. Changing a to $a + \lambda 1$ in the above equation, we see

$$\operatorname{var}_\varphi(a + \lambda 1) = \varphi((a + \lambda 1)^*(a + \lambda 1)) - |\varphi(a + \lambda 1)|^2 \leq \|a + \lambda 1\|^2.$$

The left hand side is invariant under the translation $a \to a + \lambda 1$, that is,

$$\operatorname{var}_\varphi(a + \lambda 1) = \operatorname{var}_\varphi(a).$$

This gives

$$\max\{\operatorname{var}_\varphi(a) : \varphi \in S(\mathcal{A})\} \leq \operatorname{dist}(a, \mathbb{C}1)^2. \tag{2.3.5}$$

Now $\operatorname{dist}(a, \mathbb{C}1) = \|a - \lambda_0\|$, for some $\lambda_0 \in \mathbb{C}$. Denote $a - \lambda_0$ by a_0. Then $\|a_0 + \lambda 1\| \geq \|a_0\|$ for all $\lambda \in \mathbb{C}$. By Theorem 2.3.1, there exists a state ψ on $\mathcal{A}$ such that

$$\psi(a_0^*a_0) = \|a_0\|^2 \text{ and } \psi(a_0^*) = 0. \tag{2.3.6}$$

By the first equation in (2.3.6), we get

$$\operatorname{dist}(a, \mathbb{C}1)^2 = \|a_0\|^2 = \psi(a_0^*a_0) = \psi(a^*a) - \overline{\lambda_0}\psi(a) - \lambda_0\overline{\psi(a)} + |\lambda_0|^2. \tag{2.3.7}$$

By the second equation in (2.3.6), we get $\psi(a) = \lambda_0$. Using this in (2.3.7), we obtain

$$\operatorname{dist}(a, \mathbb{C}1)^2 = \psi(a^*a) - |\psi(a)|^2 = \operatorname{var}_\psi(a) \leq \max\{\operatorname{var}_\varphi(a) : \varphi \in S(\mathcal{A})\}.$$

This together with (2.3.5) gives the desired result. $\square$

We now obtain a characterization of orthogonality in Hilbert C^*-modules. For this we require the following lemma, which is a reinterpretation of Theorem 3.4 in [19].

Lemma 2.3.3. *Let $\mathcal{E}$ be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$. Then $\mathcal{E}$ can be isometrically embedded in $\mathcal{L}(\mathcal{H}, \mathcal{K})$ for some Hilbert spaces $\mathcal{H}, \mathcal{K}$. Here $\mathcal{H}$ is a Hilbert space such that there exists a faithful representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and the isometric embedding $L : \mathcal{E} \to \mathcal{L}(\mathcal{H}, \mathcal{K})$ satisfies*

$$\langle L(e_1)h_1, L(e_2)h_2\rangle = \langle h_1, \pi(\langle e_1, e_2\rangle)h_2\rangle \text{ for all } e_1, e_2 \in \mathcal{E} \text{ and } h_1, h_2 \in \mathcal{H}.$$

Proof. Let $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be a faithful representation. Consider the space $\mathcal{E} \otimes \mathcal{H}$. Define a map $\langle \cdot, \cdot\rangle : \mathcal{E} \otimes \mathcal{H} \to \mathbb{C}$ as follows. For elementary tensors $e_1 \otimes h_1, e_2 \otimes h_2 \in \mathcal{E} \otimes \mathcal{H}$

$$\langle e_1 \otimes h_1, e_2 \otimes h_2\rangle = \langle h_1, \pi(\langle e_1, e_2\rangle)h_2\rangle .$$

Extend this definition linearly to whole of $\mathcal{E} \otimes \mathcal{H}$. We show that $\langle \cdot, \cdot\rangle$ forms a semi-inner product on the space $\mathcal{E} \otimes \mathcal{H}$. Let $x, y, z \in \mathcal{E} \otimes \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$. Clearly $\langle x, \alpha y + \beta z\rangle = \alpha \langle x, y\rangle + \beta \langle x, z\rangle$ and $\langle x, y\rangle = \overline{\langle y, x\rangle}$. Let $x = \sum_{i=1}^n e_i \otimes h_i$. Then

$$\begin{aligned}
\langle x, x\rangle &= \sum_{i,j=1}^n \langle h_i, \pi(\langle e_i, e_j\rangle)h_j\rangle \\
&= \left\langle \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} \pi(\langle e_1, e_1\rangle) & \dots & \pi(\langle e_1, e_n\rangle) \\ \vdots & \ddots & \vdots \\ \pi(\langle e_n, e_1\rangle) & \dots & \pi(\langle e_n, e_n\rangle) \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \right\rangle.
\end{aligned}$$

Note that the matrix $T = (\langle e_i, e_j\rangle)$ is a positive semidefinite matrix in $\mathbb{M}(n; \mathcal{A})$. Let

$\pi_n : \mathbb{M}(n; \mathcal{A}) \to \mathbb{M}(n; \mathcal{L}(\mathcal{H}))$ be the map defined as $\pi_n((a_{ij})) = (\pi(a_{ij}))$. Since π is a representation, so is π_n. Hence π_n is a positive map. In particular $\pi_n(T) \geq 0$. Hence $\langle x, x \rangle \geq 0$. Let $\mathcal{N} = \{x \in \mathcal{E} \otimes \mathcal{H} : \langle x, x \rangle = 0\}$. Then $\mathcal{N}$ is a closed subspace of $\mathcal{E} \otimes \mathcal{H}$. Consider the inner product on the space $\mathcal{E} \otimes \mathcal{H}/\mathcal{N}$ defined as

$$\langle x + \mathcal{N}, y + \mathcal{N} \rangle = \langle x, y \rangle.$$

Let $\mathcal{K}$ be the completion of $\mathcal{E} \otimes \mathcal{H}/\mathcal{N}$ with respect to the norm given by this inner product. For $e \in \mathcal{E}$, define $L_e : \mathcal{H} \to \mathcal{K}$ as

$$L_e(h) = e \otimes h + \mathcal{N}.$$

Note that each L_e is linear. We show that L_e is bounded. We have

$$\|L_e(h)\|^2 = \langle h, \pi(\langle e, e \rangle) h \rangle .$$

By Cauchy-Schwarz inequality, we get that

$$\begin{aligned} \|L_e(h)\|^2 &\leq \|h\|^2 \|\pi(\langle e, e \rangle)\| \\ &= \|h\|^2 \|\langle e, e \rangle\| \\ &= \|h\|^2 \|e\|^2. \end{aligned}$$

Let $L : \mathcal{E} \to \mathcal{L}(\mathcal{H}, \mathcal{K})$ be defined as

$$L(e) = L_e.$$

Then L is the required linear isometry. $\square$

Theorem 2.3.4. *Let $\mathcal{E}$ be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$. Let $e_1, e_2 \in \mathcal{E}$. Then*

$$\|e_1 + \lambda e_2\| \geq \|e_1\| \text{ for all } \lambda \in \mathbb{C} \tag{2.3.8}$$

if and only if there exists a state φ on $\mathcal{A}$ such that

$$\varphi(\langle e_1, e_1\rangle) = \|e_1\|^2 \text{ and } \varphi(\langle e_1, e_2\rangle) = 0.$$

Proof. First suppose that there exists a state φ on $\mathcal{A}$ such that $\varphi(\langle e_1, e_1\rangle) = \|e_1\|^2$ and $\varphi(\langle e_1, e_2\rangle) = 0$. Then for every $\lambda \in \mathbb{C}$,

$$\begin{aligned} \|e_1 + \lambda e_2\|^2 &= \| \langle e_1 + \lambda e_2, e_1 + \lambda e_2\rangle \| \\ &\geq |\varphi(\langle e_1, e_1\rangle) + \overline{\lambda}\varphi(\langle e_2, e_1\rangle) + \lambda\varphi(\langle e_1, e_2\rangle) + |\lambda|^2\varphi(\langle e_2, e_2\rangle)| \\ &\geq \|e_1\|^2. \end{aligned}$$

Now suppose (2.3.8) holds. Let $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be a faithful representation and let $L : \mathcal{E} \to \mathcal{L}(\mathcal{H}, \mathcal{K})$ be the isometric embedding of $\mathcal{E}$ into $\mathcal{L}(\mathcal{H}, \mathcal{K})$, as given in the previous lemma. Then (2.3.8) gives

$$\|L(e_1) + \lambda L(e_2)\| \geq \|L(e_1)\| \text{ for all } \lambda \in \mathbb{C}.$$

By Theorem 2.2.2, there exists a sequence of unit vectors $\{x_n\}$ in $\mathcal{H}$ such that $\|L(e_1)x_n\| \to \|L(e_1)\|$ and $\langle L(e_1)x_n, L(e_2)x_n\rangle \to 0$ as $n \to \infty$. Define $\varphi_n : \mathcal{A} \to \mathbb{C}$ as

$$\varphi_n(a) = \langle x_n, \pi(a)x_n\rangle.$$

Then φ_n is a state on $\mathcal{A}$. Note that $\varphi_n(\langle e_1, e_1\rangle) = \langle L(e_1)x_n, L(e_1)x_n\rangle \to \|L(e_1)\|^2$ and $\varphi_n(\langle e_1, e_2\rangle) = \langle L(e_1)x_n, L(e_2)x_n\rangle \to 0$ as $n \to \infty$. Since the collection of all states on $\mathcal{A}$ is a weak* compact subset of $\mathcal{A}^*$, $\{\varphi_n\}$ has a convergent subnet $\{\psi_\alpha\}$ which converges to some ψ in weak* topology. We have

$$\psi(\langle e_1, e_1\rangle) = \lim_\alpha \psi_\alpha(\langle e_1, e_1\rangle) = \|e_1\|^2$$

and

$$\psi(\langle e_1, e_2 \rangle) = \lim_{\alpha} \psi_\alpha(\langle e_1, e_2 \rangle) = 0.$$

$\square$

The above theorem first appeared in [3], while our work was in preparation. Our proofs are different and perhaps simpler.

Chapter 3

Orthogonality to matrix subspaces and a distance formula

Let $\mathbb{X}$ be a real or complex Banach space and let $\mathbb{W}$ be a subset of $\mathbb{X}$. Then an element x is said to be orthogonal to $\mathbb{W}$ ([32]) if

$$\|x + w\| \geq \|x\| \text{ for all } w \in \mathbb{W}. \tag{3.0.1}$$

This notion is a generalization of orthogonality, from vectors to subsets. In this chapter, our main interest is to study this concept in the space $\mathbb{M}(n)$, when $\mathbb{W}$ is a subspace of $\mathbb{M}(n)$.

The space $\mathbb{M}(n)$ is a complex Hilbert space under the inner product $\langle A, B\rangle = \operatorname{tr}(A^*B)$ and a real Hilbert space under the inner product $\langle A, B\rangle_R = \operatorname{Re}\ \operatorname{tr}(A^*B)$. Let $\mathbb{W}^\perp$ denote the orthogonal complement of $\mathbb{W}$, where the orthogonal complement is taken with respect to the usual Hilbert space orthogonality in $\mathbb{M}(n)$ with inner product $\langle\cdot,\cdot\rangle$ or $\langle\cdot,\cdot\rangle_R$, depending upon whether $\mathbb{W}$ is treated as a complex or a real subspace. With this notation, the Bhatia-Šemrl theorem (Theorem 2.1.1) says that A is orthogonal to $\mathbb{C}B$ if and only if there exists a positive semidefinite matrix P of rank one such that $\operatorname{tr} P = 1,\ \operatorname{tr} A^*AP = \|A\|^2$ and $AP \in (\mathbb{C}B)^\perp$. Positive semidefinite matrices with trace 1 are also called *density matrices*.

