Bures Distance For Completely Positive Maps

And

CP-H-Extendable Maps Between Hilbert C^*-modules

Thesis

by

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Indian Statistical Institute Bangalore Centre

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BY

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Dedicated

to

my beloved Parents

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Preface

Completely positive (CP-) maps are special kinds of positivity preserving maps on C^*-algebras. W.F. Stinespring [Sti55] obtained a structure theorem for CP-maps showing that they are closely connected with $*$-homomorphisms. W. Arveson and other operator algebraists quickly realized the importance of these maps. Presently the role of the theory of CP-maps in our understanding of C^*-algebras and von Neumann algebras is well recognised. It has been argued by physicists that CP-maps are physically more meaningful than just positive maps due to their stability under ampliations. From quantum probabilistic point of view CP-maps are quantum analogues of stochastic or sub-stochastic transition probability maps. Therefore one begins with such maps in order to construct quantum Markov processes. Recently there has been lot of interest in quantum computation and quantum information theory and here trace preserving, unital CP-maps play the role of quantum channels. This justifies detailed study of CP-maps and related concepts.

Often it is the structure theorems that makes a theory worth studying. GNS-theorem and Stinespring's theorem are the basic structure theorems for CP-maps. Our main tool to study CP-maps is the theory of *Hilbert C^*-modules*. They are objects similar to Hilbert spaces. Close connections between CP-maps and Hilbert C^*-modules are well-known ([Kas80, Mur97, Pas73]).

Given CP-maps φ_1 and φ_2 between unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, by a *common representation module* for them we mean a Hilbert $\mathcal{A}$-$\mathcal{B}$-module E where they can be represented, that is, there exists $x_i \in E$ such that $\varphi_i(\cdot) = \langle x_i, (\cdot)x_i\rangle$. We define β as the infimum of the norm differences $\|x_1 - x_2\|$ taken over all common representation modules E and representing vectors $x_i \in E$, and call it *Bures distance*. We show the existence of a sort of universal module where we can take infimum to compute the Bures distance, and thereby prove that β is a metric when the CP-maps under consideration map to a von Neumann algebra or to an injective C^*-algebra. However, β is not a metric when the range algebra is a general C^*-algebra. The definition of Bures distance is abstract and does not give us indications as to how to compute it for concrete examples. We show that Bures distance can be computed using intertwiners between two (minimal) GNS-constructions of CP-maps. We also prove a rigidity theorem, showing that GNS-representation modules ([Pas73]) of CP-maps which are close to the identity map contain a copy of the original algebra.

If $\varphi : \mathcal{A} \to \mathcal{B}$ is a linear map, then by a *φ-map* we mean a linear map $T : E \to F$ from a Hilbert $\mathcal{A}$-module E into a Hilbert $\mathcal{B}$-module F such that $\langle T(x_1), T(x_2)\rangle = \varphi(\langle x_1, x_2\rangle)$ for all $x_i \in E$, that is, T preserves the inner product up to the linear map φ. We prove that if E is *full* and if φ is bounded linear, then φ will be automatically CP. Moreover, T is completely bounded with CB-norm $\|T\|_{cb} := \sup_n \|T_n\| = \sqrt{\|\varphi\|}$. We derive a Stinespring type structure theorem for φ-maps for the case when $\mathcal{A} = \mathcal{B}(G)$ and $F = \mathcal{B}(G, H)$, where G and H are Hilbert spaces. We also find three equivalent conditions that tell us when a map $T : E \to F$ is a φ-map for some CP-map φ without knowing φ, just by looking at T. One of the important condition says that they are precisely *CP-H-extendable maps*, that is, maps $T : E \to F$ which allows a blockwise CP-extension between the *extended linking algebras* of E and $F_T := \overline{\operatorname{span}}\, T(E)\mathcal{B}$ such that the 22-corner of the CP-extension is a $*$-homomorphism. If such an extension is possible into the extended linking algebra of F we call $T : E \to F$ a *CPH-map*. CPH-maps are important if we want to talk about semigroups of CP-H-extendable maps. We also study maps $T : E \to F$ which allows a *strict* blockwise CP-extension between the linking algebras of E and F, and give a factorization theorem of such maps that generalizes those of Asadi([Asa09]) and Skeide([Ske12]).

* * * * *

Chapter 1. We begin the thesis by providing necessary background material on Hilbert C^*-modules. Our purpose in this chapter is to review some basic theory of Hilbert C^*-modules to make it accessible to non-specialists. Most of the definitions, examples, results and proofs can be found in [Lan95, Chapters 1-5, 7],[Ske01, Chapters 1-4]. We will not mention it explicitly each time. Other details can be found in the articles cited. Michael Frank's Hilbert C^*-Modules Home Page (http://www.imn.htwk-leipzig.de/ mfrank/hilmod.html) lists about 1700 references.

Chapter 2. D. Bures [Bur69] defined a notion of distance (metric) between states on von Neumann algebras and that there is a scope to generalize this to CP-maps was shown by [KSW08a]. We study this generalization using the language of Hilbert C^*-modules.

Chapter 3. We consider maps between Hilbert C^*-modules which generalizes the notion of isometries and unitaries. This study was motivated mainly by the work of [Asa09, TS07, Ske12]. First we search properties of such maps and later discuss structure theorem for such maps. In particular, we strengthen Asadi's theorem

([Asa09]) and discuss the minimality of the representations and prove the uniqueness of such representations up to unitary.

Chapter 4. We investigate maps, called *CP-H-extendable maps*, between Hilbert C^*-modules which allows for a CP-extension to a map between the associated *extended linking algebras* acting blockwise with 22-corner a $*$-homomorphism. We give different characterizations of such maps. This study is motivated by the work of Bakic-Guljas([BG02b]), Skeide ([Ske06b]) and Abbaspour-Skeide ([TS07]).

Appendix A. In appendix we give some background materials. Basic definitions and theory of C^*-algebra, von Neumann algebra, CP-maps, CB-maps, normal maps, CP-semigroups, E_0-semigroups, dilation of semigroups and operator spaces are outlined.

* * * * *

Notations and conventions: By $\mathbb{N}, \mathbb{R}, \mathbb{R}^+$ and $\mathbb{C}$ we denote the set of all positive integers, real numbers, non-negative real numbers and complex numbers, respectively. All vector spaces under consideration are over the field $\mathbb{C}$. We use $\oplus, \otimes$ to denote algebraic direct sum and algebraic tensor product of vector spaces.

We use G, H, K to denote Hilbert spaces. For denoting C^*-algebras we use $\mathcal{A}, \mathcal{B}, \mathcal{C}$. Hilbert C^*-modules are denoted by the symbols $E, F, \mathcal{E}, \mathcal{F}, etc.$ We use X, Y, Z to denote subsets, subspaces, normed spaces, operator spaces, etc. All sesquilinear maps are linear in its second variable and conjugate linear in its first variable. In particular, our inner products are linear in second variable and conjugate linear in first variable. If $(x, y) \mapsto xy$ is bilinear or sesquilinear on $X \times Y$, then XY is the set $\{xy : x \in X,\ y \in Y\}$. We do not adopt the convention that $XY = \mathsf{span}\{xy : x \in X,\ y \in Y\}$ or $XY = \overline{\mathsf{span}}\{xy : x \in X,\ y \in Y\}$. Sequences and nets in a set X are denoted as $\{x_n\}_{n \in \mathbb{N}}$, $\{x_\alpha\}_{\alpha \in \Lambda}$, respectively, where Λ is a directed set. We let $M_n(X)$ denote the set of all $n \times n$ matrices over X. Elements of $M_n(X)$ are denoted as $x = [x_{ij}]$ where $x_{ij} \in X$ is the $(i, j)^{th}$-entry of x. We use 't' to denote the transpose of a matrix.

Given two (normed) vector spaces X and Y, the space of all linear maps from X to Y is denoted by $\mathcal{L}(X, Y)$, and the space of all bounded linear maps from X to Y is denoted by $\mathcal{B}(X, Y)$. If $X = Y$, then $\mathcal{L}(X) := \mathcal{L}(X, X)$ and $\mathcal{B}(X) := \mathcal{B}(X, X)$. We may denote $\mathcal{B}(X \oplus Y)$ as $\mathcal{B}(\begin{pmatrix} X \\ Y \end{pmatrix})$ if X, Y are inner product spaces. The norm completion of a normed space X is denoted by $\overline{X}$. Also the closure of a subset Y in a topological space X is denoted by $\overline{Y}$.

* * * * *

Publications:

(1) B. V. Rajarama Bhat, K. Sumesh; *Bures Distance For Completely Positive Maps*, Infinite Dimensional Analysis, Quantum Probability and Related Topics, Vol. 16, No. 4 (2013), 1350031 (22 pages).

(2) B. V. Rajarama Bhat, G. Ramesh and K. Sumesh; *Stinespring's theorem for maps on Hilbert C^*-modules*, J. Operator Theory 68 (2012), No. 1, 173–178.

(3) Michael Skeide, K. Sumesh; *CP-H-Extendable Maps between Hilbert modules and CPH-Semigroups*, Journal of Mathematical Analysis and Applications, Vol. 414, No. 2 (2014), 886–913.

This thesis is based on the three papers listed above. Chapter 2 is essentially the paper (1). Chapter 3 and 4 are based on paper (2) and (3, Section 1-3).

Contents

Chapter 1

Introduction to Hilbert C^*-modules

Irving Kaplansky ([Kap53]) introduced the notion of Hilbert C^*-modules as a generalization of Hilbert spaces by allowing the inner product to take values in a commutative unital C^*-algebra. Subsequently W. L. Paschke [Pas73] extended this theory to noncommutative C^*-algebras. Independently, M. A. Rieffel [Rie74a] developed similar theory and applied it successfully to the study of induced representations of C^*-algebras. Hilbert C^*-modules can also be viewed as the generalization of vector bundles to noncommutative C^*-algebras. Hilbert C^*-modules arise often in operator theory, operator algebras, operator space theory, operator K-theory, group representation theory, noncommutative geometry, etc. Besides this, the theory of Hilbert C^*-modules is very rich and well studied.

In quantum dynamics, product systems of Hilbert C^*-modules were introduced by Bhat and Skeide [BS00], as a generalization of products systems of Hilbert spaces ([Arv89]). They are necessary to extend Arveson's theory from $\mathcal{B}(H)$ to general C^*-algebras. This is one of our motivations.

This introduction (including notations) is based mostly on the works of M. Skeide ([Ske00, Ske01]). We also borrow results and ideas from Lance ([Lan95]) and papers of several other authors.

1.1 Hilbert C^*-modules

1.1.1 Pre-Hilbert C^*-modules

Definition 1.1.1. Let $\mathcal{B}$ be a pre-C^*-algebra. An *inner product $\mathcal{B}$-module* (or *pre-Hilbert $\mathcal{B}$-module*) is a complex linear space E which is a right $\mathcal{B}$-module (with a compatible scalar multiplication: $\lambda(xb) = (\lambda x)b = x(\lambda b)$ for all $x \in E,\ b \in \mathcal{B},\ \lambda \in \mathbb{C}$), together with a map $\langle \cdot, \cdot \rangle : E \times E \to \mathcal{B}$ such that

(1) $\langle x, x \rangle \geq$[a] 0 $\quad (x \in E)$,

(2) $\langle x, x \rangle = 0 \Longleftrightarrow x = 0$ $\quad (x \in E)$,

(3) $\langle x, \lambda y + \lambda' z \rangle = \lambda \langle x, y \rangle + \lambda' \langle x, z \rangle$ $\quad (x, y, z \in E \text{ and } \lambda, \lambda' \in \mathbb{C})$,

[a] An element is said to be positive in a pre-C^*-algebra $\mathcal{B}$ if it is positive in the completion of $\mathcal{B}$.

(4) $\langle x, yb \rangle = \langle x, y \rangle b$ $(x, y \in E \text{ and } b \in \mathcal{B})$,

(5) $\langle x, y \rangle = \langle y, x \rangle^*$ $(x, y \in E)$.

If E satisfies all the conditions for an inner product $\mathcal{B}$-module except (2), then we call E a *semi-Hilbert $\mathcal{B}$-module*. By a *submodule* of a pre-Hilbert $\mathcal{B}$-module E we always mean a $\mathcal{B}$-submodule of E.

The map $\langle \cdot, \cdot \rangle$ will be called a *$\mathcal{B}$-valued inner product* on E. Note that condition (3) requires the inner product to be linear in its second variable. It follows from (5) that the inner product is conjugate linear in its first variable and $\langle xb, y \rangle = b^* \langle x, y \rangle$.

As in the case of inner product spaces we have the *polarization identity* given by

$$\langle x, y \rangle = \frac{1}{4} \sum_{n=0}^{3} (-i)^n \langle x + i^n y, x + i^n y \rangle \qquad \forall x, y \in E, \quad i^2 = -1.$$

Example 1.1.2. Here are some basic examples of pre-Hilbert C^*-modules.

(i) Any pre-C^*-algebra $\mathcal{B}$ is a pre-Hilbert $\mathcal{B}$-module with inner product $\langle b, b' \rangle := b^* b'$. More generally, any right ideal I in $\mathcal{B}$ can be made into a pre-Hilbert $\mathcal{B}$-module (actually a pre-Hilbert I-module) in the same way.

(ii) Let G and H be pre-Hilbert spaces and let $\mathcal{B} \subseteq \mathfrak{B}(G)$ be a $*$-algebra of bounded operators on G. Suppose $E \subseteq \mathfrak{B}(G, H)$ is a subspace such that $E\mathcal{B} \subseteq E$ and E^*E[b] $\subseteq \mathcal{B}$. Then E forms a pre-Hilbert $\mathcal{B}$-module with composition as module action and inner product given by $\langle x, y \rangle := x^* y$.

1.1.2 Cauchy-Schwarz inequality

Recall that in a semi-Hilbert space the *Cauchy-Schwarz inequality*, which asserts that $\langle h_1, h_2 \rangle \langle h_2, h_1 \rangle \leq \langle h_2, h_2 \rangle \langle h_1, h_1 \rangle$ for all elements h_1, h_2, allows to quotient out the null vectors[c]. For semi-Hilbert C^*-modules we have the following version: Let E be a semi-Hilbert C^*-module over a pre-C^*-algebra $\mathcal{B}$. Then

$$\langle x, y \rangle \langle y, x \rangle \leq \| \langle y, y \rangle \| \, \langle x, x \rangle$$

[b]For any subset X of a C^*-algebra, $X^* := \{x^* : x \in X\}$.

[c]A vector $x \in E$ is said to be a null vector if $\langle x, x \rangle = 0$.

for all $x, y \in E$. So if $x \in E$ is a null vector, then $\langle x, y\rangle = 0 = \langle y, x\rangle$ for all $y \in E$. Now given $x \in E$ define

$$\|x\| := \|\langle x, x\rangle\|^{\frac{1}{2}} \text{ and } |x| := \langle x, x\rangle^{\frac{1}{2}}.$$

Note that $|\cdot|$ may not satisfy triangular inequality.

Proposition 1.1.3. *Let E be a semi-Hilbert $\mathcal{B}$-module. Then*

(i) *$\|\cdot\|$ is a semi-norm on E, which is a norm if and only if E is a pre-Hilbert $\mathcal{B}$-module.*

(ii) *$\|\langle x, y\rangle\| \leq \|x\| \|y\|$ and $|\langle x, y\rangle| \leq \|x\| |y|$. In particular $\langle x, 0\rangle = \langle 0, y\rangle = 0$ for all $x, y \in E$.*

(iii) *$\|xb\| \leq \|x\| \|b\|$ and $|xb| \leq \|x\| |b|$ for all $x \in E$ and $b \in \mathcal{B}$.*

(iv) *If $x \in E$, then $\|x\| = \sup_{\|y\|\leq 1} \|\langle y, x\rangle\|$.*

Proposition 1.1.4. *Let E be a semi-Hilbert $\mathcal{B}$-module and $N_E := \{x \in E : \langle x, x\rangle = 0\}$. Then N_E is a closed submodule of E so that the quotient E/N_E is a right $\mathcal{B}$-module. Moreover, E/N_E inherits an inner product which turns it into a pre-Hilbert $\mathcal{B}$-module by defining*

$$\langle x + N_E, y + N_E\rangle := \langle x, y\rangle$$

for all $x, y \in E$.

Suppose E is a pre-Hilbert $\mathcal{B}$-module. Then $\langle x, y\rangle = \langle x', y\rangle$ for all $y \in E$ implies that $x = x'$. Also if $\mathcal{B}$ is unital, then $x1_{\mathcal{B}} = x$ for all $x \in E$. If $\mathcal{B}$ is not unital and $\widetilde{\mathcal{B}}$ is the unitalization of $\mathcal{B}$, then E becomes a pre-Hilbert $\widetilde{\mathcal{B}}$-module if we define $x1 := x$ for all $x \in E$.

Proposition 1.1.5. *Let E be a pre-Hilbert $\mathcal{B}$-module and $x \in E$. Then*

(i) *$xe_\alpha \xrightarrow{\alpha} x$ for any approximate unit $\{e_\alpha\}_{\alpha\in\Lambda}$ of $\mathcal{B}$.*

(ii) *$xb = 0$ for all $b \in \mathcal{B}$ implies that $x = 0$.*

Proposition 1.1.6. *Let E be a pre-Hilbert $\mathcal{B}$-module. Then*

(i) *$\overline{\mathsf{span}}\, E\mathcal{B}$[d] $= E$.*

[d]By span we always mean the $\mathcal{B}$-linear span.

(ii) $\overline{\mathsf{span}}\,\langle E, E\rangle$[e] *is a closed two-sided ideal in* $\mathcal{B}$ *and* $\overline{\mathsf{span}}\, E\langle E, E\rangle = E$.

Definition 1.1.7. The ideal $\mathcal{B}_E := \overline{\mathsf{span}}\langle E, E\rangle$ is called *range ideal*. If $\mathcal{B}_E = \mathcal{B}$, then we say E is *full*.

In general $\overline{\mathsf{span}}\,\langle E, E\rangle$ is not whole of $\mathcal{B}$, that is, E may not be full. (Recall that pre-Hilbert spaces are full pre-Hilbert $\mathbb{C}$-module.) But E can always be thought of as a full pre-Hilbert $\mathcal{B}_E$-module. If $\mathcal{B}$ is a unital C^*-algebra and if there exists a unit vector $x \in E$ (i.e., $\langle x, x\rangle = 1$), then E is a full Hilbert $\mathcal{B}$-module.

1.1.3 Hilbert C^*-modules

Definition 1.1.8. A *Hilbert C^*-module* is a pre-Hilbert module over a C^*-algebra which is complete with respect to the norm defined in Proposition 1.1.3.

Example 1.1.9. Following are some examples of Hilbert C^*-modules.

(i) A complex Hilbert space is a Hilbert $\mathbb{C}$-module under its inner product.

(ii) If Ω is a locally compact Hausdorff space and E a vector bundle over Ω with a Riemannian metric d, then the space of continuous sections of E is a Hilbert $C(\Omega)$-module. The inner product is given by $\langle f, g\rangle(x) := d(f(x), g(x))$.

(iii) A C^*-algebra is a Hilbert C^*-module over itself.

(iv) Let $\mathcal{B}$ be a C^*-algebra and H be a Hilbert space. Then the vector space tensor product $H \otimes \mathcal{B}$ is a Hilbert $\mathcal{B}$-module with right action $(h \otimes b)b' := h \otimes bb'$ and inner product $\langle h \otimes b, h' \otimes b'\rangle := \langle h, h'\rangle_H b^* b'$.

(v) If $\{E_k\}_{k=1}^n$ is a finite set of Hilbert C^*-modules over a C^*-algebra $\mathcal{B}$, then $\oplus_{k=1}^n E_k$ is a Hilbert $\mathcal{B}$-module if we define $\big\langle \{x_k\}, \{y_k\}\big\rangle := \sum_k \langle x_k, y_k\rangle$ and $\{x_k\}b := \{x_k b\}$. In particular, if $E_k = E$ for all k, then we write E^n for $\oplus_{k=1}^n E_k$. Also we write elements of E^n as column vectors rather than as row vectors.

(vi) Let $\{E_\alpha\}_{\alpha\in\Lambda}$ be an infinite set of Hilbert C^*-modules over a C^*-algebra $\mathcal{B}$. Define $\oplus_{\alpha\in\Lambda} E_\alpha := \big\{\{x_\alpha\}_{\alpha\in\Lambda} : \sum_\alpha \langle x_\alpha, x_\alpha\rangle \text{ conerges in } \mathcal{B}\big\}$, which is a Hilbert $\mathcal{B}$-module with inner product $\big\langle \{x_\alpha\}, \{y_\alpha\}\big\rangle := \sum_\alpha \langle x_\alpha, y_\alpha\rangle$ and module action

[e]If E is a pre-Hilbert $\mathcal{B}$-module, then $\langle E, E\rangle := \{\langle x, y\rangle : x, y \in E\}$ and $E\langle E, E\rangle := \{x\langle y, z\rangle : x, y, z \in E\}$.

$\{x_\alpha\}b := \{x_\alpha b\}$.

Remark 1.1.10. The following remarks enables us to assume that both the underlying pre-C^*-algebra and the pre-Hilbert C^*-module are complete.

(i) Let $\mathcal{B}$ be a C^*-algebra and E be a pre-Hilbert $\mathcal{B}$-module. Then using the completeness of the C^*-algebra $\mathcal{B}$, we can make the completion $\overline{E}$ of E into Hilbert $\mathcal{B}$-module in a natural fashion.

(ii) We can define a Hilbert $\mathcal{B}$-module over a pre-C^*-algebra $\mathcal{B}$ in exactly the same way as a Hilbert C^*-module over a C^*-algebra. Now if E be a Hilbert $\mathcal{B}$-module, then, using the continuity of the right multiplication $(x, b) \mapsto xb$ (in fact this map is jointly continuous), the module action of $\mathcal{B}$ on E can be extend to a module action of $\overline{\mathcal{B}}$ on E, and thereby to make E a Hilbert $\overline{\mathcal{B}}$-module.

(iii) Suppose $\mathcal{B}$ is a pre-C^*-algebra and E is a pre-Hilbert $\mathcal{B}$-module. Then, using the joint continuity of right multiplication, $\overline{E}$ can be made into a Hilbert $\overline{\mathcal{B}}$-module. Note that completeness of E may not imply completeness of $\mathcal{B}$.

Proposition 1.1.11 ([Ske09c, Lemma 3.2]). *Let E be a full Hilbert C^*-module over a unital C^*-algebra. Then there exists $n \in \mathbb{N}$ and $\xi \in E^n$ such that $\langle \xi, \xi \rangle = 1$.*

1.1.4 Ideal submodules

From here onwards by an ideal in a C^*-algebra $\mathcal{B}$ we always mean a closed two-sided ideal. An ideal $\mathcal{B}_0$ in $\mathcal{B}$ is said to be *essential* if there is no nonzero ideal of $\mathcal{B}$ that has zero intersection with $\mathcal{B}_0$. It is well known that for any C^*-algebra $\mathcal{B}$ there is a unique (up to isomorphism) C^*-algebra which contains $\mathcal{B}$ as an essential ideal and is maximal in the sense that any other such algebra can be embedded in it. This algebra is called the *multiplier algebra* of $\mathcal{B}$ and is denoted by $M(\mathcal{B})$. If $\mathcal{B}$ is unital, then $M(\mathcal{B}) \cong \mathcal{B}$. (See [Mur90, Ped79] for details).

In this section we discuss ideal submodules of Hilbert C^*-module. Details can be found in [BG02b].

Definition 1.1.12. Let I be an ideal in a C^*-algebra $\mathcal{B}$ and E be a Hilbert $\mathcal{B}$-module.

The *associated ideal submodule* E_I is defined by

$$E_I := \overline{\mathsf{span}}\, EI = \overline{\mathsf{span}}\{xb : x \in E,\ b \in I\}.$$

Proposition 1.1.13. *Let E be a Hilbert $\mathcal{B}$-module and I be an ideal in $\mathcal{B}$. Then*

(i) $E_I = EI = \{xb : x \in E,\ b \in I\}$.

(ii) $E_I = \{x \in E : \langle x, x\rangle \in I\} = \{x \in E : \langle x, x'\rangle \in I$ for all $x' \in E\}$.

If E is full, then E_I is full as a Hilbert I-module.

Corollary 1.1.14. *If E is a Hilbert $\mathcal{B}$-module, then $E = \{xb : x \in E,\ b \in \mathcal{B}\}$.*

Proposition 1.1.15. *Let E be a Hilbert $\mathcal{B}$-module and I be an essential ideal in $\mathcal{B}$. Then for all $x \in E$,*

(i) $\|x\| = \sup\left\{ \|xb\| : b \in I, \|b\| \leq 1\right\}$ *and*

(ii) $\|x\| = \sup\left\{ \|\langle x, x'\rangle\| : x' \in E_I, \|x'\| \leq 1\right\}$.

Conversely, if E is a full $\mathcal{B}$-module in which (i) or (ii) is satisfied with respect to (the ideal submodule associated with) some ideal I in $\mathcal{B}$, then I is an essential ideal in $\mathcal{B}$.

1.1.5 Self-duality

We have seen a substitute for Cauchy-Schwarz inequality in case of Hilbert C^*-modules. A very natural question is: Hilbert C^*-modules are self-dual or not ? We know that Hilbert spaces are self-dual, that is, all bounded linear functional are given by an inner product.

Definition 1.1.16. Let E be pre-Hilbert module over a pre-C^*-algebra $\mathcal{B}$. Define

$$E^r := \{\phi : E \to \mathcal{B} : \phi \text{ is linear and } \phi(xb) = (\phi x)b \quad \forall x \in E,\ b \in \mathcal{B},\ \ \|\phi\| < \infty\}$$

$$E^* := \{x^* : E \to \mathcal{B} : x^*(x') := \langle x, x'\rangle \quad \forall x, x' \in E\}.$$

The space E^r is called the space of all *bounded right linear $\mathcal{B}$-functionals* (or *$\mathcal{B}$-functionals*) on E and the space E^* is called the *dual module* of E.

From Proposition 1.1.3 we have $\|x^*\| = \|x\|$. Thus $x \mapsto x^*$ is an antilinear Banach space isometry from E onto E^*. Clearly $E^* \subseteq E^r \subseteq \mathfrak{B}(E, \mathcal{B})$. The containment can be even proper. Thus, in general, a $\mathcal{B}$-functional on a (pre-) Hilbert module may not be given by an inner product. So sometimes one may consider on E other $\mathcal{B}$-valued inner products defining norms equivalent to the given one ([Fra99, Man96b, Man96a]).

Definition 1.1.17. A pre-Hilbert $\mathcal{B}$-module E is said to be *self-dual* if $E^* = E^r$.

Self-dual pre-Hilbert modules over C^*-algebras are complete. The converse is not true in general. So the cases where we need self-dual Hilbert C^*-modules we consider "von Neumann modules" (Section 1.4) which are modules over von Neumann algebras.

Definition 1.1.18. The *$\mathcal{B}$-weak topology* on a pre-Hilbert $\mathcal{B}$-module E is the locally convex Hausdorff topology generated by the family $\|\langle x, \cdot\rangle\|$ $(x \in E)$ of seminorms.

Theorem 1.1.19 ([Fra99, Theorem 6.4]). *Let $\mathcal{B}$ be a C^*-algebra and E be a Hilbert $\mathcal{B}$-module. Then E is self-dual if and only if the unit ball of E is complete with respect to the $\mathcal{B}$-weak topology.*

Proposition 1.1.20 ([Pas73, Proposition 3.8]). *Let E be a self-dual Hilbert C^*-module over a W^*-algebra. Then E is a conjugate space.*

Proposition 1.1.21 ([Pas73, Proposition 3.11]). *Let E be a self-dual Hilbert C^*-module over a W^*-algebra $\mathcal{B}$. Then each $x \in E$ can be written $x = w\,|x|$, where $w \in E$ is such that $\langle w, w\rangle$ is the range projection of $|x|$. This decomposition is unique in the sense that if $x = vb$ where $0 \leq b \in \mathcal{B}$ and $\langle v, v\rangle$ is the range projection of b, then $w = v$ and $b = |x|$.*

1.2 Operators on Hilbert C^*-modules

1.2.1 Bounded adjointable operators

Definition 1.2.1. Let E and F be semi-Hilbert modules over a pre-C^*-algebra $\mathcal{B}$. A map $T : E \to F$ is said to be *adjointable*, if there exists a map $T^* : F \to E$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in E,\ y \in F$ and we call T^* an *adjoint* of T. If T is adjointable, then so is T^* with $(T^*)^* = T$.

Let $T : E \to F$ be a linear map between semi-Hilbert $\mathcal{B}$-modules. Then the operator norm of T is given by

$$\|T\| := \sup_{\|x\|\leq 1} \|Tx\| = \sup_{\|x\|\leq 1, \|y\|\leq 1} \|\langle y, Tx\rangle\| .$$

Clearly $\|Tx\| \leq \|T\| \|x\|$ for all $x \in E$ with $\|x\| \neq 0$. Now if E is a pre-Hilbert module, then the inequality holds for all $x \in E$. If T is adjointable, then by definition $\|T\| = \|T^*\|$ and $\|T^*T\| \geq \|T\|^2$. For pre Hilbert $\mathcal{B}$-modules E and F we can have $\|T^*T\| = \|T\|^2$.

We let $\mathcal{L}^a(E, F)$ and $\mathfrak{B}^r(E, F)$ denote the space of all linear adjointable and bounded right linear (i.e., $\mathcal{B}$-linear) maps from E to F respectively, and let $\mathcal{L}^a(E) = \mathcal{L}^a(E, E)$ and $\mathfrak{B}^r(E) = \mathfrak{B}^r(E, E)$. Note that $E^r = \mathfrak{B}^r(E, \mathcal{B})$.

Proposition 1.2.2. *Let E and F be semi-Hilbert $\mathcal{B}$-modules.*

(i) *Any map $T \in \mathcal{L}^a(E, F)$ gives rise to a unique element $\widetilde{T} \in \mathcal{L}^a(E/N_E, F/N_F)$.*

(ii) *Any map $T \in \mathfrak{B}^r(E, F)$ gives rise to a unique element $\widetilde{T} \in \mathfrak{B}^r(E/N_E, F/N_F)$ of the same norm.*

Proposition 1.2.3. *Let E and F be semi-Hilbert $\mathcal{B}$-modules and let $T : E \to F$ adjointable. Then*

(i) *E is pre-Hilbert $\mathcal{B}$-module implies T^* is unique.*

(ii) *F is pre-Hilbert $\mathcal{B}$-module implies T is $\mathcal{B}$-linear.*

Proposition 1.2.4. *Let E and F be pre-Hilbert $\mathcal{B}$-modules and $T : E \to F$ be an adjointable map. If either E or F is complete, then T is bounded.*

Thus all adjointable maps between Hilbert C^*-modules are bounded and right linear. But the converse is not true in general, that is, bounded right linear maps between Hilbert C^*-modules may not be adjointable.

Proposition 1.2.5 ([Pas73, Proposition 3.4]). *Let E and F be pre-Hilbert C^*-modules over the same C^*-algebra and $T : E \to F$ be a bounded right linear map. If E is self-dual, then T is adjointable.*

For pre-Hilbert $\mathcal{B}$-modules E and F we denote the space of all bounded adjointable maps from E to F by $\mathfrak{B}^a(E, F)$, and if $E = F$ then $\mathfrak{B}^a(E, E) = \mathfrak{B}^a(E)$. If one of E and F is complete, then Proposition 1.2.4 says that $\mathcal{L}^a(E, F) = \mathfrak{B}^a(E, F)$. From Proposition 1.2.3 we have $\mathfrak{B}^a(E, F) \subseteq \mathfrak{B}^r(E, F)$. Clearly any $x^* \in E^*$ is adjointable with adjoint given by $(x^*)^* : b \mapsto xb$ for all $b \in \mathcal{B}$, and thus $E^* \subseteq \mathfrak{B}^a(E, \mathcal{B})$.

Proposition 1.2.6. *Let E and F be pre-Hilbert $\mathcal{B}$-modules. Then*

(i) *E is complete implies $\mathfrak{B}^a(E, F)$ is a closed subspace of $\mathfrak{B}^r(E, F)$.*

(ii) *F is complete implies $\mathfrak{B}^r(E, F)$ is a Banach space.*

(iii) *E and F are complete implies $\mathfrak{B}^a(E, F)$ is a Banach subspace of $\mathfrak{B}^r(E, F)$.*

Corollary 1.2.7. *Let E be a pre-Hilbert $\mathcal{B}$-module. Then $\mathfrak{B}^r(E)$ forms a normed algebra and $\mathfrak{B}^a(E)$ forms a pre-C^*-algebra. If E is complete, then $\mathfrak{B}^r(E)$ is a Banach algebra and $\mathfrak{B}^a(E)$ is a C^*-algebra.*

Proposition 1.2.8 ([Pas73, Proposition 3.10]). *If E is a self-dual Hilbert C^*-module over a W^*-algebra, then $\mathfrak{B}^a(E)$ is a W^*-algebra.*

Note that $\mathfrak{B}^a(E, F)$ forms a pre-Hilbert $\mathfrak{B}^a(E)$-module with composition as the module action and with inner product given by $\langle T, T' \rangle := T^* T'$.

Example 1.2.9. Let E be a Hilbert C^*-module. Since E is complete so is E^n. Since $\mathcal{L}^a(E^n) \cong M_n(\mathcal{L}^a(E))$, from Proposition 1.2.4, we have $M_n(\mathfrak{B}^a(E)) = M_n(\mathcal{L}^a(E)) \cong \mathcal{L}^a(E^n) = \mathfrak{B}^a(E^n)$. Thus $M_n(\mathfrak{B}^a(E))$ forms a C^*-algebra.

Proposition 1.2.10. *Let E and F be pre-Hilbert $\mathcal{B}$-modules and let $\mathfrak{t} : E \times F \to \mathcal{B}$ be a bounded $\mathcal{B}$-sesquilinear form (i.e., $\|\mathfrak{t}\| := \sup\{\|\mathfrak{t}(x,y)\| : \|x\| \leq 1, \|y\| \leq 1\} < \infty$ and $\mathfrak{t}(xb, yb') = b^*\mathfrak{t}(x,y)b'$).*

(i) *If E is self-dual, then there exists a unique operator $T \in \mathcal{B}^r(F, E)$ such that $\mathfrak{t}(x, y) = \langle x, Ty\rangle$ for all $x \in E,\ y \in F$.*

(ii) *If also F is self-dual, then T is adjointable.*

In particular, for self-dual E and F, there is a one-to-one correspondence between bounded $\mathcal{B}$-sesquilinear forms $\mathfrak{t}$ on $E \times F$ and operators $T \in \mathcal{B}^a(F, E)$ such that $\mathfrak{t}(x, y) = \langle x, Ty\rangle$.

Theorem 1.2.11 ([Pas73, Theorem 2.8]). *Let E and F be Hilbert C^*-modules over a unital C^*-algebra $\mathcal{B}$. Then for a linear map $T : E \to F$ the following are equivalent:*

(i) *T is bounded and $T(xb) = (Tx)b$ for all $x \in E,\ b \in \mathcal{B}$, i.e., $T \in \mathcal{B}^r(E, F)$.*

(ii) *There exists $r \in \mathbb{R}^+$ such that $\langle T(x), T(x)\rangle \leq r\langle x, x\rangle$ for all $x \in E$.*

Corollary 1.2.12 ([Pas73, Remark 2.9]). *If E and F are Hilbert C^*-modules over a unital C^*-algebra $\mathcal{B}$ and $T \in \mathcal{B}^r(E, F)$, then*

$$\|T\| = \inf\{r^{\frac{1}{2}} : \langle T(x), T(x)\rangle \leq r\langle x, x\rangle\ \forall x \in E,\ \ r \in \mathbb{R}^+\}.$$

1.2.2 Finite-rank and compact operators

Let E and F be Hilbert C^*-modules over a C^*-algebra $\mathcal{B}$. Given $x \in E,\ y \in F$ define $|y\rangle\langle x| : E \to F$ by $x' \mapsto y\langle x, x'\rangle$ for all $x' \in E$. Then $|y\rangle\langle x| \in \mathcal{B}^a(E, F)$ with adjoint $|x\rangle\langle y|$.

Definition 1.2.13. An operator of the form $|y\rangle\langle x| \in \mathcal{B}^a(E, F)$ is called *rank-one operator*. The linear space $\mathcal{F}(E, F)$ of all rank-one operators is called the space of *finite-rank operators*, and its completion $\mathcal{K}(E, F)$ is called the Banach space of *compact operators*. If $E = F$, then $\mathcal{F}(E) := \mathcal{F}(E, E)$ and $\mathcal{K}(E) := \mathcal{K}(E, E)$.

In general, neither the finite-rank operators have finite rank in the sense of operators between linear spaces, nor the compact operators are compact in the sense of operators between Banach spaces.

Proposition 1.2.14. *Let E and F be Hilbert $\mathcal{B}$-modules. Then*

(i) $|y\rangle\langle x|\,|x'\rangle\langle y'| = |y\langle x, x'\rangle\rangle\langle y'| = |y\rangle\langle y'\langle x', x\rangle|$ *for all* $x, x' \in E,\ y, y' \in F$.

(ii) $T|x\rangle\langle y| = |Tx\rangle\langle y|$ *for all* $x \in E,\ y \in F,\ T \in \mathcal{B}^a(E)$.

(iii) $|x\rangle\langle y|S = |x\rangle\langle S^*y|$ *for all* $x \in E,\ y \in F, S \in \mathcal{B}^a(F)$.

Corollary 1.2.15. *Let E be a Hilbert $\mathcal{B}$-module. Then $\mathcal{K}(E)$ is an ideal in $\mathcal{B}^a(E)$.*

Observation 1.2.16. Suppose $\mathcal{B}$ is a C^*-algebra and E is a Hilbert $\mathcal{B}$-module.

(i) Given $x \in E$ define $r_x : \mathcal{B} \to E$ by $r_x(b) := xb$. Then $r_x \in \mathcal{B}^a(\mathcal{B}, E)$ with adjoint $x^* \in E^* \subseteq \mathcal{B}^a(E, \mathcal{B})$. Since $(xb)^* = |b^*\rangle\langle x|$ and $E\mathcal{B} = E$ we have $E^* = \{x^* : x \in E\} = \mathcal{K}(E, \mathcal{B})$. Consequently $\{r_x : x \in E\} = \mathcal{K}(\mathcal{B}, E)$. Moreover, $E \ni x \mapsto r_x \in \mathcal{B}^a(\mathcal{B}, E)$ is an isometric linear isomorphism of E onto $\mathcal{K}(\mathcal{B}, E)$. If $\mathcal{B}$ is unital, then $\mathcal{K}(\mathcal{B}, E) = \mathcal{B}^a(\mathcal{B}, E)$ and $\mathcal{K}(E, \mathcal{B}) = \mathcal{B}^a(E, \mathcal{B})$. In fact, any $T \in \mathcal{B}^a(\mathcal{B}, E)$ equals $|T(1)\rangle\langle 1|$ and any $T \in \mathcal{B}^a(E, \mathcal{B})$ equals $(T^*(1))^* = |1\rangle\langle T^*(1)|$.

(ii) Considering $\mathcal{B}$ as a Hilbert $\mathcal{B}$-module we have $\mathcal{B} \ni b \mapsto l_b \in \mathcal{B}^a(\mathcal{B})$ with $l_b(b') := bb'$ is an C^*-isomorphism of $\mathcal{B}$ onto $\mathcal{K}(\mathcal{B})$. If $\mathcal{B}$ is unital, then $\mathcal{B} \cong \mathcal{K}(\mathcal{B}) = \mathcal{B}^a(\mathcal{B})$.

Notation. From here onwards we write xy^* instead of $|x\rangle\langle y|$.

Definition 1.2.17. Suppose E is a Hilbert $\mathcal{B}$-module and $X \subseteq E$ is a subset. Then X is a *generating set* for E if $\overline{\operatorname{span}}\, X\mathcal{B} = E$. We say that E is *countably generated* if it has a countable generating set.

Proposition 1.2.18 ([Lan95, Proposition 6.7]). *A Hilbert $\mathcal{B}$-module E is countably generated if and only if the C^*-algebra $\mathcal{K}(E)$ is σ-unital.*

As in Hilbert space theory, in Hilbert C^*-module theory also there are notions called *orthonormal basis* and *orthonormal systems*. See [BG02a, Ara08, Ske00] for details.