In this chapter, we provide a generalization of the Bhatia-Šemrl theorem, and obtain a characterization of orthogonality to a subspace of $\mathbb{M}(n)$. Some known results like the one by

Andruchow, Larotonda, Recht, and Varela [5, Theorem 1] can be interpreted as special instances of this. They showed that if A is Hermitian, then $\|A + D\| \geq \|A\|$ for all $D \in \mathbb{D}(n;\mathbb{R})$, the subspace of real diagonal matrices, if and only if there exists a density matrix P such that $PA^2 = \|A\|^2 P$ and all diagonal entries of PA are zero. A Hermitian matrix A such that A is orthogonal to $\mathbb{D}(n;\mathbb{R})$ is called *minimal* in [5].

Let $\operatorname{dist}(A, \mathbb{W})$ denote the distance of a matrix A from the subspace $\mathbb{W}$. It is defined as

$$\operatorname{dist}(A, \mathbb{W}) = \min\left\{\|A - W\| : W \in \mathbb{W}\right\}.$$

For $\mathbb{W} = \mathbb{C}I$, an expression (2.3.3) for the distance has been given by Audenaert [6]. A natural question that arises is whether we can find an analogous expression for the distance when $\mathbb{C}I$ is replaced by any C^*-subalgebra of $\mathbb{M}(n)$. In this chapter, we provide an answer to this question.

3.1 Orthogonality to subspaces of $\mathbb{M}(n)$

Theorem 3.1.1. *Let $A \in \mathbb{M}(n)$ and let $m(A)$ be the multiplicity of the maximum singular value $\|A\|$ of A. Let $\mathbb{W}$ be any (real or complex) subspace of $\mathbb{M}(n)$. Then A is orthogonal to $\mathbb{W}$ if and only if there exists a density matrix P of rank at most $m(A)$ such that $A^*AP = \|A\|^2 P$ and $AP \in \mathbb{W}^\perp$. (If rank $P = \ell$, then P has the form $P = \sum_{i=1}^{\ell} \alpha_i v_{(i)} v_{(i)}^*$ where $v_{(i)}, 1 \leq i \leq \ell$, are unit vectors such that $A^*Av_{(i)} = \|A\|^2 v_{(i)}$ and $\alpha_i, 1 \leq i \leq \ell$, are such that $0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^{\ell} \alpha_i = 1$.)*

Proof. First suppose that there exists a density matrix P such that $A^*AP = \|A\|^2 P$ and $AP \in \mathbb{W}^\perp$. Then for any $W \in \mathbb{W}$,

$$\begin{aligned}\|A + W\|^2 &= \|(A + W)^*(A + W)\| \\ &= \|A^*A + W^*A + A^*W + W^*W\|.\end{aligned}$$

So by (1.3.4), we get

$$\begin{aligned} \|A+W\|^2 &\geq |\operatorname{tr}(A^*AP + W^*AP + A^*WP + W^*WP)| \\ &\geq \text{Re } \operatorname{tr}(A^*AP + W^*AP + A^*WP + W^*WP). \end{aligned} \tag{3.1.1}$$

Since $AP \in \mathbb{W}^\perp$, we have $\text{Re } \operatorname{tr}(A^*WP) = \text{Re } \operatorname{tr}(W^*AP) = 0$. The matrices W^*W and P are positive semidefinite, therefore $\operatorname{tr}(W^*WP) \geq 0$ and by our assumption, $\operatorname{tr}(A^*AP) = \|A\|^2$. Using these in (3.1.1) we get that $\|A+W\|^2 \geq \|A\|^2$.

Conversely, suppose

$$\|A+W\| \geq \|A\| \text{ for all } W \in \mathbb{W}. \tag{3.1.2}$$

Let $S : \mathbb{W} \to \mathbb{M}(n)$ be the inclusion map. Then $S^* : \mathbb{M}(n) \to \mathbb{W}$ is the orthogonal projection onto the subspace $\mathbb{W}$. Let $L : \mathbb{W} \to \mathbb{M}(n)$ be the map defined as

$$L(W) = A + S(W).$$

Let $g : \mathbb{M}(n) \to \mathbb{R}$ be the map taking an $n \times n$ matrix W to $\|W\|$. Then (3.1.2) can be rewritten as

$$(g \circ L)(W) \geq (g \circ L)(0),$$

that is, $g \circ L$ is minimized at 0. Therefore $0 \in \partial(g \circ L)(0)$. Using Proposition 1.2.12, we get

$$0 \in S^*\partial\|A\|. \tag{3.1.3}$$

By Corollary 1.4.8,

$$S^*\partial\|A\| = \text{conv}\left\{S^*(uv^*) : \|u\| = \|v\| = 1, Av = \|A\|u\right\}. \tag{3.1.4}$$

From (3.1.3) and (3.1.4), it follows that there exist unit vectors $u_{(i)}, v_{(i)}$, $1 \leq i \leq \ell$, and numbers α_i, such that $0 \leq \alpha_i \leq 1$, $\sum_{i=1}^{\ell} \alpha_i = 1$, $Av_{(i)} = \|A\| u_{(i)}$ and

$$S^* \left(\sum_{i=1}^{\ell} \alpha_i u_{(i)} v_{(i)}^* \right) = 0. \tag{3.1.5}$$

Let $P = \sum_{i=1}^{\ell} \alpha_i v_{(i)} v_{(i)}^*$. Then P is a density matrix and

$$\begin{aligned} AP &= \sum_{i=1}^{\ell} \alpha_i A v_{(i)} v_{(i)}^* \\ &= \|A\| \sum_{i=1}^{\ell} \alpha_i u_{(i)} v_{(i)}^*. \end{aligned}$$

By using (3.1.5), we get $S^*(AP) = 0$, that is, $AP \in \mathbb{W}^{\perp}$. Since each $v_{(i)}$ is a right singular vector for the singular value $\|A\|$ of A, we have $A^*Av_{(i)} = \|A\|^2 v_{(i)}$. Using this we obtain

$$\begin{aligned} A^*AP &= \sum_{i=1}^{\ell} \alpha_i A^*A v_{(i)} v_{(i)}^* \\ &= \sum_{i=1}^{\ell} \alpha_i \|A\|^2 v_{(i)} v_{(i)}^* \\ &= \|A\|^2 P. \end{aligned} \tag{3.1.6}$$

Now let $m(A) = r$. We show that if P satisfies (3.1.6), then rank $P \leq r$. First note that A^*A and P commute and therefore can be diagonalized simultaneously. So we can assume A^*A and P in (3.1.6) to be diagonal matrices. By hypothesis, r of the diagonal entries of A^*A are equal to $\|A\|^2$. Let $A^*A = \begin{bmatrix} \|A\|^2 & & & & & \\ & \ddots & & & & \\ & & \|A\|^2 & & & \\ & & & s_{r+1}^2 & & \\ & & & & \ddots & \\ & & & & & s_n^2 \end{bmatrix}$, where $s_j < \|A\|$ for all

$r+1 \leq j \leq n$. If $P = \begin{bmatrix} p_1 & & \\ & \ddots & \\ & & p_n \end{bmatrix}$, then from (3.1.6) we obtain

$$(s_j^2 - \|A\|^2)p_j = 0 \text{ for all } r+1 \leq j \leq n.$$

This gives $p_j = 0$ for all $r+1 \leq j \leq n$. Hence rank $P \leq r$.

□

Remark 3.1.2. From the proof of Theorem 3.1.1, it is clear that the condition $A^*AP = \|A\|^2 P$ can be replaced by the weaker condition $\operatorname{tr}(A^*AP) = \|A\|^2$ in the statement of Theorem 3.1.1.

Corollary 3.1.3 (The Bhatia-Šemrl theorem)**.** *We have $\|A + \lambda B\| \geq \|A\|$ for all $\lambda \in \mathbb{C}$ if and only if there is a unit vector x such that $\|Ax\| = \|A\|$ and $\langle Ax, Bx\rangle = 0$.*

Proof. By Theorem 3.1.1, A is orthogonal to $\mathbb{C}B$ if and only if there exist unit vectors $v_{(i)}$ $(1 \leq i \leq \ell)$ and numbers α_i such that $0 \leq \alpha_i \leq 1$, $\sum_{i=1}^{\ell} \alpha_i = 1$, $A^*Av_{(i)} = \|A\|^2 v_{(i)}$ and $\sum_{i=1}^{\ell} \alpha_i \langle v_{(i)}, B^*Av_{(i)}\rangle = 0$. If this is the case, then by the Hausdorff-Toeplitz theorem, we get a unit vector v such that $A^*Av = \|A\|^2 v$ and $\langle v, B^*Av\rangle = 0$. The first condition $A^*Av = \|A\|^2 v$ is stronger than what is required. □

The next corollary shows that Theorem 1 in [5] is a special case of Theorem 3.1.1.

Corollary 3.1.4. *A Hermitian matrix $A \in \mathbb{M}(n)$ is minimal if and only if there exists a positive semidefinite matrix $P \in \mathbb{M}(n)$ such that $PA^2 = \|A\|^2 P$ and all the diagonal elements of PA are zero.*

Proof. In our notation, to say that A is minimal is the same as saying that A is orthogonal to the subspace $\mathbb{D}(n;\mathbb{R})$. If A is Hermitian, then A is orthogonal to $\mathbb{D}(n;\mathbb{R})$ if and only if A is orthogonal to $\mathbb{D}(n)$, the subspace of complex diagonal matrices. By Theorem 3.1.1, there exists a density matrix P such that $A^2P = \|A\|^2 P$ and $AP \in \mathbb{D}(n)^{\perp}$. Now $\mathbb{D}(n)^{\perp}$ is the subspace of all matrices such that their diagonal entries are zero. The condition $PA^2 = \|A\|^2 P$ is the same as $A^2P = \|A\|^2 P$ and the diagonal entries of PA are the same as the complex conjugates of the diagonal entries of AP. Thus we get the required result. $\square$

The next corollary can also be seen by direct calculation (see Remark 3.1.6). A neater proof follows using Theorem 3.1.1.

Corollary 3.1.5. *Let* $\mathbb{W} = \{X : \operatorname{tr} X = 0\}$. *Then* $\{A : \|A + W\| \geq \|A\| \text{ for all } W \in \mathbb{W}\} = \mathbb{W}^{\perp} = \mathbb{C}I$.

Proof. By Theorem 3.1.1, we have that if $A \in \mathbb{W}^{\perp}$ is such that $A^*A = \|A\|^2 I$, then A is orthogonal to $\mathbb{W}$. Therefore all the scalar matrices are orthogonal to $\mathbb{W}$. Conversely, let A be such that $\|A + W\| \geq \|A\|$ for all $W \in \mathbb{W}$. By Theorem 3.1.1, there exists a density matrix P of complex rank at most $m(A)$ such that

$$A^*AP = \|A\|^2 P \tag{3.1.7}$$

and

$$AP \in \mathbb{C}I. \tag{3.1.8}$$

Let $AP = \mu I$ for some $\mu \in \mathbb{C}$. Substituting this in (3.1.7), we get

$$\|A\|^2 P = \mu A^* = \overline{\mu} A. \tag{3.1.9}$$

This shows that $\mu \neq 0$ and

$$A = \frac{\|A\|^2}{\overline{\mu}} P. \tag{3.1.10}$$

So $AP = \frac{\|A\|^2}{\overline{\mu}} P^2$. But $AP = \mu I$. Thus we obtain $P^2 = \frac{|\mu|^2}{\|A\|^2} I$, and hence $P = \frac{|\mu|}{\|A\|} I$. By substituting P in (3.1.10), we get $A = \frac{\mu\|A\|}{|\mu|} I$, which belongs to $\mathbb{W}^{\perp}$. $\square$

Remark 3.1.6. Let $\mathbb{W} = \{X : \operatorname{tr} X = 0\}$. Then we can give a proof of Corollary 3.1.5, independent of Theorem 3.1.1 . First suppose that $A \in \mathbb{W}^{\perp}$. This means $A = \mu I$ for some

$\mu \in \mathbb{C}$. Let $W \in \mathbb{W}$. Then for each X such that $\|X\|_1 = 1$, we have

$$\|\mu I + W\| \geq |\operatorname{tr}(\mu I + W)X|.$$

In particular for $X = \frac{I}{n}$, we get

$$\|\mu I + W\| \geq \frac{1}{n}|\operatorname{tr}(\mu I + W)|.$$

Since $\operatorname{tr} W = 0$, we obtain

$$\|\mu I + W\| \geq \|\mu I\|.$$

Now we show that if $A \notin \mathbb{C}I$, then there exists a matrix W with $\operatorname{tr} W = 0$ such that $\|A + W\| < \|A\|$. Let $\mathcal{D}(A)$ and $\mathcal{O}(A)$ denote the diagonal and off-diagonal parts of A, respectively. Then $\mathcal{O}(A) \in \mathbb{W}$ and $A - \mathcal{O}(A) = \mathcal{D}(A)$. We know that $\|\mathcal{D}(A)\| \leq \|A\|$. So it is enough to find $W \in \mathbb{W}$ such that $\|\mathcal{D}(A) + W\| < \|\mathcal{D}(A)\|$. Let $\mathcal{D}(A) = \operatorname{diag}(a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_k, \ldots, a_k)$, where $k \geq 2$, each a_j $(1 \leq j \leq k)$ occurs on the diagonal n_j times and $n_1 + \cdots + n_k = n$. We can assume $\|\mathcal{D}(A)\| = 1$. Take $W = \operatorname{diag}\left(\frac{a_2-a_1}{kn_1}, \ldots, \frac{a_2-a_1}{kn_1}, \frac{a_3-a_2}{kn_2}, \ldots, \frac{a_3-a_2}{kn_2}, \ldots, \frac{a_k-a_{k-1}}{kn_{k-1}}, \ldots, \frac{a_k-a_{k-1}}{kn_{k-1}}, \frac{a_1-a_k}{kn_k}, \ldots, \frac{a_1-a_k}{kn_k}\right)$. Then $\operatorname{tr} W = 0$ and $\mathcal{D}(A) + W = \operatorname{diag}\left(\frac{(kn_1-1)a_1+a_2}{kn_1}, \ldots, \frac{(kn_1-1)a_1+a_2}{kn_1}, \frac{(kn_2-1)a_2+a_3}{kn_2}, \ldots, \frac{(kn_2-1)a_2+a_3}{kn_2}, \ldots, \frac{(kn_{k-1}-1)a_{k-1}+a_k}{kn_{k-1}}, \ldots, \frac{(kn_{k-1}-1)a_{k-1}+a_k}{kn_{k-1}}, \frac{(kn_k-1)a_k+a_1}{kn_k}, \ldots, \frac{(kn_k-1)a_k+a_1}{kn_k}\right)$. It is easy to check that each entry has modulus less than 1.

Remark 3.1.7. In Theorem 3.1.1, $m(A)$ is the best possible upper bound on rank P. Consider $\mathbb{W} = \{X : \operatorname{tr} X = 0\}$. By Corollary 3.1.5, we get that if a matrix A is orthogonal to $\mathbb{W}$, then it has to be of the form $A = \lambda I$, for some $\lambda \in \mathbb{C}$. When $A \neq 0$ then $m(A) = n$. Let P be any density matrix satisfying $AP \in \mathbb{W}^{\perp}$. Then $AP = \mu I$, for some $\mu \in \mathbb{C}, \mu \neq 0$. If P also satisfies $A^*AP = \|A\|^2 P$, then we get $P = \frac{\mu}{\lambda} I$. Thus rank $P = n = m(A)$.

Remark 3.1.8. As one would expect, the set $\{A : \|A + W\| \geq \|A\| \text{ for all } W \in \mathbb{W}\}$ need not be a subspace. As an example consider the subspace $\mathbb{W} = \mathbb{C}I$ of $\mathbb{M}(3)$. Let $A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. By Theorem 3.1.1, it can be checked that A_1, A_2 are orthogonal to $\mathbb{W}$. (Take $P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ for A_1 and $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ for A_2, respectively.) Then $A_1 + A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, and $\|A_1 + A_2\| = 2$. But $\left\|A_1 + A_2 - \frac{1}{2}I\right\| = \frac{3}{2} < \|A_1 + A_2\|$. Hence $A_1 + A_2$ is not orthogonal to $\mathbb{W}$.

3.2 A distance formula

Let $\operatorname{dist}(A, \mathbb{W})$ denote the distance of a matrix A from the subspace $\mathbb{W}$, i.e.,

$$\operatorname{dist}(A, \mathbb{W}) = \min\left\{\|A - W\| : W \in \mathbb{W}\right\}.$$

It is interesting to have evaluation or estimation of distance to a subspace. It is a well known result by Stampfli [43] that

$$\operatorname{dist}(A, \mathbb{C}I) = \frac{1}{2}\max\{\|AX - XA\| : \|X\| = 1\}.$$

In [15], Bhatia and Šemrl showed that

$$\operatorname{dist}(A, \mathbb{C}I) = \max\{|\langle y, Ax\rangle| : \|x\| = \|y\| = 1, x \perp y\}.$$

They also showed that

$$\operatorname{dist}(A, \mathbb{C}I) = \frac{1}{2} \max\{\|A - UAU^*\| : U \text{ unitary}\}.$$

In [16], Bhatia and Sharma obtained estimates on $\operatorname{dist}(A, \mathbb{C}I)$. Another expression for the distance has been given by Audenaert [6]:

$$\operatorname{dist}(A, \mathbb{C}I)^2 = \max\{\operatorname{tr} A^*AP - |\operatorname{tr} AP|^2 : P \text{ positive semidefinite}, \operatorname{tr} P = 1\}. \tag{3.2.1}$$

He also showed that the maximum over P on the right hand side of (3.2.1) can be restricted to density matrices of rank 1. This is an example of a distance formula.

If $\mathbb{W} = \mathbb{C}B$, the subspace spanned by a matrix B, then Arambašić and Rajić showed that

$$\operatorname{dist}(A, \mathbb{C}B) = \max\{M_{A,B}(x) : \|x\| = 1\},$$

where $M_{A,B}(x) = \begin{cases} \|Ax\|^2 - \frac{|\langle Ax, Bx\rangle|^2}{\|Bx\|^2} & \text{if } Bx \neq 0 \\ \|Ax\|^2 & \text{if } Bx = 0 \end{cases}$.

In [41], Rieffel obtained a formula (2.3.4), analogous to (3.2.1), for unital C^*-algebras and raised the more general problem of obtaining a formula for the distance to any unital C^*-subalgebra. We provide an answer to this problem in the case of a finite dimensional C^*-algebra. This is essentially the case of $\mathbb{M}(n)$. Let $\mathcal{B}$ be any unital C^*-subalgebra of $\mathbb{M}(n)$. We know that every finite dimensional C^*-algebra is $*$-isomorphic to a direct sum of matrix algebras, so there exist $n_1, \ldots, n_k$ such that $\mathcal{B} \simeq \mathbb{M}(n_1) \oplus \cdots \oplus \mathbb{M}(n_k)$ (see [24, p. 74].

Let $\mathcal{C}_{\mathcal{B}} : \mathbb{M}(n) \to \mathcal{B}$ denote the orthogonal projection of $\mathbb{M}(n)$ onto $\mathcal{B}$. Then $\mathcal{C}_{\mathcal{B}}$ is a bimodule map:

$$\mathcal{C}_{\mathcal{B}}(BX) = B\mathcal{C}_{\mathcal{B}}(X) \text{ and } \mathcal{C}_{\mathcal{B}}(XB) = \mathcal{C}_{\mathcal{B}}(X)B \text{ for all } B \in \mathcal{B}, X \in \mathbb{M}(n). \tag{3.2.2}$$

In particular, when $\mathcal{B}$ is a subalgebra of block diagonal matrices, the matrix $\mathcal{C}_{\mathcal{B}}(X)$ is called a *pinching* of X and is denoted by $\mathcal{C}(X)$. If $X = \begin{bmatrix} X_{11} & \cdots & X_{1k} \\ X_{21} & \cdots & X_{2k} \\ \vdots & \vdots & \vdots \\ X_{k1} & \cdots & X_{kk} \end{bmatrix}$, then

$$\mathcal{C}(X) = \begin{bmatrix} X_{11} & & & \\ & X_{22} & & \\ & & \ddots & \\ & & & X_{kk} \end{bmatrix}. \tag{3.2.3}$$

Properties of pinchings are studied in detail in [9] and [10]. Our next result provides a generalization of (3.2.1) for the distance of A to any C^*-subalgebra of $\mathbb{M}(n)$.