1.2.3 Positive operators

Definition 1.2.19. Suppose $\mathcal{B}$ is a C^*-algebra and E is a Hilbert $\mathcal{B}$-module. A linear map $T : E \to E$ is said to be *positive* if $\langle x, Tx\rangle \geq 0$ for all $x \in E$, and we denote it by $T \geq 0$.

If T is positive, then T is adjointable with $T = T^*$. Given $T, S \in \mathcal{B}^a(E)$ such that $T - S \geq 0$, then we write $T \geq S$ or $S \leq T$.

Proposition 1.2.20. *For $T \in \mathcal{B}^r(E)$ the following are equivalent:*

(i) *T is positive in the C^*-algebra $\mathcal{B}^a(E)$.*

(ii) *T is positive according to definition 1.2.19.*

Proposition 1.2.21. *Let E and F be Hilbert C^*-modules over the same C^*-algebra.*

(i) *A positive operator $T \in \mathcal{B}^a(E)$ is a contraction if and only if $\langle x, Tx\rangle \leq \langle x, x\rangle$ for all $x \in E$.*

(ii) *For $T \in \mathcal{B}^a(E, F)$ and $x \in E$, $\langle Tx, Tx\rangle \leq \|T\|^2 \langle x, x\rangle$.*

Example 1.2.22. Let E be a Hilbert $\mathcal{B}$-module. By identifying $\mathcal{B}$ with $\mathcal{K}(\mathcal{B})$ we have $M_n(\mathcal{B}) = M_n(\mathcal{K}(\mathcal{B})) = \mathcal{K}(\mathcal{B}^n)$. Then $\big[\langle x_i, x_j\rangle\big]$ is positive in $M_n(\mathcal{B})$ for all $x_1, \ldots, x_n \in E$. We have seen that E^n is a Hilbert $\mathcal{B}$-module. Now for $[b_{ij}] \in M_n(\mathcal{B})$ and $x = (x_1, \cdots, x_n)^t, y = (y_1, \cdots, y_n)^t \in E^n$ define

$$\langle x, y\rangle := [\langle x_i, y_j\rangle] \quad \text{and} \quad x[b_{ij}] := (\sum_k x_k b_{k1}, \cdots, \sum_k x_k b_{kn})^t. \tag{1.2.1}$$

With these structures E^n becomes a Hilbert $M_n(\mathcal{B})$-module. The two norms (given by $\mathcal{B}$-valued and $M_n(\mathcal{B})$-valued inner products) on E^n are different, but they are equivalent, and so in particular E^n is a Hilbert $M_n(\mathcal{B})$-module, which we denote by $E_{(n)}$. We may write elements of $E_{(n)}$ as row vectors, so that operations given in (1.2.1) are very natural. (See [Lan95, Page 39].)

1.2.4 Projections and complemented submodules

Definition 1.2.23. A linear map $P : E \to E$ on a Hilbert C^*-module E is a *projection* if $P^2 = P = P^*$.

Note that by definition P is adjointable and therefore is right linear. Since $\|P\| = \|P^*P\| = \|P\|^2$, we have $\|P\| = 1$ or $\|P\| = 0$.

Example 1.2.24. If $E = \oplus_{\alpha\in\Lambda} E_\alpha$ is a direct sum of Hilbert C^*-modules, then the canonical projections P_α onto E_α is a projection in $\mathcal{B}^a(E)$.

Definition 1.2.25. For a subset X of a Hilbert C^*-module E we define the *orthogonal complement* of X as

$$X^\perp := \{x \in E : \langle x, x' \rangle = 0 \quad \forall x' \in X\}.$$

A closed submodule E_0 of E is said to be *orthogonally complemented*, in short *complemented* in E, if $E = E_0 \oplus E_0^\perp$. We say that E_0 is *topologically complemented* if there is a closed submodule E_0' of E such that $E_0 + E_0' = E$ and $E_0 \cap E_0' = \{0\}$. We say E_0 is *orthogonally closed* in E if $E_0^{\perp\perp} := (E_0^\perp)^\perp = E_0$.

Clearly $X^\perp$ is a closed submodule of E. If E_0 is orthogonally complemented, then clearly E_0 is topologically complemented; but the converse is false. Unlike Hilbert spaces, closed submodules are not complemented ($E_0^{\perp\perp}$ is usually larger than E_0) in general. If E_0 is orthogonally complemented, then it is orthogonally closed. But the converse is not necessarily true in general ([Sch99]).

Theorem 1.2.26 ([Sch99, Theorem 1]). *If E is a full Hilbert C^*-module, then every closed submodule of E is orthogonally closed if and only if every closed submodule of E is orthogonally complemented in E .*

Theorem 1.2.27 ([Mag97a, Theorem 1]). *Let $\mathcal{B}$ be a C^*-algebra. If there exists a full Hilbert $\mathcal{B}$-module in which every closed submodule is orthogonally complemented, then $\mathcal{B}$ is $*$-isomorphic to a C^*-algebra of (not necessarily all) compact operators on some Hilbert space. Consequently, all closed submodules in all Hilbert $\mathcal{B}$-modules are orthogonally complemented.*

J. Schweizer ([Sch99]), under the weaker assumption that every closed submodule in E is orthogonally closed, showed that not only $\mathcal{B}$ but also $\mathcal{K}(E)$ and E are isomorphic to a C^*-subalgebra and C^*-submodule, respectively, of the algebra of compact operators on some Hilbert space. See also [Kus05] for details on orthogonally closed modules. On the other direction, study of Hilbert C^*-modules over the C^*-algebras of compact operators is also interesting ([BG02a, Fra08, FS10]).

Proposition 1.2.28 ([Zet94, Man96a]). *Let E be a Hilbert C^*-module and let $E = E_1 \oplus E_2$ be a topological direct sum (not necessarily orthogonal) of closed submodules. Then there exists a new inner product on E equivalent to the given one with respect to which given decomposition is orthogonal.*

Proposition 1.2.29. *Let E be a Hilbert C^*-module. Then a closed submodule E_0 of E is complemented in E if and only if there exists a projection $P \in \mathcal{B}^a(E)$ onto E_0.*

Proposition 1.2.30. *Let E and F be Hilbert C^*-modules over the same C^*-algebra and suppose that $T \in \mathcal{B}^a(E, F)$ has closed range. Then*

(i) *$\mathsf{ker}(T)$ is a complemented submodule of E.*

(ii) *$\mathsf{ran}(T)$ is a complemented submodule of F.*

(iii) *$T^* \in \mathcal{B}^a(F, E)$ has closed range.*

Observation 1.2.31. Suppose E, F are Hilbert C^*-modules and $T \in \mathcal{B}^a(E, F)$.

(i) It is easy to verify that $\mathsf{ran}(T)^\perp = \mathsf{ker}(T^*)$. But it need not be the case that $\mathsf{ker}(T^*)^\perp = \overline{\mathsf{ran}}(T)$.

(ii) If T has closed range, then from proposition 1.2.30 we can get $E = \mathsf{ker}(T) \oplus^\perp \mathsf{ran}(T^*)$ and $F = \mathsf{ran}(T) \oplus^\perp \mathsf{ker}(T^*)$.

(iii) If T does not have closed range, then neither $\mathsf{ker}(T)$ nor $\overline{\mathsf{ran}}(T)$ need be complemented.

(iv) If T has closed range, then $\mathsf{ran}(T) = \mathsf{ran}(TT^*)$. Since T^* also has closed range, $\mathsf{ran}(T^*) = \mathsf{ran}(T^*T)$.

(v) In general we have $\overline{\mathsf{ran}}(T) = \overline{\mathsf{ran}}(TT^*)$ and $\overline{\mathsf{ran}}(T^*) = \overline{\mathsf{ran}}(T^*T)$.

Definition 1.2.32. A Hilbert $\mathcal{B}$-module is called *complementary*, if it is complemented in all Hilbert $\mathcal{B}$-modules where it appears as a submodule.

Proposition 1.2.33. *Self-dual Hilbert C^*-modules are complementary.*

1.2.5 Isometries and unitaries

Definition 1.2.34. Suppose $\mathcal{B}$ is a C^*-algebra and E, F are Hilbert $\mathcal{B}$-modules. An *isometry* between E and F is a map $V : E \to F$ that preserves inner products, i.e., $\langle Vx, Vx' \rangle = \langle x, x' \rangle$ for all $x, x' \in E$.

Proposition 1.2.35 ([Lan94]). *For a map $V : E \to F$ the following are equivalent:*

(i) *V is an isometry.*

(ii) *V is $\mathcal{B}$-linear and $\|Vx\| = \|x\|$ for all $x \in E$.*

In Hilbert space case a map V is an isometry if and only if $V^*V = id$. But in Hilbert C^*-module theory this is not the case. Because isometries are not adjointable in general.

Proposition 1.2.36. *An isometry $V : E \to F$ is adjointable if and only if the $\mathsf{ran}(V)$ is complemented in F.*

Corollary 1.2.37. *For a map $V : E \to F$ the following are equivalent:*

(i) *V is an isometry with complemented range.*

(ii) *$V \in \mathcal{B}^a(E, F)$ and $V^*V = id_E$.*

Definition 1.2.38. A map between Hilbert C^*-modules is called *unitary* if it is a surjective isometry. Two Hilbert $\mathcal{B}$-modules E and F are said to be isomorphic, and write $E \cong F$, if there exists a unitary $U : E \to F$.

Proposition 1.2.39. *For a map $U : E \to F$ the following are equivalent:*

(i) *U is unitary.*

(ii) *U is adjointable with $U^*U = id_E$ and $UU^* = id_F$.*

Proposition 1.2.40. *Let E and F be Hilbert C^*-modules and $T \in \mathcal{B}^a(E, F)$. If T and T^* have dense range, then $E \cong F$.*

Proposition 1.2.41 ([Lin92, Proposition 2.6]). *Let E, F be Hilbert C^*-modules. If there exists an invertible map $T \in \mathcal{B}^a(E, F)$, then $E \cong F$.*

Proposition 1.2.42 ([Lin92, Proposition 2.7]). *Let E and F be two Hilbert C^*-modules such that $\mathcal{B}^r(E) = \mathcal{B}^a(E)$. If there exists an invertible map $T \in \mathcal{B}^r(E, F)$, then $E \cong F$.*

Given Hilbert C^*-modules over a C^*-algebra $\mathcal{B}$ one may ask whether they are isomorphic as Banach $\mathcal{B}$-module or as Hilbert $\mathcal{B}$-module ([Lan94, Fra97a, Fra99, Bro85]). Recall that two Hilbert spaces are isomorphic as Banach spaces if and only if they are unitarily isomorphic if and only if they are isometrically isomorphic. L. G. Brown ([Bro85]) gave examples of Hilbert C^*-modules which are isomorphic as Banach C^*-modules but which are nonisomorphic as Hilbert C^*-modules.

Theorem 1.2.43 ([Fra97a]). *Let $\mathcal{B}$ be a C^*-algebra and E be a right Banach $\mathcal{B}$-module with two $\mathcal{B}$-valued inner products $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ which induce norms equivalent to the given one. Then the following conditions are equivalent:*

(i) *The Hilbert $\mathcal{B}$-modules $(E, \langle \cdot, \cdot \rangle_1)$ and $(E, \langle \cdot, \cdot \rangle_2)$ are isomorphic as Hilbert C^*-modules.*

(ii) *The Hilbert $\mathcal{B}$-modules $(E, \langle \cdot, \cdot \rangle_1)$ and $(E, \langle \cdot, \cdot \rangle_2)$ are isometrically isomorphic as Banach $\mathcal{B}$-modules.*

(iii) *The C^*-algebras $\mathcal{K}(E, \langle \cdot, \cdot \rangle_1)$ and $\mathcal{K}(E, \langle \cdot, \cdot \rangle_2)$ are $*$-isomorphic.*

(iv) *The unital C^*-algebras $\mathcal{B}^a(E, \langle \cdot, \cdot \rangle_1)$ and $\mathcal{B}^a(E, \langle \cdot, \cdot \rangle_2)$ are $*$-isomorphic.*

Theorem 1.2.44 ([Man96a, Theorem 2.6]). *Let E be a Hilbert C^*-module over a W^*-algebra and let $T \in \mathcal{B}^a(E)$ is such that all its powers are uniformly bounded (i.e., $\|T^n\| \leq r$ for some $r \in \mathbb{R}$ and for all $n \in \mathbb{N}$). Then there exists an inner product equivalent to the given one so that T is unitary with respect to this inner product.*

Theorem 1.2.45 ([Fra90, Theorem 2.6]). *If a Hilbert C^*-module $(E, \langle \cdot, \cdot \rangle)$ over a C^*-algebra $\mathcal{B}$ is self dual, then every $\mathcal{B}$-valued inner product $\langle \cdot, \cdot \rangle'$ on E inducing an equivalent norm to the given one fulfills the identity $\langle \cdot, \cdot \rangle = \langle T(\cdot), T(\cdot) \rangle'$ on $E \times E$ for a unique positive invertible bounded $\mathcal{B}$-linear operator T on E.*

Theorem 1.2.46 ([Fra99, Theorem 4.1]). *Suppose $\mathcal{B}$ is a C^*-algebra.*

(i) *If E is a countably generated right Banach $\mathcal{B}$-module, then every pair of $\mathcal{B}$-valued inner products on E inducing equivalent norms to the given one defines unitarily isomorphic Hilbert C^*-module structures on E.*

(ii) *Two countably generated Hilbert $\mathcal{B}$-modules are isomorphic as Hilbert $\mathcal{B}$-modules if and only if they are isomorphic as Banach $\mathcal{B}$-modules if and only if they are isometrically isomorphic as Banach $\mathcal{B}$-modules.*

Theorem 1.2.47 ([Fra99, Theorem 4.2]). *Let $\mathcal{B}$ be a C^*-algebra and E be a Hilbert $\mathcal{B}$-module. Then any two $\mathcal{B}$-valued inner products on E which induce equivalent norms are pairwise unitarily isomorphic if every bounded $\mathcal{B}$-linear operator on E possesses an adjoint operator.*

Proposition 1.2.48 ([Fra99, Proposition 5.3]). *Let $\mathcal{B}$ be a C^*-algebra and E be a Hilbert $\mathcal{B}$-module possessing two isomorphic $\mathcal{B}$-valued inner products $\langle\cdot,\cdot\rangle_1 = \langle T(\cdot), T(\cdot)\rangle_2$, where $T \in \mathcal{B}^r(E)$ is invertible.*

(i) *The operator T possesses an adjoint operator w.r.t $\langle\cdot,\cdot\rangle_1$ if and only if it has an adjoint w.r.t $\langle\cdot,\cdot\rangle_2$.*

(ii) *If T is adjointable, then the operator C^*-algebras $\mathcal{B}^a(E,\langle\cdot,\cdot\rangle_1)$ and $\mathcal{B}^a(E,\langle\cdot,\cdot\rangle_2)$, $\mathcal{K}(E,\langle\cdot,\cdot\rangle_1)$ and $\mathcal{K}(E,\langle\cdot,\cdot\rangle_2)$ coincide pairwise as sets of bounded $\mathcal{B}$-linear operators on E.*

Proposition 1.2.49 ([Fra99, Proposition 5.4],[JT91]). *Let E_1, E_2 be Hilbert C^*-modules over a C^*-algebra $\mathcal{B}$. If $E_1 \cong E_2$, then the corresponding C^*-algebras of all compact/ adjointable $\mathcal{B}$-linear operators are pairwise $*$-isomorphic. The converse is not true.*

Definition 1.2.50. Let E and F be Hilbert C^*-modules. An element $V \in \mathcal{B}^a(E,F)$ is called a *partial isometry* if $F_0 = \mathsf{ran}(V)$ is complemented in F and there exists a complemented submodule E_0 of E such that $V : E_0 \to F_0$ is unitary and $V(E_0^\perp) = \{0\}$.

Proposition 1.2.51. *For a map $V \in \mathcal{B}^a(E,F)$ the following conditions are equivalent:*

(i) *V is a partial isometry.*

(ii) *V^*V is a projection in $\mathcal{B}^a(E)$.*

(iii) *VV^* is a projection in $\mathcal{B}^a(F)$.*

(iv) *$V = VV^*V$.*

(v) *$V^* = V^*VV^*$.*

Adjointable operators between Hilbert C^*-modules do not generally have a polar decomposition. But under certain conditions we have a version of polar decomposition.

Proposition 1.2.52. *Let E and F be Hilbert C^*-modules and $T \in \mathcal{B}^a(E, F)$ be such that $\overline{\mathsf{ran}}(T)$ and $\overline{\mathsf{ran}}(T^*)$ are both complemented. Then there exists a partial isometry $V \in \mathcal{B}^a(E, F)$ such that $T = V\,|T|$.*

Proposition 1.2.53 ([Lin92, Lemma 2.4]). *Let E be a Hilbert C^*-module and $T \in \mathcal{B}^a(E)$. If T has a closed range, then $E = \mathsf{ker}(T) \oplus \mathsf{ran}(|T|)$. In particular, T has a polar decomposition $T = V\,|T|$ in $\mathcal{B}^a(E)$.*

1.3 Topology of $\mathcal{B}^a(E)$

In this section we discuss different topologies of $\mathcal{B}^a(E)$ other than the norm topology.

1.3.1 $*$-strong topology

Definition 1.3.1. Let E, F be Hilbert C^*-modules. The *$*$-strong topology* on $\mathcal{B}^a(E, F)$ is the locally convex Hausdorff topology generated by the two families

$$T \mapsto \|Tx\|\,, \quad T \mapsto \|T^*y\| \qquad (x \in E,\ y \in F)$$

of semi-norms.

Observe that a net $\{T_\alpha\}_{\alpha\in\Lambda}$ converges in $*$-strong topology if and only if $\{T_\alpha\}_{\alpha\in\Lambda}$ and $\{T_\alpha^*\}_{\alpha\in\Lambda}$ converges strongly. Since $E\langle E, E\rangle$ is total in E we can see that any approximate unit $\{Q_\alpha\}_{\alpha\in\Lambda}$ for $\mathcal{K}(E)$ converges $*$-strongly to id_E. In fact, $\{TQ_\alpha\}_{\alpha\in\Lambda}$ converges $*$-strongly to T for all $T \in \mathcal{B}^a(E)$.

Proposition 1.3.2. *Suppose E, F are Hilbert C^*-modules. Then*

(i) *$\mathcal{B}^a(E, F)$ is complete in the $*$-strong topology.*

(ii) *The unit ball of $\mathcal{K}(E, F)$ is $*$-strongly dense in the unit ball of $\mathcal{B}^a(E, F)$.*

(iii) *If $\mathcal{C}$ is a $*$-strongly dense C^*-subalgebra of $\mathcal{B}^a(E)$, then the unit ball of $\mathcal{C}$ is $*$-strongly dense in the unit ball of $\mathcal{B}^a(E)$.*

Proposition 1.3.3. *If $\mathcal{B}$ is a σ-unital C^*-algebra and E is a full Hilbert $\mathcal{B}$-module, then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in E such that $\sum\langle x_n, x_n\rangle$ converges $*$-strongly to 1 in $M(\mathcal{B})$.*

1.3.2 $\mathcal{B}^a(E)$ as a multiplier algebra

Definition 1.3.4. Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras and let E be a Hilbert $\mathcal{B}$-module. A *representation* of $\mathcal{A}$ on E is a $*$-homomorphism $\tau : \mathcal{A} \to \mathcal{B}^a(E)$, and is said to be *nondegenerate* if $\overline{\mathsf{span}}\, \tau(\mathcal{A})E = E$.

If $\mathcal{A}$ is unital, then τ is nondegenerate if and only if τ is unital. Note that $E_0 := \overline{\mathsf{span}}\, \tau(\mathcal{A})E$ is invariant under the action of $\mathcal{A}$, so that $\tau : \mathcal{A} \to \mathcal{B}^a(E_0)$ is always nondegenerate.

Proposition 1.3.5. *Suppose $\mathcal{A}_0$ is an ideal in $\mathcal{A}$ and $\tau : \mathcal{A}_0 \to \mathcal{B}^a(E)$ is a nondegenerate $*$-homomorphism. Then τ extends uniquely to a $*$-homomorphism $\widetilde{\tau} : \mathcal{A} \to \mathcal{B}^a(E)$. If τ is injective and $\mathcal{A}_0$ is essential in $\mathcal{A}$, then $\widetilde{\tau}$ is injective.*

Corollary 1.3.6. *If $\mathcal{B}$ is a C^*-algebra, then $M(\mathcal{B}) \cong \mathcal{B}^a(\mathcal{B})$ as C^*-algebras.*

Theorem 1.3.7. *Let $\mathcal{B}$ be a C^*-algebra.*

(i) *The algebra $\mathcal{B}^a(\mathcal{B})$ is an essential extension of $\mathcal{K}(\mathcal{B})$ which is maximal in the sense that if $\mathcal{K}(\mathcal{B})$ is an essential ideal in a C^*-algebra $\mathcal{C}$, then there is an injective $*$-homomorphism from $\mathcal{C}$ to $\mathcal{B}^a(\mathcal{B})$ whose restriction to $\mathcal{K}(\mathcal{B})$ is the identity map.*

(ii) *If the C^*-algebra $\mathcal{C}$ is a maximal essential extension of $\mathcal{B}$, then there is a $*$-isomorphism from $\mathcal{C}$ onto $\mathcal{B}^a(\mathcal{B})$ whose restriction to $\mathcal{B}$ is the canonical map from $\mathcal{B}$ to $\mathcal{K}(\mathcal{B})$.*

Proposition 1.3.8. *Suppose that $\tau : \mathcal{A} \to \mathcal{B}^a(E)$ is a nondegenerate injective $*$-homomorphism and let I be the idealiser of $\tau(\mathcal{A})$ in $\mathcal{B}^a(E)$; that is,*

$$I = \{T \in \mathcal{B}^a(E) : T\tau(\mathcal{A}) \subseteq \tau(\mathcal{A}) \text{ and } \tau(\mathcal{A})T \subseteq \tau(\mathcal{A})\}.$$

Then τ extends to a $*$-isomorphism between $M(\mathcal{A})$ and I.

Theorem 1.3.9 ([Kas80]). *As C^*-algebras $\mathcal{B}^a(\mathcal{K}(E)) \cong M(\mathcal{K}(E)) \cong \mathcal{B}^a(E)$. In particular the Hilbert C^*-modules E and $\mathcal{K}(E)$ have the same C^*-algebra of adjointable operators.*

Proposition 1.3.10. *For a $*$-homomorphism $\tau : \mathcal{A} \to \mathcal{B}^a(E)$, the following conditions are equivalent:*

(i) *τ is nondegenerate.*

(ii) *τ is the restriction to $\mathcal{A}$ of a unital $*$-homomorphism $\widetilde{\tau} : M(\mathcal{A}) \to \mathcal{B}^a(E)$ which is $*$-strongly continuous on the unit ball.*

(iii) *For some approximate unit $\{e_\alpha\}_{\alpha\in\Lambda}$ of $\mathcal{A}$, $\{\tau(e_\alpha)\}_{\alpha\in\Lambda}$ converges $*$-strongly to id_E.*

Observation 1.3.11. In fact, if (iii) holds for one approximate unit, then it must holds for any other approximate unit.

1.3.3 Strict topology

Suppose E is a Hilbert C^*-module. Being the multiplier algebra of $\mathcal{K}(E)$ we equip $\mathcal{B}^a(E)$ with a new topology.

Definition 1.3.12. The *strict topology* on $\mathcal{B}^a(E)$ is the locally convex Hausdorff topology generated by the two families

$$T \mapsto \|TQ\|\,, \quad T \mapsto \|T^*Q\| \qquad (Q \in \mathcal{K}(E))$$

of semi-norms.

Observation 1.3.13. The strict topology is finer than the $*$-strong topology.

Proposition 1.3.14. *The strict topology and the $*$-strong topology of $\mathcal{B}^a(E)$ coincide on bounded subsets.*

Corollary 1.3.15. *Suppose E is a Hilbert C^*-module.*

(i) *Any approximate unit for $\mathcal{K}(E)$ converges strictly to id_E.*

(ii) *(The unit ball of) $\mathcal{K}(E)$ is strictly dense in (the unit ball of) $\mathcal{B}^a(E)$.*

(iii) *$\mathcal{B}^a(E)$ is complete in the strict topology.*

Definition 1.3.16. Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras and E be a Hilbert $\mathcal{B}$-module. A CP-map $\varphi : \mathcal{A} \to \mathcal{B}^a(E)$ is said to be *strict* if $\{\varphi(e_\alpha)\}_{\alpha\in\Lambda}$ is strictly Cauchy in $\mathcal{B}^a(E)$ for some approximate unit $\{e_\alpha\}_{\alpha\in\Lambda}$ of $\mathcal{A}$.

Remark 1.3.17. The unit ball of $\mathcal{B}^a(E)$ is complete for the strict topology. So $\varphi : \mathcal{A} \to \mathcal{B}^a(E)$ is strict if and only if there is a positive element $T \in \mathcal{B}^a(E)$ with $\|T\| \leq \|\varphi\|$ such that $\{\varphi(e_\alpha)\} \longrightarrow T$ strictly.

Proposition 1.3.18. *Suppose that $\mathcal{A}, \mathcal{B}$ are C^*-algebras, E, F are Hilbert $\mathcal{B}$-modules, $\pi : \mathcal{A} \to \mathcal{B}^a(F)$ is a nondegenerate $*$-homomorphism and $W \in \mathcal{B}^a(E, F)$. Then $\varphi : \mathcal{A} \to \mathcal{B}^a(E)$ defined by $\varphi(a) := W^*\pi(a)W$ is a strict CP-map.*

Theorem 1.3.19. *Suppose that $\mathcal{A}, \mathcal{B}$ are C^*-algebras and E is a Hilbert $\mathcal{B}$-module. If $\varphi : \mathcal{A} \to \mathcal{B}^a(E)$ is a strict CP-map, then there exists a Hilbert $\mathcal{B}$-module F, a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}^a(F)$ and an element $W \in \mathcal{B}^a(E, F)$ such that $\overline{\operatorname{span}}\, \pi(\mathcal{A})WE = F$ and $\varphi(a) = W^*\pi(a)W$ for all $a \in \mathcal{A}$.*

Definition 1.3.20. The unique (up to unitary equivalence) triple (F, π, W) obtained from φ as in above theorem is called the *KSGNS-construction* associated with φ.

If $F = \mathcal{B} = \mathbb{C}$, then the KSGNS-construction reduces to the classical GNS-construction. If $\mathcal{B} = \mathbb{C}$ (so that F is a Hilbert space), then we get the Stinespring's construction. In the context of Hilbert C^*-modules the construction was given by Kasparov ([Kas80], [Mur97, Theorem 2.4]).

Corollary 1.3.21. *Suppose that $\mathcal{A}, \mathcal{B}$ are C^*-algebras, E is a Hilbert $\mathcal{B}$-module and $\varphi : \mathcal{A} \to \mathcal{B}^a(E)$ is a CP-map. Then φ is strict if and only if there is a CP-map $\widetilde{\varphi} : M(\mathcal{A}) \to \mathcal{B}^a(E)$, strictly continuous on the unit ball, whose restriction to $\mathcal{A}$ is equal to φ. Also, $\widetilde{\varphi}$ is unital if and only if $\{\varphi(e_\alpha)\}_{\alpha\in\Lambda}$ converges strictly to id_E in $\mathcal{B}^a(E)$ for some approximate unit $\{e_\alpha\}_{\alpha\in\Lambda}$ of $\mathcal{A}$.*

Proposition 1.3.22. *Suppose that $\mathcal{A}, \mathcal{B}$ are C^*-algebras and E is a Hilbert $\mathcal{B}$-module. Then for a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}^a(E)$ the following are equivalent:*

(i) *$\overline{\mathsf{span}}\, \pi(\mathcal{A})E$ is complemented submodule of E.*

(ii) *π is the restriction to $\mathcal{A}$ of a $*$-homomorphism $\widetilde{\pi} : M(\mathcal{A}) \to \mathcal{B}^a(E)$ which is strictly continuous on the unit ball.*

(iii) *π is strict.*

If these conditions hold then $\widetilde{\pi}(1)$, which is the strict limit of $\{\varphi(e_\alpha)\}_{\alpha\in\Lambda}$ for an approximate unit $\{e_\alpha\}_{\alpha\in\Lambda}$ of $\mathcal{A}$, is the projection from E onto $\overline{\mathsf{span}}\, \pi(\mathcal{A})E$.

Following Lance's convention ([Lan95]), from here onwards, by a *strict map* from $\mathcal{B}^a(E) \to \mathcal{B}^a(F)$ we always mean a bounded linear map which is strictly (and hence $*$-strongly) continuous on bounded subsets. Note that since $\mathcal{B}^a(E) \cong M(\mathcal{B}^a(E))$, from Corollary 1.3.21, this definition coincides with Definition 1.3.16.

1.4 von Neumann modules

1.4.1 Two-sided Hilbert C^*-modules

Definition 1.4.1. Suppose $\mathcal{A}, \mathcal{B}$ are C^*-algebras. A Hilbert $\mathcal{B}$-module E with a non-degenerate $*$-homomorphism $\tau : \mathcal{A} \to \mathcal{B}^a(E)$ is called *Hilbert $\mathcal{A}$-$\mathcal{B}$-module* or *$\mathcal{A}$-$\mathcal{B}$-correspondence*.

If E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module, then we may consider $\mathcal{A} \subseteq \mathcal{B}^a(E)$, and we denote $\tau(a)$ by a itself and thereby $\tau(a)x = ax$ for all $x \in E$, $a \in \mathcal{A}$. Since τ is contractive automatically, $\|ax\| \leq \|\tau\| \|a\| \|x\| \leq \|a\| \|x\|$.

Definition 1.4.2. Suppose E and F are Hilbert $\mathcal{A}$-$\mathcal{B}$-modules. A linear map $\Phi : E \to F$ is said to be *$\mathcal{A}$-$\mathcal{B}$-linear* (or *bilinear*) if $\Phi(axb) = a\Phi(x)b \quad \forall\, a \in \mathcal{A},\ b \in \mathcal{B},\ x \in E$.

The space of all bounded, adjointable and bilinear maps from E to F is denoted by $\mathcal{B}^{a,bil}(E,F)$. If $E = F$, then $\mathcal{B}^{a,bil}(E) := \mathcal{B}^{a,bil}(E,F)$. Note that $\mathcal{B}^{a,bil}(E)$ is the relative commutant of the image of $\mathcal{A}$ in $\mathcal{B}^a(E)$.

The complement of an $\mathcal{A}$-$\mathcal{B}$-submodule E is again a $\mathcal{A}$-$\mathcal{B}$-submodule. The range of a projection P is an $\mathcal{A}$-$\mathcal{B}$-submodule if and only if $P \in \mathcal{B}^{a,bil}(E)$.

Example 1.4.3. Suppose $\mathcal{A}, \mathcal{B}$ are C^*-algebras.

(i) If E is a Hilbert $\mathcal{B}$-module, then E is a Hilbert $\mathcal{B}^a(E)$-$\mathcal{B}$-module with left action given by $\tau(a)x := ax$ for all $x \in E,\ a \in \mathcal{B}^a(E)$. Moreover, E^n is a Hilbert $M_n(\mathcal{B}^a(E))$-$\mathcal{B}$-module with an obvious left action.

(ii) If E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module, then $\mathcal{A}$ has a homomorphic image in $\mathcal{B}^a(E)$. Therefore $M_n(\mathcal{A})$ has a homomorphic image in $M_n(\mathcal{B}^a(E))$ so that E^n is a Hilbert $M_n(\mathcal{A})$-$\mathcal{B}$-module.

(iii) If E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module, then $E_{(n)}$ can be made into a Hilbert $\mathcal{A}$-$M_n(\mathcal{B})$-module.

(iv) If E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module, then $M_n(E)$ is a $M_n(\mathcal{A})$-$M_n(\mathcal{B})$-module with module actions resembles usual matrix multiplication. Moreover, $M_n(E)$ is a Hilbert $M_n(\mathcal{A})$-$M_n(\mathcal{B})$-module with inner product given by $\big\langle [x_{ij}], [x'_{ij}] \big\rangle := [\sum_k \langle x_{ki}, x'_{kj}\rangle]$.

Lemma 1.4.4. *Let E and F be pre-Hilbert modules over a C^*-algebra $\mathcal{B}$. Suppose $X \subseteq E$ and $Y \subseteq F$ are subsets such that* $\operatorname{span} X\mathcal{B} = E$ *and* $\operatorname{span} Y\mathcal{B} = F$. *Suppose $a : X \to F$ and $a^* : Y \to E$ are maps such that $\langle y, ax\rangle = \langle a^*y, x\rangle$ for all $x \in X,\ y \in Y$. Then a extends to a (unique) $a \in \mathcal{L}^a(E,F)$ whose adjoint is the unique extension of $a^* \in \mathcal{L}^a(F,E)$.*

Lemma 1.4.5. *Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras and X be a subset of a pre-Hilbert $\mathcal{B}$-module E such that* $\operatorname{span} X\mathcal{B} = E$. *Suppose $\mathcal{A} \ni a \overset{\pi}{\longmapsto} \big(\pi(a) : X \to X\big)$ are well defined maps such that $\big\langle x, \pi(a)x'\big\rangle = \big\langle \pi(a^*)x, x'\big\rangle$ and $\pi(a)\pi(a') = \pi(aa')$ for all $x, x' \in E,\ a, a' \in \mathcal{A}$. Then π coextends*[f] *to a unique contractive $*$-homomorphism from $\mathcal{A} \to \mathcal{B}^a(E)$ and further from $\mathcal{A} \to \mathcal{B}^a(\overline{E})$.*

[f]Note that under the given conditions $\pi(a) : X \to X$ will be automatically linear.

Suppose E, F are Hilbert modules over the C^*-algebras $\mathcal{B}$ and $\mathcal{C}$ respectively. The above Lemma provides a method to define a left action of $\mathcal{A}$ on $\mathcal{B}^a(F)$. For the special case when $\mathcal{A} = \mathcal{B}^a(E)$, Proposition 1.3.10 and Theorem 1.3.9 asserts that: A nondegenerate left action of $\mathcal{K}(E)$ on F extends to a left action (strict unital $*$-homomorphism) of $\mathcal{B}^a(E)$ on F.

1.4.2 Representation of Hilbert C^*-modules

Definition 1.4.6. Let $\mathcal{M}$ be an algebra with subspaces $\mathcal{B}_{ij}(i, j = 1, \ldots, n)$ such that

$$\mathcal{M} = \begin{bmatrix} \mathcal{B}_{11} & \cdots & \mathcal{B}_{1n} \\ \vdots & & \vdots \\ \mathcal{B}_{n1} & \cdots & \mathcal{B}_{nn} \end{bmatrix} \qquad \text{(i.e., } \mathcal{M} = \oplus_{i,j=1}^n \mathcal{B}_{ij}).$$

We say $\mathcal{M}$ is a *generalized matrix algebra* (of order n) if the multiplication in $\mathcal{M}$ is compatible with the usual matrix multiplication, i.e., if $BB' = [\sum_{k=1}^n b_{ik} b'_{kj}]$ for all $B = [b_{ij}]$ and $B' = [b'_{ij}]$ in $\mathcal{M}$. If $\mathcal{M}$ is also a normed or a Banach algebra, then we say $\mathcal{M}$ is a *generalized normed* and a *generalized Banach matrix algebra* respectively. If $\mathcal{M}$ is also a $*$-algebra fulfilling $B^* = [b^*_{ji}]$, then we say $\mathcal{M}$ is a *generalized matrix $*$-algebra*. If $\mathcal{M}$ is also a (pre-)C^*-algebra, then we call $\mathcal{M}$ a *generalized matrix (pre-)C^*-algebra*.

Proposition 1.4.7. *Let $\mathcal{M}$ be a matrix pre-C^*-algebra. Then $\mathcal{M}$ is complete if and only if each $\mathcal{B}_{ij}$ is complete with respect to the norm induced by the norm of $\mathcal{M}$.*

If H_i, $i = 1, \ldots, n$ are Hilbert spaces, then $\mathcal{B}(\oplus H_i) = [\mathcal{B}(H_j, H_i)]$ is a matrix C^*-algebra. Now if $\Pi : \mathcal{M} \to \mathcal{B}(H)$ is a (nondegenerate) representation of a matrix $*$-algebra $\mathcal{M} = [\mathcal{B}_{ij}]$ on a pre-Hilbert space H, then H decomposes into the subspaces $H_i = \overline{\mathsf{span}}\, \Pi(\mathcal{B}_{ii})H$ and that $\Pi(\mathcal{B}_{ij}) \subseteq \mathcal{B}^a(H_j, H_i)$. Clearly, $\Pi(\mathcal{M}) = [\Pi(\mathcal{B}_{ij})]$.

Definition 1.4.8. A *matrix von Neumann algebra* on a Hilbert space $H = \oplus_{i=1}^n H_i$ is a strongly (or weakly) closed matrix $*$-subalgebra $\mathcal{M} = [\mathcal{B}_{ij}]$ of $\mathcal{B}(H)$.

Clearly, a matrix von Neumann algebra $\mathcal{M}$ is a von Neumann algebra with unit equals the sum of the units of the diagonal von Neumann subalgebras $\mathcal{B}_{ii}$.

Proposition 1.4.9. *Let $\mathcal{M} = [\mathcal{B}_{ij}]$ be a matrix pre-C^*-subalgebra of the von Neumann algebra $\mathcal{B}(\oplus_{i=1}^n H_i)$. Then $\mathcal{M}$ is strongly (weakly) closed, if and only if each $\mathcal{B}_{ij}$ is strongly (weakly) closed in $\mathcal{B}(H_j, H_i)$.*

Proposition 1.4.10. *Let $\mathcal{M} = [\mathcal{B}_{ij}]$ be a matrix von Neumann algebra on $\oplus_{i=1}^n H_i$ and let $b \in \mathcal{B}_{ij}$. There exists a unique partial isometry $v \in \mathcal{B}_{ij}$ such that $b = v\,|b|$ and $\mathit{ker}(v) = \mathit{ker}(b)$.*

Suppose $\mathcal{B}$ is a C^*-algebra and E is a Hilbert $\mathcal{B}$-module. We define

$$\mathfrak{A}(E) := \begin{bmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{bmatrix} = \left\{ \begin{bmatrix} b & x^* \\ x' & a \end{bmatrix} : b \in \mathcal{B},\ x, x' \in E,\ a \in \mathcal{B}^a(E) \right\}$$

which is clearly a vector space under entrywise operations. Similarly define

$$\mathfrak{A}^0(E) := \begin{bmatrix} \mathcal{B}_E & E^* \\ E & \mathcal{K}(E) \end{bmatrix} \quad \text{and} \quad \mathfrak{A}^1(E) := \begin{bmatrix} \mathcal{B} & E^* \\ E & \mathcal{K}(E) \end{bmatrix}$$

which are subspaces of $\mathfrak{A}(E)$. It can be seen that $\mathfrak{A}(E)$ is a $*$-algebra with multiplication and involution defined by

$$\begin{bmatrix} b_1 & x_1^* \\ x_1' & a_1 \end{bmatrix} \begin{bmatrix} b_2 & x_2^* \\ x_2' & a_2 \end{bmatrix} := \begin{bmatrix} b_1 b_2 + \langle x_1, x_2' \rangle & (x_2 b_1^* + a_2^* x_1)^* \\ x_1 b_2' + a_1 x_2' & x_1' x_2^* + a_1 a_2 \end{bmatrix} \text{ and } \begin{bmatrix} b & x^* \\ x' & a \end{bmatrix}^* := \begin{bmatrix} b^* & x'^* \\ x & a^* \end{bmatrix}$$

respectively, and $\mathfrak{A}^0(E)$ and $\mathfrak{A}^1(E)$ are matrix $*$-subalgebras of $\mathfrak{A}(E)$.

It is known that $\mathfrak{A}(E)$ has a (unique) C^*-norm extending the norm of $\mathcal{B}$ ([Ske00]). Moreover, the restriction of such a norm to E, E^* and $\mathcal{B}^a(E)$ coincide with the original norms on E, E^* and $\mathcal{B}^a(E)$, respectively. The C^*-norm can be find by extending a faithful representation of $\mathcal{B}$ to a representation Π of $\mathfrak{A}(E)$ on a Hilbert space. Moreover, such a representation decomposes this Hilbert space into subspaces G and H such that the representation maps E to a subset of $\mathcal{B}^a(G, H)$. Thus, any Hilbert module can be considered as a space of operators between two Hilbert spaces ([Mur97, Ske00]).