Theorem 3.2.1. *Let $\mathcal{B}$ be any C^*-subalgebra of $\mathbb{M}(n)$. Then*

$$\operatorname{dist}(A, \mathcal{B})^2 = \max\left\{\operatorname{tr}\left(A^*AP - \mathcal{C}_{\mathcal{B}}(AP)^*\, \mathcal{C}_{\mathcal{B}}(AP)\, \mathcal{C}_{\mathcal{B}}(P)^{-1}\right) : P \geq 0, \operatorname{tr} P = 1\right\}, \tag{3.2.4}$$

where $\mathcal{C}_{\mathcal{B}}(P)^{-1}$ denotes the Moore-Penrose inverse of $\mathcal{C}_{\mathcal{B}}(P)$. Further the maximum on the right hand side of (3.2.4) *can be restricted to density matrices P with rank $P \leq m(A)$.*

Proof. We first show that it is sufficient to prove the result when $\mathcal{B}$ is a C^*-subalgebra of block diagonal matrices in $\mathbb{M}(n)$. If $\mathcal{B}$ is any C^*-subalgebra of $\mathbb{M}(n)$, then there exist $n_1, n_2, \ldots, n_k$ such that $\mathcal{B}$ is $*$-isomorphic to $\oplus_{i=1}^k \mathbb{M}(n_i)$. Let $\varphi : \mathcal{B} \to \oplus_{i=1}^k \mathbb{M}(n_i)$ denote this $*$-isomorphism. Then there exists a unitary matrix $V \in \mathbb{M}(n)$ such that $\varphi(X) = V^*XV$ (see [23, p. 249], [24, p. 74]). By definition

$$\operatorname{dist}(A, \mathcal{B}) = \min_{W \in \mathcal{B}} \|A - W\|.$$

Let $\tilde{A}$ denote the matrix V^*AV. Since $\|\cdot\|$ is unitarily invariant, we get

$$\operatorname{dist}(A, \mathcal{B}) = \operatorname{dist}(\tilde{A}, \oplus_i \mathbb{M}(n_i)). \tag{3.2.5}$$

Next we show that for any density matrix P,

$$\max\left\{\operatorname{tr}\left(A^*AP - \mathcal{C}_{\mathbb{B}}(AP)^*\,\mathcal{C}_{\mathbb{B}}(AP)\,\mathcal{C}_{\mathbb{B}}(P)^{-1}\right) : P \geq 0, \operatorname{tr} P = 1\right\}$$
$$= \max\left\{\operatorname{tr}\left(\tilde{A}^*\tilde{A}\tilde{P} - \mathcal{C}(\tilde{A}\tilde{P})^*\,\mathcal{C}(\tilde{A}\tilde{P})\,\mathcal{C}(\tilde{P})^{-1}\right) : \tilde{P} \geq 0, \operatorname{tr} \tilde{P} = 1\right\}, \tag{3.2.6}$$

where $\mathcal{C}$ is the pinching map as defined in (3.2.3). We have

$$\operatorname{tr}\left(A^*AP - \mathcal{C}_{\mathbb{B}}(AP)^*\,\mathcal{C}_{\mathbb{B}}(AP)\,\mathcal{C}_{\mathbb{B}}(P)^{-1}\right) = \operatorname{tr}\left(V^*A^*APV - V^*\,\mathcal{C}_{\mathbb{B}}(AP)^*\,\mathcal{C}_{\mathbb{B}}(AP)\,\mathcal{C}_{\mathbb{B}}(P)^{-1}\,V\right).$$

From (3.2.2), we have that for any $X \in \mathbb{M}(n)$, $V^*\mathcal{C}_{\mathbb{B}}(X)V = \mathcal{C}(V^*XV)$. Therefore the above expression is the same as

$$\operatorname{tr}\left(\tilde{A}^*\tilde{A}\tilde{P} - \mathcal{C}(\tilde{A}\tilde{P})^*\,\mathcal{C}(\tilde{A}\tilde{P})\,\mathcal{C}(\tilde{P})^{-1}\right). \tag{3.2.7}$$

So it is enough to prove (3.2.4) when $\mathbb{B}$ is a subalgebra of block diagonal matrices and $\mathcal{C}_{\mathbb{B}}$ is pinching $\mathcal{C}$. We first show that

$$\max\left\{\operatorname{tr}\left(A^*AP - \mathcal{C}(AP)^*\,\mathcal{C}(AP)\,\mathcal{C}(P)^{-1}\right) : P \geq 0, \operatorname{tr} P = 1\right\} \leq \operatorname{dist}(A, \mathbb{B})^2. \tag{3.2.8}$$

Let P be any density matrix. Then $\operatorname{tr}(A^*AP) \leq \|A\|^2$. Therefore

$$\operatorname{tr}\left(A^*AP - \mathcal{C}(AP)^*\,\mathcal{C}(AP)\,\mathcal{C}(P)^{-1}\right) \leq \|A\|^2. \tag{3.2.9}$$

Let $B \in \mathbb{B}$. Applying the translation $A \mapsto A + B$ in (3.2.9), we get

$$\operatorname{tr}\left((A+B)^*(A+B)P - \mathcal{C}((A+B)P)^*\,\mathcal{C}((A+B)P)\,\mathcal{C}(P)^{-1}\right) \leq \|A+B\|^2. \tag{3.2.10}$$

By expanding the expression on the left hand side of (3.2.10), we get that it is equal to

$$\left(\operatorname{tr}\left(A^*AP - \mathcal{C}(AP)^*\,\mathcal{C}(AP)\,\mathcal{C}(P)^{-1}\right)\right) + \left(\operatorname{tr}\left(B^*AP - \mathcal{C}(BP)^*\,\mathcal{C}(AP)\,\mathcal{C}(P)^{-1}\right)\right)$$
$$+ \left(\operatorname{tr}\left(A^*BP - \mathcal{C}(AP)^*\,\mathcal{C}(BP)\,\mathcal{C}(P)^{-1}\right)\right) + \left(\operatorname{tr}\left(B^*BP - \mathcal{C}(BP)^*\,\mathcal{C}(BP)\,\mathcal{C}(P)^{-1}\right)\right). \tag{3.2.11}$$

We show that all the terms in (3.2.11) are zero except the first one. We begin by proving that the second term

$$\operatorname{tr}\left(B^*AP - \mathcal{C}(BP)^*\,\mathcal{C}(AP)\,\mathcal{C}(P)^{-1}\right) \tag{3.2.12}$$

is zero. The proof for the other two terms is similar.

By using (3.2.2), the expression in (3.2.12) is equal to

$$\operatorname{tr}\left(B^*\,\mathcal{C}(AP)\left(I - \mathcal{C}(P)^{-1}\mathcal{C}(P)\right)\right). \tag{3.2.13}$$

Let $Q = \mathcal{C}(P)^{-1}\mathcal{C}(P)$. We show that $\mathcal{C}(AP)\,(I-Q) = 0$. First note that the matrix $\|A\|^2P^2 - PA^*AP$ is positive semidefinite. Since P is a density matrix, we have that $P - P^2$ is a positive semidefinite matrix. This implies that $\|A\|^2P - PA^*AP$ is positive semidefinite and hence $\|A\|^2\mathcal{C}(P) - \mathcal{C}(PA^*AP)$ is positive semidefinite. We also have that $\mathcal{C}(PA^*AP) - \mathcal{C}(AP)^*\mathcal{C}(AP)$ is positive semidefinite. Combining these relations, we obtain that $\|A\|^2\mathcal{C}(P) - \mathcal{C}(AP)^*\mathcal{C}(AP)$ is a positive semidefinite matrix. So

$$\operatorname{tr}(I-Q)\mathcal{C}(AP)^*\mathcal{C}(AP)(I-Q) \leq \|A\|^2\operatorname{tr}(I-Q)\mathcal{C}(P)(I-Q). \tag{3.2.14}$$

Since $\mathcal{C}(P)(I-Q) = 0$, we get by (3.2.14) that $\mathcal{C}(AP)(I-Q) = 0$.

Thus from (3.2.10), we obtain

$$\operatorname{tr}\left(A^*AP - \mathcal{C}(AP)^*\,\mathcal{C}(AP)\,\mathcal{C}(P)^{-1}\right) \leq \|(A+B)\|^2,$$

for all $B \in \mathcal{B}$ and for all density matrices P. The required inequality (3.2.8) now follows from here.

To show equality in (3.2.8), let $\operatorname{dist}(A, \mathcal{B}) = \|A_0\|$, where $A_0 = A - B_0$ for some $B_0 \in \mathcal{B}$. Then A_0 is orthogonal to $\mathcal{B}$. By Theorem 3.1.1, there exists a density matrix P such that

$$A_0^*A_0P = \|A_0\|^2P \tag{3.2.15}$$

and

$$\mathcal{C}(A_0P) = 0, \text{ that is, } \mathcal{C}(AP) = \mathcal{C}(B_0P). \tag{3.2.16}$$

From (3.2.15) we get that

$$\begin{aligned}\|A_0\|^2 &= \operatorname{tr}((A - B_0)^*(A - B_0)P)\\ &= \operatorname{tr}(A^*AP) - \operatorname{tr}(B_0^*AP) - \operatorname{tr}(A^*B_0P) + \operatorname{tr}(B_0^*B_0P).\end{aligned}$$

By using (3.2.2), we obtain

$$\|A_0\|^2 = \operatorname{tr}(A^*AP) - \operatorname{tr}(B_0^*\,\mathcal{C}(AP)) - \operatorname{tr}(B_0\,\mathcal{C}(AP)^*) + \operatorname{tr}(B_0^*\,\mathcal{C}(B_0P)). \tag{3.2.17}$$

Substituting (3.2.16) in (3.2.17) we get

$$\|A_0\|^2 = \operatorname{tr}(A^*AP) - \operatorname{tr}(B_0^*B_0\mathcal{C}(P)). \tag{3.2.18}$$

Now $\operatorname{tr}(B_0^*\,B_0\,\mathcal{C}(P)) = \operatorname{tr}\big(B_0^*B_0\,\mathcal{C}(P)\mathcal{C}(P)^{-1}\mathcal{C}(P)\big)$, which by (3.2.16), is the same as $\operatorname{tr}\big(\mathcal{C}(AP)^*\,\mathcal{C}(AP)\,\mathcal{C}(P)^{-1}\big)$. Therefore by (3.2.18), we get

$$\operatorname{dist}(A, \mathcal{B})^2 = \|A_0\|^2 = \operatorname{tr}\left(A^*AP - \mathcal{C}(AP)^*\,\mathcal{C}(AP)\,\mathcal{C}(P)^{-1}\right).$$

$\square$

Remark 3.2.2 (The special case $n = 2$)**.** We now examine Theorem 3.1.1 and Theorem 3.2.1 for the particular case when $n = 2$ and when $\mathcal{B}$ is a C^*-subalgebra of $\mathbb{M}(2)$ containing the identity matrix. We show that

$$\{A : \|A + W\| \geq \|A\| \text{ for all } W \in \mathcal{B}\} = \mathcal{B}^{\perp},$$

and that the maximum on the right hand side of (3.2.4) can be restricted to rank one density matrices. By the same argument as in the proof of Theorem 3.2.1, it is sufficient to prove these results for $\mathbb{D}(2)$, the subalgebra of diagonal matrices with complex entries. We first show that

$$\{A : \|A+W\| \geq \|A\| \text{ for all } W \in \mathbb{D}(2)\} = \mathbb{D}(2)^{\perp}.$$

If A is an off-diagonal 2×2 matrix, that is, $A = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ then by Theorem 2.1 in [11], we obtain $\|A+W\| \geq \|A\|$ for all $W \in \mathbb{D}(2)$. Conversely, let $A \in \mathbb{M}(2)$ be such that $\|A+W\| \geq \|A\|$ for all $W \in \mathbb{D}(2)$. Then by taking $W = -\mathcal{D}(A)$, we get $A + W = \mathcal{O}(A)$. Again by using Theorem 2.1 in [11], we obtain that $\|\mathcal{O}(A)\| = \|A\|$. So A is of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $\|A\| = \max\{|b|, |c|\}$. Since norm of each row and each column is less than or equal to $\|A\|$, we get that $a = d = 0$. Hence $A \in \mathbb{D}(2)^{\perp}$.