Let π be a representation of $\mathcal{B}$ on a Hilbert space G. Define a sesquilinear form on $E \otimes G$ by $\langle x \otimes g, x' \otimes g' \rangle := \langle g, \pi(\langle x, x' \rangle) g' \rangle$, which is a semi-inner product on $E \otimes G$. Suppose $N_{E \otimes G}$ is the set of all null vectors, and H is the completion of the pre-Hilbert space $E \underline{\odot} G := E \otimes G / N_{E \otimes G}$. We let $x \odot g$ denote the equivalence class containing $x \otimes g$. To each $x \in E$ associate a linear map $L_x : g \mapsto x \odot g$ in $\mathcal{B}^a(G, H)$ with adjoint $L_x^* : x' \odot g \mapsto \pi(\langle x, x' \rangle)g$. Define maps $\eta : E \to \mathcal{B}^a(G, H)$ by $\eta(x) := L_x$ and $\eta^* : E^* \to \mathcal{B}^a(H, G)$ by $\eta^*(x^*) := L_x^*$. Note that $\pi(\langle x, x' \rangle) = \eta^*(x^*)\eta(x')$ and $\eta(xb) = \eta(x)\pi(b)$ for all $x, x' \in E,\ b \in \mathcal{B}$. If π is an isometry, then so is η.

Definition 1.4.11. The pair (H, η) is called the *Stinespring representation of E associated with π.*

To each $a \in \mathcal{B}^a(E)$ the map $x \otimes g \mapsto ax \otimes g$ on $E \otimes G$ induces a map $\rho(a) \in \mathcal{B}(H)$. Clearly, the map $\rho : a \mapsto \rho(a)$ defines a nondegenerate unital representation of $\mathcal{B}^a(E)$ on H. Moreover, $\Pi := \begin{bmatrix} \pi & \eta^* \\ \eta & \rho \end{bmatrix}$ (acting matrix element-wise) defines a (nondegenerate, if π is) representation of $\mathfrak{A}(E)$ on H. If π is isometric, then so are ρ and Π.

Definition 1.4.12. We refer to the pair (H, ρ) as the *Stinespring representation of $\mathcal{B}^a(E)$ associated with π.* If E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module, then by $\rho_{\mathcal{A}}$ we mean the representation $\mathcal{A} \to \mathcal{B}^a(E) \to \mathcal{B}^a(H)$ of $\mathcal{A}$ on H. We refer to the pair $(H, \rho_{\mathcal{A}})$ as the *Stinespring representation of $\mathcal{A}$ associated with E and π.* If we are interested in both η and ρ, then we refer also to the triple (H, η, ρ) as the *Stinespring representation.*

Note that if π is an isometric representation of $\mathcal{B}$ on G, then Π defines a isometric representation of $\mathfrak{A}(E)$ by bounded operators on $G \oplus H$, and there by $\mathfrak{A}(E)$ forms matrix pre-C^*-algebra.

Definition 1.4.13. The matrix pre-C^*-algebra $\mathfrak{A}(E)$ is called the *extended linking algebra* of E. The $*$-subalgebras $\mathfrak{A}^0(E)$ and $\mathfrak{A}^1(E)$ are called the *reduced linking algebra* and the *linking algebra* of E, respectively.

Suppose $\mathcal{B}$ is a C^*-algebra and E is a Hilbert $\mathcal{B}$-module. As in Observation 1.2.16, we may consider $\mathcal{B} \subseteq \mathcal{B}^a(\mathcal{B})$ and $E \subseteq \mathcal{B}^a(\mathcal{B}, E)$ via the identifications $b \mapsto l_b$ and $x \mapsto r_x$, respectively. Then

$$\mathfrak{A}^1(E) = \begin{bmatrix} \mathcal{B} & E^* \\ E & \mathcal{K}(E) \end{bmatrix} = \begin{bmatrix} \mathcal{K}(\mathcal{B}) & \mathcal{K}(E, \mathcal{B}) \\ \mathcal{K}(\mathcal{B}, E) & \mathcal{K}(E) \end{bmatrix} = \mathcal{K}(\mathcal{B} \oplus E)$$

and

$$\mathfrak{A}(E) = \begin{bmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{bmatrix} \subseteq \begin{bmatrix} \mathcal{B}^a(\mathcal{B}) & \mathcal{B}^a(E,\mathcal{B}) \\ \mathcal{B}^a(\mathcal{B},E) & \mathcal{B}^a(E) \end{bmatrix} = \mathcal{B}^a(\mathcal{B}\oplus E).$$

If $\mathcal{B}$ is unital, then $\mathfrak{A}(E) = \mathcal{B}^a(\mathcal{B}\oplus E)$.

1.4.3 von Neumann modules

In this Section $\mathcal{B} \subseteq \mathcal{B}(G)$ is always a von Neumann algebra acting nondegenerately on a Hilbert space G, unless stated otherwise explicitly. For a Hilbert $\mathcal{B}$-module E we denote by H the completion $E \odot G$ of $E \underline{\odot} G$. We always identify $x \in E$ with $L_x \in \mathcal{B}(G,H)$ and $a \in \mathcal{B}^a(E)$ with $\rho(a) \in \mathcal{B}(H)$, and thereby consider $E \subseteq \mathcal{B}(G,H)$ and $\mathcal{B}^a(E) \subseteq \mathcal{B}(H)$.

Definition 1.4.14. A *von Neumann $\mathcal{B}$-module* is a pre-Hilbert $\mathcal{B}$-module E for which $\mathfrak{A}(E)$ is a matrix von Neumann algebra on $G \oplus H$. The strong topology on E is the relative strong topology of $\mathfrak{A}(E)$.

Example 1.4.15. Let $\mathcal{B} = \mathcal{B}(G)$. Then a von Neumann $\mathcal{B}$-module E is necessarily all of $\mathcal{B}(G,H)$. Moreover, $\mathcal{B}^a(E) = \mathcal{B}(H)$.

Proposition 1.4.16. *A pre-Hilbert $\mathcal{B}$-module E is a von Neumann $\mathcal{B}$-module if and only if E is strongly closed in $\mathcal{B}(G,H) \subseteq \mathcal{B}(G\oplus H)$. In particular, if E is strongly closed, then $\mathcal{B}^a(E)$ is a von Neumann algebra.*

Proposition 1.4.17. *The $\mathcal{B}$-functionals are strongly continuous maps from $E \to \mathcal{B}$. For all $x \in E$ the map $\mathcal{B}^a(E) \ni a \mapsto ax \in E$ is strongly continuous. For all $a \in \mathcal{B}^a(E)$ the map $E \ni x \mapsto ax \in E$ is strongly continuous.*

Proposition 1.4.18. *The unit-ball of $\mathcal{F}(E)$ is strongly dense in the unit-ball of $\mathcal{B}^a(E)$.*

Definition 1.4.19. Let $\{E_\alpha\}_{\alpha\in\Lambda}$ be a family of von Neumann $\mathcal{B}$-modules and denote $E = \oplus_{\alpha\in\Lambda} E_\alpha$. Then setting $H_\alpha = E_\alpha \odot G$ and $H = E \odot G$, we have $H = \overline{\oplus}_{\alpha\in\Lambda} H_\alpha$ in an obvious manner. By the *von Neumann module direct sum* $\overline{E}^s = \overline{\oplus}^s_{\alpha\in\Lambda} E_\alpha$ we mean the strong closure of E in $\mathcal{B}(G, H)$.

Theorem 1.4.20. *Any von Neumann $\mathcal{B}$-module E is self-dual.*

Corollary 1.4.21. *A subset X of a von Neumann module E is strongly total, if and only if $\langle x', x\rangle = 0$ for all $x' \in X$ implies $x = 0$.*

Proposition 1.4.22 ([Ske00, Proposition 5.1]). *Let E_0 be a strongly dense submodule of a von Neumann $\mathcal{B}$-module E. Then any $\mathcal{B}$-functional ϕ on E_0 extends to a (unique) $\mathcal{B}$-functional $\widetilde{\phi}$ on E. Moreover, $\|\widetilde{\phi}\| = \|\phi\|$.*

Theorem 1.4.23 ([Ske00, Theorem 5.2],[Lin92, Theorem 3.8]). *Any $\mathcal{B}$-functional ϕ on a $\mathcal{B}$-submodule E_0 of a von Neumann $\mathcal{B}$-module E may be extended norm preserving and uniquely to a $\mathcal{B}$-functional on E vanishing on $E_0^{\perp}$.*

Corollary 1.4.24 ([Ske00, Corollary 5.3]). *Let E, F be von Neumann $\mathcal{B}$-modules and E_0 a submodule of E. Then any map in $\mathcal{B}^r(E_0, F)$ extends uniquely to a map in $\mathcal{B}^a(E, F)$ having the same norm and vanishing on $E_0^{\perp}$.*

Proposition 1.4.25. *A von Neumann $\mathcal{B}$-module has a pre-dual.*

Theorem 1.4.26. *Let E be a pre-Hilbert module over a W^*-algebra $\mathcal{B}$. For any normal representation π of $\mathcal{B}$ on G denote by η_π the Stinespring representation associated with π. Then the following conditions are equivalent:*

(i) *$\eta_\pi(E)$ is a von Neumann $\pi(\mathcal{B})$-module for some faithful normal representation π of $\mathcal{B}$.*

(ii) *$\eta_\pi(E)$ is a von Neumann $\pi(\mathcal{B})$-module for every faithful normal representation π of $\mathcal{B}$.*

(iii) *E is self-dual.*

Corollary 1.4.27. *Let E be Hilbert C^*-module over a W^*-algebra $\mathcal{B}$. Then E^r is a self-dual Hilbert $\mathcal{B}$-module.*

1.4.4 Two-sided von Neumann modules

Suppose $\mathcal{A}$ is a C^*-algebra and $\mathcal{B} \subseteq \mathcal{B}(G)$ is a von Neumann algebra acting nondegenerately on a Hilbert space G.

Definition 1.4.28. A von Neumann $\mathcal{B}$-module E with a nondegenerate $*$-homomorphism from $\mathcal{A} \to \mathcal{B}^a(E)$ is called a *von Neumann $\mathcal{A}$-$\mathcal{B}$-module.*

From Proposition 1.4.17 we know that the action of any operator $a \in \mathcal{A}$ on a von Neumann $\mathcal{B}$-module E is strongly continuous, so that the action of $a \in \mathcal{A}$ from a strongly dense subset of E can be extend to all of E (Proposition 1.4.22).

Definition 1.4.29. Suppose $\mathcal{A}$ is a von Neumann algebra. A von Neumann $\mathcal{A}$-$\mathcal{B}$-module E such that $\mathcal{A} \ni a \mapsto \langle x, ax \rangle \in \mathcal{B}$ is a normal map for all $x \in E$ is called *two sided von Neumann $\mathcal{A}$-$\mathcal{B}$-module.*

Lemma 1.4.30. *Suppose E is a von Neumann $\mathcal{B}$-module, $\mathcal{A}$ is a von Neumann algebra and there exists a nondegenerate $*$-homomorphism from $\mathcal{A} \to \mathcal{B}^a(E)$. Then the following conditions are equivalent:*

(i) *E is a two-sided von Neumann $\mathcal{A}$-$\mathcal{B}$-module.*

(ii) *Maps $\mathcal{A} \ni a \mapsto \langle x, ax' \rangle \in \mathcal{B}$ are σ-weakly continuous for all $x, x' \in E$.*

(iii) *The canonical representation $\rho_{\mathcal{A}}$ of $\mathcal{A}$ on $H = E \odot G$ is normal.*

Definition 1.4.31. Suppose $\mathcal{B}$ is a C^*-algebra. The $\mathcal{B}$-*center* of a Hilbert $\mathcal{B}$-$\mathcal{B}$-module E is the linear subspace

$$C_{\mathcal{B}}(E) := \{x \in E : xb = bx \text{ for all } b \in \mathcal{B}\}$$

of E. In particular, $C_{\mathcal{B}}(\mathcal{B})$ is the *center* of $\mathcal{B}$.

Proposition 1.4.32. *Suppose $\mathcal{B}$ is a C^*-algebra and E is a Hilbert $\mathcal{B}$-$\mathcal{B}$-module. Then*

$$\big\langle C_{\mathcal{B}}(E), C_{\mathcal{B}}(E) \big\rangle \subseteq C_{\mathcal{B}}(\mathcal{B}).$$

Corollary 1.4.33. *If E is a Hilbert $\mathcal{B}$-module (respectively, a von Neumann $\mathcal{B}$-module),* *then* $C_{\mathcal{B}}(E)$ *is a Hilbert* $C_{\mathcal{B}}(\mathcal{B})$*-module (respectively, a von Neumann* $C_{\mathcal{B}}(\mathcal{B})$*-module).*

Corollary 1.4.34. *Each element in the* $\mathcal{B}$*-linear span of* $C_{\mathcal{B}}(E)$ *commutes with each element in* $C_{\mathcal{B}}(\mathcal{B})$*.*

1.5 Tensor product of Hilbert C^*-modules

1.5.1 Interior tensor product

Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be C^*-algebras. Given a Hilbert $\mathcal{B}$-module E and a Hilbert $\mathcal{B}$-$\mathcal{C}$-module F consider the vector space tensor product $E \otimes F$, with the module action given by $(x \otimes y)c := x \otimes yc$ for $x \in E,\ y \in F,\ c \in \mathcal{C}$, and define

$$\big\langle x_1 \otimes y_1, x_2 \otimes y_2\big\rangle := \big\langle y_1, \langle x_1, x_2\rangle y_2\big\rangle \qquad x_i \in E,\ y_i \in F. \tag{1.5.1}$$

Then $\langle \cdot, \cdot \rangle$ is a sesquilinear form which makes $E \otimes F$ into a semi-inner product $\mathcal{C}$-module. Set $N_{E\otimes F} = \{z \in E \otimes F : \langle z, z\rangle = 0\}$. We let $x \odot y$ denotes the equivalence class in $E \underline{\odot}_{\mathcal{B}} F := (E \otimes F)/N_{E\otimes F}$ containing the element $x \otimes y$. The completion $E \odot_{\mathcal{B}} F$ of the inner product $\mathcal{C}$-module $E \underline{\odot}_{\mathcal{B}} F$ is called the *interior tensor product* of E and F. We may simply write $E \underline{\odot} F$ and $E \odot F$ instead of $E \underline{\odot}_{\mathcal{B}} F$ and $E \odot_{\mathcal{B}} F$ respectively, if there is no ambiguity about $\mathcal{B}$. In the above situation if E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module, then $E \odot F$ is a Hilbert $\mathcal{A}$-$\mathcal{C}$-module with left action given by $a(x \odot y) := ax \odot y$ (see Corollary 1.5.6).

We define the algebraic tensor product $E \otimes_{\mathcal{B}} F$ of E and F over $\mathcal{B}$ as the quotient of $E \otimes F$ by the subspace $N_{\mathcal{B}}$ generated by elements of the form $xb \otimes y - x \otimes by$ where $x \in E,\ y \in F,\ b \in \mathcal{B}$. It can be shown that $N_{E\otimes F} = N_{\mathcal{B}}$ (see for example [Lan95, Chapter 4]). Thus $E \otimes_{\mathcal{B}} F = (E \otimes F)/N_{E\otimes F}$ as vector spaces. Therefore (1.5.1) defines an inner product on $E \otimes_{\mathcal{B}} F$, the resulting inner product $\mathcal{C}$-module is nothing but $E \underline{\odot}_{\mathcal{B}} F$. So $E \odot F$ can be also thought of as the completion of $E \otimes_{\mathcal{B}} F$ under the norm induced by the inner product.

Observation 1.5.1. The interior tensor product is associative. More precisely $(E_1 \odot E_2) \odot E_3 \cong E_1 \odot (E_2 \odot E_3)$ via $(x_1 \odot x_2) \odot x_3 \mapsto x_1 \odot (x_2 \odot x_3)$. Also it is distributive over addition, i.e., $(E_1 \oplus E_2) \odot F \cong (E_1 \odot F) \oplus (E_2 \odot F)$ via $(x_1 \oplus x_2) \odot y \mapsto (x_1 \odot y) \oplus (x_2 \odot y)$.

Observation 1.5.2. For unital $\mathcal{B}$, we identify always $E \underline{\odot} \mathcal{B}$ and E (via $x \odot b \mapsto xb$), and we identify always $\mathcal{B} \underline{\odot} F$ and F (via $b \odot y \mapsto by$). For nonunital $\mathcal{B}$, observe that $E \odot \mathcal{B} = E$ and $\mathcal{B} \odot F = F$. Also via the identification $x^* \odot x' \mapsto \langle x, x' \rangle$ we have $E^* \underline{\odot}_{\mathcal{B}^a(E)} E = \mathsf{span}\langle E, E \rangle$ and $E^* \odot_{\mathcal{B}^a(E)} E = \overline{\mathsf{span}}\langle E, E \rangle = \mathcal{B}_E$.

Observation 1.5.3. Note that $\mathcal{B}$ does not appear explicitly in the inner product (1.5.1). So, if $\mathcal{B}'$ is another pre-C^*-algebra containing $\mathcal{B}_E$ as an ideal, and acting on F via a representation such that the action of the elements of $\mathcal{B}_E$ is the same, then $E \odot_{\mathcal{B}'} F$ is the same Hilbert C^*-module $E \odot_{\mathcal{B}} F$.

Proposition 1.5.4. *Let E_1, E_2 be Hilbert $\mathcal{B}$-modules and $a \in \mathcal{B}^a(E_1, E_2)$. Then $a \odot id : x \odot y \mapsto ax \odot y$ defines an operator on $E_1 \odot F \to E_2 \odot F$ with adjoint $a^* \odot id$ and $\|a \odot id\| \leq \|a\|$. Moreover, the map $a \mapsto a \odot id$ is a unital $*$-homomorphism from $\mathcal{B}^a(E)$ into $\mathcal{B}^a(E \odot F)$ which is strictly continuous on the unit ball of $\mathcal{B}^a(E)$.*

Corollary 1.5.5. *Suppose $x \in E \subseteq \mathcal{B}^a(\mathcal{B}, E)$. Then $x \odot id : y \mapsto x \odot y$ is a map from $F = \widetilde{\mathcal{B}} \odot F \to E \odot F$, and $x^* \odot id : x' \odot y \mapsto \langle x, x' \rangle y$ is its adjoint. If x is a unit vector, then $x \odot id$ is an isometry. In particular, $(x^* \odot id)(x \odot id) = x^*x \odot id = id_F$ and $(x \odot id)(x^* \odot id) = xx^* \odot id$ is a projection onto the range of $x \odot id$. Also $\|x \odot id\| = \|\tau(|x|)\| \leq \|x\|$ where τ is the left action of $\mathcal{B}$ on F.*

Corollary 1.5.6. *If E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module and F is a Hilbert $\mathcal{B}$-$\mathcal{C}$-module, then $E \odot F$ is a Hilbert $\mathcal{A}$-$\mathcal{C}$-module with left action $a(x \odot y) := ax \odot y$.*

Example 1.5.7. $M_{nl}(E) \odot M_{lm}(F) \cong M_{nm}(E \odot F)$ via the identification $[x_{ij}] \odot [y_{ij}] \mapsto [\sum_k x_{ik} \odot y_{kj}]$.

Theorem 1.5.8. *Suppose $\mathcal{B}$ is a unital C^*-algebra, E is a Hilbert $\mathcal{B}$-module with a unit vector, and F is a Hilbert $\mathcal{B}$-$\mathcal{C}$-module. Then for each $a \in \mathcal{B}^{a,bil}(F)$ the map $x \odot y \mapsto x \odot ay$ extends as a well-defined map $id \odot a \in \mathcal{B}^a(E \odot F)$. Moreover, the map $a \mapsto id \odot a$ is an isometric isomorphism from $\mathcal{B}^{a,bil}(F)$ onto the relative commutant of $\mathcal{B}^a(E) \odot id$ in $\mathcal{B}^a(E \odot F)$. In other words, $(\mathcal{B}^a(E) \odot id)' = id \odot \mathcal{B}^{a,bil}(F) \cong \mathcal{B}^{a,bil}(F)$.*

Proposition 1.5.9. *Suppose $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are von Neumann algebras, E is a two-sided von Neumann $\mathcal{A}$-$\mathcal{B}$-module, and let F be a two-sided von Neumann $\mathcal{B}$-$\mathcal{C}$-module where $\mathcal{C}$ acts on a Hilbert space K. Then the strong closure $E \overline{\odot}^s F$ of $E \underline{\odot} F$ in $\mathcal{B}^a(K, E \odot F \odot K)$ is a two-sided von Neumann $\mathcal{A}$-$\mathcal{C}$-module.*

Theorem 1.5.10 ([MSS06, Theorem 1.4]). *Let $\mathcal{B}, \mathcal{C}$ be C^*-algebras, E be a Hilbert $\mathcal{B}$-module, F be a Hilbert $\mathcal{C}$-module and let $\vartheta : \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ be a unital $*$-homomorphism which is strictly continuous on bounded subsets. Then $F_\vartheta := E^* \odot F$ is a correspondence from $\mathcal{B}$ to $\mathcal{C}$ and the formula $U\big(x_1 \odot (x_2^* \odot y)\big) := \vartheta(x_1 x_2^*) y$ defines a unitary $U : E \odot F_\vartheta \to F$ such that $\vartheta(a) = U(a \odot id_{F_\vartheta}) U^*$ for all $a \in \mathcal{B}^a(E)$.*

Remark 1.5.11. The multiplicity correspondence in the above *representation theorem* is unique provided E is full ([MSS06, Theorem 1.8]).

See [Rie74a, Lan95, Ske00, Ble97a] for details on interior tensor product.

1.5.2 Haagerup tensor product

Suppose X and Y are two operator spaces. Given $x = [x_{ij}] \in M_{n,k}(X)$ and $y = [y_{ij}] \in M_{k,n}(Y)$ we let $x \boxdot y$ denotes the $n \times n$ matrix $[\sum_{k=1}^n x_{ik} \otimes y_{kj}]$ in $M_n(X \otimes Y)$. Note that $x \boxdot (\lambda y) = (x\lambda) \boxdot y$ for all scalar matrices λ. Given $z \in M_n(X \otimes Y)$ define

$$\begin{aligned}\|z\|_n &:= \inf\Big\{ \sum_{i=i}^m \|x_i\| \, \|y_i\| : z = \sum_{i=1}^m x_i \boxdot y_i, x_i \in M_{n,k_i}(X), y_i \in M_{k_i,n}(Y), m, k_i \in \mathbb{N} \Big\} \\ &= \inf\Big\{ \|x\| \, \|y\| : z = x \boxdot y, x \in M_{n,k}(X), y \in M_{k,n}(Y), k \in \mathbb{N} \Big\}. \qquad (1.5.2)\end{aligned}$$

(See [BP91, Lemma 3.2] which states that sums appearing in the definition can be avoided . In fact, the infimum in (1.5.2) is attained [ER91, Proposition 3.5]). If $n = 1$, that is, if $z \in X \otimes Y$, then

$$\|z\|_1 = \inf\Big\{ \Big\|\sum_{i=1}^k x_i x_i^*\Big\|^{\frac{1}{2}} \Big\|\sum_{i=1}^k y_i^* y_i\Big\|^{\frac{1}{2}} : z = \sum_{i=1}^k x_i \otimes y_i \in X \otimes Y, k \in \mathbb{N} \Big\}$$

where $\big\|\sum_{i=1}^k x_i x_i^*\big\|^{\frac{1}{2}}$ denotes the norm of $[x_1, \cdots, x_k] \in M_{1,k}(X)$ and $\big\|\sum_{i=1}^k y_i^* y_i\big\|^{\frac{1}{2}}$ denotes the norm of $[y_1, \cdots, y_k]^t \in M_{k,1}(Y)$. Note that these expressions makes sense when X and Y are C^*-algebras. Usually $\|\cdot\|_1$ is denoted by $\|\cdot\|_h$. It is known that $\|\cdot\|_n$ is a norm on $M_n(X \otimes Y)$ for all $n \in \mathbb{N}$ and satisfies Ruan's axioms, so that $(X \otimes Y, \{\|\cdot\|_n\}_{n\in\mathbb{N}})$ is an operator space. The completion is known as the *Haagerup tensor product* of X and Y and is denoted by $X \odot_h Y$.

Observation 1.5.12. If X_i $(i = 1, 2, 3)$ are operator spaces, then $(X_1 \odot_h X_2) \odot_h X_3 \cong X_1 \odot_h (X_2 \odot_h X_3)$ completely isometrically, i.e., Haagerup tensor product is associative. Also we have a natural isometry $M_{m,n}(X_1 \odot_h X_2) \cong M_{m,1}(X_1) \odot_h M_{1,n}(X_2)$ for all $m, n \in \mathbb{N}$.

Observation 1.5.13. If X_i, Y_i are operator spaces and if $T_i : X_i \to Y_i$ are completely bounded, then the mapping $x_1 \otimes x_2 \mapsto T_1(x_1) \otimes T_2(x_2)$ on $X_1 \otimes X_2$ induces a CB-map $T_1 \odot T_2 : X_1 \odot_h Y_1 \to X_2 \odot_h Y_2$ such that $\|T_1 \odot T_2\|_{cb} \leq \|T_1\|_{cb} \|T_2\|_{cb}$.

Suppose $\mathcal{A}$ is a C^*-algebra. An *operator module* over $\mathcal{A}$ is an operator space X which is also a module over $\mathcal{A}$ such that the action is a completely contractive bilinear map. Hilbert C^*-modules are operator modules. A left $\mathcal{A}$-operator module X is said to be *essential* if $\overline{\mathsf{span}}\, \mathcal{A}X = X$, and similarly for right modules. Using Cohen's factorization theorem ([Coh59],[Rie67, Proposition 3.4]), we can have $\overline{\mathsf{span}}\, \mathcal{A}X = \{ax : a \in \mathcal{A}, x \in X\} = \{x \in X : e_\alpha x \longrightarrow x\}$, where $\{e_\alpha\}_{\alpha\in\Lambda}$ is any approximate unit for $\mathcal{A}$.

Lemma 1.5.14. *If X is an (essential) left $\mathcal{A}$-operator module and if Y is any operator space, then $X \odot_h Y$ is an (essential) left $\mathcal{A}$-operator module. Similarly, if X is an (essential) right $\mathcal{A}$-operator module, then $Y \odot_h X$ is an (essential) right $\mathcal{A}$-operator module.*

Suppose X is a right $\mathcal{A}$-operator module and let Y be a left $\mathcal{A}$-operator module. A bilinear map $\psi : X \times Y \to Z$ is said to be *balanced* if $\psi(xa, y) = \psi(x, ay)$ for all $x \in X,\ y \in Y,\ a \in \mathcal{A}$.

Theorem 1.5.15 ([BMP00, Theorem 2.3]). *Let X be an right $\mathcal{A}$-operator module and let Y be a left $\mathcal{A}$-operator module. Up to complete isometric isomorphism, there exists a unique pair $(\mathcal{Z}, \odot_{h\mathcal{A}})$, where $\mathcal{Z}$ is an operator space and $\odot_{h\mathcal{A}} : X \times Y \to \mathcal{Z}$ is* *a completely contractive balanced bilinear map whose range densely spans $\mathcal{Z}$, with the following universal property: Given any operator space Z and a completely bounded bilinear balanced map $\psi : X \times Y \to Z$, there is a unique completely bounded linear map $\widetilde{\psi} : \mathcal{Z} \to Z$ with $\|\widetilde{\psi}\|_{cb} = \|\psi\|_{cb}$ such that $\widetilde{\psi} \circ \odot_{h\mathcal{A}} = \psi$.*

We write $X \odot_{h\mathcal{A}} Y$ for $\mathcal{Z}$, and continue to write $\|\cdot\|_h$ for the norm on $X \odot_{h\mathcal{A}} Y$. We call $X \odot_{h\mathcal{A}} Y$ the *module Haagerup tensor product* of X and Y over $\mathcal{A}$.

The existence of $X \odot_{h\mathcal{A}} Y$ is proved by setting $\mathcal{Z}$ to be the quotient $X \odot_h Y/N$ where N is the closure of the operator module subspace of $X \odot_h Y$ spanned by terms of the form $xa \odot y - x \odot ay$. Alternatively, we can define $X \odot_{h\mathcal{A}} Y$ as follows: Consider the algebraic tensor product $X \otimes_{\mathcal{A}} Y$ over $\mathcal{A}$ and define the sequence of matrix seminorms by the formula (1.5.2), and take quotient by the nullspace of the seminorm that we get.

Observation 1.5.16. Suppose $\mathcal{A}$ and $\mathcal{B}$ are operator algebras, X is a right $\mathcal{A}$-operator module, Y is a $\mathcal{A}$-$\mathcal{B}$-operator bimodule, and Z is a left $\mathcal{B}$-operator module. Then $(X \odot_{h\mathcal{A}} Y) \odot_{h\mathcal{B}} Z \cong X \odot_{h\mathcal{A}} (Y \odot_{h\mathcal{B}} Z)$ completely isometrically isomorphic. Thus module Haagerup tensor product is also associative.

Observation 1.5.17. Suppose X_1, Y_1 are right $\mathcal{A}$-operator modules, X_2, Y_2 are left $\mathcal{A}$-operator modules, and $T_i : X_i \to Y_i$ are completely bounded $\mathcal{A}$-module maps. Then the map $T_1 \odot T_2$ on $X_1 \odot_h X_2$ descends to the quotient space $X_1 \odot_{h\mathcal{A}} X_2$ and maps it into $Y_1 \odot_{h\mathcal{A}} Y_2$. Obviously, $\|T_1 \odot_{\mathcal{A}} T_2\|_{cb} \leq \|T_1\|_{cb} \|T_2\|_{cb}$.

Observation 1.5.18. It is easily shown, using Cohen's factorization theorem, that for an Hilbert $\mathcal{B}$-module E we have $E \odot_{h\mathcal{B}} \mathcal{B} \cong E$.

Theorem 1.5.19 ([Ble97a, Theorem 4.1]). *The interior tensor product of Hilbert C^*-modules is completely isometrically isomorphic to their module Haagerup tensor product.*

Theorem 1.5.20 ([Brü99, Theorem 3]). *Let E be a Banach space which is also a right $\mathcal{B}$-module for a C^*-algebra $\mathcal{B}$. Suppose that $\mathcal{B}$ is faithfully and nondegenerately* *represented on a Hilbert space G. Then E is a Hilbert $\mathcal{B}$-module (with its Hilbert C^*-module norm coinciding with the original norm) if and only if the following conditions hold:*

(i) *The Haagerup tensor product $E \odot_{h\mathcal{B}} G^c$ is a Hilbert space*[g].

(ii) *The map $\psi : E \to \mathcal{B}(G, E \odot_{h\mathcal{B}} G^c)$ given by $\psi(x)(g) := x \odot g$ is a (complete) isometry.*

(iii) *$\psi(x)^*\psi(x) \in \mathcal{B}$ for all $x \in E$.*

If these conditions hold, the (unique) inner product on E is given by $\langle x, x' \rangle = \psi(x)^*\psi(x')$.

See [BP91, Ble97a, BMP00, ER91, PS87, Heo99] for details on Haagerup tensor product.

1.5.3 More tensor products

Other than interior and Haagerup tensor product there are more Hilbert C^*-module tensor products, namely *exterior tensor product, spatial tensor product*, etc. Blecher ([Ble97a, Theorem 4.2]) proved that exterior tensor product of Hilbert C^*-modules is completely isometrically isomorphic to their spatial tensor product. Since we are not going to deal with them we skip the details here. For details see, for example, [Ble97a, Rie74a, Lan95].

1.6 Structure theorem for CP and CB-maps

Theorem 1.6.1 ([Pas73]). *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a CP-map between unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. Then there exists a Hilbert $\mathcal{A}$-$\mathcal{B}$-module E with a vector $x \in E$ such that $\varphi(a) = \langle x, ax \rangle$ for all $a \in \mathcal{A}$.*

Note that x is a unit vector if and only if φ is unital.

Definition 1.6.2. A pair (E, x) obtained as in the Theorem 1.6.1 is called a *GNS-construction for* φ, and E is called a *GNS-module*. Such a pair is said to be a *minimal* if $x \in E$ is a *cyclic vector* (i.e., $E = \overline{\operatorname{span}}\, \mathcal{A}x\mathcal{B}$).

[g] G^c denotes the Hilbert column space $\mathcal{B}(\mathbb{C}, G)$ with its natural operator space structure.

Remark 1.6.3. The GNS-construction (E, x) obtained in Theorem 1.6.1 can be chosen to be minimal. Moreover, if (E', x') is another such pair, then $x \mapsto x'$ extends as a two-sided isomorphism from $E \to E'$. Thus, minimal GNS-constructions are unique up to isomorphism, and henceforth, we call such a pair *the GNS-construction.*

Remark 1.6.4. Suppose $\mathcal{B} = \mathcal{B}(G)$ for some Hilbert space G. Then from Section 1.4.2 we have the triple $(H, \rho_{\mathcal{A}}, L_x)$, where $H = E \odot G$, $\rho : \mathcal{A} \to \mathcal{B}^a(E) \to \mathcal{B}(H)$ is a unital representation and $L_x = \eta(x) \in \mathcal{B}(G, H)$, such that

$$\varphi(a) = \langle x, ax \rangle = L_x^* \rho_{\mathcal{A}}(a) L_x \quad \text{and} \quad H = \overline{\mathsf{span}}\, \rho_{\mathcal{A}} L_x G.$$

If φ is unital, then L_x is an isometry. Thus $(H, \rho_{\mathcal{A}}, L_x)$ is the usual *Stinespring representation.*

Proposition 1.6.5 ([Pas73]). *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{B}$ be a von Neumann algebra and let $\varphi_1 \geq \varphi_2$ be completely positive maps from $\mathcal{A} \to \mathcal{B}$. If (E, x) is the GNS-construction for φ_1, then there exists $D \in \mathcal{A}' \subseteq \mathcal{B}^a(\overline{E}^s)$ such that $\varphi_2(a) = \langle x, Dax \rangle$ for all $a \in \mathcal{A}$.*

Observation 1.6.6. Suppose $\mathcal{A}$ is a C^*-algebra and $\mathcal{B}$ is a von Neumann algebra acting nondegenerately on a Hilbert space G, and x, x' are elements from the strong closure $\overline{E}^s$ of the GNS-module $E \subseteq \mathcal{B}(G, E \odot G)$. Suppose x, x' are the strong limits of the nets $\{x_\alpha\}_{\alpha \in \Lambda}, \{x_{\alpha'}\}_{\alpha' \in \Lambda'}$, respectively, with $x_\alpha, x_{\alpha'} \in E$. Then for all $a \in \mathcal{A},\ b \in \mathcal{B}$,

$$\begin{aligned} \langle x, x' \rangle &:= \operatorname*{s.lim}_{\alpha'} \big(\operatorname*{s.lim}_{\alpha} \langle x_{\alpha'}, x_\alpha \rangle \big)^* \in \mathcal{B}, \\ ax &:= \operatorname*{s.lim}_{\alpha} a x_\alpha \in E, \\ xb &:= \operatorname*{s.lim}_{\alpha} x_\alpha b \in E \end{aligned}$$

are well defined elements.

Proposition 1.6.7. *Let $\mathcal{A}$ be a unital C^*-algebra, $\mathcal{B}$ be a von Neumann algebra and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a CP-map with the GNS-module E.*

(i) *Then $\overline{E}^s$ is a von Neumann $\mathcal{A}$-$\mathcal{B}$-module.*

(ii) *If $\mathcal{A}$ is also a von Neumann algebra and φ is a normal CP-map, then $\overline{E}^s$ is a* *two-sided von Neumann $\mathcal{A}$-$\mathcal{B}$-module.*

Observation 1.6.8. Suppose $\varphi_1 : \mathcal{A} \to \mathcal{B}$ and $\varphi_2 : \mathcal{B} \to \mathcal{C}$ are CP-maps between C^*-algebras with GNS-constructions (E_1, x_1) and (E_2, x_2), respectively. Then

$$(\varphi_2 \circ \varphi_1)(a) = \big\langle x_2, \langle x_1, a x_1 \rangle x_2 \big\rangle = \big\langle x_1 \odot x_2, a(x_1 \odot x_2) \big\rangle$$

so that $(E_1 \odot E_2, x_1 \odot x_2)$ is a GNS-construction for $\varphi_2 \circ \varphi_1$. If (E_i, x_i) are minimal GNS-constructions for φ_i, then

$$\overline{\mathsf{span}}\, \mathcal{A}(x_1 \odot x_2)\mathcal{C} \subseteq \overline{\mathsf{span}}\, (\mathcal{A} x_1 \mathcal{B} \odot x_2 \mathcal{C}) = \overline{\mathsf{span}}\, (\mathcal{A} x_1 \mathcal{B} \odot \mathcal{B} x_2 \mathcal{C}) = E_1 \odot E_2.$$

So $(E_1 \odot E_2, x_1 \odot x_2)$ may not be minimal for $\varphi_2 \circ \varphi_1$ even though (E_i, x_i) are minimal for φ_i. A similar observation can be made for normal CP-maps between von Neumann algebras.

Theorem 1.6.9 ([Heo99, Theorem 1.1]). *Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras with $\mathcal{B}$ injective. If $\varphi : \mathcal{A} \to \mathcal{B}$ is a CB-map, then there exists a Hilbert $\mathcal{B}$-module E, a $*$-homomorphism $\tau : \mathcal{A} \to \mathcal{B}^a(E)$ and vectors $x_1, x_2 \in E$ with the properties:*

(i) *$\varphi(a) = \langle x_1, \tau(a) x_2 \rangle$ for all $a \in \mathcal{A}$.*

(ii) *$\overline{\mathsf{span}}\{\tau(a)(x_i b) : a \in \mathcal{A},\ b \in \mathcal{B},\ i = 1, 2\} = E$.*

Proposition 1.6.10 ([Heo99, Proposition 2.2]). *Let $\mathcal{A}_2, \mathcal{A}_2, \mathcal{B}$ be C^*-algebras with $\mathcal{B}$ injective. If $\varphi : \mathcal{A}_1 \underline{\odot}_h \mathcal{A}_2 \to \mathcal{B}$ is a CB-map, then there exists a Hilbert $\mathcal{B}$-module E, $*$-homomorphisms $\tau_i : \mathcal{A}_i \to \mathcal{B}^a(E)$ and vectors $x_1, x_2 \in E$ such that $\varphi(a_1 \otimes a_2) = \langle x_1, \tau_1(a_1)\tau_2(a_2) x_2 \rangle$ for all $a_i \in \mathcal{A}_i,\ i = 1, 2$.*

1.7 Product system of Hilbert C^*-modules

Definition 1.7.1. Let $\mathcal{B}$ be a C^*-algebra. A family $E^{\odot} = \{E_t\}_{t \in \mathbb{R}^+}$ of pre-Hilbert $\mathcal{B}$-$\mathcal{B}$-modules is called a *tensor product system of pre-Hilbert modules* or shortly a *product system*, if $E_0 = \mathcal{B}$ and if there exists a family $\{U_{s,t}\}_{s,t \in \mathbb{R}^+}$ of two-sided unitaries $U_{s,t} : E_s \underline{\odot} E_t \to E_{s+t}$ satisfying

$$U_{r,s+t}(id \odot U_{s,t}) = U_{r+s,t}(U_{r,s} \odot id) \qquad \forall\, r, s, t \in \mathbb{R}^+,$$

where $U_{s,0}, U_{0,t}$ are the identifications given in Observation 1.5.2. A product system is said to be *full* if each E_t is full.

Once, $U_{s,t}$ is given, we always use the identification $E_s \underline{\odot} E_t = E_{s+t}$. A *product subsystem* of a product system $E^{\underline{\odot}} = \{E_t\}_{t\in\mathbb{R}^+}$ is a family $E'^{\underline{\odot}} = \{E'_t\}_{t\in\mathbb{R}^+}$ of $\mathcal{B}$-$\mathcal{B}$-submodules E'_t of E_t such that $E'_s \underline{\odot} E'_t = E'_{s+t}$. We also define tensor *product system* of *two-sided Hilbert C^*-modules* $E^{\odot}$ and *von Neumann modules* $E^{\overline{\odot}^s}$, if $E_s \odot E_t = E_{s+t}$ and $E_s \overline{\odot}^s E_t = E_{s+t}$, respectively.