Now we show that

$$\operatorname{dist}(A, \mathbb{D}(2))^2 = \max_{\|x\|=1} \left(\|Ax\|^2 - \operatorname{tr} \Delta(Axx^*)^* \Delta(Axx^*) \Delta(xx^*)^{-1}\right), \tag{3.2.19}$$

where Δ is the orthogonal projection of $\mathbb{M}(2)$ onto $\mathbb{D}(2)$. By Theorem 3.2.1, we have

$$\max_{\|x\|=1} \left(\|Ax\|^2 - \operatorname{tr} \Delta(Axx^*)^* \Delta(Axx^*) \Delta(xx^*)^{-1}\right) \leq \operatorname{dist}(A, \mathbb{D}(2))^2.$$

To prove the other side inequality, first note that

$$\operatorname{dist}(A, \mathbb{D}(2)) \leq \|\mathcal{O}(A)\|. \tag{3.2.20}$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Without loss of generality assume that $|b| \geq |c|$. Then $\|\mathcal{O}(A)\| = |b|$. For $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$\|Ax\|^2 - \operatorname{tr} \Delta(Axx^*)^* \Delta(Axx^*) \Delta(xx^*)^{-1} = \|\mathcal{O}(A)\|^2.$$

Combining this with (3.2.20), we obtain

$$\operatorname{dist}(A, \mathbb{D}(2))^2 \leq \max_{\|x\|=1} \left(\|Ax\|^2 - \operatorname{tr} \Delta(Axx^*)^* \Delta(Axx^*) \Delta(xx^*)^{-1} \right).$$

Chapter 4

Derivatives of some multilinear functions and their norms

We now turn our attention to a different direction. In the previous chapters, our main results were obtained by computing the subdifferential of the norm functions on $\mathbb{M}(n)$. In this chapter, we study the higher order derivatives of some interesting multilinear operators and functions. The famous Jacobi formula gives the derivative of the determinant function on $\mathbb{M}(n)$ as

$$\mathrm{D}\det(A)(X) = \operatorname{tr}(\operatorname{adj}(A)X), \tag{4.0.1}$$

where the symbol $\operatorname{adj}(A)$, stands for the *adjugate* (or the *classical adjoint*) of A. Formulas for higher order derivatives of $\det$ were obtained in [14]. We obtain a formula analogous to (4.0.1) for the derivative of the permanent function on $\mathbb{M}(n)$, and then obtain expressions for its higher order derivatives.

Let $\wedge^k : \mathbb{M}(n) \to \mathbb{M}\left(\binom{n}{k}\right)$ be the map that takes a matrix A to its kth antisymmetric tensor power, $\wedge^k(A)$. The problem of evaluating higher order derivatives of the map $\wedge^k$ was studied in [31], and norms of these derivatives were obtained. We have another look at this problem thinking of $\wedge^k(A)$ as restriction of the tensor power $\otimes^k(A)$ to an invariant subspace. This has several advantages. The proofs are more transparent, the formulas are seen to be valid for infinite dimensional operators and the path to studying the same problem for other symmetry classes of tensors becomes clearer. We compute the norms of these derivatives by proving a multilinear version of a famous theorem of Russo and Dye.

4.1 A Russo-Dye Theorem for multilinear maps

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. A linear map Φ from $\mathcal{L}(\mathcal{H})$ into $\mathcal{L}(\mathcal{K})$ is said to be *positive* if $\Phi(A)$ is a positive semidefinite operator whenever A is positive semidefinite. A famous theorem of Russo and Dye [10, p.42] says that if Φ is a positive linear map, then $\|\Phi\| = \|\Phi(I)\|$.

The norm of a multilinear map Φ from $\mathcal{L}(\mathcal{H})^m$ into $\mathcal{L}(\mathcal{K})$ is defined as

$$\|\Phi\| = \sup_{\|X^1\|=\cdots=\|X^m\|=1} \|\Phi(X^1, \ldots, X^m)\|. \tag{4.1.1}$$

We say Φ is *positive* if $\Phi(X^1, \ldots, X^m)$ is a positive semidefinite operator whenever $X^1, \ldots, X^m$ are positive semidefinite. We prove:

Theorem 4.1.1. *Let Φ be a positive multilinear map. Then*

$$\|\Phi\| = \|\Phi(I, I, \ldots, I)\|. \tag{4.1.2}$$

Proof. First let $\mathcal{H}$ and $\mathcal{K}$ be finite dimensional. We imitate the proof for positive linear maps given in [10, p.41].

Let $U^1, U^2, \ldots, U^m$ be unitary matrices and let

$$U^i = \sum_{j=1}^{r_i} \lambda_j^i \; P_j^i, \quad 1 \leq i \leq m, \tag{4.1.3}$$

be their spectral resolutions. Here λ_j^i are the distinct eigenvalues of U^i, and P_j^i the corresponding eigenprojections. In particular, $|\lambda_j^i| = 1$, P_j^i are positive semidefinite, and

$$\sum_{j=1}^{r_i} P_j^i = I, \quad 1 \leq i \leq m. \tag{4.1.4}$$

Since Φ is multilinear, we have

$$\Phi(U^1, \ldots, U^m) = \sum_{j_1=1}^{r_1} \sum_{j_2=1}^{r_2} \cdots \sum_{j_m=1}^{r_m} \lambda_{j_1}^1 \lambda_{j_2}^2 \ldots \lambda_{j_m}^m \; \Phi\left(P_{j_1}^1, P_{j_2}^2, \ldots, P_{j_m}^m\right),$$

and

$$\Phi(I, \ldots, I) = \sum_{j_1=1}^{r_1} \sum_{j_2=1}^{r_2} \cdots \sum_{j_m=1}^{r_m} \Phi\left(P_{j_1}^1, P_{j_2}^2, \ldots, P_{j_m}^m\right).$$

Since Φ is positive, the operators $\Phi\left(P_{j_1}^1, P_{j_2}^2, \ldots, P_{j_m}^m\right)$ are positive semidefinite.

Using these two relations, we see that

$$\begin{aligned} &\begin{bmatrix} \Phi(I, \ldots, I) & \Phi(U^1, \ldots, U^m) \\ \Phi(U^1, \ldots, U^m)^* & \Phi(I, \ldots, I) \end{bmatrix} \\ &= \sum_{j_1=1}^{r_1} \cdots \sum_{j_m=1}^{r_m} \begin{bmatrix} 1 & \lambda_{j_1}^1 \cdots \lambda_{j_m}^m \\ \overline{\lambda_{j_1}^1 \cdots \lambda_{j_m}^m} & 1 \end{bmatrix} \otimes \Phi(P_{j_1}^1, \ldots, P_{j_m}^m). \end{aligned}$$

This is a sum of tensor products of positive semidefinite matrices, and is therefore, positive semidefinite. It follows from Proposition 1.3.2 in [10] that

$$\|\Phi(U^1, \ldots, U^m)\| \leq \|\Phi(I, \ldots, I)\|. \tag{4.1.5}$$

Now let X^i, $1 \leq i \leq m$, be matrices with $\|X^i\| = 1$. Then there exist unitary matrices U^i and V^i such that $X^i = \frac{1}{2}(U^i + V^i)$ (see [10, p.42]).

By the multilinearity of Φ

$$\Phi(X^1, \ldots, X^m) = \frac{1}{2^m} \sum \Phi(W^1, \ldots, W^m),$$

where the summation is over 2^m terms obtained by choosing each of the W^i to be either U^i or V^i, $1 \leq i \leq m$. It follows from (4.1.5) that

$$\|\Phi(X^1, \ldots, X^m)\| \leq \|\Phi(I, \ldots, I)\|.$$

Hence $\|\Phi\| = \|\Phi(I, \ldots, I)\|$. This establishes Theorem 4.1.1 when $\mathcal{H}$ and $\mathcal{K}$ are finite dimensional.

Now let $\mathcal{H}$ and $\mathcal{K}$ be infinite dimensional Hilbert spaces. Our proof invokes the well-known fact that if A and B are positive operators on a Hilbert space, then $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ is positive if and only if there exists a contraction K such that $X = A^{1/2}\ K\ B^{1/2}$. (See Theorem I.1 in [4].) To prove (4.1.2), we have to show that if $X^1, \ldots, X^m$ are operators with $\|X^i\| \leq 1$, then

$$\|\Phi(X^1, \ldots, X^m)\| \leq \|\Phi(I, \ldots, I)\|. \tag{4.1.6}$$

Consider first the case when

$$X^i = \sum_{j=1}^{r_i} \lambda_j^i\ P_j^i, \quad 1 \leq i \leq m, \tag{4.1.7}$$

where P_j^i are mutually orthogonal projection operators with $\sum\limits_{j=1}^{r_i}\ P_j^i = I$, and $|\lambda_j^i| = 1$. It can be seen that $\|X^i\| \leq 1$. (See [40, p.11].) Arguing as in the finite dimensional case, we see that the inequality (4.1.6) holds in this case. Now if U^i, $1 \leq i \leq m$, are unitary operators, then by the spectral theorem, each U^i is a limit of a sequence of operators of the form (4.1.7). This shows that the inequality (4.1.6) holds when X^i are unitary. From here one can see that the inequality continues to hold if each X^i is a convex combination of unitary operators. Finally, since the closed unit ball in $\mathcal{L}(\mathcal{H})$ is the closed convex hull of unitary operators (see [29, p.75]) the inequality is valid when X^i are any operators with $\|X^i\| \leq 1$. $\square$

4.2 Formulas for $\mathrm{D}^m \otimes^k (A),\ \mathrm{D}^m \wedge^k (A),\ \mathrm{D}^m \vee^k (A)$ and their norms

Let $\otimes^k \mathcal{H}$ be the k-fold tensor power $\mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$. Let $\wedge^k \mathcal{H}$ and $\vee^k \mathcal{H}$ be the subspaces of $\otimes^k \mathcal{H}$ consisting of antisymmetric tensors and symmetric tensors, respectively. If $\dim \mathcal{H} = n$, then $\dim \wedge^k \mathcal{H} = \binom{n}{k}$ for $1 \leq k \leq n$, and $\dim \vee^k \mathcal{H} = \binom{n+k-1}{k}$ for $k \geq 1$. For $k > n$, the space $\wedge^k \mathcal{H}$ is taken to be zero. For every A in $\mathcal{L}(\mathcal{H})$, we denote by $\otimes^k A$ its k-fold tensor power

$A \otimes A \otimes \cdots \otimes A$. This is an operator on $\otimes^k \mathcal{H}$ that leaves invariant the subspaces $\wedge^k \mathcal{H}$ and $\vee^k \mathcal{H}$. The restriction of $\otimes^k A$ to these subspaces are denoted by $\wedge^k A$, the kth antisymmetric tensor power of A and $\vee^k A$, the kth symmetric tensor power of A, respectively.

Bhatia and Friedland [12] studied the problem of finding the norm of $\mathrm{D} \wedge^k (A)$. They showed that

$$\|\mathrm{D} \wedge^k (A)\| = p_{k-1}(s_1(A), \ldots, s_k(A)), \tag{4.2.1}$$

where $p_{k-1}(s_1(A), \ldots, s_k(A))$ denotes the $(k-1)$th elementary symmetric polynomial in $s_1(A), \ldots, s_k(A)$.

Later, Bhatia [8] proved that

$$\|\mathrm{D} \vee^k (A)\| = k\|A\|^{k-1}. \tag{4.2.2}$$

We describe the mth derivatives of the maps $A \mapsto \otimes^k A$, $A \mapsto \wedge^k A$, and $A \mapsto \vee^k A$, from $\mathcal{L}(\mathcal{H})$ into $\mathcal{L}(\otimes^k \mathcal{H}),\ \mathcal{L}(\wedge^k \mathcal{H})$, and $\mathcal{L}(\vee^k \mathcal{H})$, respectively, and use them to compute their norms.