Definition 1.7.2. A *unit* for a product system $E^{\underline{\odot}} = \{E_t\}_{t\in\mathbb{R}^+}$ is a family $\xi^{\underline{\odot}} = \{\xi_t\}_{t\in\mathbb{R}^+}$ of elements $\xi_t \in E_t$ such that $\xi_s \odot \xi_t = \xi_{s+t}$ in the identification $E_s \underline{\odot} E_t = E_{s+t}$ and $\xi_0 = 1 \in \mathcal{B} = E_0$. A unit is *unital*, *contractive* and *central*, if $\langle \xi_t, \xi_t \rangle = 1, \langle \xi_t, \xi_t \rangle \leq 1$ and $\xi_t \in C_{\mathcal{B}}(E_t)$, respectively for all $t \in \mathbb{R}^+$.

Definition 1.7.3. A *left dilation (left semi-dilation)* of a full product system $E^{\odot}$ to a full Hilbert $\mathcal{B}$-module E is a family of unitaries $U_t : E \odot E_t \to E$ such that $(xy_s)z_t = x(y_s z_t)$, where we define $xy_t := U_t(x \odot y_t)$ for all $t \in \mathbb{R}^+$. If E is not full, then $\{U_t\}_{t\in\mathbb{R}^+}$ is called a *left quasi-dilation (left quasi-semidilation).*

It is known that ([Ske09a, Proposition 6.3]) product system and left (semi-) dilation are essentially "unique". By setting $\vartheta_t^U(a) := U_t(a \odot id_{E_t})U_t^*$, every left dilation gives rise to a strict E_0-semigroup (i.e., semigroup of strict unital endomorphisms) $\vartheta^{\odot U} = \{\vartheta_t^U\}_{t\in\mathbb{R}^+}$ on $\mathcal{B}^a(E)$. Conversely, a strict E_0-semigroup $\vartheta^{\odot}$ on $\mathcal{B}^a(E)$ with E a full Hilbert $\mathcal{B}$-module give rise to a full product system $E^{\odot}$ of $\mathcal{B}$-correspondences and a left dilation $\{U_t\}_{t\in\mathbb{R}^+}$ such that $\vartheta^{\odot} = \vartheta^{\odot U}$. Two strict E_0-semigroups on the same $\mathcal{B}^a(E)$ have isomorphic product systems if and only if they are "*cocycle conjugate*"; see [Ske02, Ske09c, Ske09b] for details.

Definition 1.7.4. Let $\varphi^{\odot} = \{\varphi_t\}_{t\in\mathbb{R}^+}$ be a unital CP-semigroup on a unital C^*-algebra $\mathcal{B}$. A *dilation* of $\varphi^{\odot}$ on a Hilbert C^*-module is a quadruple $(E, \vartheta^{\odot}, \mathfrak{i}, \xi)$ consisting of a Hilbert $\mathcal{B}$-module E, an E_0-semigroup $\vartheta^{\odot} = \{\vartheta_t\}_{t\in\mathbb{R}^+}$ on $\mathcal{B}^a(E)$, an injective $*$-homomorphism $\mathfrak{i} : \mathcal{B} \to \mathcal{B}^a(E)$, and a unit vector $\xi \in E$ such that the following diagram commutes for all $t \in \mathbb{R}^+$.

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\varphi_t} & \mathcal{B} \\ \downarrow{\scriptstyle \mathsf{i}} & & \uparrow{\scriptstyle \langle \xi, (\cdot)\xi \rangle} \\ \mathcal{B}^a(E) & \xrightarrow[\vartheta_t]{} & \mathcal{B}^a(E) \end{array}$$

Definition 1.7.5. A *weak dilation* on a Hilbert C^*-module is a triple $(E, \vartheta^\odot, \xi)$ such that the following diagram commutes for all $t \in \mathbb{R}^+$.

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\varphi_t} & \mathcal{B} \\ {\scriptstyle \xi(\cdot)\xi^*}\downarrow & & \uparrow{\scriptstyle \langle \xi, (\cdot)\xi \rangle} \\ \mathcal{B}^a(E) & \xrightarrow[\vartheta_t]{} & \mathcal{B}^a(E) \end{array}$$

Product system of two-sided Hilbert C^*-modules appeared first probably in [BS00]. For a (unital) CP-semigroup $\varphi^\odot = \{\varphi_t\}_{t \in \mathbb{R}^+}$ on a (unital) C^*-algebra $\mathcal{B}$, Bhat and Skeide ([BS00, Section 4]) provide the following:

- A product system $E^\odot = \{E_t\}_{t \in \mathbb{R}^+}$ of Hilbert $\mathcal{B}$-$\mathcal{B}$-modules.
- A (unital) unit $\xi^\odot = \{\xi_t\}_{t \in \mathbb{R}^+}$ such that $\varphi_t(\cdot) = \langle \xi_t, (\cdot)\xi_t \rangle$ and the smallest product subsystem of $E^\odot$ containing $\xi^\odot$ is $E^\odot$. The pair $(E^\odot, \xi^\odot)$ is determined by these properties up to unit preserving isomorphism, and is called the *GNS-construction* for $\varphi^\odot$ with *GNS-system* $E^\odot$ and *cyclic unit* $\xi^\odot$.
- If $E^\odot$ is not minimal, then the sub-correspondences

$$E'_t := \overline{\mathsf{span}}\{b_n \xi_{t_n} \odot \cdots \odot b_1 \xi_{t_1} b_0 : b_i \in \mathcal{B},\ t_1 + \cdots + t_n = t,\ n \in \mathbb{N}\}$$

  of E_t form a product subsystem of $E^\odot$ that is isomorphic to the GNS-system.
- A left dilation $U_t : E \odot E_t \to E$ of $E^\odot$ to a (by definition full) Hilbert $\mathcal{B}$-module E. So the maps $\vartheta : a \mapsto U_t(a \odot id_t)U_t^*$ define a strict E_0-semigroup on $\mathcal{B}^a(E)$.
- A unit vector $\xi \in E$ such that $\xi \xi_t = \xi$. It is readily verified that the triple $(E, \vartheta^\odot, \xi)$ is a weak dilation of $\varphi^\odot$. (In [Ske02] Skeide showed how to construct a tensor product system of Hilbert $\mathcal{B}$-$\mathcal{B}$-modules from a weak dilation, at least, when the endomorphisms ϑ_t are strict.)

Product system of Hilbert C^*-modules (or correspondence) appeared in many contexts. See [Ske08] for a survey on product systems of Hilbert C^*-modules.

Chapter 2

Bures Distance for completely positive maps

Given a state ϕ on a unital C^*-algebra $\mathcal{A}$ we have the familiar GNS-triple (H, π, x), where H is a Hilbert space, $\pi : \mathcal{A} \to \mathcal{B}(H)$ is a unital $*$-homomorphism and $x \in H$ is a vector such that $\phi(\cdot) = \langle x, \pi(\cdot)x \rangle$. Now it is a natural question to ask: If two states ϕ_1, ϕ_2 are close in some metric, whether the associated triples are close in some sense? Keeping this idea in mind, D. Bures ([Bur69]) defines a distance between two states ϕ_1, ϕ_2 on $\mathcal{A}$, as

$$\beta(\phi_1, \phi_2) := \inf \|x_1 - x_2\|,$$

where the infimum is taken over all GNS-triples with common representation spaces: $(H, \pi, x_1), (H, \pi, x_2)$ of ϕ_1, ϕ_2. D. Bures showed that β is indeed a metric. The notion has found uses in many areas ([AZ09, AP00, Ara72, Dit98]).

D. Kretschmann, D. Schlingemann and R. F. Werner ([KSW08a]) extended this notion at first to CP-maps from a unital C^*-algebra $\mathcal{A}$ into $\mathcal{B}(G)$ for some Hilbert space G and then to more general range C^*-algebras using an alternative definition of the Bures distance. They use Stinespring representation ([Sti55]) for the initial definition, which in the usual formulation requires the range space to be the whole algebra $\mathcal{B}(G)$. Here we develop the theory using Hilbert C^*-module language, which allows the range algebra to be any C^*-algebra, and the definition of the metric is a very natural extension of the definition given by Bures for states. Working with C^*-modules has several advantages. The results we get are of course same as that of [KSW08a], when the range algebra is a von Neumann algebra or an injective C^*-algebra. However, we show that one may not even get a metric (triangle inequality may fail) when the range algebra is a general C^*-algebra.

There have been several papers ([Akh07, Dit99, Hüb92]) on different methods to make exact computations of the Bures metric for states. We provide several examples with explicit computations of the Bures distance for CP-maps. In particular, we show that the infimum in the definition of Bures metric may not be attained in all common representation modules, answering a question raised in [KSW08b, KSW08a]. It turns out that the example is quite simple involving CP-maps on 2×2 matrix algebra.

In the last Section we prove a rigidity theorem, which says that on von Neumann algebras, if a CP-map is strictly within unit distance (in Bures metric) from the identity map, then the GNS-module of the CP-map contains a copy of the original von Neumann algebra as a direct summand.

2.1 Bures distance

In this Chapter all C^*-algebras under consideration are assumed to be unital. Given two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, we let $CP(\mathcal{A}, \mathcal{B})$ denote the set of all nonzero CP-maps from $\mathcal{A}$ into $\mathcal{B}$.

Definition 2.1.1. A Hilbert $\mathcal{A}$-$\mathcal{B}$-module E is said to be a *common representation module* for $\varphi_1, \varphi_2 \in CP(\mathcal{A}, \mathcal{B})$ if both of them can be represented in E, that is, there exist $x_i \in E$ such that $\varphi_i(a) = \langle x_i, ax_i \rangle, \; i = 1, 2$.

Note that we are demanding no minimality for a common representation module. So we can always have such a module. For, if $(\hat{E}_i, \hat{x}_i)$ is the minimal GNS-construction for φ_i, then take $E = \hat{E}_1 \oplus \hat{E}_2, x_1 = \hat{x}_1 \oplus 0$ and $x_2 = 0 \oplus \hat{x}_2$. For a common representation module E, define $S(E, \varphi_i)$ to be the set of all $x \in E$ such that $\varphi_i(a) = \langle x, ax \rangle$ for all $a \in \mathcal{A}$.

Definition 2.1.2. Let E be a common representation module for $\varphi_1, \varphi_2 \in CP(\mathcal{A}, \mathcal{B})$. Define

$$\beta_E(\varphi_1, \varphi_2) := \inf \big\{ \|x_1 - x_2\| : x_i \in S(E, \varphi_i), i = 1, 2 \big\}$$

and the *Bures distance*

$$\beta(\varphi_1, \varphi_2) := \inf_E \beta_E(\varphi_1, \varphi_2)$$

where the infimum is taken over all common representation module E.

We have called β as a 'distance' in anticipation. Later we will show that it is indeed a metric under most situations, for instance, when $\mathcal{B}$ is a von Neumann algebra. But surprisingly β is not a metric in general.

Our first job is to show that the definition here matches with that of [KSW08a]. We see it as follows. Suppose $\mathcal{B} = \mathscr{B}(G)$. If E is a common representation module and $x_i \in S(E, \varphi_i)$, then $(\rho, L_{x_i}, E \odot G)$ is a Stinespring representation for φ_i with $\|x_1 - x_2\| = \|L_{x_1} - L_{x_2}\|$. On the other way if (π', V_i, H') is a Stinespring representation for φ_i, then $E := \mathcal{B}(G, H')$ is a Hilbert[h] $\mathcal{A}$-$\mathcal{B}(G)$-module with inner product $\langle x_1, x_2 \rangle := x_1^* x_2$, composition as the right module action and left action given by $ax := \pi'(a)x$ for all $a \in \mathcal{A},\ x \in E$. Clearly (E, V_i) is a GNS-construction for φ_i. Note that $\overline{\operatorname{span}}\, EG = H'$. We have $H := E \odot G$ is a Hilbert space with inner product $\langle x \odot g, x' \odot g' \rangle = \langle g', x^* x' g' \rangle = \langle xg, x'g' \rangle$. Thus $x \odot g \mapsto xg$ defines a unitary $U : H \to H'$. Note that $UL_{V_i} = V_i$ and $U\rho(a)U^* = \pi'(a)$ for all $a \in \mathcal{A}$. Identifying H with H' through U, we get $\pi' = \rho$ and $L_{V_i} = V_i$. Therefore $(\pi', V_i, H') = (\rho, L_{V_i}, H)$. Thus there exists a one-one correspondence between the GNS-constructions $\{(E, x_1), (E, x_2)\}$ and the Stinespring representations $\{(\pi', V_1, H'), (\pi', V_2, H')\}$ such that $\|x_1 - x_2\| = \|V_1 - V_2\|$. Hence $\beta(\varphi_1, \varphi_2)$ coincides with the definition given in [KSW08a]. In particular, if $\mathcal{B} = \mathcal{B}(\mathbb{C}) = \mathbb{C}$, then $\beta(\varphi_1, \varphi_2)$ is the Bures distance given in [Bur69].

The following proposition says that $\beta(\varphi_1, \varphi_2)$ coincide with the alternative definition, given in [KSW08a], of Bures distance for CP-maps between arbitrary C^*-algebras. Subsequently will not be needing this definition and we present it here for the sake of completeness.

Proposition 2.1.3. *With notation as above,*

$$\beta(\varphi_1, \varphi_2) = \inf_{\varphi} \|\varphi_{11}(1) + \varphi_{22}(1) - \varphi_{12}(1) - \varphi_{21}(1)\|^{\frac{1}{2}}$$

where the infimum is taken over all CP-extensions $\varphi : \mathcal{A} \to M_2(\mathcal{B})$ of the form $\varphi = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}$ with completely bounded maps $\varphi_{ij} : \mathcal{A} \to \mathcal{B}$ satisfying $\varphi_{ii} = \varphi_i$.

Proof. Let E be a common representation module and $x_i \in S(E, \varphi_i)$. Define $\varphi : \mathcal{A} \to M_2(\mathcal{B})$ by $a \mapsto [\varphi_{ij}(a)]$, where $\varphi_{ij}(a) := \langle x_i, a x_j \rangle$. Then φ is a CP-map with

$$\begin{aligned} \|x_1 - x_2\|^2 &= \|\langle x_1, x_1 \rangle + \langle x_2, x_2 \rangle - \langle x_1, x_2 \rangle - \langle x_2, x_1 \rangle\| \\ &= \|\varphi_{11}(1) + \varphi_{22}(1) - \varphi_{12}(1) - \varphi_{21}(1)\|. \end{aligned}$$

[h]If $\mathcal{A}$ is a von Neumann algebra and π is normal, then $a \mapsto \langle x, ay \rangle = \langle x, \pi(a)y \rangle$ is normal map from $\mathcal{A} \to \mathcal{B}$ for all $x, y \in E$. Thus E can be made into a two-sided von Neumann $\mathcal{A}$-$\mathcal{B}(G)$-module.

Since E is arbitrary $\beta(\varphi_1, \varphi_2) \geq \inf_{\varphi} \|\varphi_{11}(1) + \varphi_{22}(1) - \varphi_{12}(1) - \varphi_{21}(1)\|^{\frac{1}{2}}$. To get the reverse inequality, assume that $\varphi = [\varphi_{ij}] : \mathcal{A} \to M_2(\mathcal{B})$ is a CP-map with $\varphi_{ii} = \varphi_i$. Let $(\hat{E}, \hat{x})$ be a GNS-construction of φ. Note that $\hat{E}$ is a Hilbert $\mathcal{A}$-$M_2(\mathcal{B})$-module. Given $b \in \mathcal{B}, x \in \hat{E}$ define $xb := x(bI)$, where $I \in M_2(\mathcal{B})$ is the identity matrix. Under this action $\hat{E}$ becomes a right $\mathcal{B}$-module. Now for $x_1, x_2 \in \hat{E}$ define $\langle x_1, x_2\rangle' := \sum_{i,j}\langle x_1, x_2\rangle_{ij}$, where $\langle x_1, x_2\rangle_{ij}$ is the $(i,j)^{\text{th}}$ entry of $\langle x_1, x_2\rangle \in M_2(\mathcal{B})$. Then $\langle\cdot,\cdot\rangle'$ is a $\mathcal{B}$-valued inner product on $\hat{E}$. Denote the resulting inner product $\mathcal{B}$-module by E_0. The left action of $\mathcal{A}$ on $\hat{E}$ induce a nondegenerate left action of $\mathcal{A}$ on E_0. Complete E_0 to get the Hilbert $\mathcal{A}$-$\mathcal{B}$-module E. Set $x_i = \hat{x}e_{ii}$, where $\{e_{ij}\},\ 1 \leq i,j \leq 2$ are matrix units of $M_2(\mathcal{B})$. Then $x_i \in S(E, \varphi_i)$ and

$$\begin{aligned}
\|x_1 - x_2\|^2 &= \|\langle x_1 - x_2, x_1 - x_2\rangle'\| \\
&= \|\langle x_1, x_1\rangle' + \langle x_2, x_2\rangle' - \langle x_1, x_2\rangle' - \langle x_2, x_1\rangle'\| \\
&= \|\langle \hat{x}, \hat{x}\rangle_{11} + \langle \hat{x}, \hat{x}\rangle_{22} - \langle \hat{x}, \hat{x}\rangle_{12} - \langle \hat{x}, \hat{x}\rangle_{21}\| \\
&= \|\varphi_{11}(1) + \varphi_{22}(1) - \varphi_{12}(1) - \varphi_{21}(1)\|.
\end{aligned}$$

Since φ is arbitrary $\beta(\varphi_1, \varphi_2) \leq \inf_{\varphi} \|\varphi_{11}(1) + \varphi_{22}(1) - \varphi_{12}(1) - \varphi_{21}(1)\|^{\frac{1}{2}}$. □

The following proposition says that Bures distance is stable under taking ampliations.

Proposition 2.1.4. *Suppose $\varphi, \psi \in CP(\mathcal{A}, \mathcal{B})$. Then $\beta(\varphi, \psi) = \beta(\varphi_n, \psi_n)$ where $\varphi_n, \psi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ are the amplifications of φ, ψ respectively for $n \geq 1$.*

Proof. Fix $n \geq 1$. Suppose E is a common representation module for φ, ψ and $x_1 \in S(E, \varphi), x_2 \in S(E, \psi)$. Then $\operatorname{diag}(x_1, \cdots, x_1) \in S(M_n(E), \varphi_n)$ and $\operatorname{diag}(x_2, \cdots, x_2) \in S(M_n(E), \psi_n)$, and hence

$$\beta(\varphi_n, \psi_n) \leq \|\operatorname{diag}(x_1 - x_2, \cdots, x_1 - x_2)\| = \|x_1 - x_2\|.$$

Since x_1, x_2 and E are arbitrary $\beta(\varphi_n, \psi_n) \leq \beta(\varphi, \psi)$. Conversely, suppose F is a common representation module for φ_n, ψ_n and $y_1 \in S(F, \varphi_n), y_2 \in S(F, \psi_n)$. If $\{e_{ij}\}, \{f_{ij}\},\ 1 \leq i,j \leq n$ are matrix units of $M_n(\mathcal{A}), M_n(\mathcal{B})$ respectively, then $E := \{e_{11}Ff_{11}\}$ is a common representation module for φ, ψ in the natural way and moreover, $e_{11}y_1f_{11} \in S(E,\varphi)$ and $e_{11}y_2f_{11} \in S(E,\psi)$. Also,

$$\|e_{11}y_1f_{11} - e_{11}y_2f_{11}\|^2 = \|f_{11}\langle e_{11}(y_1-y_2), e_{11}(y_1-y_2)\rangle f_{11}\| \leq \|y_1-y_2\|^2.$$

Therefore $\beta(\varphi,\psi) \leq \beta(\varphi_n,\psi_n)$. □

Proposition 2.1.5. *Let $\mathcal{A},\mathcal{B}$ and $\mathcal{C}$ be unital C^*-algebras. Then for $\varphi_i \in CP(\mathcal{A},\mathcal{B})$ and $\psi_i \in CP(\mathcal{B},\mathcal{C})$, $i=1,2$,*

$$\beta(\psi_1\circ\varphi_1, \psi_2\circ\varphi_2) \leq \|\varphi_1\|^{\frac{1}{2}}\beta(\psi_1,\psi_2) + \|\psi_2\|^{\frac{1}{2}}\beta(\varphi_1,\varphi_2).$$

In particular,

$$\beta(\psi_2\circ\varphi_1, \psi_2\circ\varphi_2) \leq \|\psi_2\|^{\frac{1}{2}}\beta(\varphi_1,\varphi_2).$$

Proof. Suppose E,F are common representation modules for φ_i,ψ_i respectively, and $x_i \in S(E,\varphi_i), y_i \in S(F,\psi_i)$, $i=1,2$. Then $x_i\odot y_i \in S(E\odot F, \psi_i\circ\varphi_i)$, and hence

$$\begin{aligned}\beta(\psi_1\circ\varphi_1, \psi_2\circ\varphi_2) &\leq \|x_1\odot y_1 - x_2\odot y_2\| \\ &\leq \|x_1\odot y_1 - x_1\odot y_2 + x_1\odot y_2 - x_2\odot y_2\| \\ &\leq \|x_1\|\,\|y_1-y_2\| + \|x_1-x_2\|\,\|y_2\| \\ &= \|\varphi_1\|^{\frac{1}{2}}\|y_1-y_2\| + \|x_1-x_2\|\,\|\psi_2\|^{\frac{1}{2}}.\end{aligned}$$

Since x_i, y_i, E and F are arbitrary the results holds. □

Proposition 2.1.6. *Let $\varphi_1,\varphi_2 \in CP(\mathcal{A},\mathcal{B})$. Then*

(i) $\beta(\varphi_1,\varphi_1+\varphi_2) \leq \|\varphi_2\|^{\frac{1}{2}}$.

(ii) $\left|\beta(\varphi_1,\varphi_2) - \beta(\varphi_1,\epsilon\varphi_1+(1-\epsilon)\varphi_2)\right| \leq \epsilon^{\frac{1}{2}}(\|\varphi_1\|^{\frac{1}{2}} + \|\varphi_2\|^{\frac{1}{2}})$ *for* $0\leq\epsilon\leq 1$.

(iii) *If* $\varphi_i(1)\leq 1$, *then* $\left|\|\varphi_1\| - \|\varphi_2\|\right| \leq 2\beta(\varphi_1,\varphi_2)$.

Proof. (i) Suppose (E_i,x_i) is a GNS-construction for φ_i, $i=1,2$. Then $z_1 := x_1\oplus 0 \in S(E_1\oplus E_2,\varphi_1)$ and $z_2 := x_1\oplus x_2 \in S(E_1\oplus E_2, \varphi_1+\varphi_2)$, and hence

$$\beta(\varphi_1,\varphi_1+\varphi_2) \leq \|z_1-z_2\| = \|x_2\| = \|\varphi_2\|^{\frac{1}{2}}.$$

(ii) Using triangle inequality and part (i),

$$\left|\beta(\varphi_1,\varphi_2) - \beta(\varphi_1,\epsilon\varphi_1+(1-\epsilon)\varphi_2)\right|$$

$$\begin{aligned}
&\leq \beta(\varphi_2, \epsilon\varphi_1 + (1-\epsilon)\varphi_2) \\
&\leq \beta(\varphi_2, (1-\epsilon)\varphi_2) + \beta((1-\epsilon)\varphi_2, \epsilon\varphi_1 + (1-\epsilon)\varphi_2) \\
&\leq \beta((1-\epsilon)\varphi_2, \varphi_2) + \beta((1-\epsilon)\varphi_2, \epsilon\varphi_1 + (1-\epsilon)\varphi_2) \\
&\leq \beta((1-\epsilon)\varphi_2, (1-\epsilon)\varphi_2 + \epsilon\varphi_2) + \beta((1-\epsilon)\varphi_2, \epsilon\varphi_1 + (1-\epsilon)\varphi_2) \\
&\leq \|\epsilon\varphi_2\|^{\frac{1}{2}} + \|\epsilon\varphi_1\|^{\frac{1}{2}} \\
&\leq \epsilon^{\frac{1}{2}}(\|\varphi_1\|^{\frac{1}{2}} + \|\varphi_2\|^{\frac{1}{2}}).
\end{aligned}$$

(iii) Let E be a common representation module for φ_1, φ_2 and $x_i \in S(E, \varphi_i)$. Then

$$\begin{aligned}
\big|\|\varphi_1\| - \|\varphi_2\|\big| &= \big|\|x_1\|^2 - \|x_2\|^2\big| \\
&= \big|(\|x_1\| + \|x_2\|)(\|x_1\| - \|x_2\|)\big| \\
&= (\|x_1\| + \|x_2\|)\big|\|x_1\| - \|x_2\|\big| \\
&\leq 2\,\|x_1 - x_2\|\,.
\end{aligned}$$

Since x_1, x_2 and E are arbitrary the result follows. □

2.2 Bures distance: von Neumann algebras

As is well-known one of the problems in dealing with Hilbert C^*-modules in contrast to Hilbert spaces is that in general submodules are not complemented, that is, there is a problem in taking orthogonal complements and writing the whole space as a direct sum. This problem is not there for von Neumann modules. Here we generalize almost all the results of [KSW08a], where the results stated mainly for the case when the range algebra is the algebra of all bounded operators on a Hilbert space. The proofs are similar, though we have also taken some ideas from [Bur69]. We also give several examples and answer a question of [KSW08a] in the negative.

In this Section we assume that $\mathcal{A}$ is a C^*-algebra, $\mathcal{B} \subseteq \mathcal{B}(G)$ is a von Neumann algebra and $\varphi_1, \varphi_2 \in CP(\mathcal{A}, \mathcal{B})$.

2.2.1 Metric property

To begin with we have the following proposition.

Proposition 2.2.1. *If $\mathcal{B} \subseteq \mathcal{B}(G)$ is a von Neumann algebra, then*

$$\beta(\varphi_1, \varphi_2) = \inf_{\mathsf{E}} \beta_{\mathsf{E}}(\varphi_1, \varphi_2) \tag{2.2.1}$$

where the infimum is taken over all common representation modules E which are von Neumann $\mathcal{A}$-$\mathcal{B}$-module.

Proof. Since von Neumann $\mathcal{B}$-modules are Hilbert $\mathcal{B}$-modules we have $\beta(\varphi_1, \varphi_2) \leq \inf \beta_{\mathsf{E}}(\varphi_1, \varphi_2)$. To get the reverse inequality, assume that E is a common representation module for φ_1, φ_2. Then $\mathsf{E} := \overline{E}^s \subseteq \mathcal{B}(G, E \odot G)$ forms a von Neumann $\mathcal{A}$-$\mathcal{B}$-module. Since $E \subseteq \mathsf{E}$ we have E is a common representation module for φ_1, φ_2, and hence $\inf \beta_{\mathsf{E}}(\varphi_1, \varphi_2) \leq \beta(\varphi_1, \varphi_2)$. □

As we have taken $\mathcal{B}$ as von Neumann algebra for this Section, we may use (2.2.1) as the definition of Bures distance. Also by a common representation module and GNS-module we will mean a von Neumann $\mathcal{A}$-$\mathcal{B}$-module. However, note that for all the results here, the algebra $\mathcal{A}$ can be a general C^*-algebra and the left action by $\mathcal{A}$ need not be normal. So we do not need that φ_1, φ_2 to be normal.

The following result shows the existence of a sort of universal module where we can take infimum to compute the Bures distance.

Proposition 2.2.2. *There exists a von Neumann $\mathcal{A}$-$\mathcal{B}$-module $\mathcal{E}$ such that:*

(i) *For all $\varphi_1, \varphi_2 \in CP(\mathcal{A}, \mathcal{B})$, $\beta(\varphi_1, \varphi_2) = \beta_{\mathcal{E}}(\varphi_1, \varphi_2)$.*

(ii) *For a fixed $\varphi_1 \in CP(\mathcal{A}, \mathcal{B})$ there exists $\xi_1 \in S(\mathcal{E}, \varphi_1)$ such that*

$$\beta(\varphi_1, \varphi_2) = \inf \left\{ \|\xi_1 - \xi_2\| : \xi_2 \in S(\mathcal{E}, \varphi_2) \right\}$$

for all $\varphi_2 \in CP(\mathcal{A}, \mathcal{B})$.

Proof. For each $\varphi \in CP(\mathcal{A}, \mathcal{B})$ fix a GNS-construction (E_φ, x_φ). Set $H_\varphi = E_\varphi \odot G$ and $H = \oplus H_\varphi$. Then $\mathcal{E}_0 := \overline{\oplus}^s E_\varphi \subseteq \mathcal{B}(G, H)$ is a von Neumann $\mathcal{A}$-$\mathcal{B}$-module. Note that $S(\mathcal{E}_0, \varphi)$ is nonempty for all $\varphi \in CP(\mathcal{A}, \mathcal{B})$. Take $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_0$ which is a von Neumann $\mathcal{A}$-$\mathcal{B}$-module.

(i) Suppose $\varphi_1, \varphi_2 \in CP(\mathcal{A}, \mathcal{B})$ and E is a common representation module. We will prove that $\beta_{\mathcal{E}}(\varphi_1, \varphi_2) \leq \beta_E(\varphi_1, \varphi_2)$. For that, it is enough to show that for all $x_i \in S(E, \varphi_i)$ there exists $\xi_i \in S(\mathcal{E}, \varphi_i)$ such that $\|\xi_1 - \xi_2\| \leq \|x_1 - x_2\|$. Take $\xi_1' \in S(\mathcal{E}_0, \varphi_1)$. Let $U : \overline{\mathsf{span}}^s \mathcal{A}\xi_1'\mathcal{B} \to \overline{\mathsf{span}}^s \mathcal{A}x_1\mathcal{B}$ be the bilinear unitary satisfying $U(a\xi_1'b) = ax_1b$. Let P be the bilinear projection of E onto $\overline{\mathsf{span}}^s \mathcal{A}x_1\mathcal{B}$. Set

$$\begin{aligned} x_2' &:= Px_2 \in \overline{\mathsf{span}}^s \mathcal{A}x_1\mathcal{B} \subseteq E, \\ x_2'' &:= (1-P)x_2 \in (\overline{\mathsf{span}}^s \mathcal{A}x_1\mathcal{B})^\perp \subseteq E, \\ \varphi_2'(\cdot) &:= \langle x_2', (\cdot)x_2'\rangle \text{ and} \\ \varphi_2''(\cdot) &:= \langle x_2'', (\cdot)x_2''\rangle. \end{aligned}$$

Clearly $\varphi_2 = \varphi_2' + \varphi_2''$. Let $\xi_2' = U^*(x_2') \in \overline{\mathsf{span}}^s \mathcal{A}\xi_1'\mathcal{B} \subseteq \mathcal{E}_0$. Then

$$\langle \xi_2', a\xi_2'\rangle = \langle U^*x_2', aU^*x_2'\rangle = \langle U^*x_2', U^*(ax_2')\rangle = \langle x_2', ax_2'\rangle = \varphi_2'(a).$$

Let $\xi_2'' \in S(\mathcal{E}_0, \varphi_2'')$. Set $\xi_1 = \xi_1' \oplus 0$ and $\xi_2 = \xi_2' \oplus \xi_2''$. Then $\xi_i \in S(\mathcal{E}, \varphi_i)$ with

$$\begin{aligned} \|\xi_1 - \xi_2\|^2 &= \|\langle \xi_1, \xi_1\rangle + \langle \xi_2, \xi_2\rangle - 2\mathsf{Re}(\langle \xi_1, \xi_2\rangle)\| \\ &= \|\langle \xi_1', \xi_1'\rangle + \langle \xi_2', \xi_2'\rangle + \langle \xi_2'', \xi_2''\rangle - 2\mathsf{Re}(\langle \xi_1', \xi_2'\rangle)\| \\ &= \|\langle \xi_1' - \xi_2', \xi_1' - \xi_2'\rangle + \langle \xi_2'', \xi_2''\rangle\| \\ &= \|\langle U(\xi_1' - \xi_2'), U(\xi_1' - \xi_2')\rangle + \langle \xi_2'', \xi_2''\rangle\| \\ &= \|\langle x_1 - x_2', x_1 - x_2'\rangle + \langle x_2'', x_2''\rangle\| \\ &= \|\langle x_1, x_1\rangle + \langle x_2', x_2'\rangle - 2\mathsf{Re}(\langle x_1, x_2'\rangle) + \langle x_2'', x_2''\rangle\| \\ &= \|\langle x_1, x_1\rangle + \langle x_2, Px_2\rangle - 2\mathsf{Re}(\langle x_1, x_2'\rangle) + \langle x_2, (1-P)x_2\rangle\| \\ &= \|\langle x_1, x_1\rangle + \langle x_2, x_2\rangle - 2\mathsf{Re}(\langle x_1, x_2'\rangle)\| \\ &= \|\langle x_1 - x_2, x_1 - x_2\rangle\| \qquad (x_1 = x_1 \oplus 0, x_2 = x_2' \oplus x_2'' \text{ in } E) \\ &= \|x_1 - x_2\|^2. \end{aligned}$$

Since x_1, x_2 and E are arbitrary $\beta_{\mathcal{E}}(\varphi_1, \varphi_2) \leq \beta(\varphi_1, \varphi_2)$.

(ii) Note that $\xi_1 \in S(\mathcal{E}, \varphi_1)$ is independent of E and φ_2. If we denote ξ_2 obtained in part(i) by $\xi_2(x_1, x_2)$, then

$$\begin{aligned} \beta_{\mathcal{E}}(\varphi_1, \varphi_2) &= \inf\left\{ \|\xi - \xi'\| : \xi \in S(\mathcal{E}, \varphi_1), \xi' \in S(\mathcal{E}, \varphi_2)\right\} \\ &\leq \inf\left\{ \|\xi_1 - \xi'\| : \xi' \in S(\mathcal{E}, \varphi_2)\right\} \\ &\leq \inf\left\{ \|\xi_1 - \xi_2(x_1, x_2)\| : x_i \in S(E, \varphi_i)\right\} \\ &= \inf\left\{ \|x_1 - x_2\| : x_i \in S(E, \varphi_i)\right\} \end{aligned}$$

$$= \beta_E(\varphi_1, \varphi_2).$$

Since this is true for all common representation module E, we get

$$\beta(\varphi_1, \varphi_2) \leq \beta_{\mathcal{E}}(\varphi_1, \varphi_2) \leq \inf \left\{ \|\xi_1 - \xi'\| : \xi' \in S(\mathcal{E}, \varphi_2) \right\} \leq \beta(\varphi_1, \varphi_2).$$

This completes the proof. □

Theorem 2.2.3. *β is a metric on $CP(\mathcal{A}, \mathcal{B})$.*

Proof. *Positive definiteness:* Let $\varphi_1, \varphi_2 \in CP(\mathcal{A}, \mathcal{B})$. Take $\mathcal{E}$ and $\xi_1 \in S(\mathcal{E}, \varphi_1)$ as in Proposition 2.2.2(ii). By definition $\beta(\varphi_1, \varphi_2) \geq 0$. Now if $\beta(\varphi_1, \varphi_2) = 0$, then

$$\inf \left\{ \|\xi_1 - \xi_2\| : \xi_2 \in S(\mathcal{E}, \varphi_2) \right\} = 0.$$

Since $S(\mathcal{E}, \varphi_2)$ is a norm closed subset of $\mathcal{E}$, above equality implies that $\xi_1 \in S(\mathcal{E}, \varphi_2)$. Therefore $\varphi_1 = \varphi_2$.
Symmetry: Clear from the definition.
Triangle inequality: Let $\varphi_1, \varphi_2, \varphi_3 \in CP(\mathcal{A}, \mathcal{B})$. Suppose $\mathcal{E}$ and $\xi_1 \in S(\mathcal{E}, \varphi_1)$ are as in Proposition 2.2.2(ii). Then

$$\begin{aligned} \beta(\varphi_2, \varphi_3) &= \inf \left\{ \|\xi_2 - \xi_3\| : \xi_i \in S(\mathcal{E}, \varphi_i), i = 2, 3 \right\} \\ &\leq \inf \left\{ \|\xi_2 - \xi_1\| : \xi_2 \in S(\mathcal{E}, \varphi_2) \right\} + \inf \left\{ \|\xi_1 - \xi_3\| : \xi_3 \in S(\mathcal{E}, \varphi_3) \right\} \\ &= \beta(\varphi_2, \varphi_1) + \beta(\varphi_1, \varphi_3). \end{aligned}$$

Thus β is a metric. □

2.2.2 Intertwiners and computation of Bures distance

The definition of Bures distance is abstract and does not give us indications as to how to compute it for concrete examples. In this Section, motivated by the work of [KSW08a], we show that Bures distance can be computed using intertwiners between two (minimal) GNS-constructions of CP-maps.

Suppose E is a common representation module for φ_i and $x_i \in S(E, \varphi_i),\ i = 1, 2$. Then $\|x_1 - x_2\|^2 = \|\langle x_1 - x_2, x_1 - x_2 \rangle\| = \|\varphi_1(1) + \varphi_2(1) - 2\mathsf{Re}(\langle x_1, x_2 \rangle)\|$. Thus $\beta(\varphi_1, \varphi_2)$ is completely determined by the subsets $\{\langle x_1, x_2 \rangle : x_i \in S(E, \varphi_i)\} \subseteq \mathcal{B}$.

This observation leads to the following definition.

Definition 2.2.4. Given a common representation module E for φ_1 and φ_2 define

$$N_E(\varphi_1, \varphi_2) := \big\{\langle x_1, x_2\rangle : x_i \in S(E, \varphi_i)\big\}$$

and

$$N(\varphi_1, \varphi_2) := \bigcup_E N_E(\varphi_1, \varphi_2)$$

where the union is taken over all common representation module E.

Note that $N(\varphi_1, \varphi_2) \subseteq \mathcal{B}$ is always nonempty. Also if E is a common representation module for φ_1 and φ_2, then

$$\beta_E(\varphi_1, \varphi_2) = \inf_{N \in N_E(\varphi_1, \varphi_2)} \|\varphi_1(1) + \varphi_2(1) - 2\mathsf{Re}(N)\|^{\frac{1}{2}} \tag{2.2.2}$$

with $\varphi_1(1) + \varphi_2(1) - 2\mathsf{Re}(N) = \langle x_1 - x_2, x_1 - x_2\rangle \geq 0$ for some $x_i \in S(E, \varphi_i)$.

Definition 2.2.5. Let (E_i, x_i) be a GNS-construction for φ_i, $i = 1, 2$. Then define

$$M(\varphi_1, \varphi_2) := \big\{\langle x_1, \Phi x_2\rangle : \Phi \in \mathfrak{B}^{a,bil}(E_2, E_1), \|\Phi\| \leq 1\big\}.$$

Lemma 2.2.6. *The set $M(\varphi_1, \varphi_2) \subseteq \mathcal{B}$ depends only on the CP-maps φ_i and not on the GNS-constructions (E_i, x_i).*

Proof. We show that $M(\varphi_1, \varphi_2)$ defined via (E_i, x_i) coincides with $\hat{M}(\varphi_1, \varphi_2)$ which is defined via the minimal GNS-construction $(\hat{E}_i, \hat{x}_i)$. Let $U_i : \hat{E}_i \to \overline{\mathsf{span}}^s \mathcal{A}x_i\mathcal{B}$ be the bilinear unitary satisfying $U_i(a\hat{x}_i b) = ax_i b$ for all $a \in \mathcal{A}, b \in \mathcal{B}$. Since $\overline{\mathsf{span}}^s \mathcal{A}x_i\mathcal{B} \subseteq E_i$ is a complemented $\mathcal{B}$-submodule, $U_i \in \mathfrak{B}^{a,bil}(\hat{E}_i, E_i)$ is an adjointable isometry (Corollary 1.2.37). Note that $U_i(\hat{x}_i) = x_i$ and $U_i^*(x_i) = \hat{x}_i$. Now suppose $\langle x_1, \Phi x_2\rangle \in M(\varphi_1, \varphi_2)$, where $\Phi \in \mathfrak{B}^{a,bil}(E_2, E_1)$ with $\|\Phi\| \leq 1$. Set $\hat{\Phi} = U_1^*\Phi U_2$. Then $\hat{\Phi} \in \mathfrak{B}^{a,bil}(\hat{E}_2, \hat{E}_1)$ with $\|\hat{\Phi}\| \leq 1$. Also

$$\langle x_1, \Phi x_2\rangle = \langle U_1\hat{x}_1, \Phi U_2\hat{x}_2\rangle = \langle \hat{x}_1, U_1^*\Phi U_2\hat{x}_2\rangle = \langle \hat{x}_1, \hat{\Phi}\hat{x}_2\rangle \in \hat{M}(\varphi_1, \varphi_2).$$

Hence $M(\varphi_1, \varphi_2) \subseteq \hat{M}(\varphi_1, \varphi_2)$. To get the reverse inclusion start with a $\hat{\Phi} \in \mathfrak{B}^{a,bil}(\hat{E}_2, \hat{E}_1)$ and set $\Phi = U_1\hat{\Phi}U_2^* \in \mathfrak{B}^{a,bil}(E_2, E_1)$. □

Proposition 2.2.7. *If (E_i, x_i) is a GNS-construction for φ_i, $i = 1, 2$, then*

(i) $M(\varphi_1, \varphi_2) = N(\varphi_1, \varphi_2) = N_{E_1 \oplus E_2}(\varphi_1, \varphi_2)$ *and*

(ii) $\beta(\varphi_1, \varphi_2) = \inf_{M \in M(\varphi_1,\varphi_2)} \|\varphi_1(1) + \varphi_2(1) - 2\mathit{Re}(M)\|^{\frac{1}{2}}$.

Proof. (i) Suppose E is a common representation module and $\langle z_1, z_2\rangle \in N_E(\varphi_1, \varphi_2)$. Set $E_1 = E_2 = E$ and $\Phi = \mathrm{id}_E$. Then, from above Lemma, $\langle z_1, z_2\rangle = \langle z_1, \Phi z_2\rangle \in M(\varphi_1, \varphi_2)$. Since z_1, z_2 and E are arbitrary $N(\varphi_1, \varphi_2) \subseteq M(\varphi_1, \varphi_2)$. In particular, $M(\varphi_1, \varphi_2)$ is nonempty. For the reverse inclusion, let $\langle x_1, \Phi x_2\rangle \in M(\varphi_1, \varphi_2)$. Set $z_1 = x_1 \oplus 0$ and $z_2 = \Phi x_2 \oplus \sqrt{\mathrm{id}_{E_2} - \Phi^*\Phi}x_2$ in $E_1 \oplus E_2$. Then $\langle z_1, az_1\rangle = \langle x_1, ax_1\rangle = \varphi_1(a)$ and

$$\begin{aligned}
\langle z_2, az_2\rangle &= \langle \Phi x_2 \oplus \sqrt{\mathrm{id}_{E_2} - \Phi^*\Phi}x_2, a(\Phi x_2) \oplus a\sqrt{\mathrm{id}_{E_2} - \Phi^*\Phi}x_2\rangle \\
&= \langle \Phi x_2, \Phi(ax_2)\rangle + \langle \sqrt{\mathrm{id}_{E_2} - \Phi^*\Phi}x_2, \sqrt{\mathrm{id}_{E_2} - \Phi^*\Phi}ax_2\rangle \\
&= \langle x_2, \Phi^*\Phi(ax_2)\rangle + \langle x_2, (\mathrm{id}_{E_2} - \Phi^*\Phi)ax_2\rangle \\
&= \langle x_2, ax_2\rangle \\
&= \varphi_2(a)
\end{aligned}$$

for all $a \in \mathcal{A}$. Thus $(E_1 \oplus E_2, z_i)$ is a GNS-construction for φ_i. Note that $\langle x_1, \Phi x_2\rangle = \langle z_1, z_2\rangle \in N_{E_1\oplus E_2}(\varphi_1, \varphi_2)$. Hence $M(\varphi_1, \varphi_2) \subseteq N_{E_1\oplus E_2}(\varphi_1, \varphi_2)$. Thus $N(\varphi_1, \varphi_2) \subseteq M(\varphi_1, \varphi_2) \subseteq N_{E_1\oplus E_2}(\varphi_1, \varphi_2) \subseteq N(\varphi_1, \varphi_2)$.

(ii) Follows from equation (2.2.2). □

Corollary 2.2.8. *If (E_i, x_i) is a GNS-construction for φ_i, $i = 1, 2$, then*

$$\beta(\varphi_1, \varphi_2) = \beta_{E_1\oplus E_2}(\varphi_1, \varphi_2) = \inf\left\{ \|x_1 \oplus 0 - y_1 \oplus y_2\| : y_1 \oplus y_2 \in S(E_1 \oplus E_2, \varphi_2)\right\}.$$

Proof. Suppose $\langle x_1, \Phi x_2\rangle \in M(\varphi_1, \varphi_2)$. Then, from the proof of Proposition 2.2.7, we have $\langle x_1, \Phi x_2\rangle = \langle z_1, z_2\rangle$, where $z_i \in S(E_1 \oplus E_2, \varphi_i)$ with $z_1 = x_1 \oplus 0$. Denote the z_2 obtained by $z_2(\Phi)$. Then, from proposition 2.2.7(ii),

$$\begin{aligned}
&\beta(\varphi_1, \varphi_2) \\
&\quad = \inf\left\{ \|\varphi_1(1) + \varphi_2(1) - 2\mathsf{Re}(M)\|^{\frac{1}{2}} : M \in M(\varphi_1, \varphi_2)\right\} \\
&\quad = \inf\left\{ \|\varphi_1(1) + \varphi_2(1) - 2\mathsf{Re}(\langle x_1 \oplus 0, z_2(\Phi)\rangle)\|^{\frac{1}{2}} : \Phi \in \mathcal{B}^{a,bil}(E_2, E_1), \|\Phi\| \leq 1\right\} \\
&\quad \geq \inf\left\{ \|\varphi_1(1) + \varphi_2(1) - 2\mathsf{Re}(\langle x_1 \oplus 0, y_1 \oplus y_2\rangle)\|^{\frac{1}{2}} : y_1 \oplus y_2 \in S(E_1 \oplus E_2, \varphi_2)\right\}
\end{aligned}$$

$$= \inf\left\{ \|x_1 \oplus 0 - y_1 \oplus y_2\| : y_1 \oplus y_2 \in S(E_1 \oplus E_2, \varphi_2)\right\}$$
$$\geq \beta_{E_1 \oplus E_2}(\varphi_1, \varphi_2).$$

□

Example 2.2.9. Let $(X, \mathbb{F}, \mu)$ be a measure space and let $\mathcal{A} = L^\infty(X, \mu)$. Consider the states $\varphi_i : \mathcal{A} \to \mathbb{C}$ given by $\varphi_i(f) = \int f d\mu_i$, where μ_1 and μ_2 are two equivalent (i.e., absolutely continuous each other) probability measures on $(X, \mathbb{F})$ such that $\mu_i << \mu, i = 1, 2$. Let h be a positive function (Radon Nikodym derivative) on X such that $d\mu_1 = h d\mu_2$. Clearly $E_i = L^2(X, \mu_i)$ is a von Neumann $\mathcal{A}$-$\mathbb{C}$-module with left multiplication as the left action. Also $(E_i, 1)$ is a GNS-construction for φ_i. Suppose $g_1 \oplus g_2 \in S(E_1 \oplus E_2, \varphi_2)$. Then

$$\begin{aligned}\int f d\mu_2 &= \langle g_1 \oplus g_2, f(g_1 \oplus g_2)\rangle \\ &= \int |g_1|^2 f d\mu_1 + \int |g_2|^2 f d\mu_2 \\ &= \int (|g_1|^2 h + |g_2|^2) f d\mu_2\end{aligned}$$

for all $f \in \mathcal{A}$, and hence $|g_1|^2 h + |g_2|^2 = 1$ a.e., μ_2. Therefore

$$\begin{aligned}\beta(\varphi_1, \varphi_2) &= \inf\left\{ \|1 \oplus 0 - g_1 \oplus g_2\| : g_1 \oplus g_2 \in S(E_1 \oplus E_2, \varphi_2)\right\} \\ &= \inf\left\{(\langle 1 - g_1, 1 - g_1\rangle + \langle g_2, g_2\rangle)^{\frac{1}{2}} : |g_1|^2 h + |g_2|^2 = 1 \text{ a.e., } \mu_2\right\} \\ &= \inf\left\{(2 - 2\mathsf{Re}(\int g_1 d\mu_1))^{\frac{1}{2}} : |g_1|^2 h \leq 1 \text{ a.e., } \mu_2\right\} \\ &= \sqrt{2} \inf\left\{(1 - \int g_1 h d\mu_2)^{\frac{1}{2}} : g_1 \geq 0 \text{ and } 0 \leq g_1^2 h \leq 1 \text{ a.e., } \mu_2\right\} \\ &= \sqrt{2}(1 - \int \sqrt{h} d\mu_2)^{\frac{1}{2}}.\end{aligned}$$

In particular, if we take $X = \{1, 2, \ldots, n\}, \mu$ the counting measure, $\mu_1(i) = p_i$ and $\mu_2(i) = q_i$, where $0 < p_i, q_i < 1$ such that $\sum p_i = \sum q_i = 1$, then $\beta(\varphi_1, \varphi_2) = \sqrt{2}(1 - \sum \sqrt{p_i q_i})^{\frac{1}{2}}$.

Here we compute the Bures distance for homomorphisms and for some other special cases.

Corollary 2.2.10. *Let $\varphi_1, \varphi_2 : \mathcal{A} \to \mathcal{B}$ be two unital $*$-homomorphisms.*

(i) $\beta(\varphi_1, \varphi_2) = \sqrt{2} \inf \big\{ \|1 - \mathit{Re}(b)\|^{\frac{1}{2}} : b \in \mathcal{B}, \|b\| \le 1, \varphi_1(a)b = b\varphi_2(a)\ \forall a \in \mathcal{A} \big\}.$

(ii) *If $\mathcal{A} = \mathcal{B}$ and $\varphi_2(a) = u^*\varphi_1(a)u$ for some unitary $u \in \mathcal{B}$, then*

$$\beta(\varphi_1, \varphi_2) = \sqrt{2} \inf \big\{ \|1 - \mathit{Re}(b'u)\|^{\frac{1}{2}} : b' \in \varphi_1(\mathcal{A})', \|b'\| \le 1 \big\}.$$

(iii) *If $u \in M_n(\mathbb{C})$ is a unitary and $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is the $*$-homomorphism $\varphi(a) = u^*au$, then*

$$\beta(id, \varphi) = \sqrt{2} \inf \big\{ \|1 - \mathit{Re}(\lambda u)\|^{\frac{1}{2}} : \lambda \in \mathbb{C}, |\lambda| \le 1 \big\}.$$

Proof. (i) Let E_i be the von Neumann $\mathcal{A}$-$\mathcal{B}$-module $\mathcal{B}$ with left action $ax := \varphi_i(a)x$ for all $a \in \mathcal{A}$, $x \in E_i$. Then $(E_i, 1)$ is the minimal GNS-construction for φ_i. Suppose $\Phi \in \mathfrak{B}^{a,bil}(E_2, E_1)$. Then

$$\varphi_1(a)\Phi(1) = a\Phi(1) = \Phi(a1) = \Phi(\varphi_2(a)) = \Phi(1)\varphi_2(a)$$

for all $a \in \mathcal{A}$. Clearly, for a fixed $b_0 \in \mathcal{B}$ satisfying $\varphi_1(a)b_0 = b_0\varphi_2(a)$, the map $b \mapsto b_0 b$ is an element of $\mathfrak{B}^{a,bil}(E_2, E_1)$. Thus

$$\begin{aligned}
&\beta(\varphi_1, \varphi_2)\\
&\quad = \inf \big\{ \|\varphi_1(1) + \varphi_2(1) - 2\mathrm{Re}(M)\|^{\frac{1}{2}} : M \in M(\varphi_1, \varphi_2) \big\}\\
&\quad = \inf \big\{ \big\|2 - 2\mathrm{Re}(\langle 1, \Phi(1)\rangle)\big\|^{\frac{1}{2}} : \Phi \in \mathfrak{B}^{a,bil}(E_2, E_1), \|\Phi\| \le 1 \big\}\\
&\quad = \sqrt{2} \inf_{\Phi \in \mathfrak{B}(E_2, E_1)} \big\{ \|1 - \mathrm{Re}(\Phi(1))\|^{\frac{1}{2}} : \Phi(1)\varphi_2(a) = \varphi_1(a)\Phi(1)\ \forall a \in \mathcal{A}, \|\Phi\| \le 1 \big\}\\
&\quad = \sqrt{2} \inf \big\{ \|1 - \mathrm{Re}(b)\|^{\frac{1}{2}} : b \in \mathcal{B}, \|b\| \le 1, \varphi_1(a)b = b\varphi_2(a)\ \forall a \in \mathcal{A} \big\}.
\end{aligned}$$

(ii) Suppose $b \in \mathcal{B}$. Then $\varphi_1(a)b = b\varphi_2(a)$ for all $a \in \mathcal{A}$ implies that $bu^* \in \varphi_1(\mathcal{A})'$, and hence $b = b'u$ for some $b' \in \varphi_1(\mathcal{A})' \subseteq \mathcal{B}$.

(iii) This follows from (ii), since $M_n' = \mathbb{C}I$. □

In [KSW08a] it is shown that the Bures distance is comparable with completely bounded norm when $\mathcal{B} = \mathfrak{B}(G)$, and the following bounds were obtained.

Theorem 2.2.11 ([KSW08a]). *For* $\varphi_1, \varphi_2 \in CP(\mathcal{A}, \mathcal{B}(G))$,

$$\frac{\|\varphi_1 - \varphi_2\|_{cb}}{\sqrt{\|\varphi_1\|_{cb}} + \sqrt{\|\varphi_2\|_{cb}}} \leq \beta(\varphi_1, \varphi_2) \leq \sqrt{\|\varphi_1 - \varphi_2\|_{cb}}.$$

Moreover, there exists a common representation module E *and corresponding GNS-construction* (E, x_i) *for* φ_i *such that* $\beta(\varphi_1, \varphi_2) = \beta_E(\varphi_1, \varphi_2) = \|x_1 - x_2\|$.

In fact, from the the standard properties of operator norm, it follows that the lower bound holds even for an arbitrary unital C^*-algebra $\mathcal{B}$.

Proposition 2.2.12. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* C^**-algebras and* $\varphi_1, \varphi_2 \in CP(\mathcal{A}, \mathcal{B})$. *Then*

$$\frac{\|\varphi_1 - \varphi_2\|_{cb}}{\sqrt{\|\varphi_1\|_{cb}} + \sqrt{\|\varphi_2\|_{cb}}} \leq \beta(\varphi_1, \varphi_2).$$

Proof. Let E be a common representation module for φ_1, φ_2 and let $x_i \in S(E, \varphi_i)$. Let $A = [a_{ij}] \in M_n(\mathcal{A})$ and $X_i = \text{diag}(L_{x_i}, \cdots, L_{x_i}) \in M_n(\mathcal{B}(G, E \odot G))$ for $n \in \mathbb{N}$. Then

$$\begin{aligned}
\|(\varphi_1 - \varphi_2)_n(A)\| &= \|[\langle x_1, a_{ij}x_1\rangle] - [\langle x_2, a_{ij}x_2\rangle]\| \\
&= \left\|[L_{x_1}^*(a_{ij} \odot id_G)L_{x_1}] - [L_{x_2}^*(a_{ij} \odot id_G)L_{x_2}]\right\| \\
&= \|X_1^*(A \odot id_G)X_1 - X_2^*(A \odot id_G)X_2\| \\
&\leq \|X_1^*(A \odot id_G)(X_1 - X_2) + (X_1^* - X_2^*)(A \odot id_G)X_2\| \\
&\leq \|X_1 - X_2\| \left(\|X_1\| + \|X_2\|\right) \|A \odot id_G\| \\
&= \|x_1 - x_2\| \left(\|x_1\| + \|x_2\|\right) \|A\| \\
&= \|x_1 - x_2\| \left(\sqrt{\|\varphi_1\|_{cb}} + \sqrt{\|\varphi_2\|_{cb}}\right) \|A\|, \quad \forall n \in \mathbb{N}.
\end{aligned}$$

Hence $\|\varphi_1 - \varphi_2\|_{cb} \leq \|x_1 - x_2\| \left(\sqrt{\|\varphi_1\|_{cb}} + \sqrt{\|\varphi_2\|_{cb}}\right)$. Since E is arbitrary the results follows from above inequality. □

Example 2.2.13. In general, the upper bound given in Theorem 2.2.11 may fails to hold if the cb-norm is replaced by the operator norm. For example, consider the

CP-maps $\varphi_i : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ given by

$$\varphi_1([a_{ij}]) := \begin{bmatrix} a_{11}+2a_{22} & a_{21} \\ a_{12} & a_{22}+2a_{11} \end{bmatrix} \quad \text{and} \quad \varphi_2([a_{ij}]) := \begin{bmatrix} 2a_{22} & 0 \\ 0 & 2a_{11} \end{bmatrix}.$$

Let $E = M_{8\times 2}(\mathbb{C})$ which is a von Neumann $M_2(\mathbb{C})$-$M_2(\mathbb{C})$-module with module actions given by

$$axb := \begin{bmatrix} ax_1b \\ ax_2b \\ ax_3b \\ ax_4b \end{bmatrix} \qquad \forall x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in E \text{ and } a, b, x_i \in M_2(\mathbb{C}).$$

Then E is a common representation module with

$$z_1 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 0 & 0 & 1 & \frac{\sqrt{3}}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}^{\mathrm{t}} \in S(E, \varphi_1)$$

and

$$z_2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}^{\mathrm{t}} \in S(E, \varphi_2).$$

Note that if $x \oplus y = [x_{ij}] \oplus [y_{ij}] \in S(E \oplus E, \varphi_2)$, then by evaluating φ_2 at matrix units, we see that $x_{i1} = y_{i1} = 0 = x_{k2} = y_{k2},\ i = 1,3,5,7,\ k = 2,4,6,8$ and

$$\left.\begin{aligned} &\sum_{i=2,4,6,8} (\overline{x_{i1}}x_{i-1,2} + \overline{y_{i1}}y_{i-1,2}) = 0, \\ &\sum_{i=2,4,6,8} (|x_{i1}|^2 + |y_{i1}|^2) = 2 = \sum_{i=1,3,5,7} (|x_{i2}|^2 + |y_{i2}|^2). \end{aligned}\right\} \qquad (*)$$

Hence

$$\begin{aligned} \beta(\varphi_1, \varphi_2) &= \inf \left\{ \|z_1 \oplus 0 - x \oplus y\| : x \oplus y \in E_1 \oplus E_2 \text{ satisfying } (*) \right\} \\ &= \inf_{\substack{x\oplus y \in E_1 \oplus E_2 \\ \text{satisfying } (*)}} \left\| \varphi_1(1) + \varphi_2(1) - \mathsf{Re}(\langle z_1 \oplus 0, x \oplus y \rangle) \right\|^{\frac{1}{2}} \\ &= \inf_{\substack{x\oplus y \in E_1 \oplus E_2 \\ \text{satisfying } (*)}} \left\| \begin{bmatrix} 5 - \mathsf{Re}(\sqrt{6}x_{61} - \sqrt{2}x_{81}) & -x_{12} - \overline{x_{41}} \\ -\overline{x_{12}} - x_{41} & 5 - \mathsf{Re}(\sqrt{6}x_{52} + \sqrt{2}x_{72}) \end{bmatrix} \right\|^{\frac{1}{2}} \end{aligned}$$

$$\geq \inf_{\substack{x\oplus y\in E_1\oplus E_2\\ \text{satisfying } (*)}} \left\| \begin{bmatrix} 5-\mathrm{Re}(\sqrt{6}x_{61}-\sqrt{2}x_{81}) & 0 \\ 0 & 5-\mathrm{Re}(\sqrt{6}x_{52}+\sqrt{2}x_{72}) \end{bmatrix} \right\|^{\frac{1}{2}}$$

$$= \inf_{\substack{|x_{61}|^2+|x_{81}|^2\leq 2\\ |x_{52}|^2+|x_{72}|^2\leq 2\\ \overline{x_{61}x_{52}+x_{81}x_{72}}=0}} \left\| \begin{bmatrix} 5-\mathrm{Re}(\sqrt{6}x_{61}-\sqrt{2}x_{81}) & 0 \\ 0 & 5-\mathrm{Re}(\sqrt{6}x_{52}+\sqrt{2}x_{72}) \end{bmatrix} \right\|^{\frac{1}{2}}$$

$$= \inf_{\substack{0\leq x_{52},x_{61},x_{72}\\ x_{81}\leq 0\\ x_{61}^2+x_{81}^2\leq 2\\ x_{52}^2+x_{72}^2\leq 2\\ x_{61}x_{52}+x_{81}x_{72}=0}} \left\| \begin{bmatrix} 5-\sqrt{6}x_{61}+\sqrt{2}x_{81} & 0 \\ 0 & 5-\sqrt{6}x_{52}-\sqrt{2}x_{72} \end{bmatrix} \right\|^{\frac{1}{2}}$$

$$= \sqrt{5-\sqrt{2}-\sqrt{6}}.$$

Note that $\|z_1 - z_2\| = \sqrt{5-\sqrt{2}-\sqrt{6}}$, and hence $\beta(\varphi_1,\varphi_2) = \sqrt{5-\sqrt{2}-\sqrt{6}} > 1$. But $\varphi_1 - \varphi_2$ is the transpose map. Therefore $1 = \|\varphi_1-\varphi_2\| < \beta(\varphi_1,\varphi_2)^2 < \|\varphi_1-\varphi_2\|_{cb} = 2$ (see [Pau02] for the computation of cb-norm for transpose map).

Theorem 2.2.11 guarantees the existence of a common representation module, where Bures distance is attained. It is a natural question as to whether Bures distance is attained in every common representation module. This is true for states ([Ara72]). The question in the general case was asked by [KSW08a, KSW08b]. Here we resolve it in the negative through a simple counter example.

Example 2.2.14. Consider the (normal) CP-maps $\varphi_i : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ given by $\varphi_i(a) := a_i^* a a_i$, where $a_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $a_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $(\hat{E}_i, \hat{x}_i) := (M_2(\mathbb{C}), a_i)$ is the minimal GNS-construction for φ_i. Set $x_1 = \hat{x}_1 \oplus 0$ and $x_2 = 0 \oplus \hat{x}_2$. Then $x_i \in S(\hat{E}_1 \oplus \hat{E}_2, \varphi_i)$ and

$$\beta(\varphi_1,\varphi_2) = \beta_{\hat{E}_1\oplus\hat{E}_2}(\varphi_1,\varphi_2) \leq \|x_1 - x_2\| = \|I\| = 1.$$

Clearly, $E := M_2(\mathbb{C})$ is a common representation module. If $x_i \in S(E,\varphi_i)$, then $x_i^* a x_i = a_i^* a a_i$ for all $a \in M_2(\mathbb{C})$. In particular taking $a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ we see that $S(E,\varphi_i) = \{\lambda a_i : \lambda \in \mathbb{C}, |\lambda| = 1\}$. Now for any $x_i = \lambda_i a_i \in S(E,\varphi_i)$,

$$\|x_1 - x_2\|^2 = \left\| \begin{bmatrix} 1 & -\overline{\lambda}_1\lambda_2 \\ -\overline{\lambda}_2\lambda_1 & 1 \end{bmatrix} \right\| = \sup\left\{ |\lambda| : \lambda \in \sigma\left(\begin{bmatrix} 1 & -\overline{\lambda}_1\lambda_2 \\ -\overline{\lambda}_2\lambda_1 & 1 \end{bmatrix} \right) \right\} = 2.$$

Hence $\beta_E(\varphi_1, \varphi_2) = \sqrt{2} > 1 \geq \beta(\varphi_1, \varphi_2)$. Note that here $\beta(\varphi_1, \varphi_2) \leq 1 = \sqrt{\|\varphi_1 - \varphi_2\|}$.

Conjecture. *If $\varphi, \psi \in CP(\mathcal{A}, \mathcal{B})$, then $\beta(\varphi, \psi) = \sup_{\phi, n} \beta(\phi \circ \varphi_n, \phi \circ \psi_n)$ where the supremum is taken over all states $\phi : M_n(\mathcal{B}) \to \mathbb{C}, n \in \mathbb{N}$.*

From Proposition 2.1.4 and 2.1.5 we have $\beta(\phi \circ \varphi_n, \phi \circ \psi_n) \leq \beta(\varphi_n, \psi_n) = \beta(\varphi, \psi)$ for all states $\phi : M_n(\mathcal{B}) \to \mathbb{C}, n \geq 1$. If the conjecture can be proved directly, then using the upper bound for states [Bur69, KSW08a] we get an alternative proof of the upper bound for Bures metric:

$$\beta(\varphi, \psi) = \sup_{\phi, n} \beta(\phi \circ \varphi_n, \phi \circ \psi_n) \leq \sup_{\phi, n} \sqrt{\|\phi \circ \varphi_n - \phi \circ \psi_n\|} = \sqrt{\|\varphi - \psi\|_{cb}}.$$

2.3 Bures distance: C^*-algebras

This Section consists mostly of counter examples. But results similar to the last section do hold for injective C^*-algebras.

2.3.1 Counter examples

We saw that if the range algebras are von Neumann algebras, then the Bures metric can be computed using intertwiners. It was crucial that the space of intertwiners was independent of the choice of GNS-constructions (Lemma 2.2.6). The first example here shows that this is no longer the case for some range C^*-algebras. We have another example to show that the upper bound computed for β in Theorem 2.2.11 may not hold for general range C^*-algebras. Finally, as a worst case scenario we have a tricky example to show that even the triangle inequality may fail to hold.

Example 2.3.1. If φ_1 and φ_2 are CP-maps between C^*-algebras, then $M(\varphi_1, \varphi_2)$ may depends on the GNS-construction. For example, consider the CP-maps $\varphi_i : C([0, 2\pi]) \to C([0, 2\pi])$ given by $\varphi_i(f) := g_i f$, where $g_i(t) = |sin(t)|^i$ for all $t \in [0, 2\pi]$, $i = 1, 2$. Set $\hat{x}_i = \sqrt{g_i}$ and

$$\hat{E}_i = \overline{\mathsf{span}} \{\sqrt{g_i} f : f \in C([0, 2\pi])\}$$

$$= \{f \in C([0, 2\pi]) : f(0) = f(\pi) = f(2\pi) = 0\}.$$

Then $(\hat{E}_i, \hat{x_i})$ is the minimal GNS-construction for φ_i. Define the adjointable bilinear map $\hat{\Phi} : \hat{E}_2 \to \hat{E}_1$ by $\hat{\Phi}(f) = gf$, where

$$g(t) = \begin{cases} \frac{1}{2} & \text{if} \quad 0 \leq t < \pi, \\ 1 & \text{if} \quad \pi \leq t \leq 2\pi. \end{cases}$$

Since $\hat{\Phi}$ is a contraction $\langle \hat{x_1}, \hat{\Phi}\hat{x_2} \rangle \in \hat{M}(\varphi_1, \varphi_2)$. We have $(E_i, x_i) := (C([0, 2\pi]), \hat{x_i})$ is also a GNS-construction for φ_i. Now if $\Phi : E_2 \to E_1$ is an adjointable bilinear map, then $\Phi(f) = \Phi(1)f$ for all $f \in C([0, 2\pi])$. Thus $\mathcal{B}^{a,bil}(E_2, E_1) = \{f \mapsto hf : h \in C([0, 2\pi])\}$. Hence if $\langle \hat{x_1}, g\hat{x_2} \rangle = \langle \hat{x_1}, \hat{\Phi}\hat{x_2} \rangle \in M(\varphi_1, \varphi_2)$, then $\langle \hat{x_1}, g\hat{x_2} \rangle = \langle \hat{x_1}, h\hat{x_2} \rangle$ for some $h \in C([0, 2\pi])$; i.e.,

$$\begin{aligned} \hat{x_1}(t)g(t)\hat{x_2}(t) &= \hat{x_1}(t)h(t)\hat{x_2}(t), \quad \forall t \in [0, 2\pi] \\ \Rightarrow \qquad g(t) &= h(t), \qquad\qquad\qquad \forall t \in [0, 2\pi] \setminus \{0, \pi, 2\pi\} \end{aligned}$$

which is not possible since h is continuous on $[0, 2\pi]$. So $\langle \hat{x_1}, \hat{\Phi}\hat{x_2} \rangle \notin M(\varphi_1, \varphi_2)$.

Example 2.3.2. Suppose H is an infinite dimensional Hilbert space and $p \in \mathcal{B}(H)$ is an orthogonal projection such that both p and $q := (1 - p)$ have infinite rank. Let $\mathcal{A} = C^*\{\mathcal{K}(H) \cup \{I\}\}$ and let $u = \lambda p + \overline{\lambda} q$, where $\lambda = e^{i\theta}$ is a scalar with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Note that $u \in \mathcal{B}(H)$ is a unitary. Define $*$-homomorphisms $\varphi_i : \mathcal{A} \to \mathcal{A}$ by $\varphi_1(a) := a$ and $\varphi_2(a) := u^*au$. Now suppose E is a common representation module for φ_1, φ_2 and $x_i \in S(E, \varphi_i)$. Since $\|ax_i - x_i\varphi_i(a)\| = 0$, we get $ax_i = x_i\varphi_i(a)$ for all $a \in \mathcal{A}$. Then

$$a\langle x_1, x_2 \rangle = \varphi_1(a)\langle x_1, x_2 \rangle = \langle x_1, x_2 \rangle \varphi_2(a) = \langle x_1, x_2 \rangle u^*au$$

for all $a \in \mathcal{A}$, and hence $\langle x_1, x_2 \rangle u^* \in \mathcal{A}'$. Therefore $\langle x_1, x_2 \rangle = \lambda' u$ for some $\lambda' \in \mathbb{C}$. Since $\langle x_1, x_2 \rangle \in \mathcal{A}$ and $u \notin \mathcal{A}$ we have $\lambda' = 0$, whence $\langle x_1, x_2 \rangle = 0$. Also since E and $x_i \in S(E, \varphi_i)$ are arbitrary

$$\beta(\varphi_1, \varphi_2) = \inf_{E, x_i} \|x_1 - x_2\| = \|\varphi_1(1) + \varphi_2(1)\|^{\frac{1}{2}} = \sqrt{2}.$$

Now we prove that $\sqrt{\|\varphi_1 - \varphi_2\|_{cb}} < \beta(\varphi_1, \varphi_2)$. For $a = [a_{ij}] \in \mathcal{B}(H) = \mathcal{B}(H_p \oplus H_p^\perp)$, where $H_p = \mathsf{ran}(p)$,

$$\begin{aligned}\|\varphi_1(a) - \varphi_2(a)\| &= \|a - u^*au\| \\ &= \left\| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix}^* \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} 0 & (1-\overline{\lambda}^2)a_{12} \\ (1-\lambda^2)a_{21} & 0 \end{bmatrix} \right\| \\ &= \max\left\{ \left\|(1-\overline{\lambda}^2)a_{12}\right\|, \left\|(1-\lambda^2)a_{21}\right\| \right\} \\ &\leq \left|1-\lambda^2\right| \|a\|\end{aligned}$$

so that $\|\varphi_1 - \varphi_2\| \leq |1-\lambda^2|$. But $a = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ is of norm one and $\|(\varphi_1 - \varphi_2)(a)\| = |1-\lambda^2|$, whence $\|\varphi_1 - \varphi_2\| = |1-\lambda^2| = \left|\lambda(\overline{\lambda} - \lambda)\right| = \left|\overline{\lambda} - \lambda\right|$. Now for all $n \geq 1$, if we let U_n, P_n and Q_n denote the $n \times n$ diagonal matrix with diagonal u, p and q respectively, then $U_n = \lambda P_n + \overline{\lambda} Q_n$ and $(\varphi_1 - \varphi_2)_n(A) = A - U_n^* A U_n$ for all $A \in M_n(\mathcal{A})$. Then, as above, we get $\|(\varphi_1 - \varphi_2)_n\| = \left|\overline{\lambda} - \lambda\right|$. Thus

$$\sqrt{\|\varphi_1 - \varphi_2\|} = \sqrt{\|\varphi_1 - \varphi_2\|_{cb}} = \sqrt{\left|\overline{\lambda} - \lambda\right|} < \sqrt{2} = \beta(\varphi_1, \varphi_2).$$

Now if φ_i is considered as a map into $\mathcal{B}(H)$ denote it by $\widetilde{\varphi}_i$. Then $b \in \widetilde{\varphi}_1(\mathcal{A})' \subseteq \mathcal{B}(H)$ implies that $ba = ab$ for all $a \in \mathcal{K}(H) \subseteq \mathcal{A}$, so that $b = \lambda_b I$ for some $\lambda_b \in \mathbb{C}$. From Corollary 2.2.10,

$$\begin{aligned}\beta(\widetilde{\varphi}_1, \widetilde{\varphi}_2) &= \sqrt{2} \inf\left\{ \|1 - \mathsf{Re}(\lambda' u)\|^{\frac{1}{2}} : \lambda' \in \mathbb{C}, |\lambda'| \leq 1 \right\} \\ &\leq \sqrt{2}\ \|1 - \mathsf{Re}(u)\|^{\frac{1}{2}} \\ &= \sqrt{2}\,|1 - \mathsf{Re}(\lambda)|^{\frac{1}{2}} \\ &< \sqrt{2} \\ &= \beta(\varphi_1, \varphi_2).\end{aligned}$$

Example 2.3.3. Let H be an infinite dimensional Hilbert space. Consider the unital C^*-subalgebra

$$\mathcal{A} := C^*\left\{ \mathcal{K}(H \oplus H) \cup \left\{ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right\} \right\}$$

$$= \left\{ \begin{bmatrix} \lambda_1 I + a_{11} & a_{12} \\ a_{21} & \lambda_2 I + a_{22} \end{bmatrix} : \lambda_i \in \mathbb{C},\, a_{ij} \in \mathcal{K}(H) \right\}$$

of $\mathcal{B}(H \oplus H)$. Suppose $u \in \mathcal{B}(H)$ is a unitary and $1 < r \in \mathbb{R}$. Set

$$z_1 = \begin{bmatrix} 0 & u \\ 0 & rI \end{bmatrix},\ z_2 = \begin{bmatrix} 0 & 0 \\ 0 & rI \end{bmatrix} \text{ and } z_3 = \begin{bmatrix} 0 & I \\ 0 & rI \end{bmatrix}$$

in $\mathcal{B}(H \oplus H)$. Define CP-maps $\varphi_i : \mathcal{A} \to \mathcal{A}$ by $\varphi_i(a) := z_i^* a z_i$, $i = 1, 2, 3$. Note that each φ_i has the form, $\varphi_i(\cdot) = \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}$. Let

$$E_{12} = \left\{ \begin{bmatrix} x_{11} & \lambda_1 u + x_{12} \\ x_{21} & \lambda_2 I + x_{22} \end{bmatrix} : \lambda_i \in \mathbb{C},\, x_{ij} \in \mathcal{K}(H) \right\}$$

which is a Hilbert $\mathcal{A}$-$\mathcal{A}$-module with a natural inner product and bimodule structure. Note that $z_i \in S(E_{12}, \varphi_i)$, $i = 1, 2$, and hence $\beta(\varphi_1, \varphi_2) \leq \|z_1 - z_2\| = 1$. Similarly

$$E_{23} = \left\{ \begin{bmatrix} x_{11} & \lambda_1 I + x_{12} \\ x_{21} & \lambda_2 I + x_{22} \end{bmatrix} : \lambda_i \in \mathbb{C},\, x_{ij} \in \mathcal{K}(H) \right\}$$

is a Hilbert $\mathcal{A}$-$\mathcal{A}$-module with $z_i \in S(E_{23}, \varphi_i)$, $i = 2, 3$, and $\beta(\varphi_2, \varphi_3) \leq \|z_2 - z_3\| = 1$. Now we will show that $\beta(\varphi_1, \varphi_3) > 2 \geq \beta(\varphi_1, \varphi_2) + \beta(\varphi_2, \varphi_3)$ so that β fails to satisfy triangle inequality. Suppose E is a common representation module for φ_1, φ_3. We prove that $\langle x_1, x_3 \rangle = 0$ for all $x_i \in S(E, \varphi_i)$. If we proved this, then E and $x_i \in S(E, \varphi_i)$ arbitrary implies that

$$\beta(\varphi_1, \varphi_3) = \inf_{E, x_i} \|x_1 - x_3\| = \|\varphi_1(1) + \varphi_3(1)\|^{\frac{1}{2}} = \sqrt{2(1 + r^2)} > 2.$$

Suppose $\langle x_1, x_3 \rangle = [a_{ij}]$. Since $0 \leq \begin{bmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_3 \rangle \\ \langle x_3, x_1 \rangle & \langle x_3, x_3 \rangle \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & a_{11} & a_{12} \\ 0 & * & a_{21} & a_{22} \\ \hline a_{11}^* & a_{21}^* & 0 & 0 \\ a_{12}^* & a_{22}^* & 0 & * \end{array}\right]$ we have $a_{11} = a_{12} = a_{21} = 0$. Also for all $a \in \mathcal{K}(H)$, we get

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} x_1 = x_1 \begin{bmatrix} 0 & 0 \\ 0 & u^* a u \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} x_3 = x_3 \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}.$$

(Simply look at the norm of the difference.) Hence

$$\begin{bmatrix} 0 & 0 \\ 0 & u^*au \end{bmatrix} \langle x_1, x_3 \rangle = \langle x_1, x_3 \rangle \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix};$$

i.e.,

$$\begin{bmatrix} 0 & 0 \\ 0 & u^*au \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$$

which implies that $u^*aua_{22} = a_{22}a$; i.e., $aua_{22} = ua_{22}a$ for all $a \in \mathcal{K}(H)$. Hence $ua_{22} = \lambda I$ for some $\lambda \in \mathbb{C}$. Thus $a_{22} = \lambda u^*$. Since $a_{22} \in \mathcal{K}(H)$ and $u^* \notin \mathcal{K}(H)$ we have $\lambda = 0$, and hence $a_{22} = 0$ and $\langle x_1, x_3 \rangle = 0$.

2.3.2 Injective C^*-algebras

Recall that a C^*-algebra $\mathcal{B}$ is an injective C^*-algebra if, whenever $\mathcal{C}$ is a C^*-algebra, $\mathcal{S}$ an operator system contained in $\mathcal{C}$, and $\varphi : \mathcal{S} \to \mathcal{B}$ is a completely positive contraction, then φ extends to a completely positive contraction $\widetilde{\varphi} : \mathcal{C} \to \mathcal{B}$. Further, this is equivalent to saying that there is a faithful representation π of $\mathcal{B}$ on a Hilbert space G, such that there is a conditional expectation from $\mathcal{B}(G)$ onto $\pi(\mathcal{B})$. See [Arv69a, Pau02, Tak03] for details.

Proposition 2.3.4. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{B}$ be an injective C^*-algebra with a faithful representation $\pi : \mathcal{B} \to \mathcal{B}(G)$ on a Hilbert space G. Then $\beta(\varphi_1, \varphi_2) = \beta(\pi \circ \varphi_1, \pi \circ \varphi_2)$ for all $\varphi_1, \varphi_2 \in CP(\mathcal{A}, \mathcal{B})$.*

Proof. Since $\mathcal{B}$ is injective there exists a completely positive conditional expectation $P : \mathcal{B}(G) \to \pi(\mathcal{B})$. Take $\varphi = \pi^{-1} \circ P : \mathcal{B}(G) \to \mathcal{A}$. Then φ is a contractive CP-map. Moreover, $\varphi \circ \pi \circ \varphi_i = \varphi_i,\ i = 1, 2$. Now by Proposition 2.1.5,

$$\beta(\varphi_1, \varphi_2) = \beta(\varphi \circ \pi \circ \varphi_1, \varphi \circ \pi \circ \varphi_2) \le \beta(\pi \circ \varphi_1, \pi \circ \varphi_2) \le \beta(\varphi_1, \varphi_2).$$

□

From Proposition 2.1.5, we know that $\beta(\pi \circ \varphi_1, \pi \circ \varphi_2) \le \beta(\varphi_1, \varphi_2)$ even for an arbitrary C^*-algebra $\mathcal{B}$. But, in general, equality may not holds. See example 2.3.2.

The following bounds were first obtained in [KSW08a].

Corollary 2.3.5. *If $\mathcal{B}$ is an injective unital C^*-algebra, then β is a metric on $CP(\mathcal{A}, \mathcal{B})$ and*

$$\frac{\|\varphi_1 - \varphi_2\|_{cb}}{\sqrt{\|\varphi_1\|_{cb}} + \sqrt{\|\varphi_2\|_{cb}}} \leq \beta(\varphi_1, \varphi_2) \leq \sqrt{\|\varphi_1 - \varphi_2\|_{cb}}.$$

Further, there exists a common representation module E and corresponding GNS-construction (E, x_i) for φ_i such that $\beta(\varphi_1, \varphi_2) = \beta_E(\varphi_1, \varphi_2) = \|x_1 - x_2\|$.