Given $A^1, \ldots, A^k$ in $\mathcal{L}(\mathcal{H})$, we define their *symmetrised tensor product* as

$$A^1 \widetilde{\otimes} A^2 \widetilde{\otimes} \cdots \widetilde{\otimes} A^k = \frac{1}{k!} \sum_{\sigma \in S_k} A^{\sigma(1)} \otimes A^{\sigma(2)} \otimes \cdots \otimes A^{\sigma(k)}, \tag{4.2.3}$$

where S_k is the set of all permutations on $\{1, \ldots, k\}$. The operator (4.2.3) on $\otimes^k \mathcal{H}$ leaves invariant the subspaces $\wedge^k \mathcal{H}$ and $\vee^k \mathcal{H}$. The restriction of the symmetrised tensor product to $\wedge^k \mathcal{H}$ and $\vee^k \mathcal{H}$ will be denoted by

$$A^1 \wedge A^2 \wedge \cdots \wedge A^k \text{ and } A^1 \vee A^2 \vee \cdots \vee A^k, \tag{4.2.4}$$

respectively and called the *symmetrised antisymmetric tensor product* and the *symmetrised symmetric tensor product* of $A^1, A^2, \ldots, A^k$. The operator $A^1 \wedge A^2 \wedge \cdots \wedge A^k$ acts on product vectors $u_1 \wedge u_2 \wedge \cdots \wedge u_k$ as

$$(A^1 \wedge A^2 \wedge \cdots \wedge A^k)(u_1 \wedge u_2 \wedge \cdots \wedge u_k)$$

$$= \frac{1}{k!}\sum_{\sigma\in S_k}(A^{\sigma(1)}u_1)\wedge(A^{\sigma(2)}u_2)\wedge\cdots\wedge(A^{\sigma(k)}u_k). \tag{4.2.5}$$

Similarly the operator $A^1\vee A^2\vee\cdots\vee A^k$ acts on $u_1\vee u_2\vee\cdots u_k$ as

$$\begin{aligned}&(A^1\vee A^2\vee\cdots\vee A^k)(u_1\vee u_2\vee\cdots\vee u_k)\\ &= \frac{1}{k!}\sum_{\sigma\in S_k}(A^{\sigma(1)}u_1)\vee(A^{\sigma(2)}u_2)\vee\cdots\vee(A^{\sigma(k)}u_k).\end{aligned} \tag{4.2.6}$$

With the above notations we have:

Theorem 4.2.1. *Let $1\le m\le k$. The mth derivatives of the maps $\otimes^k, \wedge^k$ and $\vee^k$ are given by the formulas*

$$\mathrm{D}^m\otimes^k(A)(X^1,\ldots,X^m)=\frac{k!}{(k-m)!}\underbrace{A\widetilde{\otimes}\cdots\widetilde{\otimes}A}_{k-m\ copies}\widetilde{\otimes}X^1\widetilde{\otimes}X^2\widetilde{\otimes}\cdots\widetilde{\otimes}X^m, \tag{4.2.7}$$

$$\mathrm{D}^m\wedge^k(A)(X^1,\ldots,X^m)=\frac{k!}{(k-m)!}\underbrace{A\wedge\cdots\wedge A}_{k-m\ copies}\wedge X^1\wedge X^2\wedge\cdots\wedge X^m, \tag{4.2.8}$$

and

$$\mathrm{D}^m\vee^k(A)(X^1,\ldots,X^m)=\frac{k!}{(k-m)!}\underbrace{A\vee\cdots\vee A}_{k-m\ copies}\vee X^1\vee X^2\vee\cdots\vee X^m. \tag{4.2.9}$$

If $m>k$, then all the derivatives are zero.

Proof. By the formula (1.1.8),

$$\begin{aligned}&\mathrm{D}^m\otimes^k(A)(X^1,\ldots,X^m)\\ &= \left.\frac{\partial^m}{\partial t_1\cdots\partial t_m}\right|_{t_1=\cdots=t_m=0}\otimes^k(A+t_1X^1+\cdots+t_mX^m).\end{aligned} \tag{4.2.10}$$

To evaluate this we expand the k-fold tensor product on the right hand side. The resulting expansion is a polynomial in the variables $t_1,\ldots,t_m$. The derivative in (4.2.10) is evidently the coefficient of the term $t_1t_2\cdots t_m$ in this polynomial. One can check that this is given by the expression (4.2.7).

Next we prove (4.2.8) using (4.2.7). The proof for (4.2.9) is similar. The chain rule of differentiation for a composite function is given by Proposition 1.1.3. If L is a real linear map, then its derivative is equal to L, and in this case

$$\mathrm{D}(L \circ f)(a)(x) = L(\mathrm{D}f(a)(x)).$$

Repeating this argument one sees that if f is m times differentiable, then

$$\mathrm{D}^m(L \circ f)(a)(x^1, \ldots, x^m) = L(\mathrm{D}^m f(a)(x^1, \ldots, x^m)). \tag{4.2.11}$$

Now let $Q_k : \wedge^k\mathcal{H} \to \otimes^k\mathcal{H}$ be the inclusion map. Then $Q_k^* : \otimes^k\mathcal{H} \to \wedge^k\mathcal{H}$ is the projection given by

$$Q_k^*(x_1 \otimes \cdots \otimes x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \varepsilon_\sigma x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)},$$

where $\varepsilon_\sigma = \pm 1$, depending on whether σ is an even or an odd permutation. Define $\tilde{Q}_k : \mathcal{L}(\otimes^k\mathcal{H}) \to \mathcal{L}(\wedge^k\mathcal{H})$ by

$$\tilde{Q}_k(T) = Q_k^* T Q_k. \tag{4.2.12}$$

Also $\wedge^k : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\wedge^k\mathcal{H})$ factors through $\otimes^k : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\otimes^k\mathcal{H})$ via $\tilde{Q}_k$ as

$$\wedge^k A = \tilde{Q}_k(\otimes^k A) \quad \forall\ A \in \mathcal{L}(\mathcal{H}).$$

Since $\tilde{Q}_k$ is linear, by (4.2.11) we have

$$\mathrm{D}^m \wedge^k (A) = \tilde{Q}_k \circ \mathrm{D}^m \otimes^k (A). \tag{4.2.13}$$

Using this we obtain the expression (4.2.8).

From (4.2.7) we see that

$$\mathrm{D}^k \otimes^k (A)(X^1, \ldots, X^k) = k!\ X^1 \widetilde{\otimes} X^2 \widetilde{\otimes} \cdots \widetilde{\otimes} X^k.$$

This expression does not involve A. Hence $\mathrm{D}^m \otimes^k (A) = 0$ if $m > k$. Similarly $\mathrm{D}^m \wedge^k (A) = 0$ and $\mathrm{D}^m \vee^k (A) = 0$ if $m > k$. $\square$

Remark 4.2.2. By putting $k = n$ in (4.2.8), we get

$$\mathrm{D}^m \det(A)(X^1, \ldots, X^m) = \frac{n!}{(n-m)!} \underbrace{A \wedge \cdots \wedge A}_{n-m \text{ copies}} \wedge X^1 \wedge X^2 \wedge \cdots \wedge X^m. \tag{4.2.14}$$

Let $T_{[i]}$ denote the ith column of the matrix T. Then the mixed discriminant of n matrices $T^1, \ldots, T^n$ is defined as

$$\mathcal{D}(T^1, \ldots, T^n) = \frac{1}{n!} \sum_{\sigma \in S_n} \det[T^{\sigma(1)}_{[1]}, \ldots, T^{\sigma(n)}_{[n]}].$$

Equation (4.2.14) can also be written as

$$\mathrm{D}^m \det(A)(X^1, \ldots, X^m) = \frac{n!}{(n-m)!} \mathcal{D}(A, \ldots, A, X^1, \ldots, X^m).$$

This is the same as Theorem 1 in [14].

From these formulas we obtain the values of the norms of these derivatives. We separate the cases of $\|\mathrm{D}^m \otimes^k (A)\|$ and $\|\mathrm{D}^m \vee^k (A)\|$. The evaluation of these norms is independent of Theorem 4.1.1, whereas we make essential use of this theorem in calculating $\|\mathrm{D}^m \wedge^k (A)\|$.

Theorem 4.2.3. *For $1 \leq m \leq k$, we have*

$$\|\mathrm{D}^m \otimes^k (A)\| = \frac{k!}{(k-m)!} \, \|A\|^{k-m} \tag{4.2.15}$$

and

$$\|\mathrm{D}^m \vee^k (A)\| = \frac{k!}{(k-m)!} \, \|A\|^{k-m}. \tag{4.2.16}$$

Proof. To compute the norm $\|\mathrm{D}^m \otimes^k (A)\|$, we first see that by the definition of the symmetrised tensor product (4.2.3) and by the triangle inequality, we get

$$\|\underbrace{A\widetilde{\otimes}\cdots\widetilde{\otimes}A}_{k-m\ copies}\widetilde{\otimes}X^1\widetilde{\otimes}X^2\widetilde{\otimes}\cdots\widetilde{\otimes}X^m\| \leq \frac{1}{k!}\sum_{\sigma\in S_k}\|Y^{\sigma(1)}\otimes Y^{\sigma(2)}\otimes\cdots\otimes Y^{\sigma(k)}\|,$$

where $k-m$ of the Y's are equal to A and the rest are $X^1, X^2, \ldots, X^m$. Each of the terms in the summation is equal to $\|A\|^{k-m}\|X^1\|\|X^2\|\cdots\|X^m\|$. By the definition of the norm of a multilinear map (4.1.1), we obtain

$$\|\mathrm{D}^m \otimes^k (A)\| \leq \frac{k!}{(k-m)!}\|A\|^{k-m}.$$

Also note that

$$\left\|\mathrm{D}^m \otimes^k (A)\left(\frac{A}{\|A\|}, \frac{A}{\|A\|}, \ldots, \frac{A}{\|A\|}\right)\right\| = \frac{k!}{(k-m)!}\|A\|^{k-m}.$$

This shows that

$$\|\mathrm{D}^m \otimes^k (A)\| \geq \frac{k!}{(k-m)!}\|A\|^{k-m}.$$

Hence we obtain (4.2.15). This argument works equally well in infinite dimensions.