Proof. Suppose $\pi : \mathcal{B} \to \mathcal{B}(G)$ is a faithful representation of $\mathcal{B}$. Now the first part follows from Theorem 2.2.3 and Proposition 2.3.4. Also from Theorem 2.2.11 and Proposition 2.3.4, we have

$$\beta(\varphi_1, \varphi_2) = \beta(\pi \circ \varphi_1, \pi \circ \varphi_2) \leq \sqrt{\|\pi \circ \varphi_1 - \pi \circ \varphi_2\|_{cb}} = \sqrt{\|\varphi_1 - \varphi_2\|_{cb}}.$$

Now, from Theorem 2.2.11, we know that there exists a von Neumann $\mathcal{A}$-$\mathcal{B}(G)$-module F with $y_i \in S(F, \pi \circ \varphi_i)$ such that $\|y_1 - y_2\| = \beta(\pi \circ \varphi_1, \pi \circ \varphi_2)$. Given $b \in \mathcal{B}$, $y \in F$ define $yb := y\pi(b)$. Under this action, F forms a right $\mathcal{B}$-module, denoted by E_0. Let $P : \mathcal{B}(G) \to \pi(\mathcal{B})$ be a completely positive conditional expectation satisfying $P(b_1 a b_2) = b_1 P(a) b_2$ for all $b_i \in \pi(\mathcal{B})$, $a \in \mathcal{B}(G)$. Now define a $\mathcal{B}$-valued semi-inner product on E_0 by $\langle x_1, x_2\rangle' := \pi^{-1}P(\langle x_1, x_2\rangle)$. Let E be the completion of the $\mathcal{B}$-valued inner product space E_0/N, where $N := \{x \in E_0 : \langle x, x\rangle' = 0\}$. Then E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module with left action induced by that of $\mathcal{A}$ on F. Note that $x_i := y_i + N \in S(E, \varphi_i)$, $i = 1, 2$, are such that

$$\begin{aligned}
\beta_E(\varphi_1, \varphi_2) &\leq \|x_1 - x_2\| \\
&= \left\|\pi^{-1}P(\langle y_1 - y_2, y_1 - y_2\rangle)\right\|^{\frac{1}{2}} \\
&\leq \|y_1 - y_2\| \\
&= \beta(\pi \circ \varphi_1, \pi \circ \varphi_2) \\
&= \beta(\varphi_1, \varphi_2).
\end{aligned}$$

Thus $\beta(\varphi_1, \varphi_2) = \beta_E(\varphi_1, \varphi_2) = \|x_1 - x_2\|$. □

2.4 Bures distance and a rigidity theorem

Observe that for the identity map on a unital C^*-algebra $\mathcal{B}$ the GNS-module is $\mathcal{B}$ itself. Here we show that if a CP-map on a von Neumann algebra $\mathcal{B}$ is close to the identity map in Bures distance then the GNS-module has a copy of $\mathcal{B}$.

Suppose $\mathcal{B} \subseteq \mathcal{B}(G)$ is a von Neumann algebra and $\varphi : \mathcal{B} \to \mathcal{B}$ is a CP-map.

Proposition 2.4.1. *If (E, x) is the minimal GNS-construction for φ, then the following are equivalent:*

(i) *The center $C_{\mathcal{B}}(E) := \{y \in E : by = yb \quad \forall b \in \mathcal{B}\}$ contains a unit vector.*

(ii) *$E \cong \mathcal{B} \oplus F$ for some von Neumann $\mathcal{B}$-$\mathcal{B}$-module F.*

(iii) *There exists an element $c \in \mathcal{B}$ such that the two sided (strongly closed) ideal generated by c is $\mathcal{B}$, and a CP-map $\psi : \mathcal{B} \to \mathcal{B}$ such that $\varphi(b) = c^*bc + \psi(b)$ for all $b \in \mathcal{B}$.*

Proof. $(i) \Rightarrow (ii)$: Let $z \in C_{\mathcal{B}}(E)$ be a unit vector. The two sided $\mathcal{B}$-$\mathcal{B}$-module generated by z is naturally isomorphic to $\mathcal{B}$ by $bz \mapsto b$, and let us denote it by $\mathcal{B}z$. Then E decomposes as $\mathcal{B}z \oplus (\mathcal{B}z)^{\perp}$.

$(ii) \Rightarrow (iii)$: Without loss of generality, we may take $E = \mathcal{B} \oplus F$. Then $x \in E$ decomposes as $x = c \oplus y$ with $c \in \mathcal{B},\ y \in F$. Clearly, $\varphi(b) = \langle x, bx\rangle = c^*bc + \langle y, by\rangle$, and we can take $\psi(b) = \langle y, by\rangle$ for all $b \in \mathcal{B}$. Since $\mathcal{B} \oplus F = E = \overline{\operatorname{span}}^{\,s}\, \mathcal{B}x\mathcal{B} = \overline{\operatorname{span}}^{\,s}\, (\mathcal{B}c\mathcal{B} \oplus \mathcal{B}y\mathcal{B})$ we have $\mathcal{B}$ is the two sided (strongly closed) ideal generated by c.

$(iii) \Rightarrow (i)$: Note that the CP-map $b \mapsto c^*bc$ is dominated by the CP-map φ, and hence there exists a vector $z \in E$ (Proposition 1.6.5) such that $c^*bc = \langle z, bz\rangle$ for all $b \in \mathcal{B}$. Note that, for elements $a, a', b, d, d' \in \mathcal{B}$, $(acd)^*b(a'cd') = d^*(c^*a^*ba'c)d' = d^*\langle z, a^*ba'z\rangle d' = \langle azd, ba'zd'\rangle$. It follows that for any element d in the (strongly closed) ideal generated by c, there exists an element $z_d \in E$ such that $d^*bd = \langle z_d, bz_d\rangle$. Taking $d = 1$, we have an element $w \in E$ such that $b = \langle w, bw\rangle$ for all $b \in \mathcal{B}$. Observe that w is a unit vector. Direct computation yields $\langle bw - wb, bw - wb\rangle = 0$, hence w is in the center $C_{\mathcal{B}}(E)$. □

Theorem 2.4.2. *Let $\varphi : \mathcal{B} \to \mathcal{B}$ be a CP-map such that $\beta(id, \varphi) < 1$. Let (E, x) be a GNS-construction for φ. Then $E \cong \mathcal{B} \oplus F$ for some von Neumann $\mathcal{B}$-$\mathcal{B}$-module F.*

Proof. Without loss of generality, assume that (E, x) is the minimal GNS-construction for φ. Let $\varepsilon > 0$ be such that $\beta(id, \varphi) + \varepsilon < 1$. Since the identity map has $(\mathcal{B}, 1)$ as its GNS-construction, from Theorem 2.2.8, there exists $z_1 = 1 \oplus 0$, $z_2 = c \oplus y$ in $\mathcal{B} \oplus E$

such that $\|z_1 - z_2\| \leq \beta(id, \varphi) + \varepsilon < 1$ and $\varphi(b) = \langle z_2, bz_2 \rangle = c^*bc + \langle y, by \rangle$. Further, as $\|1 - c\| \leq \|z_1 - z_2\| < 1$ we note that c is invertible. Therefore the ideal generated by c is whole of $\mathcal{B}$. Now the result follows from the previous Proposition. □

2.5 Some applications of Bures metric

In [Kos83] Kosaki obtained certain expressions for Bures metric between normal states, which clarify their importance in theoretical physics. According to him (but in our notation): "When a physical system is described by a von Neumann algebra $\mathcal{B}$, each self adjoint element $b = \int_{-\infty}^{\infty} \lambda de(\lambda)$ in $\mathcal{B}$ is considered as an observable. Then, for a state φ on $\mathcal{B}$ and a partition $\mathbb{R} = \cup_{i=1}^{n} X_i$ (of $\mathbb{R}$ into disjoint Borel subsets), $\varphi(p_i) = \int_{X_i} d_\varphi(e(\lambda))$ (with $p_i = \int_{X_i} de(\lambda)$) is interpreted as the probability that a measurement of b performed on the system in the state φ yields a result lying in X_i. Thus, $\beta(\varphi_1, \varphi_2) \leq \varepsilon$ for a small $\varepsilon > 0$ means that two states φ_1, φ_2 give almost similar measurements for any observable b in the sense that $\sum_{i=1}^{n} (\varphi_1(p_i)^{\frac{1}{2}} - \varphi_2(p_i)^{\frac{1}{2}})^2 \leq \varepsilon^2$ (for any partition $\mathbb{R} = \cup_{i=1}^{n} X_i$). Therefore, the Bures distance is quite suitable to describe a distance between two (physical) states".

In [Hüb92] M. Hubner gives explicit computation of Bures distance for density matrices. He proves the following theorem: The set of two-dimensional normalized density matrices equipped with the Bures metric is isometric to one closed half of the three-sphere with radius $\frac{1}{2}$.

In quantum information theory, a *quantum channels(QC)* is a communication channel which can transmit quantum information. Formally, they are trace preserving CP-maps between spaces of operators. Any QC arises from a unitary evolution on a larger system. In [KSW08b] D.Kretschmann, D.Schlingemann and R.F.Werner proved that if two QCs are close in cb-norm, then there exists unitary implementations which are close in operator norm, and derive a formulation of the information-disturbance tradeoff in terms of QCs. Also pointed out further implications for quantum cryptography, thermalization processes, etc.

We consider Theorem 2.4.2 as the most important positive result of this chapter and we expect that the result will have further applications in the study of CP-maps, CP-semigroups and the associated product system of Hilbert C^*-modules.

Chapter 3

Stinespring type theorem for maps between Hilbert C^*-modules

The question whether given Hilbert C^*-modules are isomorphic or not is always interesting. Two Hilbert C^*-modules are said to be identical if there exists a unitary (i.e., surjective isometry) between them. Recall that isometries preserve not only the inner product but also the module action. Thus isometries are the structure preserving maps between Hilbert C^*-modules. For a Hilbert C^*-module the C^*-valued inner product is uniquely determined by the module structure and Banach space structure. The inner product can be recovered from the norm and the module structure by

$$\langle x, x\rangle = \sup\big\{\phi(x)^*\phi(x): \;\; \phi : E \to \mathcal{B} \text{ is an } \mathcal{B}\text{-module map with } \|\phi\| \leq 1\big\}.$$

Using polarization identity we can get $\langle x_1, x_2\rangle$ for $x_i \in E$. See [Lan95, Theorem], [Ble97a, Theorem 3.1 and 3.2], [Fra97b, Theorem 5]) and [Fra99, Proposition 3.3] for details.

Often, in applications, we come across Hilbert C^*-modules over different C^*-algebras and have to consider maps between them. In such situations, we ask what can replace the notion of isometries and unitaries. Muhly and Solel considered such cases and proved a generalized version ([MS00, Lemma 5.10]) of Lance-Blecher theorem: If E is a Hilbert $\mathcal{B}$-module, F is a Hilbert C-module and $T : E \to F$ is a Banach space isometry, and if there exists a $*$-isomorphism $\pi : \mathcal{B} \to \mathcal{C}$ such that $T(xb) = T(x)\pi(b)$, then T satisfies $\langle T(x_1), T(x_2)\rangle = \pi(\langle x_1, x_2\rangle)$. That is, T preserves the inner product up to the $*$-isomorphism π. In [Sol01] Solel asked to what extent it is possible to recover the C^*-module structure from the Banach space structure only. More precisely, given an surjective linear norm preserving map T from a Hilbert $\mathcal{B}$-module E onto a Hilbert $\mathcal{C}$-module F, can we find a $*$-isomorphism $\pi : \mathcal{B} \to \mathcal{C}$ such that $T(xb) = T(x)\pi(b)$? If we can, then we have $\langle T(x_1), T(x_2)\rangle = \pi(\langle x_1, x_2\rangle)$. He observed that: To say that T preserves the C^*-module structure amounts to saying that T can be extended to a $*$-isomorphism of $\mathfrak{A}^1(E)$ onto $\mathfrak{A}^1(F)$. He proved that if E and F are full, then T can always be extended to an isometry of $\mathfrak{A}^1(E)$ onto

$\mathfrak{A}^1(F)$ ([Sol01, Theorem 3.2]).

Linear maps between Hilbert C^*-modules which preserve the inner product up to a $*$-homomorphism have been studied in different contexts ([TS07, Ske06b, BG02b, BG03, Brü04, Ara05]). Here we study the theory in a more general case, namely, we consider maps between Hilbert C^*-modules, possibly over different C^*-algebras, which preserves inner product up to a (bounded) linear map between the underlying C^*-algebras. First we determine properties of such maps. We prove that if the map between the underlying C^*-algebras is bounded linear, then it will be automatically CP-map on the range ideal and as a consequence the module map on full Hilbert C^*-modules will be completely bounded. We strengthen B. Asadi's ([Asa09]) analogue of Stinespring's theorem for module maps on Hilbert C^*-modules and illustrate this with an example.

3.1 Module maps

Suppose E, F are Hilbert C^*-modules over C^*-algebras $\mathcal{B}, \mathcal{C}$ respectively.

Definition 3.1.1. Let $\varphi : \mathcal{B} \to \mathcal{C}$ be a linear map. A map $T : E \to F$ is said to be a φ-map if

$$\langle T(x), T(x')\rangle = \varphi(\langle x, x'\rangle) \tag{$*$}$$

for all $x, x' \in E$.

If φ is a $*$-homomorphism between the C^*-algebras, then T has been called *φ-isometry* or *φ-morphism* in [TS07, Ske06b, BG02b, BG03]. If $\mathcal{C}$ is the algebra of bounded linear operators on a Hilbert space, then they are known as *φ-representation* in literature ([Asa09, Ara05]). Note that φ-isometries arise naturally when we realise Hilbert C^*-modules as a submodule of the module $\mathcal{B}(G, H)$ of bounded operators between two Hilbert spaces G and H. Such maps are called a *representation* of a Hilbert C^*-module in [Ske00, Ara05].

Remark 3.1.2. Suppose $\varphi : \mathcal{B} \to \mathcal{C}$ is a linear map and $T : E \to F$ is a φ-map.

(i) Then T is automatically linear. This follows because $\langle T(x + \lambda x') - T(x) - \lambda T(x'), T(x + \lambda x') - T(x) - \lambda T(x')\rangle = 0$ for all $x, x' \in E$ and $\lambda \in \mathbb{C}$.

(ii) Suppose φ is a $*$-homomorphism. Using polarization identity, one immediately concludes that T is a φ-isometry if and only if $\langle T(x), T(x)\rangle = \varphi(\langle x, x\rangle)$ for all $x \in E$. By calculating the norm of $T(xb) - (Tx)\varphi(b)$ we find that $T(xb) = (Tx)\varphi(b)$ for all $x \in E,\ b \in \mathcal{B}$.

(iii) The inflation $T_n : M_n(E) \to M_n(F)$ of T (i.e., T acting element-wise on the matrix) is a φ_n-map for the inflation $\varphi_n : M_n(\mathcal{B}) \to M_n(\mathcal{C})$ of φ. Also $T^n : E^n \to F^n$ given by

$$T^n(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}) := \begin{pmatrix} T(x_1) \\ \vdots \\ T(x_n) \end{pmatrix}$$

is a φ-map for all $n \in \mathbb{N}$.

(iv) If φ is bounded linear, then, from ($*$), T is bounded linear with $\|T\| \leq \sqrt{\|\varphi\|}$. Also then, T^n is bounded linear with $\|T^n\| \leq \sqrt{\|\varphi\|}$. Since φ_n is bounded (see [Pau02, Exercise 3.10]) we have T_n is bounded linear with $\|T_n\| \leq \sqrt{\|\varphi_n\|}$ for all $n \geq 1$. But the converse namely, T bounded implies φ bounded, may not be true.

Example 3.1.3. Let $H \neq \{0\}$ be a Hilbert space with ONB $\{e_i\}_{i\in I}$. For E we choose the full Hilbert $\mathcal{K}(H)$-module H^* (with inner product $\langle x'^*, x^*\rangle := x'x^*$). For F we choose H. So, $\mathcal{B} = \mathcal{K}(H)$ and $\mathcal{C} = \mathbb{C}$. Let T be the *transpose map* with respect to the ONB. That is, T sends the "row vector" $x^t = \sum_i x_i e_i^*$ in E to the "column vector" $x = (x^t)^t = \sum_i x_i e_i$ in F. Of course, $\|T\| = 1$.

A linear map $\varphi : \mathcal{K}(H) \to \mathbb{C}$ turning T into a φ-map, would send $e_i e_j^*$ to $\varphi(e_i e_j^*) = \varphi(\langle e_i^*, e_j^*\rangle) = \langle T(e_i^*), T(e_j^*)\rangle = \langle e_i, e_j\rangle = \delta_{i,j}$. So, on $\mathcal{F}(H)$ the map φ is bound to be the (non-normalized) *trace* $\mathsf{Tr}(\cdot) := \sum_i \langle e_i, (\cdot)e_i\rangle$. Recall that $\|\mathsf{Tr}\| = \dim H$. This shows several things:

(i) Suppose H is infinite-dimensional. Then φ cannot be bounded. Since positive maps are bounded, there cannot be whatsoever positive map φ turning T into a φ-map. (Of course, we can extend $\varphi = \mathsf{Tr}$ by brute-force linear algebra from $\mathcal{F}(E)$ to $\mathcal{K}(E)$, so that T is still a φ-map with unbounded and nonpositive φ.)

(ii) Suppose H is n-dimensional (so that, in particular, $\mathcal{K}(H) = M_n(\mathbb{C})$ is unital). The column vector x^{*n} in H^{*n} with entries $e_1^*, \cdots, e_n^*$ has square modulus $\langle x^{*n}, x^{*n}\rangle = \sum_{i=1}^n e_i e_i^*$. So, $\|x^{*n}\| = \sqrt{\|\sum_{i=1}^n e_i e_i^*\|} = 1$. However, the norm of the column vector y^n with entries $T(e_1^*) = e_1, \cdots, T(e_n^*) = e_n$ is $\sqrt{\sum_{i=1}^n \langle e_i, e_i \rangle} = \sqrt{n}$. Since $M_n(H^*) \supset M_{n,1}(H^*) = H^{*n}$, we find $\|T\|_{cb} \geq \|T_n\| \geq \sqrt{n}$. Thus $\|T\| = 1 \neq \|T\|_{cb}$ for $n \geq 2$. (From theorem 3.1.5, we can have, $\|T\|_{cb} \leq \sqrt{\|\varphi\|} = \sqrt{n}$. Therefore, $\|T\|_{cb} = \sqrt{n}$.)

Lemma 3.1.4. *Let $\varphi : \mathcal{B} \to \mathcal{C}$ be a bounded linear map fulfilling $(*)$ for some map $T : E \to F$. Then φ is positive on $\mathcal{B}_E$.*

Proof. Let $0 \leq bb^* \in \mathcal{B}_E$. We prove that $\varphi(bb^*) \geq 0$. Let $\{b_\alpha\}_{\alpha\in\Lambda}$ be an approximate unit for $\mathcal{B}_E$ consisting of elements $b_\alpha = \sum_{i=1}^{n_\alpha} \langle x_i^\alpha, y_i^\alpha \rangle \in \mathcal{B}_E$. Defining the elements $x_\alpha \in E^{n_\alpha}$ with entries x_i^α and, similarly, y_α, we get $b_\alpha = \langle x_\alpha, y_\alpha \rangle$. Let $a_\alpha \in \mathcal{K}(E^{n_\alpha})$ be the positive square root of the rank-one operator $x_\alpha bb^* x_\alpha^* = (x_\alpha b)(x_\alpha b)^*$. Since $T^{n_\alpha} : E^{n_\alpha} \to F^{n_\alpha}$ is a φ-map we get,

$$\begin{aligned}
\varphi(b_\alpha^* bb^* b_\alpha) &= \varphi(\langle y_\alpha, x_\alpha \rangle bb^* \langle x_\alpha, y_\alpha \rangle) \\
&= \varphi(y_\alpha^* x_\alpha bb^* x_\alpha^* y_\alpha) \\
&= \varphi(y_\alpha^* a_\alpha^2 y_\alpha) \\
&= \varphi(\langle a_\alpha y_\alpha, a_\alpha y_\alpha \rangle) \\
&= \langle T^{n_\alpha}(a_\alpha y_\alpha), T^{n_\alpha}(a_\alpha y_\alpha) \rangle \\
&\geq 0.
\end{aligned}$$

Since $b_\alpha^* bb^* b_\alpha \longrightarrow bb^*$ in norm, and since φ is bounded, we get $\varphi(bb^*) \geq 0$. □

Theorem 3.1.5. *Suppose E is a full Hilbert $\mathcal{B}$-module and $\varphi : \mathcal{B} \to \mathcal{C}$ is a bounded linear map. If $T : E \to F$ is a φ-map, then φ is completely positive. Moreover, T is completely bounded with $\|T\|_{cb} = \sqrt{\|\varphi\|}$.*

Proof. Since φ_n is bounded and T_n is a φ_n-map, from Remark 3.1.2 and Lemma 3.1.4, φ_n is positive on $M_n(\mathcal{B}) = M_n(\mathcal{B}_E)$ for all $n \in \mathbb{N}$. Thus φ is a CP-map. Also since $\|T_n\|^2 \leq \|\varphi_n\| \leq \|\varphi\|_{cb} = \|\varphi\|$ for all $n \geq 1$, we get $\|T\|_{cb} \leq \sqrt{\|\varphi\|}$.

To prove the reverse inequality assume that $\varepsilon > 0$. Let bb^* be in the unit ball of $\mathcal{B}$ such that $\|\varphi\| \leq \|\varphi(bb^*)\| + \frac{\varepsilon}{2}$. Choose $b_\alpha, x_\alpha, y_\alpha$ and a_α as in the proof of Lemma

3.1.4. Set $z_\alpha = a_\alpha y_\alpha \in E^{n_\alpha}$. Then

$$\langle z_\alpha, z_\alpha\rangle = \langle y_\alpha, x_\alpha bb^* x_\alpha^* y_\alpha\rangle = \big\langle \langle x_\alpha, y_\alpha\rangle, bb^*\langle x_\alpha, y_\alpha\rangle\big\rangle = b_\alpha^* bb^* b_\alpha \longrightarrow bb^*.$$

Hence

$$\begin{aligned}\|\langle T^{n_\alpha}(z_\alpha), T^{n_\alpha}(z_\alpha)\rangle - \varphi(bb^*)\| &= \|\varphi(\langle z_\alpha, z_\alpha\rangle) - \varphi(bb^*)\| \\ &\leq \|\varphi\| \, \|\langle z_\alpha, z_\alpha\rangle - bb^*\| \\ &\longrightarrow 0,\end{aligned}$$

and so, $\|\langle T^{n_\alpha}(z_\alpha), T^{n_\alpha}(z_\alpha)\rangle\| \longrightarrow \|\varphi(bb^*)\|$. Choose $n = n_{\alpha_0}$ such that $\|\varphi(bb^*)\| - \frac{\varepsilon}{2} \leq \|\langle T^n(z_{\alpha_0}), T^n(z_{\alpha_0})\rangle\|$. Note that $\|z_{\alpha_0}\|^2 = \|\langle z_{\alpha_0}, z_{\alpha_0}\rangle\| = \left\|b_{\alpha_0}^* bb^* b_{\alpha_0}\right\| \leq \|b_{\alpha_0}\|^2 \|bb^*\| \leq 1$. Then

$$\begin{aligned}\|\varphi\| &\leq \|\varphi(bb^*)\| + \frac{\varepsilon}{2} \\ &\leq \|\langle T^n(z_{\alpha_0}), T^n(z_{\alpha_0})\rangle\| + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &\leq \|T^n\|^2 \|z_{\alpha_0}\|^2 + \varepsilon \\ &\leq \|T\|_{cb}^2 + \varepsilon.\end{aligned}$$

By letting $\varepsilon \longrightarrow 0$ we get $\|\varphi\| \leq \|T\|_{cb}^2$. □

Remark 3.1.6. Suppose $\varphi : \mathcal{B} \to \mathcal{C}$ is linear and $T : E \to F$ is a φ-map. If $\mathcal{B}_E$ is unital, then by considering E as a $\mathcal{B}_E$-module, from Proposition 1.1.11, there exists $x = (x_1, \cdots, x_n)^t \in E^n$ such that $\langle x, x\rangle = 1$. Then for $b \in \mathcal{B}_E$,

$$\varphi(b^*b) = \varphi(b^*\langle x, x\rangle b) = \varphi(\sum_i \langle x_i b, x_i b\rangle) = \sum_i \langle T(x_i b), T(x_i b)\rangle \geq 0.$$

Thus φ is positive (and hence bounded) on $\mathcal{B}_E$. From Theorem 3.1.5, $\varphi : \mathcal{B}_E \to \mathcal{C}$ is a CP-map. Thus if E is full module over a unital C^*-algebra $\mathcal{B}$ and $(*)$ holds for some map $T : E \to F$ and $\varphi : \mathcal{B} \to \mathcal{C}$ linear, then φ is bounded automatically and hence also CP.

It is, in general, not true that $\|T\|_{cb} = \|T\|$, not even if $\mathcal{B}$ and $\mathcal{C}$ are unital. See example 3.1.3. It is true, if E has a unit vector x. For, then E is full and hence

$$\|T\|_{cb} \leq \sqrt{\|\varphi\|} = \sqrt{\|\varphi(1)\|} = \sqrt{\|\varphi(\langle x, x\rangle)\|} = \sqrt{\|\langle T(x), T(x)\rangle\|} \leq \|T\|.$$

3.2 Stinespring type theorem for module maps

In this section we discuss a structure theorem for φ-maps for the special case when φ is a CP-map from a unital C^*-algebra $\mathcal{A}$ into the algebra $\mathcal{B}(H_1)$ of bounded linear maps on a Hilbert space H_1. In [Asa09], Asadi presented a theorem, which looks like Stinespring's theorem, for φ-maps T from a Hilbert $\mathcal{A}$-module E into the Hilbert $\mathcal{B}(H_1)$-module $F = \mathcal{B}(H_1, H_2)$, where H_2 is another Hilbert space.

Theorem 3.2.1 ([Asa09]). *Suppose $\varphi : \mathcal{A} \to \mathcal{B}(H_1)$ is a unital CP-map and $T : E \to \mathcal{B}(H_1, H_2)$ is a φ-map with the additional property $T(x_0)T(x_0)^* = id_{H_2}$ for some $x_0 \in E$. Then there exist Hilbert spaces K_1, K_2, isometries $V : H_1 \to K_1$, $W : H_2 \to K_2$, a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(K_1)$ and a π-representation $S : E \to \mathcal{B}(K_1, K_2)$ such that $\varphi(a) = V^*\pi(a)V$* and *$T(x) = W^*S(x)V$ for all $x \in E$, $a \in \mathcal{A}$.*

The proof of this Theorem as given in [Asa09] is erroneous as the sesquilinear form defined there on $E \otimes H_2$ is not positive definite. This can be fixed by interchanging the indices i, j in the definition of this form. However such a modification yields a 'nonminimal' representation. Moreover, the technical condition to have $T(x_0)T(x_0)^* = id_{H_2}$ for some $x_0 \in E$ is completely unnecessary.

Here we strengthen this result by removing the technical condition of Asadi's theorem. We also remove the assumption of unitality on maps under consideration. Further we prove uniqueness up to unitary equivalence for minimal representations, which is an important ingredient of structure theorems like GNS-theorem and Stinespring's theorem. Now the result looks even more like Stinespring's theorem.

Theorem 3.2.2. *Suppose $\varphi : \mathcal{A} \to \mathcal{B}(H_1)$ is a CP-map and $T : E \to \mathcal{B}(H_1, H_2)$ is a φ-map. Then there exists a pair of triples (K_1, π, V) and (K_2, S, W), where*

(i) *K_1 and K_2 are Hilbert spaces;*

(ii) *$\pi : \mathcal{A} \to \mathcal{B}(K_1)$ is a unital $*$-homomorphism and $S : E \to \mathcal{B}(K_1, K_2)$ is a π-representation;*

(iii) *$V : H_1 \to K_1$ and $W : H_2 \to K_2$ are bounded linear operators such that $\varphi(a) = V^*\pi(a)V$ and $T(x) = W^*S(x)V$ for all $a \in \mathcal{A}$, $x \in E$.*

Proof. We prove the theorem in two steps.

Step 1: Existence of K_1, π and V: This is the content of Stinespring's theorem ([Pau02, Theorem 4.1]). In fact we can choose a minimal Stinespring representation (K_1, π, V) for φ. That is, $K_1 = \overline{\mathsf{span}}\, \pi(\mathcal{A})VH_1$.

Step 2: Construction of K_2, S and V: Let $K_2 := \overline{\mathsf{span}}\, T(E)H_1$. For $x \in E$, define $S(x) : \mathsf{span}\, \pi(\mathcal{A})VH_1 \to K_2$ by

$$S(x)\big(\pi(a)Vh\big) := T(xa)h, \quad \forall\, a \in \mathcal{A},\ h \in H_1.$$

Since

$$\begin{aligned}
\Big\|S(x)\Big(\sum_{i=1}^{n} \pi(a_i)Vh_i\Big)\Big\|^2 &= \Big\langle \sum_i T(xa_i)h_i, \sum_j T(xa_j)h_j \Big\rangle \\
&= \sum_{i,j} \Big\langle h_i, \langle T(xa_i), T(xa_j)\rangle h_j \Big\rangle \\
&= \sum_{i,j} \Big\langle h_i, \varphi(\langle xa_i, xa_j\rangle)h_j \Big\rangle \\
&= \sum_{i,j} \Big\langle h_i, V^*\pi(a_i^*\langle x, x\rangle a_j)Vh_j \Big\rangle \\
&= \Big\langle \sum_i \pi(a_i)Vh_i, \pi(\langle x, x\rangle)\Big(\sum_j \pi(a_j)Vh_j\Big) \Big\rangle \\
&\leq \big\|\pi(\langle x, x\rangle)\big\| \Big\|\sum_{i=1}^{n} \pi(a_i)Vh_i\Big\|^2 \\
&\leq \|x\|^2 \Big\|\sum_{i=1}^{n} \pi(a_i)Vh_i\Big\|^2,
\end{aligned}$$

$S(x)$ is well defined and bounded. Hence it can be extended to whole of K_1. This gives the required $S : E \to \mathcal{B}(K_1, K_2)$. To prove that S is a π-representation, let $x, y \in E,\ a_i, a_i' \in \mathcal{A},\ h_i, h_i' \in H_1,\ i = 1, 2, \ldots n,\ n \in \mathbb{N}$. Then

$$\begin{aligned}
\Big\langle \sum_{i=1}^{n} \pi(a_i)Vh_i, S(x)^*S(y)\Big(\sum_{j=1}^{n} \pi(a_j')Vh_j'\Big) \Big\rangle &= \Big\langle \sum_i T(xa_i)h_i, \sum_j T(ya_j')h_j' \Big\rangle \\
&= \Big\langle \sum_{i=1}^{n} \pi(a_i)Vh_i, \pi(\langle x, y\rangle)\Big(\sum_{j=1}^{n} \pi(a_j')Vh_j'\Big) \Big\rangle.
\end{aligned}$$

Thus $S(x)^*S(y) = \pi(\langle x, y\rangle)$ on the dense set $\mathsf{span}\, \pi(A)VH_1$ and hence they are equal on K_1. Note that $K_2 \subseteq H_2$. Let $W := P_{K_2}$, the orthogonal projection onto K_2. Then $W^* : K_2 \to H_2$ is the inclusion map. Hence $WW^* = id_{K_2}$. That is W is a co-isometry. Now for $x \in E$ and $h \in H_1$, we have $W^*S(x)Vh = S(x)Vh =$

$S(x)(\pi(1)Vh) = T(x)h.$ □

Definition 3.2.3. Let φ and T be as in Theorem 3.2.2. We say that a pair of triples $\big((K_1, \pi, V), (K_2, S, W)\big)$ is a *Stinespring representation* for (φ, T) if the conditions (i)-(iii) of Theorem 3.2.2 are satisfied. Such a representation is said to be *minimal* if $K_1 = \overline{\mathrm{span}}\ \pi(A)VH_1$ and $K_2 = \overline{\mathrm{span}}\ S(E)VH_1$.

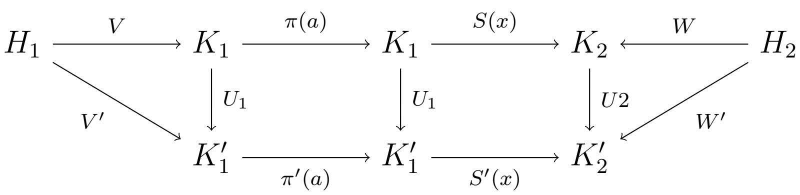
Remark 3.2.4. The pair $\big((K_1, \pi, V), (K_2, S, W)\big)$ obtained in the proof of Theorem 3.2.2 is a minimal representation for (φ, T).

Theorem 3.2.5. *Let φ and T be as in Theorem 3.2.2. Let $\big((K_1, \pi, V), (K_2, S, W)\big)$ and $\big((K_1', \pi', V'), (K_2', S', W')\big)$ be two minimal representations for (φ, T). Then there exist unitary operators $U_1 : K_1 \to K_1'$ and $U_2 : K_2 \to K_2'$ such that*

(i) *$U_1V = V'$, $U_1\pi(a) = \pi'(a)U_1$ and*

(ii) *$U_2W = W'$, $U_2S(x) = S'(x)U_1$.*

That is, the following diagram commutes, for $a \in \mathcal{A}$ and $x \in E$.



Proof. Define $U_1 : \mathrm{span}\ \pi(\mathcal{A})VH_1 \to \mathrm{span}\ \pi'(\mathcal{A})V'H_1$ by $U_1(\pi(a)Vh) := \pi'(a)V'h$ for all $a \in \mathcal{A},\ h \in H_1$, which can be seen to be an onto isometry and the unitary extension of this is the required map $U_1 : K_1 \to K_2$ ([Pau02, Theorem 4.2]). Now define $U_2 : \mathrm{span}\ S(E)VH_1 \to \mathrm{span}\ S'(E)V'H_1$ by $U_2(S(x)Vh) := S'(x)V'h$ for all $x \in E,\ h \in H_1$. Consider

$$\begin{aligned}
\Big\|\sum_{i=1}^{n} S'(x_i)V'h_i\Big\|^2 &= \Big\langle \sum_i S'(x_i)V'h_i, \sum_j S'(x_j)V'h_j \Big\rangle \\
&= \sum_{i,j} \Big\langle h_i, V'^*\big\langle S'(x_i), S'(x_j)\big\rangle V'h_j \Big\rangle \\
&= \sum_{i,j=1}^{n} \Big\langle h_i, V'^*\pi'(\langle x_i, x_j\rangle)V'h_i \Big\rangle
\end{aligned}$$

$$= \sum_{i,j=1}^{n} \Big\langle h_i, \varphi(\langle x_i, x_j\rangle) h_i \Big\rangle$$
$$= \sum_{i,j=1}^{n} \Big\langle h_i, V^*\pi(\langle x_i, x_j\rangle) V h_j \Big\rangle$$
$$= \Big\|\sum_{i=1}^{n} S(x_i) V h_i\Big\|^2.$$

Thus U_2 is well defined and an isometry and can be extended to whole of K_2, call the extension U_2 itself, and being onto it is a unitary. Since $\big((K_1, \pi, V), (K_2, S, W)\big)$ and $\big((K_1', \pi', V'), (K_2', S', W')\big)$ are representations for (φ, T), it follows that $T(x) = W^*S(x)V = W'^*S'(x)V' = W'^*U_2S(x)V$ and hence $(W^* - W'^*U_2)S(x)V = 0$. Since $\overline{\text{span}}\ S(E)VH_1 = K_2$, it follows that $W^* - W'^*U_2 = 0$, that is, $U_2W = W'$. As S is a π-representation and S' is a π'-representation, it can be shown that

$$\begin{aligned} U_2S(x)\big(\sum_{i=1}^{n} \pi(a_i)Vh_i\big) &= \sum_i U_2S(xa_i)Vh_i \\ &= \sum_i S'(xa_i)Vh_i \\ &= \sum_i S'(x)\pi'(a)V'h_j \\ &= S'(x)U_1\big(\sum_{i=1}^{n} \pi(a_i)Vh_i\big), \end{aligned}$$

for all $x \in E,\ a_i \in \mathcal{A},\ h \in H_1,\ 1 \le i \le n,\ n \in \mathbb{N}$, concluding $U_2S(x) = S'(x)U_1$. □

Remark 3.2.6. Let $\big((K_1, \pi, V), (K_2, S, W)\big)$ be a Stinespring representation for (φ, T). If φ is unital, then V is an isometry. If the representation is minimal, then W is a co-isometry by the proof of Theorem 3.2.2 and Theorem 3.2.5(ii). Conversely if W is a co-isometry, then $T(\cdot) := W^*S(\cdot)V$ defines a φ-map where $\varphi(\cdot) = V^*\pi(\cdot)V$.

Example 3.2.7. Let $\mathcal{A} = M_2(\mathbb{C}), H_1 = \mathbb{C}^2, H_2 = \mathbb{C}^8$ and $E = \mathcal{A} \oplus \mathcal{A}$. Let $b = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$. Define $\varphi : \mathcal{A} \to \mathcal{B}(H_1)$ by $\varphi(a) = a \circ b$, for all $a \in \mathcal{A}$, here $\circ$ denote the Schur product. As b is positive, φ is a CP-map (see [Pau02, Theorem 3.7]). Let $b_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $b_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$. Let $K_1 = \mathbb{C}^4$ and $K_2 = H_2$. Define $T : E \to \mathcal{B}(H_1, H_2)$ and

$S : E \to \mathcal{B}(K_1, K_2)$ by

$$T(a_1 \oplus a_2) := \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{2}} a_1 b_1 \\ \frac{\sqrt{3}}{\sqrt{2}} a_2 b_1 \\ \frac{1}{\sqrt{2}} a_1 b_2 \\ \frac{1}{\sqrt{2}} a_2 b_2 \end{bmatrix}, \quad S(a_1 \oplus a_2) := \begin{bmatrix} a_1 & 0 \\ a_2 & 0 \\ 0 & a_1 \\ 0 & a_2 \end{bmatrix} \qquad \forall\, a_1, a_2 \in \mathcal{A}.$$

It can be verified that T is a φ-map. Define $V : H_1 \to K_1$ and $\pi : \mathcal{A} \to \mathcal{B}(K_1)$ by

$$V = \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{2}} b_1 \\ \frac{1}{\sqrt{2}} b_2 \end{bmatrix}, \quad \pi(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \qquad \forall\, a \in \mathcal{A}.$$

Clearly S is a π-representation and $T(a_1 \oplus a_1) = W^* S(a_1 \oplus a_2)V$, where $W = id_{H_2}$.

This example illustrates Theorem 3.2.2. Note that in this example, there does not exists an $x_0 \in E$ with the property that $T(x_0)T(x_0)^* = id_{H_2}$, which is an assumption in Theorem 3.2.1.

3.3 Recent developments

In [Joi11] M. Joita gave a covariant version of Theorem 3.2.2 and using that proved a Radon-Nikodym type theorem for φ-maps (where φ is CP) on Hilbert C^*-modules (see [Joi12]). In [Pli12] M. Pliev proved an analogue of Theorem 3.2.2 for a finite family of maps on Hilbert C^*-modules. In [HJ11] Heo and Ji studied semigroups, called *φ-quantum dynamical semigroup*, of φ-maps on Hilbert C^*-modules. The reconstruction theorem for quantum stochastic processes from a pair (φ_t, T_t) of families of such maps is investigated.

M. Skeide proved a very generalized version of Theorem 3.2.2. We state the result here for further use and also for the completeness of this chapter.

Theorem 3.3.1 ([Ske12]). *Let E and F be Hilbert C^*-modules over unital C^*-algebras $\mathcal{B}$ and $\mathcal{C}$, respectively. Then for every linear map $T : E \to F$ the following conditions are equivalent:*

(i) *T is a φ-map for some CP-map $\varphi : \mathcal{B} \to \mathcal{C}$.*

(ii) *There exists a pair $(\mathcal{F}, \zeta)$ of a $\mathcal{B}$-$\mathcal{C}$-correspondence $\mathcal{F}$ and a vector $\zeta \in \mathcal{F}$, and* *there exists an isometry* $v : E \odot \mathfrak{F} \to F$ *such that* $T(x) = v(x \odot \zeta)$ *for all* $x \in E$.

This factorization theorem helped a lot in further studies of φ-maps. Some of them we discuss in the next Chapter. Motivated by the work of Tabadkan-Skeide ([TS07]), Bakic-Guljas ([BG02b]) and Solel ([Sol01]) one may ask which maps between Hilbert C^*-modules allows for a CP-extension to a map acting blockwise between the associated (extended) linking algebras. In next chapter we investigate in particular those CP-extendable maps where the 22-corner of the extension can be chosen to be a $*$-homomorphism. We show that they coincide with the maps considered by Asadi ([Asa09]), Bhat-Ramesh-Sumesh ([BRS12]) and Skeide ([Ske12]).

CHAPTER 4

CP-H-EXTENDABLE MAPS BETWEEN HILBERT C^*-MODULES

In the previous Chapter we have seen that if φ is a bounded linear map and $T : E \to F$ is a φ-map, then φ is CP on $\mathcal{B}_E$. In this Chapter we find a criteria that tells us when a map $T : E \to F$ is a φ-map for some CP-map φ without knowing φ, just by looking at T. The case, when a possible φ is required to be a $*$-homomorphism has been resolved by Tabadkan and Skeide ([TS07]). For full E, [TS07, Theorem 2.1] asserts: T is a φ-map for some $*$-homomorphism φ if and only if T is linear[i] and fulfills $T\big(x_1\langle x_2, x_3\rangle\big) = T(x_1)\big\langle T(x_2), T(x_3)\big\rangle$, that is, if T is a *ternary homomorphism*. Another equivalent criterion is that T extends as a $*$-homomorphism

$$\begin{bmatrix} \varphi & T^* \\ T & \vartheta \end{bmatrix} : \begin{bmatrix} \mathcal{B} & E^* \\ E & \mathcal{K}(E) \end{bmatrix} \to \begin{bmatrix} \mathcal{C} & F^* \\ F & \mathcal{K}(F) \end{bmatrix}$$

acting blockwise between the linking algebras of E and of F. We would call such maps *H-extendable*.

It is always a good idea to look at properties of Hilbert C^*-modules in terms of properties of their linking algebras. (For instance, recall that, Skeide [Ske00] defined a Hilbert C^*-module E over a von Neumann algebra to be a von Neumann module if its extended linking algebra is a von Neumann algebra in a canonically associated representation.) Likewise, it is a good idea to look at properties of maps between Hilbert C^*-modules in terms of how they may be extended to blockwise maps between their linking algebras. (For instance, many maps between von Neumann modules are σ-weakly continuous if and only if they allow for a normal (that is, order continuous) blockwise extension to a map between the linking algebras.) In addition to the usual linking algebra of a Hilbert C^*-module, it is sometimes useful to look at the reduced linking algebra or at the extended linking algebra. It would be tempting to see if φ-maps (where φ is CP) are precisely the *CP-extendable maps*, that is, maps that allow for some blockwise CP-extension between some sort

[i] We should emphasize that, contrary to the statement in [TS07], linearity of T cannot be dropped. The map $T : E \to \mathcal{C}$ defined as $T(x) = 1$ is a counter example. Indeed, without linearity, the map φ defined in the proof of [TS07, Theorem 2.1] is a well-defined multiplicative $*$-map; but it may fail to be linear.

of linking algebras. Unfortunately, this is not so: There are more CP-extendable maps than φ-maps; see Section 4.3. We, therefore, strongly object to use the name *CP-maps* between Hilbert C^*-modules as meaning φ-maps (where φ is CP), which was proposed recently by several authors; see, for instance, [HJ11] or [Joi12].

But if CP-extendable is not the right condition, what is the right condition? And what is the right "intrinsic condition" replacing the ternary condition for φ-isometries?

4.1 CP-H-extendable maps

Through out this section we assume that E and F are Hilbert C^*-modules over C^*-algebras $\mathcal{B}$ and $\mathcal{C}$, respectively. Also for a linear map $T : E \to F$ we let $F_T := \overline{\operatorname{span}}\, T(E)\mathcal{C}$.

As a main result of this Section we prove the following theorem.

Theorem 4.1.1. *Suppose E is a full Hilbert $\mathcal{B}$-module. Then for a linear map $T : E \to F$ the following conditions are equivalent:*

1. *There exists a (unique) CP-map $\varphi : \mathcal{B} \to \mathcal{C}$ such that T is a φ-map.*
2. *T extends to a blockwise CP-map $\mathfrak{T} = \begin{bmatrix} \varphi & T^* \\ T & \vartheta \end{bmatrix} : \begin{bmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{bmatrix} \to \begin{bmatrix} \mathcal{C} & F_T^* \\ F_T & \mathcal{B}^a(F_T) \end{bmatrix}$ with ϑ a $*$-homomorphism.*
3. *T is a CB-map and F_T can be turned into a $\mathcal{B}^a(E)$-$\mathcal{C}$-correspondence in such a way that T is left $\mathcal{B}^a(E)$-linear, i.e., $T(ax) = aT(x)$ for all $x \in E,\ a \in \mathcal{B}^a(E)$.*
4. *T is a CB-map fulfilling*
$$\Big\langle T(x_1), T(x_2\langle x_3, x_4\rangle)\Big\rangle = \Big\langle T(x_3\langle x_2, x_1\rangle), T(x_4)\Big\rangle. \qquad (**)$$
for all $x_i \in E,\ i = 1, \cdots, 4$.

A more readable version of (**) is $\big\langle T(x_1), T(x_2x_3^*x_4)\big\rangle = \big\langle T(x_3x_2^*x_1), T(x_4)\big\rangle$. This *quaternary condition* is the intrinsic condition we were seeking, and which generalizes the ternary condition guaranteeing that T is a φ-isometry.

Observation 4.1.2. While proving the theorem we will make the following observations.

(i) The homomorphism ϑ in (2) coincides with the left action in (3); see Remark 4.1.10.

(ii) It is routine to show that ($**$) defines a nondegenerate action of $\mathcal{F}(E)$. So, the same argument also shows that (3) and (4) are equivalent.

(iii) Clearly, Example 3.1.3(i) shows that the condition on T to be completely bounded in (3) and (4), may not be dropped. However, if E is full over a unital C^*-algebra, then T just linear is sufficient; see Observation 4.1.15 .

Remark 4.1.3. It should be noted that the CP-map φ in (2) need not coincide with the map φ in (1) making T a φ-map. We can add an arbitrary CP-map from $\mathcal{B} \to \mathcal{C}$ to the latter.

Remark 4.1.4. Unlike for φ-isometries, for more general φ-maps the $*$-homomorphism ϑ in (2) will only rarely map $\mathcal{K}(E)$ into $\mathcal{K}(F_T)$. So, in (2) it is forced that we pass to the extended linking algebras. Also considerations about the strict topology cannot be avoided completely.

Remark 4.1.5. We already know that a φ-map T is linear, so linearity of T may be dropped from (1). But the example in Footnote [i] shows that linearity cannot be dropped from (4), not even if T fulfills the stronger ternary condition. Linearity may be dropped from (3), if E contains a unit vector ξ, for in that case we have $T(x) = T(x\xi^*\xi) = (x\xi^*)T(\xi)$, which is linear in x. However, linearity of T cannot be dropped from (4) even if E is a full module over a unital $\mathcal{B}$.

Proof of Theorem 4.1.1

We shall follow the order (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and (3)$\Leftrightarrow$(4). In Section 4.3, we present an alternative direct proof of (2) $\Rightarrow$ (1), which avoids using arguments originating in operator spaces as involved in the proof (3) $\Rightarrow$ (1). Since we also wish to make comments on the mechanisms of some steps or how parts of the proof are applicable in more general situations, we put each of the steps into an own subsection and indicate by "$\boxslash$" where the part specific to Theorem 4.1.1 ends.

Proof of (1) implies (2)

CASE 1: We first consider the case where $\mathcal{B}$ and $\mathcal{C}$ are unital, but without requiring that E is full. So let $\varphi : \mathcal{B} \to \mathcal{C}$ be a CP-map between unital C^*-algebras, and let $T : E \to F$ be a φ-map from an arbitrary Hilbert $\mathcal{B}$-module E to a Hilbert $\mathcal{C}$-module F. Note that, in this case, the extended linking algebras of E and F equals $\mathcal{B}^a(\mathcal{B} \oplus E)$ and $\mathcal{B}^a(\mathcal{C} \oplus F)$, respectively.

Suppose $(\mathcal{F}, \zeta)$ is the minimal GNS-construction for φ. Define $v : E \odot \mathcal{F} \to F$ by $x \odot (b\zeta c) \mapsto T(xb)c$. Since $\langle x \odot \xi, x' \odot \xi\rangle = \big\langle \xi, \langle x, x'\rangle \xi\big\rangle = \varphi(\langle x, x'\rangle) = \langle T(x), T(x')\rangle$ the map v defines an isometry. Note that, T factors as $T(\cdot) = v\big((\cdot) \odot \zeta\big)$. (We just have reproduced the proof of the "only if" direction of the theorem in [Ske12].) Now, v is obviously a unitary onto F_T. So $\vartheta(\cdot) := v\big((\cdot) \odot \mathsf{id}_{\mathcal{F}}\big)v^*$ defines a (unital and strict) $*$-homomorphism from $\mathcal{B}^a(E) \to \mathcal{B}^a(F_T)$. Identifying $\mathcal{F}$ with $\mathcal{B}^a(\mathcal{C}, \mathcal{F})$ via $y \mapsto (l_y : c \mapsto yc)$ and identifying $\mathcal{B} \odot \mathcal{F}$ with $\mathcal{F}$ via $b \odot y \mapsto by$, we may define a map

$$\Xi := \begin{bmatrix} \zeta & 0 \\ 0 & v^* \end{bmatrix} \in \mathcal{B}^a(\begin{pmatrix} \mathcal{C} \\ F_T \end{pmatrix}, \begin{pmatrix} \mathcal{B} \odot \mathcal{F} \\ E \odot \mathcal{F} \end{pmatrix}) = \mathcal{B}^a(\begin{pmatrix} \mathcal{C} \\ F_T \end{pmatrix}, \begin{pmatrix} \mathcal{B} \\ E \end{pmatrix} \odot \mathcal{F}).$$

Obviously, the map $\mathcal{T} : \mathcal{B}^a(\mathcal{B} \oplus E) \to \mathcal{B}^a(\mathcal{C} \oplus F_T)$ defined by $\mathcal{T}(\cdot) := \Xi^*\big((\cdot) \odot \mathsf{id}_{\mathcal{F}}\big)\Xi$ is completely positive. Also for all $\begin{bmatrix} b & x^* \\ x' & a \end{bmatrix} \in \mathcal{B}^a(\mathcal{B} \oplus E)$ we have,

$$\begin{aligned}
\mathcal{T}(\begin{bmatrix} b & x^* \\ x' & a \end{bmatrix}) &= \begin{bmatrix} \zeta^* & 0 \\ 0 & v \end{bmatrix} \left(\begin{bmatrix} b & x^* \\ x' & a \end{bmatrix} \odot id_{\mathcal{F}} \right) \begin{bmatrix} \zeta & 0 \\ 0 & v^* \end{bmatrix} \\
&= \begin{bmatrix} \zeta^* & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} b \odot id_{\mathcal{F}} & x^* \odot id_{\mathcal{F}} \\ x' \odot id_{\mathcal{F}} & a \odot id_{\mathcal{F}} \end{bmatrix} \begin{bmatrix} \zeta & 0 \\ 0 & v^* \end{bmatrix} \\
&= \begin{bmatrix} \zeta^*(b \odot id_{\mathcal{F}})\zeta & \zeta^*(x^* \odot id_{\mathcal{F}})v^* \\ v(x' \odot id_{\mathcal{F}})\zeta & v(a \odot id_{\mathcal{F}})v^* \end{bmatrix} \\
&= \begin{bmatrix} \zeta^*(b \odot id_{\mathcal{F}})\zeta & (v(x \odot \zeta))^* \\ v(x' \odot \zeta) & v(a \odot id_{\mathcal{F}})v^* \end{bmatrix}.
\end{aligned}$$

Thus $\mathcal{T} = \begin{bmatrix} \varphi & T^* \\ T & \vartheta \end{bmatrix}$, where $T^*(x^*) := T(x)^*$, is a blockwise CP-map. This proves (1)$\Rightarrow$(2) for unital C^*-algebras but not necessarily full E.

CASE 2: Now suppose $\mathcal{B}$ is not necessarily unital. Nonunital $\mathcal{C}$ may always be "repaired" by appropriate use of approximate units. The following is *folklore*.

Lemma 4.1.6. *If $\varphi : \mathcal{B} \to \mathcal{C}$ is a CP-map, then the map $\widetilde{\varphi} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{C}}$ between the unitalizations of $\mathcal{B}$ and $\mathcal{C}$, defined by*

$$\widetilde{\varphi}|_{\mathcal{B}} := \varphi, \quad \widetilde{\varphi}(\widetilde{1}) := \|\varphi\|\,\widetilde{1},$$

is a CP-map.

Proof. Denote by $\delta : \widetilde{\mathcal{B}} \to \mathbb{C}$ the unique character vanishing on $\mathcal{B}$, and choose a contractive approximate unit $\{b_\alpha\}_{\alpha\in\Lambda}$ for $\mathcal{B}$. Then the maps

$$\varphi_\alpha(\cdot) := \varphi\big(b_\alpha^*(\cdot)b_\alpha\big) + \big(\,\|\varphi\|\,\widetilde{1} - \varphi(b_\alpha^* b_\alpha)\big)\delta(\cdot)$$

are CP-maps (as sum of CP-maps) and converge point-wise to $\widetilde{\varphi}$. Therefore, $\widetilde{\varphi}$ is a CP-map. □

Note that, E and F are modules over the unitalizations too, with $x1_{\mathcal{B}} := x$, $y1_{\mathcal{C}} := y$ for all $x \in E$, $y \in F$. Also $T : E \to F$ is a $\widetilde{\varphi}$-map. Since in the first part E was not required full, we may apply the result to get the blockwise CP-extension $\widetilde{\mathfrak{T}} : \begin{bmatrix} \widetilde{\mathcal{B}} & E^* \\ E & \mathcal{B}^a(E) \end{bmatrix} \to \begin{bmatrix} \widetilde{\mathcal{C}} & F^* \\ F & \mathcal{B}^a(F) \end{bmatrix}$ of T, which restricts to the desired CP-map $\mathfrak{T}$. This concludes the proof (1) ⇒(2). ☑

Definition 4.1.7. A linear map $T : E \to F$ is said to be *CP-H-extendable* if it extends to a blockwise CP-map $\mathfrak{T} = \begin{bmatrix} \varphi & T^* \\ T & \vartheta \end{bmatrix} : \begin{bmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{bmatrix} \to \begin{bmatrix} \mathcal{C} & F_T^* \\ F_T & \mathcal{B}^a(F_T) \end{bmatrix}$ with ϑ a $*$-homomorphism.

Observation 4.1.8. Obviously, the proof shows that the conclusion (1)⇒(2) holds in general, even if E is not full. Thus all φ-maps are CP-H-extendable.

Proof of (2) implies (3)

Let $T : E \to F$ be a linear map from a Hilbert $\mathcal{B}$-module E to a Hilbert $\mathcal{C}$-module F. Suppose we find a CP-map $\varphi : \mathcal{B} \to \mathcal{C}$ and a $*$-homomorphism $\vartheta : \mathcal{B}^a(E) \to \mathcal{B}^a(F_T)$ such that $\mathfrak{T} = \begin{bmatrix} \varphi & T^* \\ T & \vartheta \end{bmatrix} : \begin{bmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{bmatrix} \to \begin{bmatrix} \mathcal{C} & F_T^* \\ F_T & \mathcal{B}^a(F_T) \end{bmatrix}$ is a CP-map. Since $\mathfrak{T}$ preserves adjoint $T^* : E^* \to F^*$ should be the map $x^* \mapsto T(x)^*$. Also being the corner of a

CP-map T is a CB-map.

Lemma 4.1.9. *Let $\theta : \mathcal{B} \to \mathcal{C}$ be a CP-map between C^*-algebras $\mathcal{B}$ and $\mathcal{C}$. Suppose $\mathcal{A} \subset \mathcal{B}$ is a C^*-subalgebra of $\mathcal{B}$ with unit $1_{\mathcal{A}}$ such that the restriction $\vartheta := \theta|_{\mathcal{A}}$ of θ to $\mathcal{A}$ is a $*$-homomorphism. Then*

$$\theta(ab) = \vartheta(a)\theta(1_{\mathcal{A}}b), \quad \theta(ba) = \theta(b1_{\mathcal{A}})\vartheta(a)$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Proof. Assume that $\mathcal{B}$ and $\mathcal{C}$ are unital. (Otherwise, unitalize as explained in Lemma 4.1.6 and observe that also the unitalization $\widetilde{\theta}$ fulfills the hypotheses for $\mathcal{A} \subset \widetilde{\mathcal{B}}$ with the same ϑ. If the statement is true for $\widetilde{\theta}$, then so it is for $\theta = \widetilde{\theta}|_{\mathcal{B}}$.) Let $(\mathcal{F}, \zeta)$ denote the GNS-construction for θ. Then for all $a \in \mathcal{A}$, by the stated properties,

$$\begin{aligned}
&\big\langle a\zeta - 1_{\mathcal{A}}\zeta\vartheta(a), a\zeta - 1_{\mathcal{A}}\zeta\vartheta(a)\big\rangle \\
&\qquad = \langle \zeta, a^*a\zeta\rangle - \langle \zeta, a^*\zeta\rangle\vartheta(a) - \vartheta(a^*)\langle \zeta, a\zeta\rangle + \vartheta(a^*)\langle \zeta, 1_{\mathcal{A}}\zeta\rangle\vartheta(a) \\
&\qquad = \theta(a^*a) - \theta(a^*)\vartheta(a) - \vartheta(a^*)\theta(a) + \vartheta(a^*)\theta(1_{\mathcal{A}})\vartheta(a) \\
&\qquad = 0,
\end{aligned}$$

since $\theta|_{\mathcal{A}} = \vartheta$. Thus $a\zeta = 1_{\mathcal{A}}\zeta\vartheta(a)$ and hence

$$\theta(ab) = \langle \zeta, ab\zeta\rangle = \langle a^*\zeta, 1_{\mathcal{A}}b\zeta\rangle = \vartheta(a)\langle 1_{\mathcal{A}}\zeta, 1_{\mathcal{A}}b\zeta\rangle = \vartheta(a)\theta(1_{\mathcal{A}}b)$$

and $\theta(ba) = \theta(a^*b^*)^* = \theta(b1_{\mathcal{A}})\vartheta(a)$ for all $a \in \mathcal{A},\ b \in \mathcal{B}$. □

By applying Lemma 4.1.9 to the CP-map $\mathfrak{T} : \begin{bmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{bmatrix} \to \begin{bmatrix} \mathcal{C} & F_T^* \\ F_T & \mathcal{B}^a(F_T) \end{bmatrix}$ with the subalgebra $\mathcal{A} = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{B}^a(E) \end{bmatrix} \ni \begin{bmatrix} 0 & 0 \\ 0 & id_E \end{bmatrix} = 1_{\mathcal{A}}$, we get

$$\begin{bmatrix} 0 & 0 \\ T(ax) & 0 \end{bmatrix} = \mathfrak{T}\Big(\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} \Big) = \begin{bmatrix} 0 & 0 \\ 0 & \vartheta(a) \end{bmatrix} \mathfrak{T}\Big(\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} \Big) = \begin{bmatrix} 0 & 0 \\ \vartheta(a)T(x) & 0 \end{bmatrix}.$$

Thus $T(ax) = \vartheta(a)T(x)$ for all $x \in E,\ a \in \mathcal{B}^a(E)$. In other words ϑ defines a left action of $\mathcal{B}^a(E)$ on F_T with respect to which T is left $\mathcal{B}^a(E)$-linear. This proves (2) $\Rightarrow$ (3). ☑

Remark 4.1.10. Suppose ϑ' is any other left action of $\mathcal{B}^a(E)$ on F_T with respect to which T is left $\mathcal{B}^a(E)$-linear. Then $\vartheta'(a)T(x) = T(ax) = \vartheta(a)T(x)$ for all $x \in E$, hence $\vartheta'(a) = \vartheta(a)$ for all $a \in \mathcal{B}^a(E)$. Thus ϑ in (2) coincides with the left action in (3). In fact, since, the set $T(E)$ generates the Hilbert $\mathcal{C}$-module F_T, the left action in (3) (and, consequently, also ϑ in (2)) is uniquely determined by $(xy^*)T(z) = T(xy^*z)$. This formula shows that $\mathcal{F}(E)$ act nondegenerately on F_T, so there is a unique extension to all of $\mathcal{B}^a(E)$. Moreover, this unique extension is strict and unital (Proposition 1.3.10).

Observation 4.1.11. Here also we did not require that E is full. So (2) $\Rightarrow$ (3) is true for all CP-H-extendable maps.

Proof of (3) if and only if (4)

Clearly (3)$\Rightarrow$(4). Now, suppose $T : E \to F$ is a bounded linear map satisfying ($**$). Then for all $a = \sum_{i=1}^n x_i' x_i^* \in \mathcal{F}(E)$ and $x, x' \in E$ we have

$$\big\langle T(ax), T(x') \big\rangle = \sum \big\langle T(x_i' x_i^* x), T(x') \big\rangle = \sum \big\langle T(x), T(x_i x_i'^* x') \big\rangle = \big\langle T(x), T(a^* x') \big\rangle.$$

Also if $\mathcal{K}(E) \ni a = \lim a_n$ with $a_n \in \mathcal{F}(E)$, then, since T is bounded,

$$\big\langle T(ax), T(x') \big\rangle = \lim \big\langle T(a_n x), T(x') \big\rangle = \lim \big\langle T(x), T(a_n^* x') \big\rangle = \big\langle T(x), T(a^* x') \big\rangle.$$

Now for each $a \in \mathcal{K}(E)$ define $\pi(a) : T(E) \to T(E)$ by $T(x) \mapsto T(ax)$. Note that if $T(x) = T(x')$, then

$$\begin{aligned}
&\big\langle \pi(a)T(x) - \pi(a)T(x'), \pi(a)T(x) - \pi(a)T(x') \big\rangle \\
&\quad = \big\langle T(ax), T(ax) \big\rangle - \big\langle T(ax), T(ax') \big\rangle - \big\langle T(ax'), T(ax) \big\rangle + \big\langle T(ax'), T(ax') \big\rangle \\
&\quad = \big\langle T(x), T(a^*ax) \big\rangle - \big\langle T(x), T(a^*ax') \big\rangle - \big\langle T(x'), T(a^*ax) \big\rangle + \big\langle T(x'), T(a^*ax') \big\rangle \\
&\quad = \big\langle T(x) - T(x'), T(a^*ax) \big\rangle - \big\langle T(x) - T(x'), T(a^*ax') \big\rangle \\
&\quad = 0
\end{aligned}$$

and hence $\pi(a)T(x) = \pi(a)T(x')$. Thus $\pi(a)$ is well defined for all $a \in \mathcal{K}(E)$. Clearly $\big\langle \pi(a)T(x), T(x') \big\rangle = \big\langle T(x), \pi(a^*)T(x') \big\rangle$ and $\pi(a)\pi(a') = \pi(aa')$ for all $x, x' \in E$, $a, a' \in \mathcal{K}(E)$. Then from Lemma 1.4.4 and Lemma 1.4.5 we get a $*$-homomorphism $\pi : \mathcal{K}(E) \to \mathcal{B}^a(F_T)$. Since $\overline{\mathsf{span}}\, \pi(\mathcal{F}(E))F_T = F_T$ we have π

is nondegenerate. Then, by Proposition 1.3.10, it further extends uniquely to a strict unital $*$-homomorphism, denote again by π, from $\mathcal{B}^a(E) \to \mathcal{B}^a(F_T)$. Since $\mathcal{K}(E)$ is strictly dense in $\mathcal{B}^a(E)$ the extension π satisfies $\pi(a)T(x) = T(ax)$ for all $a \in \mathcal{B}^a(E)$, $x \in E$. Thus π defines a left action of $\mathcal{B}^a(E)$ on F_T such that T is left $\mathcal{B}^a(E)$-linear. This proves (4)⇒(3). ☑

Proof of (3) implies (1)

Given a CB-map $T : E \to F$ and a left action of $\mathcal{B}^a(E)$ on F_T such that $aT(x) = T(ax)$ for all $x \in E$, $a \in \mathcal{B}^a(E)$, our scope is to define φ by ($*$). So, in this part it is essential that E is full. We will show that the hypotheses of (3), which showed already to be necessary, are also sufficient.

Suppose E is a full Hilbert $\mathcal{B}$-module. Now if there exists a map φ such that T is a φ-map, then φ appears to be the unique map $\langle x, x'\rangle \mapsto \langle T(x), T(x')\rangle$. Since T is left $\mathcal{B}^a(E)$-linear the map $\langle x, x'\rangle \mapsto \langle T(x), T(x')\rangle$ is balanced[j] over $\mathcal{B}^a(E)$. We prove that the map φ assigning the value $\langle T(x), T(x')\rangle \in \mathcal{C}_{F_T} = F_T^* \odot_{\mathcal{B}^a(F_T)} F_T$ to each element $\langle x, x'\rangle \in \mathcal{B}_E = E^* \odot_{\mathcal{B}^a(E)} E$ is a well defined bounded linear map. Once φ is bounded, Theorem 3.1.5 asserts that φ is completely positive.

The proof of boundedness can be done by appealing to the module Haagerup tensor product and Blecher's result (Theorem 1.5.19) that the internal tensor product of correspondences is completely isometrically the same as their module Haagerup tensor product. Note that $F_T^* \odot_{\mathcal{B}^a(F_T)} F_T \subseteq F^* \odot_{\mathcal{B}^a(F)} F$. And if F is a correspondence making T left $\mathcal{B}^a(E)$-linear, then, by definition of left $\mathcal{B}^a(E)$-linear, F_T is a correspondence making T left $\mathcal{B}^a(E)$-linear, too. (Also strictness does not play any role here.) So, it does not really matter if we require the property in (3) for F_T or for F, because the latter implies the former. So, let F be a $\mathcal{B}^a(E)$-$\mathcal{C}$-correspondence such that T is left $\mathcal{B}^a(E)$-linear. Then, $T^* := * \circ T \circ *$ is a right $\mathcal{B}^a(E)$-linear map for the corresponding $\mathcal{B}^a(E)$-module structures of E^* and F^*. (Note that E^* is the *dual* Hilbert $\mathcal{B}^a(E)$-module of E with inner product $\langle x'^*, x^*\rangle = x'x^*$. But F^* is not a Hilbert $\mathcal{B}^a(E)$-module, it is a Banach right $\mathcal{B}^a(E)$-module.) Consider the map

$$T^* \odot T : E^* \odot_{h\mathcal{B}^a(E)} E \longrightarrow F^* \odot_{h\mathcal{B}^a(E)} F$$

[j]For $a \in \mathcal{B}^a(E)$ we have $\langle ax, x'\rangle = \langle x, a^*x'\rangle$. So if the map $\langle x, x'\rangle \mapsto \langle T(x), T(x')\rangle$ is well defined, then $\langle T(ax), T(x)\rangle = \langle T(x), T(a^*x)\rangle$.

between the module Haagerup tensor products over $\mathcal{B}^a(E)$. Indeed, since T is CB, the universal property of the module Haagerup tensor product guarantees that the map $T^* \odot T$ is completely bounded with $\|T^* \odot T\|_{cb} \leq \|T^*\|_{cb} \|T\|_{cb}$. The Haagerup seminorm on $F^* \otimes F$ with amalgamation over $\mathcal{B}^a(E)$, which is homomorphic to a subset of $\mathcal{B}^a(F)$, is bigger than the Haagerup seminorm with amalgamation over $\mathcal{B}^a(F)$. So, together with Blecher's result we get that, as map between the internal tensor products $E^* \odot_{\mathcal{B}^a(E)} E = \mathcal{B}_E = \mathcal{B}$ and $F^* \odot_{\mathcal{B}^a(F)} F = \mathcal{C}_F \subseteq \mathcal{C}$ the map $T^* \odot T$ (equals φ and) is of CB-norm not bigger than $\|T^*\|_{cb} \|T\|_{cb}$. Thus φ is bounded. But we prefer to give a direct independent proof of boundedness of φ. Actually, our method will provide us with a quick proof of Blecher's result.

We have $E^* \underline{\odot}_{\mathcal{B}^a(E)} E = \operatorname{span} \langle E, E \rangle$ as subset of $E^* \odot_{\mathcal{B}^a(E)} E = \mathcal{B}_E = \mathcal{B}$. Once $\varphi : E^* \underline{\odot}_{\mathcal{B}^a(E)} E \to \mathcal{C}$ is bounded (for the norm of the internal tensor product $E^* \odot_{\mathcal{B}^a(E)} E$ on $E^* \underline{\odot}_{\mathcal{B}^a(E)} E \subseteq E^* \odot_{\mathcal{B}^a(E)} E$), then so is the extension to $\mathcal{B} = E^* \odot_{\mathcal{B}^a(E)} E$. So it remains to show that φ is bounded on $E^* \underline{\odot}_{\mathcal{B}^a(E)} E$. Let $z = \sum_{i=1}^n x_i^* \odot y_i = \sum_{i=1}^n \langle x_i, y_i \rangle \in E^* \underline{\odot}_{\mathcal{B}^a(E)} E = \operatorname{span}\langle E, E \rangle$. For the elements x and y in E^n with entries x_i and y_i, respectively, this reads $z = \langle x, y \rangle$. We get $\varphi(z) = \langle T^n(x), T^n(y) \rangle$. Consequently,

$$\|\varphi(z)\| = \|\langle T^n(x), T^n(y) \rangle\| \leq \|T^n\|^2 \|x\| \|y\| \leq \|T\|_{cb}^2 \|x\| \|y\| .$$

Now if, for any $\varepsilon > 0$, we can find x_ε and y_ε in E^n such that $\langle x_\varepsilon, y_\varepsilon \rangle = z$ and $\|x_\varepsilon\| \|y_\varepsilon\| \leq \|z\| + \varepsilon$, then we obtain

$$\|\varphi(z)\| \leq \|T\|_{cb}^2 \|x_\varepsilon\| \|y_\varepsilon\| \leq \|T\|_{cb}^2 (\|z\| + \varepsilon),$$

and further $\|\varphi\| \leq \|T\|_{cb}^2$, by letting $\varepsilon \to 0$.

For showing that this is possible, we recall the following well-known result. (See, for instance, [Lan95, Lemma 4.4].)

Lemma 4.1.12. *For every element x in a Hilbert $\mathcal{B}$-module E and for every $r \in (0, 1)$ there is an element $w_r \in E$ such that $x = w_r |x|^r$.*

The proof in [Lan95] shows that w_r can be chosen in the Hilbert $C^*(|x|)$-module $\overline{xC^*(|x|)}$, which is isomorphic to $C^*(|x|)$ via the bilinear unitary $u : x \mapsto |x|$. The element $w_r \in \overline{xC^*(|x|)}$ is unique. For, suppose $x = w |x|^r$ for some $w \in \overline{xC^*(|x|)}$, then $(w - w_r) |x|^r = 0$ and hence $u(w - w_r) |x|^r = u\big((w - w_r) |x|^r\big) = 0$. Since $|x|^r$ is positive in the C^*-algebra $C^*(|x|)$, we get $u(w - w_r) = 0$[k], thus $w = w_r$. Obviously, when represented in $C^*(|x|)$, w_r is $|x|^{1-r}$.

Corollary 4.1.13. *Let E be a Hilbert $\mathcal{B}$-module and let F be a Hilbert-$\mathcal{B}$-$\mathcal{C}$-module. Choose $x \in E$, $y \in F$ and put $z := x \odot y \in E \otimes_{\mathcal{B}} F$. Then for every $\varepsilon > 0$, there exist $x_\varepsilon \in E$ and $y_\varepsilon \in F$ such that $x_\varepsilon \odot y_\varepsilon = z$ and $\|x_\varepsilon\| \|y_\varepsilon\| \leq \|z\| + \varepsilon$, that is,*

$$\|x \odot y\| = \inf \{\|x'\| \|y'\| : x' \in E, y' \in F, x' \odot y' = x \odot y\}.$$

Proof. For each $r \in (0,1)$ let $w_r \in \overline{xC^*(|x|)} \subseteq E$ be such that $x = w_r |x|^r$. Since $\|w_r\| = \left\||x|^{1-r}\right\| = \sup_{\lambda \in [0,\|x\|]} \lambda^{1-r} = \|x\|^{1-r} \longrightarrow 1$, and since $|x|^r$ converges in norm to $|x|$ we have $\|w_r\| \||x|^r y\| \xrightarrow{r \to 1} 1 \||x| y\| = \|x \odot y\| = \|z\|$. So given $\varepsilon > 0$ there exists $r' \in (0,1)$ such that $z = x \odot y = w_{r'} \odot |x|^{r'} y$ with $\|z\| \leq \|w_{r'}\| \left\||x|^{r'} y\right\| \leq \|z\| + \varepsilon$. ☑

With the proof of this corollary we did not only conclude the proof of (3) ⇒ (1), but also the proof of Theorem 4.1.1. ☑

Corollary 4.1.14 ([Ble97a, Theorem 4.3])**.** *Let E be a Hilbert $\mathcal{B}$-module and let F be a Hilbert-$\mathcal{B}$-$\mathcal{C}$-module. The internal tensor product norm of $z \in E \underline{\odot} F$ is*

$$\|z\| = \inf \left\{ \|x_n\| \|y^n\| : x_n \in E_{(n)},\ y^n \in F^n,\ x_n \odot y^n = z,\ n \in \mathbb{N} \right\}, \qquad (\star)$$

with the row space $E_{(n)} = M_{1,n}(E)$ and the internal tensor product $x_n \odot y^n$ over $M_n(\mathcal{B})$. That is, the internal tensor product norm coincides with the module Haagerup tensor product norm. Moreover, since $M_n(E \odot F)$ is isomorphic to the internal tensor product $M_n(E) \odot M_n(F)$, the internal tensor product is completely isometrically isomorphic to the module Haagerup tensor product.

Proof. First observe that $E_{(n)} \underline{\odot}_{M_n(\mathcal{B})} F^n \cong E \underline{\odot}_{\mathcal{B}} F$ under the isometric isomorphism

$$(x_1, x_2, \cdots, x_n) \odot (y_1, y_2, \cdots, y_n)^t \mapsto \sum x_i \odot y_i.$$

[k]Let $g \in C(\sigma(|x|))$ is such that $g(|x|) = u(w - w_r) \in C^*(|x|) \cong C(\sigma(|x|))$. Then $u(w - w_r)\,|x|^r = 0$ implies that $g(t)t^r = 0$ for all $t \in \sigma(|x|)$. Therefore $g(t) = 0$ for all $t \in \sigma(|x|) \setminus \{0\}$, thus, $g = 0$ since g is continuous.

Suppose $z = \sum_{i=1}^{n} x_i \odot y_i \in E \otimes_{\mathcal{B}} F$. If $x := (x_1, x_2, \ldots, x_n) \in E_{(n)}$ and $y := (y_1, y_2, \ldots, y_n)^t \in E^n$, then $z = x \odot y \in E_{(n)} \underline{\odot}_{M_n(\mathcal{B})} F^n = E \underline{\odot} F$. So, given any $\varepsilon > 0$, there exists $x_\varepsilon \in E_{(n)}$ and $y_\varepsilon \in F^n$ such that $z = x_\varepsilon \odot y_\varepsilon$ and $\|x_\varepsilon\| \, \|y_\varepsilon\| \leq \|z\| + \varepsilon$. Therefore

$$\inf \left\{ \|x_n\| \, \|y^n\| : x_n \in E_{(n)}, \ y^n \in F^n, \ z = x_n \odot y^n, \ n \in \mathbb{N} \right\} \leq \|z\| .$$

The reverse inequality is trivial. But RHS of ($\star$) is $\|z\|_h$, and thus $\|z\| = \|z\|_h$ on $E \otimes_{\mathcal{B}} F$. Therefore the completions $E \odot F$ and $E \odot_{h\mathcal{B}} F$ are isometrically isomorphic. Replacing E, F by $M_n(E), M_n(F)$ respectively, we get,

$$\begin{aligned} M_n(E \odot F) &= M_n(E) \odot M_n(F) \cong M_n(E) \odot_h M_n(F) \\ &= M_{n,1}(E_{(n)}) \odot_h M_{1,n}(F^n) = M_n(E_{(n)} \odot_h F^n) \\ &= M_n(E \odot_h F) \end{aligned}$$

for all $n \in \mathbb{N}$, and hence the result holds. □

After this digression on the Haagerup tensor product, let us return to maps fulfilling (3). However, we weaken the conditions a bit. Firstly, we replace F_T with F, so that now F is a $\mathcal{B}^a(E)$-$\mathcal{C}$-correspondence fulfilling $T(ax) = aT(x)$. We still may define the map $T^* \odot T$ on $E^* \underline{\odot} E = \mathsf{span}\langle E, E \rangle$, and if T is CB, everything goes as before. Secondly, we wish to weaken the boundedness condition on T. We know from Example 3.1.3 that if $\mathcal{B}_E$ is nonunital, the CB-condition is indispensable. So, suppose that E is full and that $\mathcal{B} = \mathcal{B}_E$ is unital.

Observation 4.1.15. In the prescribed situation, suppose E has a unit vector ξ. In that case, $\varphi := T^* \odot T$ is defined on all $\mathcal{B} = \langle \xi, \xi \rangle \mathcal{B} \subseteq \mathsf{span}\langle E, E \rangle = E^* \underline{\odot} E \subseteq \mathcal{B}$. Since $\varphi(b^*b) = \varphi(b^* \langle \xi, \xi \rangle b) = \langle \langle T(\xi b), T(\xi b) \rangle \geq 0$ we have φ is positive and hence is bounded by $\|\varphi(1)\|$. From $T(x) = T(x\langle \xi, \xi \rangle) = (x\xi^*)T(\xi)$, we conclude that $\|T(x)\|^2 = \|\langle T(\xi), \xi x^* x \xi^* T(\xi) \rangle\| \leq \|x\|^2 \left\| \left\langle T(\xi), T(\xi) \right\rangle \right\| = \|x\|^2 \|\varphi(1)\|$. (This is the same trick in Remark 4.1.5 that allowed to show that a map $T : E \to F$ fulfilling (3) without boundedness and linearity, is linear provided E has a unit vector ξ.)

Even if E has no unit vector but $\mathcal{B} = \mathcal{B}_E$ still is unital, then there is a number $n \in \mathbb{N}$ such that E^n has a unit vector, say, $\xi^n = \left(\xi_1, \ldots, \xi_n \right)^t$ (Proposition 1.1.11).

Again, since

$$\varphi(b^*b) = \varphi\big(b^*\langle\xi^n, \xi^n\rangle b\big) = \varphi\Big(\sum_{i=1}^{n}\langle\xi_i b, \xi_i b\rangle\Big) = \sum_{i=1}^{n}\big\langle(T(\xi_i b), T(\xi_i b)\big\rangle \geq 0$$

we have $\varphi := T^* \odot T$ is positive and bounded by $\varphi(1)$. Since T is linear, $T^n : E^n \to F^n$ is left $M_n(\mathcal{B}^a(E))$-linear. So, for all $z \in E^n$, we have

$$\begin{aligned}\|T^n(z)\|^2 &= \|\langle T^n(\xi^n), \xi^n z^* z \xi^{n*} T^n(\xi^n)\rangle\| \\ &\leq \|z\|^2 \, \|\langle T^n(\xi^n), T^n(\xi^n)\rangle\| \\ &= \|z\|^2 \Big\|\sum \langle T(\xi_i), T(\xi_i)\rangle\Big\| \\ &= \|z\|^2 \Big\|\sum \varphi(\langle\xi_i, \xi_i\rangle)\Big\| \\ &= \|z\|^2 \big\|\varphi(\langle\xi^n, \xi^n\rangle)\big\|.\end{aligned}$$

Thus T^n, and a *fortiori* T, is bounded by $\sqrt{\|\varphi(1)\|}$ with the same φ as obtained from T. Finally, since $M_m(E^n)$ has a unit vector (with entries ξ^n in the diagonal) and $(T^n)_m = T_{mn,m} : M_m(E^n) \to M_m(F^n)$ is left $M_{mn}\big(\mathcal{B}^a(E)\big)$-linear[1], as above, we have $(T^n)_m$ is bounded by $\|\varphi_m(1_m)\| = \|\varphi(1)\|$ for all $m \geq 1$. So, T^n, and *a fortiori* T, is completely bounded by $\sqrt{\|\varphi\|} = \sqrt{\|\varphi(1)\|}$.