Let $R_k : \vee^k\mathcal{H} \to \otimes^k\mathcal{H}$ be the inclusion map. Define $\tilde{R}_k : \mathcal{L}(\otimes^k\mathcal{H}) \to \mathcal{L}(\vee^k\mathcal{H})$ by

$$\tilde{R}_k(T) = R_k^* T R_k. \tag{4.2.17}$$

Then $\|\tilde{R}_k\| \leq 1$. Arguments similar to those in the proof of Theorem 4.2.1 lead to an expression similar to (4.2.13):

$$\mathrm{D}^m \vee^k (A) = \tilde{R}_k \circ \mathrm{D}^m \otimes^k (A).$$

It follows that

$$\|\mathrm{D}^m \vee^k (A)\| \leq \|\tilde{R}_k\|\|\mathrm{D}^m \otimes^k (A)\| \leq \frac{k!}{(k-m)!}\|A\|^{k-m}. \tag{4.2.18}$$

Let us now consider the case when $\mathcal{H}$ is an n dimensional space. Let $A = U|A|$ be a polar decomposition of A. Since $UU^* = I$, we have

$$\begin{aligned} &\vee^k(A + t_1X^1 + t_2X^2 + \cdots + t_mX^m) \\ &= \vee^k \left(U(|A| + t_1U^*X^1 + t_2U^*X^2 + \cdots + t_mU^*X^m)\right) \\ &= (\vee^k U)\ \vee^k \left(|A| + t_1U^*X^1 + t_2U^*X^2 + \cdots + t_mU^*X^m\right). \end{aligned}$$

So from (1.1.8) we obtain

$$\begin{aligned} &\mathrm{D}^m \vee^k (A)(X^1, \ldots, X^m) \\ &= (\vee^k U)\, \mathrm{D}^m \vee^k (|A|)(U^*X^1, \ldots, U^*X^m). \end{aligned} \tag{4.2.19}$$

Now $\vee^k U$ is unitary and the norm is unitarily invariant. So we have

$$\begin{aligned} &\|\mathrm{D}^m \vee^k (A)(X^1, \ldots, X^m)\| \\ &= \|\mathrm{D}^m \vee^k (|A|)(U^*X^1, \ldots, U^*X^m)\|. \end{aligned}$$

The condition $\|X^j\| = 1$ is equivalent to $\|U^*X^j\| = 1$ for $1 \leq j \leq m$. So we have proved that

$$\|\mathrm{D}^m \vee^k (A)\| = \|\mathrm{D}^m \vee^k (|A|)\|. \tag{4.2.20}$$

Now assume A is positive semidefinite and let u be an eigenvector corresponding to its maximal eigenvalue $\|A\|$. Consider the vector $w = u \vee u \vee \cdots \vee u$ in $\vee^k \mathcal{H}$. If $T = Y^1 \vee Y^2 \vee \cdots \vee Y^k$ is an operator in which $k - m$ of the Y's are equal to A and the rest of them are equal to I, then $Tw = \|A\|^{k-m}w$. It then follows from (4.2.9) that

$$(\mathrm{D}^m \vee^k (A)(I, \ldots, I))w = \frac{k!}{(k-m)!}\ \|A\|^{k-m}w.$$

This shows that

$$\|\mathrm{D}^m \vee^k (A)\| \geq \frac{k!}{(k-m)!}\|A\|^{k-m}.$$

We have already noted the reverse inequality in (4.2.18). So we have (4.2.16) in the case when A is positive semidefinite. The relation (4.2.20) then shows that (4.2.16) is valid for all A.

We now indicate the modifications needed in this proof to handle the infinite dimensional case. In this case A has a *maximal polar representation* $A = U|A|$ in which U is either an isometry ($U^*U = I$) or a coisometry ($UU^* = I$) ([29, p.75]). When $\mathcal{H}$ is finite dimensional these two conditions are equivalent and U is unitary. Our argument using the polar decomposition for proving (4.2.20) can be modified. A very similar idea is used in [44] and we refer the reader to that paper for details.

To prove (4.2.16) in the infinite dimensional case we may, therefore, again assume that A is a positive operator. If A has pure point spectrum, then the arguments given for the finite dimensional case serve equally well here. In particular, (4.2.16) is valid for compact operators. Every positive operator is a limit of a sequence of positive operators with pure point spectrum. Using this fact one can see that (4.2.16) is valid for all operators. $\square$

Note that for the above proof no use of Theorem 4.1.1 has been made. The formula (4.2.21) for $\|\mathrm{D}^m \wedge^k (A)\|$ is more interesting, and to prove it we do need to invoke Theorem 4.1.1.

Theorem 4.2.4. *We have*

$$\|\mathrm{D}^m \wedge^k (A)\| = m!\, p_{k-m}(s_1(A), \ldots, s_k(A)), \tag{4.2.21}$$

where p_{k-m} is the $(k-m)$th elementary symmetric polynomial.

Proof. To compute $\|\mathrm{D}^m \wedge^k (A)\|$ we first note that the symmetrised antisymmetric product of positive semidefinite operators is positive semidefinite. It follows from (4.2.8) that if A is positive semidefinite, then the map $\mathrm{D}^m \wedge^k (A)$ from $\mathcal{L}(\mathcal{H})^m$ into $\mathcal{L}(\wedge^k\mathcal{H})$ is a positive multilinear map. So, we have from Theorem 4.1.1

$$\|\mathrm{D}^m \wedge^k (A)\| = \|\mathrm{D}^m \wedge^k (A)(I, \ldots, I)\|. \tag{4.2.22}$$

Arguments similar to the ones used in the proof of Theorem 4.2.3 show that

$$\|\mathrm{D}^m \wedge^k (A)\| = \|\mathrm{D}^m \wedge^k (|A|)\|. \tag{4.2.23}$$

So we assume A to be positive semidefinite. By (4.2.22), we have

$$\begin{aligned} \|\mathrm{D}^m \wedge^k (A)\| &= \|\mathrm{D}^m \wedge^k (A)(I, \ldots, I)\| \\ &= \left\| \left.\frac{\partial^m}{\partial t_1 \cdots \partial t_m}\right|_{t_1=\cdots=t_m=0} \wedge^k (A + t_1 I + \cdots + t_m I) \right\|. \end{aligned}$$

By the spectral theorem there exists a unitary W such that $A = WDW^*$, where D is the diagonal matrix whose diagonal entries are $\alpha_1 \geq \cdots \geq \alpha_n (\geq 0)$, the eigenvalues of A. The matrix $\wedge^k W$ is again unitary, and our norm is unitarily invariant. So in the right hand side of the equation above we can replace A by D. Now

$$\left.\frac{\partial^m}{\partial t_1 \cdots \partial t_m}\right|_{t_1=\cdots=t_m=0} \wedge^k (D + t_1 I + \cdots + t_m I)$$

is a diagonal matrix of order $\begin{pmatrix} n \\ k \end{pmatrix}$. Its norm is equal to its top diagonal entry, which is

$$\left.\frac{\partial^m}{\partial t_1 \cdots \partial t_m}\right|_{t_1=\cdots=t_m=0} \prod_{j=1}^{k} (\alpha_j + t_1 + \cdots + t_m).$$

A calculation shows that this is equal to

$$m!\, p_{k-m}\, (\alpha_1, \ldots, \alpha_k).$$

This establishes (4.2.21) in the case when A is positive semidefinite. The general case follows from (4.2.23).

Theorem 4.2.4 can be modified for infinite dimensional operators. The statement of this theorem involves the sequence $s_1(A) \geq s_2(A) \geq \cdots$. If we stretch the definitions and interpret a point of the essential spectrum of $|A|$ as an eigenvalue of infinite multiplicity, then Theorem 4.2.4 is valid for infinite dimensional operators too. The proof is similar to the proof for symmetric tensor powers. $\square$

4.3 Formulas for $\mathrm{D}^m \operatorname{per}(A)$

The *permanent* of $A = (a_{ij}) \in \mathbb{M}(n)$, written as $\operatorname{per} A$, is defined by

$$\operatorname{per} A = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}. \tag{4.3.1}$$

Since the definitions of per and $\det$ are similar, it is natural to expect a formula for $\mathrm{D}\operatorname{per}(A)$ similar to the Jacobi formula (4.0.1). By (1.1.2), we see that $\mathrm{D}\operatorname{per}(A)(X)$ is the coefficient of t in the polynomial $\operatorname{per}(A + tX)$. For $1 \leq j \leq n$, let $A(j; X)$ be the matrix obtained from A by replacing the j^{th} column of A by the j^{th} column of X and keeping the rest of the columns unchanged. Since per is a linear function in each of the columns, we get

$$\mathrm{D}\operatorname{per}(A)(X) = \sum_{j=1}^{n} \operatorname{per} A(j; X). \tag{4.3.2}$$

To give a formula analogous to the Jacobi formula, define the *permanental adjoint* of A as the $n \times n$ matrix whose (i, j)-entry is $\operatorname{per} A(i|j)$ (see [38, p.237]). Note that the adjugate of A, $\operatorname{adj} A$, is defined as the transpose of the matrix whose (i, j)-entry is $(-1)^{i+j} \det A(i|j)$, whereas in the definition of padj, the transpose is not taken. This is just a matter of convention. The expression (4.3.2) can be rewritten as follows.

Theorem 4.3.1. *For each* $X \in \mathbb{M}(n)$*,*

$$\mathrm{D}\operatorname{per}(A)(X) = \operatorname{tr}(\operatorname{padj}(A)^t X). \tag{4.3.3}$$

Our next aim is to obtain higher order derivatives of the permanent function. The expressions obtained are analogous to the ones for the $\det$ function given in [14]. Applying (1.1.8) to the per function, we see that $\mathrm{D}^m \operatorname{per} A(X^1, \ldots, X^m)$ is the coefficient of $t_1 \cdots t_m$ in the expansion of $\operatorname{per}(A + t_1X^1 + \cdots + t_mX^m)$. To write an explicit expression for this, we require some notations.

Let $Q_{m,n} = \{(i_1, \ldots, i_m) |\, i_1, \ldots, i_m \in \mathbb{N}, 1 \leq i_1 < \cdots < i_m \leq n\}$. For $m > n$, $Q_{m,n} = \varnothing$ by convention. Let $G_{m,n} = \{(i_1, \ldots, i_m) |\, i_1, \ldots, i_m \in \mathbb{N}, 1 \leq i_1 \leq \cdots \leq i_m \leq n\}$. Note that for $m \leq n$, $Q_{m,n}$ is a subset of $G_{m,n}$. For $\mathcal{J} = (j_1, \ldots, j_m) \in Q_{m,n}$, we denote by $A(\mathcal{J}; X^1, \ldots, X^m)$, the matrix obtained from A by replacing the j_p^{th} column of A by the j_p^{th} column of X^p for $1 \leq p \leq m$, and keeping the rest of the columns unchanged. Expanding $\operatorname{per}(A + t_1X^1 + \cdots + t_mX^m)$ by using the fact that per is a linear function in each of the columns, we obtain an expression for $\mathrm{D}^m \operatorname{per} A(X^1, \ldots, X^m)$. This is a generalisation of (4.3.2).

Theorem 4.3.2. *For* $1 \leq m \leq n$,

$$\mathrm{D}^m \operatorname{per}(A)(X^1, \ldots, X^m) = \sum_{\sigma \in S_m} \sum_{\mathcal{J} \in Q_{m,n}} \operatorname{per} A(\mathcal{J}; X^{\sigma(1)}, X^{\sigma(2)}, \ldots, X^{\sigma(m)}). \tag{4.3.4}$$

In particular,

$$\mathrm{D}^m \operatorname{per}(A)(X, \ldots, X) = m! \sum_{\mathcal{J} \in Q_{m,n}} \operatorname{per} A(\mathcal{J}; X, \ldots, X).$$

The *Laplace expansion theorem* for permanents [39, p.16] says that for any $1 \leq m \leq n$, and for any $\mathcal{I} \in Q_{m,n}$,

$$\operatorname{per} A = \sum_{\mathcal{J} \in Q_{m,n}} \operatorname{per} A[\mathcal{I}|\mathcal{J}] \operatorname{per} A(\mathcal{I}|\mathcal{J}), \tag{4.3.5}$$

where $A[\mathcal{I}|\mathcal{J}]$ denotes the $m \times m$ submatrix obtained from A by picking rows $\mathcal{I}$ and columns $\mathcal{J}$ and $A(\mathcal{I}|\mathcal{J})$ denotes the $(n-m) \times (n-m)$ submatrix obtained from A by deleting rows $\mathcal{I}$ and columns $\mathcal{J}$. In particular, for any $i, 1 \leq i \leq n$,

$$\operatorname{per} A = \sum_{j=1}^{n} a_{ij} \operatorname{per} A(i|j). \tag{4.3.6}$$

Using this, equation (4.3.2) can be rewritten as

$$\mathrm{D} \operatorname{per}(A)(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \operatorname{per} A(i|j). \tag{4.3.7}$$

We obtain a generalisation of this expression for higher order derivatives. Let σ be a permutation on $\{1, \ldots, m\}$, then by $Y^{\sigma}_{[\mathcal{J}]}$, we mean the matrix in which $Y^{\sigma}_{[j_p]} = X^{\sigma(p)}_{[j_p]}$ for $1 \leq p \leq m$ and $Y^{\sigma}_{[\ell]} = 0$ if ℓ does not occur in $\mathcal{J}$. By using the Laplace expansion (4.3.5) for each term in the summation of (4.3.4), we obtain the following expression for $\mathrm{D}^m \operatorname{per}(A)(X^1, \ldots, X^m)$.

Theorem 4.3.3. *For* $1 \leq m \leq n$,

$$\mathrm{D}^m \operatorname{per}(A)(X^1, \ldots, X^m) = \sum_{\sigma \in S_m} \sum_{\mathcal{I}, \mathcal{J} \in Q_{m,n}} \operatorname{per} A(\mathcal{I}|\mathcal{J}) \operatorname{per} Y^{\sigma}_{[\mathcal{J}]}[\mathcal{I}|\mathcal{J}]. \tag{4.3.8}$$

In particular,

$$\mathrm{D}^m \operatorname{per}(A)(X, \ldots, X) = m! \sum_{\mathcal{I}, \mathcal{J} \in Q_{m,n}} \operatorname{per} A(\mathcal{I}|\mathcal{J}) \operatorname{per} X[\mathcal{I}|\mathcal{J}].$$

Note that

$$\mathrm{D}^n \operatorname{per}(A)(X, \ldots, X) = n! \ \operatorname{per} X, \tag{4.3.9}$$

and

$$\mathrm{D}^m \operatorname{per}(A)(X^1, \ldots, X^m) = 0 \text{ for all } m > n. \tag{4.3.10}$$

We now describe a generalisation of (4.3.3) for higher order derivatives of the per function. Given an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of an n dimensional Hibert space $\mathcal{H}$, a useful orthonormal basis for the space $\vee^m \mathcal{H}$ is constructed as follows. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in G_{m,n}$, define

$$e_{(\alpha)} = e_{\alpha_1} \vee \cdots \vee e_{\alpha_m}.$$

If α consists of ℓ distinct indices $\alpha_1, \ldots, \alpha_\ell$ with multiplicities $m_1, \ldots, m_\ell$ respectively, put $m(\alpha) = m_1! \cdots m_\ell!$. Then the set $\{m(\alpha)^{-1/2}\ e_\alpha : \alpha \in G_{m,n}\}$ is an orthonormal basis of

$\vee^m\mathcal{H}$. (See [9, p.17] for details.) Let P_m be the canonical projection of $\vee^m\mathcal{H}$ onto the subspace spanned by $\{e_\alpha : \alpha \in Q_{m,n}\}$. Then there is a permutation of the above orthonormal basis of $\vee^m\mathcal{H}$ in which $P_m = \begin{bmatrix} I & O \\ O & O \end{bmatrix}$ and the matrix T_m, defined by $T_m = (\text{per}\, A[\alpha|\beta])_{\alpha,\beta\in Q_{m,n}}$, is the upper left corner of $\vee^m A$, that is,

$$P_m\,(\vee^m A)\,P_m = \begin{bmatrix} T_m & O \\ O & O \end{bmatrix}.$$

Let U be the $\binom{n}{m}\times\binom{n}{m}$ unitary matrix given by $U = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix}$. For $\alpha, \beta \in Q_{n-m,n}$, the (α,β)-entry of $U^* T_m U$ is per $A(\alpha|\beta)$. Let $\widetilde{U}$ be the $\binom{n+m-1}{m} \times \binom{n+m-1}{m}$ matrix given by $\widetilde{U} = \begin{bmatrix} U & O \\ O & I \end{bmatrix}$. Let $\widetilde{\vee}^m A$ denotes the matrix $\widetilde{U}^*\,(\vee^m A)^t\,\widetilde{U}$. Then

$$P_m\left(\widetilde{\vee}^m A\right)P_m = \begin{bmatrix} U^* T_m^t U & O \\ O & O \end{bmatrix}. \tag{4.3.11}$$

In particular for $m = n-1$,

$$P_{n-1}\left(\widetilde{\vee}^{n-1} A\right)P_{n-1} = \begin{bmatrix} (\text{padj}\, A)^t & O \\ O & O \end{bmatrix}.$$

Identifying an $n\times n$ matrix X with $\binom{2n-2}{n-1}\times\binom{2n-2}{n-1}$ matrix $\begin{bmatrix} X & O \\ O & O \end{bmatrix}$, equation (4.3.3) can be rewritten as

$$\mathrm{D}\,\text{per}\,(A)(X) = \text{tr}\left(P_{n-1}\left(\widetilde{\vee}^{n-1} A\right)P_{n-1}\right)X. \tag{4.3.12}$$

Its generalisation for higher order derivatives is given as follows.

Theorem 4.3.4. *For* $1 \leq m \leq n$,

$$\mathrm{D}^m \operatorname{per}(A)(X^1, \ldots, X^m) = m! \operatorname{tr}\left[\left(P_{n-m}\left(\widetilde{\vee}^{n-m} A\right) P_{n-m}\right)\left(P_m\left(X^1 \vee \cdots \vee X^m\right) P_m\right)\right]. \tag{4.3.13}$$

In particular,

$$\mathrm{D}^m \operatorname{per}(A)(X, \ldots, X) = m! \operatorname{tr}\left[\left(P_{n-m}(\widetilde{\vee}^{n-m} A) P_{n-m}\right)\left(P_m\left(\vee^m X\right) P_m\right)\right].$$

Proof. To prove (4.3.13), we first describe the notion of the *mixed permanent* of $m \times m$ matrices $T^1, \ldots, T^m$. (This was first introduced by Bapat in [7].) It is denoted by $\Delta_p(T^1, \ldots, T^m)$, and is defined as

$$\Delta_p(T^1, \ldots, T^m) = \frac{1}{m!} \sum_{\sigma \in S_m} \operatorname{per}\left[T_{[1]}^{\sigma(1)}, \ldots, T_{[m]}^{\sigma(m)}\right].$$

When all $T^j = T$, then $\Delta_p(T, \ldots, T) = \operatorname{per} T$. Observe that for $\mathcal{I}, \mathcal{J} \in Q_{m,n}$,

$$\sum_{\sigma \in S_m} \operatorname{per} Y_{[\mathcal{J}]}^{\sigma}[\mathcal{I}|\mathcal{J}] = m! \, \Delta_p(X^1[\mathcal{I}|\mathcal{J}], \ldots, X^m[\mathcal{I}|\mathcal{J}]). \tag{4.3.14}$$

Using this, Theorem 4.3.3 can be rewritten as follows:

$$\begin{aligned} & \mathrm{D}^m \operatorname{per}(A)(X^1, \ldots, X^m) \\ = \;& m! \sum_{\mathcal{I}, \mathcal{J} \in Q_{m,n}} \operatorname{per} A(\mathcal{I}|\mathcal{J}) \, \Delta_p(X^1[\mathcal{I}|\mathcal{J}], \ldots, X^m[\mathcal{I}|\mathcal{J}]). \end{aligned} \tag{4.3.15}$$

Next we note that for $\mathcal{I}, \mathcal{J} \in G_{m,n}$, the $(\mathcal{I}, \mathcal{J})$-entry of $X^1 \vee \cdots \vee X^m$ is

$$(m(\mathcal{I}) m(\mathcal{J}))^{-1/2} \Delta_p(X^1[\mathcal{I}|\mathcal{J}], \ldots, X^m[\mathcal{I}|\mathcal{J}]). \tag{4.3.16}$$

In particular, if $\mathcal{I}, \mathcal{J} \in Q_{m,n}$, then the $(\mathcal{I}, \mathcal{J})$-entry of $X^1 \vee \cdots \vee X^m$ is $\Delta_p(X^1[\mathcal{I}|\mathcal{J}], \ldots, X^m[\mathcal{I}|\mathcal{J}])$. The $(\mathcal{J}, \mathcal{I})$-entry of $P_{n-m}(\widetilde{\vee}^{n-m} A) P_{n-m}$ is $\operatorname{per} A(\mathcal{I}|\mathcal{J})$. The expression (4.3.13) can now be easily seen as a reformulation of (4.3.15). □

Remark 4.3.5. An alternative proof of Theorem 4.3.2 can be given using (4.2.9). We know that

$\mathrm{D}^m \operatorname{per}(A)(X^1, \ldots, X^m)$ is the (α, α)-entry of $\mathrm{D}^m \vee^n A(X^1, \ldots, X^m)$ for $\alpha = (1, \ldots, n)$, which by (4.2.9) and (4.3.16) is $\frac{n!}{(n-m)!}\Delta_p(X^1, \ldots, X^m, A, \ldots, A)$. This is the same as

$$\frac{1}{(n-m)!} \sum_{\tau \in S_n} \operatorname{per}\left[Y_{[1]}^{\tau(1)}, \ldots, Y_{[n]}^{\tau(n)}\right],$$

where $k - m$ of the Y's are equal to A and the rest are $X^1, X^2, \ldots, X^m$. For any given $\mathcal{J} \in Q_{m,n}$ and $\sigma \in S_m$, there are $(n-m)!$ terms in this summation which are equal to $\operatorname{per} A(\mathcal{J}; X^{\sigma(1)}, X^{\sigma(2)}, \ldots, X^{\sigma(m)})$. This gives Theorem 4.3.2.

Remark 4.3.6. An upper bound for norms of the derivatives of the permanent function can be obtained by using (4.2.16). By using the fact that $\mathrm{D}^m \operatorname{per}(A)(X^1, \ldots, X^m)$ is one of the entries of the matrix $\mathrm{D}^m \vee^n A(X^1, \ldots, X^m)$, we obtain

$$\|\mathrm{D}^m \operatorname{per}(A)\| \leq \frac{n!}{(n-m)!}\|A\|^{n-m}. \tag{4.3.17}$$

While we have equality in (4.2.16), we may have strict inequality here. For example, let $A = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}$. Then A is a positive semidefinite matrix. So $\mathrm{D} \operatorname{per}(A)$ is a positive linear functional. By the Russo-Dye Theorem, we have

$$\|\mathrm{D} \operatorname{per}(A)\| = |\mathrm{D} \operatorname{per}(A)(I)|, \tag{4.3.18}$$

which is equal to 1, by (4.3.7). But the right hand side of (4.3.17) is equal to 2.

Remark 4.3.7. We have limited ourselves to tensor powers, symmetric tensor powers and antisymmetric tensor powers. There are other symmetry classes of tensors, and the corresponding problems for these classes have been studied by Carvalho and Freitas in [21] and [22]. These results can also be found in the doctoral dissertation of Carvalho [20]. Norms of first order derivatives of the operators induced on the symmetry classes of tensors had been computed earlier by Bhatia and Da Silva [17]. The work in [22] extends this to higher order derivatives. The work of Carvalho and Frietas was done simultaneously with ours and supplements our work in this Chapter.

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