4.2 CPH-maps

We have seen in Theorem 4.1.1 that the submodule F_T of F generated by $T(E)$ plays a distinguished role. (If T is a φ-isometry, then $T(E)$ is already a closed $\varphi(\mathcal{B})$-submodule of F.) It is natural to ask to what extent the condition in (2) can be satisfied if we write F instead of F_T. In developing semigroup versions ([SS14, Section 4,5]), this situation becomes so important that we prefer to use the acronym CPH for that case, and leave for the equivalent of φ-maps the rather contorted term CP-H-extendable.

Definition 4.2.1. A *CPH-map* from E to F is a linear map that extends as a blockwise CP-map between the extended linking algebras of E and of F such that the 22-corner is a $*$-homomorphism. A CPH-map is *strictly CPH* if the homomorphism can be

[1] Note that, $M_m(E^n) = M_{mn,m}(E)$ is a $M_{mn}\big(\mathcal{B}^a(E)\big)$-$M_m\big(\mathcal{B}\big)$ correspondence in an obvious way.

chosen strict. A (strictly) CPH-map is a (*strictly*) *CPH*$_0$-*map* if the homomorphism can be chosen unital.

Observation 4.2.2. Effectively, in the proof of (2)⇒(3), for the conclusion $T(ax) = \vartheta(a)T(x)$, we did not even need that $\mathfrak{T}$ maps into the linking algebra of F_T. The conclusion remains true for all CPH-maps, so that for a CPH-map the subspace F_T of F reduces ϑ.

Corollary 4.2.3. *A CPH-map $T : E \to F$ is CP-H-extendable.*

Proof. Suppose $\mathfrak{T} = \begin{bmatrix} \varphi & T^* \\ T & \vartheta \end{bmatrix} : \begin{bmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{bmatrix} \to \begin{bmatrix} \mathcal{C} & F^* \\ F & \mathcal{B}^a(F) \end{bmatrix}$ is a blockwise CP-map such that $\vartheta : \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ is a $*$-homomorphism. Then from Lemma 4.1.9 we have $\vartheta(a)T(x) = T(ax)$ and thus $\vartheta : \mathcal{B}^a(E) \to \mathcal{B}^a(F_T)$ defines a strict unital $*$-homomorphism. So $\mathfrak{T}' = \begin{bmatrix} \varphi & T^* \\ T & \vartheta \end{bmatrix} : \begin{bmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{bmatrix} \to \begin{bmatrix} \mathcal{C} & F_T^* \\ F_T & \mathcal{B}^a(F_T) \end{bmatrix}$ is a CP-H extension of T, once we prove that it is CP.

Let $T(x) \in F_T \subseteq F = \overline{\mathsf{span}}\, F\langle F, F\rangle = F\mathcal{C}_F$. (Recall Corollary 1.1.14.) Suppose $c_m = \sum_{k=1}^{l_m} \langle z_{mk}, w_{mk}\rangle \in \mathsf{span}\,\langle F, F\rangle$ is such that $T(x) = y(\lim_m c_m)$. Then for all $c \in \mathcal{C},\ \mathfrak{a}_i \in \mathfrak{A}(E)$ and $\begin{bmatrix} c_i & Tx_i \\ Tx_i' & d_i \end{bmatrix} \in \mathfrak{A}(F_T),\ i = 1, 2, \cdots, n$ we have

$$\begin{aligned}
&\left\langle \begin{pmatrix} c \\ Tx \end{pmatrix}, \sum_{i,j=1}^{n} \begin{bmatrix} c_i & Tx_i^* \\ Tx_i' & d_i \end{bmatrix}^* \mathfrak{T}'(\mathfrak{a}_i{}^*\mathfrak{a}_j) \begin{bmatrix} c_j & Tx_j^* \\ Tx_j' & d_j \end{bmatrix} \begin{pmatrix} c \\ Tx \end{pmatrix} \right\rangle \\
&= \lim_m \sum_{k=1}^{l_m} \sum_{i,j=1}^{n} \left\langle \begin{bmatrix} c_i & Tx_i^* \\ Tx_i' & d_i \end{bmatrix} \begin{pmatrix} c \\ yz_{mk}^* w_{mk} \end{pmatrix}, \mathfrak{T}'(\mathfrak{a}_i{}^*\mathfrak{a}_j) \begin{bmatrix} c_j & Tx_j^* \\ Tx_j' & d_j \end{bmatrix} \begin{pmatrix} c \\ yz_{mk}^* w_{mk} \end{pmatrix} \right\rangle \\
&= \lim_m \sum_{k=1}^{l_m} \sum_{i,j=1}^{n} \left\langle \begin{bmatrix} c_i & (z_{mk}\langle y, Tx_i\rangle)^* \\ Tx_i' & d_i y z_{mk}^* \end{bmatrix} \begin{pmatrix} c \\ w_{mk} \end{pmatrix}, \mathfrak{T}(\mathfrak{a}_i{}^*\mathfrak{a}_j) \begin{bmatrix} c_j & (z_{mk}\langle y, Tx_j\rangle)^* \\ Tx_j' & d_j y z_{mk}^* \end{bmatrix} \begin{pmatrix} c \\ w_{mk} \end{pmatrix} \right\rangle \\
&= \lim_m \sum_{k=1}^{l_m} \left\langle \begin{pmatrix} c \\ w_{mk} \end{pmatrix}, \sum_{i,j=1}^{n} \begin{bmatrix} c_i & (z_{mk}\langle y, Tx_i\rangle)^* \\ Tx_i' & d_i y z_{mk}^* \end{bmatrix}^* \mathfrak{T}(\mathfrak{a}_i{}^*\mathfrak{a}_j) \begin{bmatrix} c_j & (z_{mk}\langle y, Tx_j\rangle)^* \\ Tx_j' & d_j y z_{mk}^* \end{bmatrix} \begin{pmatrix} c \\ w_{mk} \end{pmatrix} \right\rangle \\
&\geq 0
\end{aligned}$$

since $\begin{bmatrix} c_j & (z_{mk}\langle y, Tx_j\rangle)^* \\ Tx_j' & d_j y z_{mk}^* \end{bmatrix} \in \mathcal{B}^a(\mathcal{C} \oplus F)$ and $\mathfrak{T}$ is CP. Thus $\mathfrak{T}'$ is also a CP-map. □

Observation 4.2.4. If E is full, then the above corollary also follows via CPH ⇒(3)⇒

(1) $\Rightarrow$ (2).

Observation 4.2.5. If F_T is complemented in F, then $T : E \to F$ is a CPH-map if and only if it is CP-H-extendable. In that case, $\mathcal{B}^a(F_T)$ is the corner $\begin{bmatrix} \mathcal{B}^a(F_T) & 0 \\ 0 & 0 \end{bmatrix}$ of $\mathcal{B}^a(F) = \mathcal{B}^a(F_T \oplus F_T^{\perp})$, so that ϑ may be considered a map into $\mathcal{B}^a(F)$. But this condition is not at all necessary, nor natural; see Section 4.4.1.

Despite the fact that there are fewer CPH-maps than CP-H-extendable maps, looking at CPH-maps is particularly crucial if we wish to look at semigroups of CP-H-extendable maps T_t on E. Obviously, for full E, the associated CP-maps φ_t form a CP-semigroup. But the same question for the homomorphisms ϑ_t, a priori, has no meaning. The extensions ϑ_t map $\mathcal{B}^a(E)$ into $\mathcal{B}^a(E_{T_t})$, not into $\mathcal{B}^a(E)$. And if E_{T_t} is not complemented in E, then it is not possible to interpret $\mathcal{B}^a(E_{T_t})$ as a subset of $\mathcal{B}^a(E)$, to which ϑ_s could be applied in order to make sense out of $\vartheta_s \circ \vartheta_t$.

Observation 4.2.6. Adding the obvious statement that for each $\mathcal{B}$-$\mathcal{C}$-correspondence $\mathcal{F}$ and for each vector $\zeta \in \mathcal{F}$, an isometry $v : E \odot \mathcal{F} \to F$ gives rise to a φ-map $T(\cdot) := v\big((\cdot) \odot \zeta\big)$ for the CP-map $\varphi(\cdot) := \big\langle \zeta, (\cdot)\zeta \big\rangle$, we also get the "if" direction of the theorem in [Ske12]. For this it is not necessary that $\mathcal{F}$ is the minimal GNS-correspondence of φ. This observation provides us with many CPH-maps. It also plays a role in developing the theory of CPH-semigroups ([SS14, Section 4]).

4.3 CP-extendable maps

In (1) $\Rightarrow$ (2) we have written down the (strict unital) $*$-homomorphism $\vartheta : \mathcal{B}^a(E) \to \mathcal{B}^a(F_T)$ in the form $\vartheta(\cdot) := v\big((\cdot) \odot \mathsf{id}_{\mathcal{F}}\big)v^*$ with the unitary $v : E \odot \mathcal{F} \to F_T$ granted by the theorem in [Ske12]. Then we have shown that the blockwise map $\mathfrak{T} := \begin{bmatrix} \varphi & T^* \\ T & \vartheta \end{bmatrix}$ is completely positive, by writing it as $\Xi^*\big((\cdot) \odot \mathsf{id}_{\mathcal{F}}\big)\Xi$ with a diagonal map $\Xi \in \mathcal{B}^a(\begin{pmatrix} \mathcal{C} \\ F_T \end{pmatrix}, \begin{pmatrix} \mathcal{B} \\ E \end{pmatrix} \odot \mathcal{F})$. (Recall that it was necessary to unitalize φ if $\mathcal{B}$ was nonunital.) We wish to illustrate that these forms for ϑ and $\mathfrak{T}$ are not accidental, but it actually holds for all strictly CP-extendable maps T.

Lemma 4.3.1. *Let E be a Hilbert $\mathcal{B}$-module, F be a Hilbert $\mathcal{C}$-module, and let $\mathfrak{T} : \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ be a CP-map with the minimal GNS-construction $(\mathcal{E}, \Xi)$. The following conditions are equivalent:*

(i) *$\mathfrak{T}$ is strict, that is, bounded strictly converging nets in $\mathcal{B}^a(E)$ are sent to strictly converging nets in $\mathcal{B}^a(F)$.*

(ii) *The action of $\mathcal{K}(E)$ on the $\mathcal{B}^a(E)$-$\mathcal{C}$-correspondence $\mathcal{E} \odot F$ is nondegenerate.*

(iii) *The left action of $\mathcal{B}^a(E)$ on the $\mathcal{B}^a(E)$-$\mathcal{C}$-correspondence $\mathcal{E} \odot F$ defines a strict $*$-homomorphism.*

Proof. (i) $\Rightarrow$ (ii). Suppose $\mathfrak{T}$ is strict, and choose a bounded approximate unit $\{u_\alpha\}_{\alpha\in\Lambda}$ for $\mathcal{K}(E)$. Then $u_\alpha a \longrightarrow a$ strictly for all $a \in \mathcal{B}^a(E)$ (see for instance [Lan95, proof of Proposition 1.3]). Now for every element $a\Xi \odot y$ from the total subset $\mathcal{B}^a(E)\Xi \odot F$ of $\mathcal{E} \odot F$, we have

$$\begin{aligned}
\left\|(u_\alpha a - a)\Xi \odot y\right\|^2 &= \left\|\big\langle(u_\alpha a - a)\Xi \odot y, (u_\alpha a - a)\Xi \odot y\big\rangle\right\| \\
&= \left\|\big\langle y, \big\langle(u_\alpha a - a)\Xi, (u_\alpha a - a)\Xi\big\rangle y\big\rangle\right\| \\
&= \left\|\big\langle y, \mathfrak{T}\big((u_\alpha a - a)^*(u_\alpha a - a)\big)y\big\rangle\right\| \\
&\longrightarrow 0,
\end{aligned}$$

so that $\lim(u_\alpha \odot \mathsf{id}_F)(a\Xi \odot y) = \lim u_\alpha a\Xi \odot y = a\Xi \odot y$. Therefore $\overline{\mathsf{span}}\,\mathcal{K}(E)(\mathcal{E} \odot F) = \mathcal{E} \odot F$.

(ii) $\Leftrightarrow$ (iii). Recall that a correspondence, by definition, has nondegenerate left action. It is well-known (and easy to show) that (ii) and (iii) are equivalent for every $\mathcal{B}^a(E)$-$\mathcal{C}$-correspondence. (Indeed, since a bounded approximate unit for $\mathcal{K}(E)$ converges strictly to id_E, for a strict left action the compacts must act nondegenerately. And if $\mathcal{K}(E)$ acts nondegenerately, then this action extends to a unique action of all $\mathcal{B}^a(E)$ that is strict, automatically. See [Lan95, Proposition 5.8] or the proof of [MSS06, Corollary 1.20].) Recall, also, that on bounded subsets, strict and $*$-strong topology coincide (Proposition 1.3.14).

(iii) $\Rightarrow$ (i). Suppose $\{a_\alpha\}_{\alpha\in\Lambda}$ is a bounded net in $\mathcal{B}^a(E)$ converging strictly to $a \in \mathcal{B}^a(E)$. If the left action of $\mathcal{E} \odot F$ is strict, then we have $\{(a_\alpha \odot \mathsf{id}_F)(\Xi \odot y)\}_{\alpha\in\Lambda}$ converges to $(a \odot \mathsf{id}_F)(\Xi \odot y)$, and likewise for $\{a_\alpha^*\}_{\alpha\in\Lambda}$. Therefore

$$\mathfrak{T}(a_\alpha)y = \big\langle\Xi, a_\alpha\Xi\big\rangle y$$

$$
\begin{aligned}
&= (\Xi \odot id_F)^*\big(a_\alpha \odot id_F\big)(\Xi \odot y)\\
&\longrightarrow (\Xi \odot id_F)^*\big(a \odot id_F\big)(\Xi \odot y)\\
&= \big\langle \Xi, a\Xi \big\rangle y\\
&= \mathfrak{T}(a)y
\end{aligned}
$$

and similarly $\mathfrak{T}(a_\alpha^*)y \longrightarrow \mathfrak{T}(a^*)y$ for all $y \in F$. In other words, $\{\mathfrak{T}(a_\alpha)\}_{\alpha\in\Lambda}$ converges $*$-strongly, hence, strictly to $\mathfrak{T}(a)$. □

Theorem 4.3.2. *Let E be a Hilbert $\mathcal{B}$-module, F be a Hilbert $\mathcal{C}$-module, and suppose that $\mathfrak{T} : \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ is a strict CP-map. Then there exist a $\mathcal{B}$-$\mathcal{C}$-correspondence $\mathcal{F}$ and a map $\Xi \in \mathcal{B}^a(F, E \odot \mathcal{F})$ such that $\mathfrak{T}(\cdot) = \Xi^*\big((\cdot) \odot \mathsf{id}_{\mathcal{F}}\big)\Xi$.*

Proof. Let $(\mathcal{E}, \Xi)$ be the minimal GNS-construction for $\mathfrak{T}$. Like every Hilbert $\mathcal{B}^a(F)$-module, we may embed $\mathcal{E}$ into $\mathcal{B}^a(F, \mathcal{E} \odot F)$ by identifying $z \in \mathcal{E}$ with the map $z \odot \mathsf{id}_F : y \mapsto z \odot y$ having adjoint $z^* \odot \mathsf{id}_F : z' \odot y \mapsto \langle z, z'\rangle y$. So, $\mathfrak{T}(a) = \Xi^*(a \odot \mathsf{id}_F)\Xi$ where $a \in \mathcal{B}^a(E)$ acts by the canonical left action on the factor $\mathcal{E}$ of $\mathcal{E} \odot F$. Define the $\mathcal{B}$-$\mathcal{C}$-correspondence $\mathcal{F} := E^* \odot \mathcal{E} \odot F$[m]. If $\mathfrak{T}$ is strict, so that $\mathcal{K}(E) \cong E \odot E^*$ acts nondegenerately on $\mathcal{E} \odot F$, then the string

$$\mathcal{E} \odot F = \overline{\mathsf{span}}\, \mathcal{K}(E)(\mathcal{E} \odot F) \cong \mathcal{K}(E) \odot (\mathcal{E} \odot F) \cong (E \odot E^*) \odot (\mathcal{E} \odot F) = E \odot (E^* \odot \mathcal{E} \odot F) = E \odot \mathcal{F}$$

of (canonical) identifications proves that the map $(x'x^*)(z \odot y) \mapsto x' \odot (x^* \odot z \odot y)$ defines an isomorphism $\mathcal{E} \odot F \to E \odot \mathcal{F}$ of ($\mathcal{K}(E)$-$\mathcal{C}$ and hence) $\mathcal{B}^a(E)$-$\mathcal{C}$-correspondences. Thus $\Xi \in \mathcal{E} \subseteq \mathcal{B}^a(F, \mathcal{E} \odot F) = \mathcal{B}^a(F, E \odot \mathcal{F})$ is such that $\mathfrak{T}(\cdot) = \Xi^*\big((\cdot) \odot \mathsf{id}_{\mathcal{F}}\big)\Xi$. □

Remark 4.3.3. For $E = \mathcal{B}$ so that $\mathcal{B}^a(\mathcal{B}) = M(\mathcal{B})$, the multiplier algebra of $\mathcal{B}$, this result is known as *KSGNS-construction* for a strict CP-map from $\mathcal{B}$ into $\mathcal{B}^a(F)$ ([Kas80], [Lan95, Theorem 5.6]). One may consider Theorem 4.3.2 as a consequence of the KSGNS-construction applied to $\mathfrak{T}|_{\mathcal{K}(E)}$ and the representation theory of $\mathcal{B}^a(E)$

[m]This way to construct the $\mathcal{B}$-$\mathcal{C}$-correspondence $\mathcal{F}$ from a $\mathcal{B}^a(E)$-$\mathcal{B}^a(F)$-correspondence is, actually, from [BLS08, Section 3]. There, however, it is incorrectly claimed that the GNS-correspondence of a strict CP-map has strict left action. (This is false, in general, as the maps $\mathfrak{T} = id_{\mathcal{B}^a(E)}$ shows. The results in [BLS08] are, however, correct, as strictness is never used for $\mathcal{E}$ but always only in the combination as tensor product $\mathcal{E} \odot F$.) For that reason, we preferred to discuss this here carefully, including also the precise statements in Lemma 4.3.1.

(Theorem 1.5.10). Effectively, when $\mathfrak{T}$ is a strict unital $*$-homomorphism, so that $\mathcal{E} := {}_{\mathfrak{T}}\mathcal{B}^a(F)$[n] is the GNS-module for $\mathfrak{T}$ and $\mathcal{F} := E^* \odot \mathcal{E} \odot F = E^* \odot_{\mathfrak{T}} F$, the theorem (and its proof) specialize to [MSS06, Theorem 1.4] (and its proof). We like to view Theorem 4.3.2 as a joint generalization of the KSGNS-construction and of the representation theory, and the rapid joint proof shows that this point of view is an advantage.

Observation 4.3.4. Like with all GNS and Stinespring type constructions, also here we have suitable uniqueness statements. The GNS-correspondence $\mathcal{E}$ together with the cyclicity condition $\mathcal{E} = \overline{\operatorname{span}}\, \mathcal{B}^a(E)\Xi\mathcal{B}^a(F)$ is unique up to isomorphism of correspondences. In that case $\Xi \in \mathcal{E} \subseteq \mathcal{B}^a(F, \mathcal{E} \odot F) = \mathcal{B}^a(F, E \odot \mathcal{F})$ obtained in the proof satisfies $\overline{\operatorname{span}}\, \mathcal{B}^a(E)\Xi(F) = \mathcal{E} \odot F = E \odot \mathcal{F}$. Under this assumption $\mathcal{F}$ is unique up to isomorphism if E is full. For, suppose there exists a $\mathcal{B}$-$\mathcal{C}$-correspondence $\mathcal{F}'$ and $\Xi' \in \mathcal{B}^a(F, E \odot \mathcal{F}')$ with $\overline{\operatorname{span}}\, \mathcal{B}^a(E)\Xi'(F) = E \odot \mathcal{F}'$ such that $\mathfrak{T}(\cdot) = \Xi'^*((\cdot) \odot \mathsf{id}_{\mathcal{F}'})\Xi'$. Then

$$\langle \Xi(y), \Xi(y') \rangle = \langle y, \Xi^*\Xi(y') \rangle = \langle y, \mathfrak{T}(1)y' \rangle = \langle y, \Xi'^*\Xi'(y') \rangle = \langle \Xi'(y), \Xi'(y') \rangle$$

for all $y, y \in F$, so that $\Xi(y) \mapsto \Xi'(y)$ extends to a two-sided isomorphism from $E \odot \mathcal{F} \to E \odot \mathcal{F}'$. Therefore,

$$\mathcal{F} \cong \mathcal{B} \odot \mathcal{F} \cong E^* \odot E \odot \mathcal{F} \cong E^* \odot E \odot \mathcal{F}' \cong \mathcal{B} \odot \mathcal{F}' \cong \mathcal{F}'$$

as $\mathcal{B}$-$\mathcal{C}$-correspondences.

Corollary 4.3.5. *Suppose $E = E_1 \oplus E_2$ and $F = F_1 \oplus F_2$. Then a strict CP-map $\mathfrak{T}$ acts blockwise from $\mathcal{B}^a(E) = \begin{bmatrix} \mathcal{B}^a(E_1) & \mathcal{B}^a(E_2, E_1) \\ \mathcal{B}^a(E_1, E_2) & \mathcal{B}^a(E_2) \end{bmatrix}$ to $\mathcal{B}^a(F) = \begin{bmatrix} \mathcal{B}^a(F_1) & \mathcal{B}^a(F_2, F_1) \\ \mathcal{B}^a(F_1, F_2) & \mathcal{B}^a(F_2) \end{bmatrix}$ if and only if the map Ξ in Theorem 4.3.2 has the diagonal form $\Xi = \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix}$.*

Proof. If $\Xi = \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix}$, then by evaluating $\mathfrak{T}$ at $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ we get $\xi_{12} = \xi_{21} = 0$. □

[n] By ${}_{\mathfrak{T}}\mathcal{B}^a(F)$ we mean the Hilbert $\mathcal{B}^a(E)$-$\mathcal{B}^a(F)$-module $\mathcal{B}^a(F)$ with the left action of $\mathcal{B}^a(E)$ given by $a.b := \mathfrak{T}(a)b$ for all $a \in \mathcal{B}^a(E)$ and $b \in \mathcal{B}^a(F)$.

Now, suppose $\mathfrak{T} = \begin{bmatrix} \varphi & T^* \\ T & \vartheta \end{bmatrix} : \begin{bmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{bmatrix} \to \begin{bmatrix} \mathcal{C} & F^* \\ F & \mathcal{B}^a(F) \end{bmatrix}$ is a blockwise CP-map with strict 22-corner ϑ. There is no harm in assuming that $\mathcal{C}$ is unital. And if $\mathcal{B}$ is not unital, unitalize φ. For unital $\mathcal{B}$, the extended linking algebra is $\mathcal{B}^a(\mathcal{B} \oplus E)$ and the strict topology of all corners but $\mathcal{B}^a(E)$, coincides with the norm topology[o]. Therefore, $\mathfrak{T}$ is strict. So, except for the possibly necessary unitalization, we see that the form we used in the proof (1) $\Rightarrow$ (2) to establish that the constructed $\mathfrak{T}$ is completely positive, actually, is also necessary. (If unitalization is necessary, then ξ_1 is an element of a $\widetilde{\mathcal{B}}$-$\widetilde{\mathcal{C}}$-correspondence.) We arrive at the factorization theorem for strictly CP-extendable maps, which is the analogue to the Theorem 3.3.1.

Theorem 4.3.6. *Let $\mathcal{B}$ and $\mathcal{C}$ be unital C^*-algebras. Then for a map T from a Hilbert $\mathcal{B}$-module E to a Hilbert $\mathcal{C}$-module F the following conditions are equivalent:*

(i) *T admits a strict blockwise extension to a CP-map $\mathfrak{T} = \begin{bmatrix} \varphi & T^* \\ T & \vartheta \end{bmatrix} : \begin{bmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{bmatrix} \to \begin{bmatrix} \mathcal{C} & F^* \\ F & \mathcal{B}^a(F) \end{bmatrix}$.*

(ii) *There exists a $\mathcal{B}$-$\mathcal{C}$-correspondence $\mathfrak{F}$, an element $\xi_1 \in \mathfrak{F}$ and a map $\xi_2 \in \mathcal{B}^a(F, E \odot \mathfrak{F})$ such that $T(\cdot) = \xi_2^*\big((\cdot) \odot \xi_1\big)$.*

Proof. $(i) \Rightarrow (ii)$ Suppose T admits a strict blockwise extension to a CP-map $\mathfrak{T} = \begin{bmatrix} \varphi & T^* \\ T & \vartheta \end{bmatrix}$. Then, from Theorem 4.3.2 and Corollary 4.3.5, there exists a $\mathcal{B}$-$\mathcal{C}$-correspondence $\mathfrak{F}$ and $\Xi = \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix} \in \mathcal{B}^a(\begin{pmatrix} \mathcal{C} \\ F \end{pmatrix}, \begin{pmatrix} \mathcal{B} \\ E \end{pmatrix} \odot \mathfrak{F}) = \mathcal{B}^a(\begin{pmatrix} \mathcal{C} \\ F \end{pmatrix}, \begin{pmatrix} \mathcal{B} \odot \mathfrak{F} \\ E \odot \mathfrak{F} \end{pmatrix})$ such that

$$\begin{aligned} \begin{bmatrix} \varphi & T^* \\ T & \vartheta \end{bmatrix} (\begin{bmatrix} b & x^* \\ x' & a \end{bmatrix}) &= \begin{bmatrix} \xi_1^* & 0 \\ 0 & \xi_2^* \end{bmatrix} \left(\begin{bmatrix} b & x^* \\ x' & a \end{bmatrix} \odot id_{\mathfrak{F}} \right) \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix} \qquad (\star\star) \\ &= \begin{bmatrix} \xi_1^*(b \odot id_{\mathfrak{F}})\xi_1 & \xi_1^*(x^* \odot id_{\mathfrak{F}})\xi_2 \\ \xi_2^*(x' \odot id_{\mathfrak{F}})\xi_1 & \xi_2^*(a \odot id_{\mathfrak{F}})\xi_2 \end{bmatrix} \\ &= \begin{bmatrix} \xi_1^*(b \odot id_{\mathfrak{F}})\xi_1 & (\xi_2^*(x \odot \xi_1))^* \\ \xi_2^*(x' \odot \xi_1) & \xi_2^*(a \odot id_{\mathfrak{F}})\xi_2 \end{bmatrix}. \end{aligned}$$

[o]Suppose $\left[\begin{smallmatrix} b_\alpha & x'^*_\alpha \\ x_\alpha & a_\alpha \end{smallmatrix}\right] \xrightarrow{strictly} \left[\begin{smallmatrix} b & x'^* \\ x & a \end{smallmatrix}\right]$. Then $\left\| \left(\begin{smallmatrix} b_\alpha \\ x_\alpha \end{smallmatrix}\right) - \left(\begin{smallmatrix} b \\ x \end{smallmatrix}\right) \right\| = \left\| \left[\begin{smallmatrix} b_\alpha & x'^*_\alpha \\ x_\alpha & a_\alpha \end{smallmatrix}\right] \left(\begin{smallmatrix} 1_{\mathcal{B}} \\ 0 \end{smallmatrix}\right) - \left[\begin{smallmatrix} b & x'^* \\ x & a \end{smallmatrix}\right] \left(\begin{smallmatrix} 1_{\mathcal{B}} \\ 0 \end{smallmatrix}\right) \right\| \longrightarrow 0$, that is, $b_\alpha \longrightarrow b$ and $x_\alpha \longrightarrow x$ in norm. Similarly by considering the adjoint we get $x'_\alpha \longrightarrow x'$ in norm.

Thus $T(x) = \xi_2^*(x' \odot \xi_1)$ where $\xi_1 \in \mathcal{B}^a(\mathcal{C}, \mathcal{B} \odot \mathcal{F}) = \mathcal{B}^a(\mathcal{C}, \mathcal{F}) = \mathcal{F}$ and $\xi_2 \in \mathcal{B}^a(F, E \odot \mathcal{F})$.

$(ii) \Rightarrow (i)$ Suppose there exists a $\mathcal{B}$-$\mathcal{C}$-correspondence $\mathcal{F}$, an element $\xi_1 \in \mathcal{F} = \mathcal{B}^a(\mathcal{C}, \mathcal{B} \odot \mathcal{F})$ and a map $\xi_2 \in \mathcal{B}^a(F, E \odot \mathcal{F})$ such that $T(\cdot) = \xi_2^*\big((\cdot) \odot \xi_1\big)$. Then ($\star\star$) defines the required extension. □

As for a criterion that consists in looking just at T, we reluctant to expect too much. Clearly, such a T must be completely bounded. By appropriate application of [Pau86, Lemma 7.1], T should extend to the operator system $\begin{bmatrix} \mathbb{C}1 & E^* \\ E & \mathbb{C}id_E \end{bmatrix} \subset \begin{bmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{bmatrix}$. But to extend this further, we would have to tackle problems like extending CP-maps from an operator systems to the C^*-algebra containing it. We do not know if the special algebraic structure will allow to find a solution to out specific problem. But, in general, existence of such extensions is only granted if the codomain is an injective C^*-algebra.

We think that it is the class of strictly CP-extendable maps that truly merits to be called **CP-maps** between Hilbert modules, and not the more restricted class of CP-H-extendable maps.

We close this section with an direct proof of (2) $\Rightarrow$ (1) of Theorem 4.1.1. First we prove the following lemmas.

Lemma 4.3.7. *Let $\mathcal{B}$ and $\mathcal{C}$ be C^*-algebras and F be a Hilbert $\mathcal{B}$-$\mathcal{C}$-module. Then for any full Hilbert $\mathcal{B}$-module E the relative commutant of $\mathcal{B}^a(E) \odot id_F$ in $\mathcal{B}^a(E \odot F)$ is $id_E \odot \mathcal{B}^{a,bil}(F)$.*

Proof. If $\Phi \in \mathcal{B}^{a,bil}(F)$, then $id_E \odot \Phi \in \mathcal{B}^a(E \odot F)$ commutes with all elements of the form $a \odot id_F$ for all $a \in \mathcal{B}^a(E)$ and hence we have $id_E \odot \mathcal{B}^{a,bil}(F) \subseteq (\mathcal{B}^a(E) \odot id_F)'$. For the reverse inclusion assume that $\mathfrak{a} \in (\mathcal{B}^a(E) \odot id_F)'$. Since E is full we have $F = E^* \odot E \odot F$ under the identification $\langle x_1, x_2 \rangle y \mapsto x_1^* \odot x_2 \odot y$. Set $\Phi = id_{E^*} \odot \mathfrak{a} \in \mathcal{B}^{a,bil}(F)$. Then, since $E \odot E^* \cong \mathcal{K}(E)$ via $x \odot x'^* \mapsto xx'^*$, we get

$$
\begin{aligned}
(id_E \odot \Phi)(x_1 \odot \langle x_2, x_3 \rangle y) &= x_1 \odot x_2^* \odot \mathfrak{a}(x_3 \odot y) \\
&= (x_1 x_2^* \odot id_F)\mathfrak{a}(x_3 \odot y) \\
&= \mathfrak{a}(x_1 x_2^* \odot id_F)(x_3 \odot y)
\end{aligned}
$$

$$= \mathfrak{a}(x_1 \odot \langle x_2, x_3 \rangle y).$$

Thus $\Phi \in \mathcal{B}^{a,bil}(F)$ is such that $\mathfrak{a} = id_E \odot \Phi$. Hence $(\mathcal{B}^a(E) \odot id_F)' \subseteq id_E \odot \mathcal{B}^{a,bil}(F)$. □

Lemma 4.3.8. *Let $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras with $\mathcal{A} \subseteq \mathcal{B}$. Suppose $p \in \mathcal{B}$ is a projection and $\pi : \mathcal{A} \to \mathcal{B}$ given by $a \mapsto pap$ is a $*$-homomorphism. Then $pa = ap$ for all $a \in \mathcal{A}$, i.e., $p \in \mathcal{A}' \subseteq \mathcal{B}$.*

Proof. Since π is a $*$-homomorphism $pa^*pap = pa^*ap$, hence, $(ap-pap)^*(ap-pap) = 0$ for all $a \in \mathcal{A}$, i.e., $ap = pap$. Thus $ap = pap = (pa^*p)^* = (a^*p)^* = pa$. □

Suppose $T : E \to F$ is a linear map from a full Hilbert $\mathcal{B}$-module E to a Hilbert $\mathcal{C}$-module F, which extends to a blockwise CP-map $\mathfrak{T} = \begin{bmatrix} \varphi & T^* \\ T & \vartheta \end{bmatrix}$ between the linking algebras of E and F_T such that the 22-corner ϑ is a $*$-homomorphism. We may assume that $\mathcal{B}$ and $\mathcal{C}$ are unital C^*-algebras. (Otherwise replace φ by $\widetilde{\varphi}$, the extension to the unitalization of $\mathcal{B}$ and $\mathcal{C}$. Note that the resulting blockwise map is again CP.) Thus $\mathfrak{T}$ is a map from $\mathcal{B}^a(\mathcal{B} \oplus E)$ into $\mathcal{B}^a(\mathcal{C} \oplus F_T)$which is strict automatically. From theorem 4.3.2 there exists a Hilbert $\mathcal{B}$-$\mathcal{C}$ module $\mathcal{F}$ and an isometry $\Xi = \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix} \in \mathcal{B}^a(\begin{pmatrix} \mathcal{C} \\ F_T \end{pmatrix}, \begin{pmatrix} \mathcal{B} \\ E \end{pmatrix} \odot \mathcal{F})$ such that

$$\begin{bmatrix} \varphi(b) & T(x)^* \\ T(x') & \vartheta(a) \end{bmatrix} = \Xi^* \Big(\begin{bmatrix} b & x^* \\ x' & a \end{bmatrix} \odot id_{\mathcal{F}} \Big) \Xi = \begin{bmatrix} \xi_1^*(b \odot id_{\mathcal{F}})\xi_1 & \xi_1^*(x^* \odot id_{\mathcal{F}})\xi_2 \\ \xi_2^*(x' \odot id_{\mathcal{F}})\xi_1 & \xi_2^*(a \odot id_{\mathcal{F}})\xi_2 \end{bmatrix}.$$

Since ϑ is a unital homomorphism ξ_2 is an isometry, hence $\xi_2\xi_2^*$ is a projection, and $\xi_2^*(a_1 \odot id_{\mathcal{F}})\xi_2\xi_2^*(a_2 \odot id_{\mathcal{F}})\xi_2 = \xi_2^*(a_1 \odot id_{\mathcal{F}})(a_2 \odot id_{\mathcal{F}})\xi_2$ for all $a_1, a_2 \in \mathcal{B}^a(E)$. Then $a \odot id_{\mathcal{F}} \mapsto \xi_2\xi_2^*(a \odot id_{\mathcal{F}})\xi_2\xi_2^*$ defines a $*$-homomorphism from $\mathcal{B}^a(E) \odot id_{\mathcal{F}} \to \mathcal{B}^a(E \odot \mathcal{F})$. From Lemmas 4.3.7 and 4.3.8 we have $\xi_2\xi_2^* = id_E \odot \Phi$ for some projection $\Phi \in \mathcal{B}^{a,bil}(\mathcal{F})$. So

$$\begin{aligned}
\big\langle T(x_1), T(x_2) \big\rangle &= \xi_1^*(x_1^* \odot id_{\mathcal{F}})\xi_2\xi_2^*(x_2 \odot id_{\mathcal{F}})\xi_1 \\
&= \xi_1^*(x_1^* \odot id_{\mathcal{F}})(id_E \odot \Phi)(id_E \odot \Phi)(x_2 \odot id_{\mathcal{F}})\xi_1 \\
&= \xi_1^*(id_{\mathcal{B}} \odot \Phi)(x_1^* \odot id_{\mathcal{F}})(x_2 \odot id_{\mathcal{F}})(id_{\mathcal{B}} \odot \Phi)\xi_1 \\
&= \xi_1^*(id_{\mathcal{B}} \odot \Phi)\big(\langle x_1, x_2 \rangle \odot id_{\mathcal{F}}\big)(id_{\mathcal{B}} \odot \Phi)\xi_1
\end{aligned}$$

for all $x_1, x_2 \in E$. Note that $\zeta := (id_{\mathcal{B}} \odot \Phi)\xi_1 \in \mathcal{B}^a(\mathcal{C}, \mathcal{B} \odot \mathcal{F}) = \mathcal{B}^a(\mathcal{C}, \mathcal{F}) = \mathcal{F}$ which is a Hilbert $\mathcal{B}$-$\mathcal{C}$-module. Then $\varphi'(\cdot) := \langle \zeta, (\cdot)\zeta \rangle$ is a CP-map from $\mathcal{B} \to \mathcal{C}$ such that T is a φ'-map. (Note that the case when $\mathcal{B}$ and $\mathcal{C}$ are nonunital, φ' will be a map from $\widetilde{\mathcal{B}} \to \widetilde{\mathcal{C}}$. But, since E full and $\varphi'(\langle x_1, x_2 \rangle) = \langle T(x_1), T(x_2) \rangle \in \mathcal{C}$ we have $\varphi'|_{\mathcal{B}} \subseteq \mathcal{C}$.)

We may abbreviate this proof to the following: Recall that the proof (2) $\Rightarrow$ (3) shows us that ϑ is unital and strict. Unitalizing if necessary, we get ξ_1 and ξ_2. Since ϑ is a unital $*$-homomorphism, ξ_2 must be an isometry with $\xi_2\xi_2^*$ commuting with all $a \odot \mathsf{id}_{\mathcal{F}}$. Since our specific ξ_2 fulfills $\overline{\mathsf{span}}\,(\mathcal{B}^a(E) \odot id_{\mathcal{F}})\xi_2 F_T = E \odot \mathcal{F}$, it is unitary. We get

$$\|\langle T^n(X^n), T^n(X'^n)\rangle\| = \|\langle X^n \odot \xi_1, X'^n \odot \xi_1\rangle\| \leq \|\varphi\| \, \|\langle X^n, X'^n\rangle\|^2,$$

so $T^* \odot T$ is bounded.

*****

We wish to underline that **all** results above can be formulated for von Neumann algebras, von Neumann modules, and von Neumann correspondences, replacing also the tensor product of C^*-correspondences with that of von Neumann correspondences, replacing *full* with *strongly full* (i.e., $\overline{\mathcal{B}_E}^s = \mathcal{B}$), and adding to all maps between von Neumann objects the word *normal* (or σ-weak). We do not give any details, because the proofs either generalize word by word or are simple adaptations of the C^*-proofs. We emphasize, however, that all problems regarding adjointability of maps or complementability of F_T in F disappear. Therefore, for a map between von Neumann modules, CPH and CP-H-extendable is the same thing and they do no longer depend on (strong) fullness.

4.4 Recent Developments

In [SS14] some possible applications of the theory of φ-maps are hinted. In [SS14, Section 4], Skeide studied semigroups of CP-H-extendable maps, so-called *CPH-semigroups*, and examined how the results of the previous sections may be generalized or reformulated. These results depend essentially on the theory of tensor product systems of correspondences initiated Bhat and Skeide [BS00]. In [SS14, Section 5] he introduced the new concept of *CPH-dilation* of a CP-map or a CP-semigroup. It generalizes the concept of *weak dilation* and is intimately related to CPH-maps or CPH-semigroups.

Here we add some of the definitions and results from [SS14, Section 4,5] without any details or discussion on the theory.

4.4.1 CPH-semigroups

Definition 4.4.1. Let $\mathcal{B}$ be a C^*-algebra. A semigroup $T^\odot = \{T_t\}_{t\in\mathbb{R}^+}$ of maps $T_t : E \to E$ on a Hilbert $\mathcal{B}$-module E is a *CP-H-extendable semigroup* if each each T_t is CP-H-extendable.

Theorem 4.4.2. *Let $\mathcal{B}$ be a unital C^*-algebra and let $T^\odot = \{T_t\}_{t\in\mathbb{R}^+}$ be a family of maps on a Hilbert $\mathcal{B}$-module E. Then the following are equivalent:*

(i) *$T^\odot$ is a CP-H-extendable semigroup.*

(ii) *There are a product system $E^\odot = \{E_t\}_{t\in\mathbb{R}^+}$ of $\mathcal{B}$-correspondences, a unit $\xi^\odot$ for $E^\odot$, and a family of (not necessarily adjointable) isometries $v_t : E \odot E_t \to E$ fulfilling $(xy_s)z_t = x(y_s z_t)$, such that $T_t(x) = v_t(x \odot \xi_t)$ for all $x \in E,\ t \in \mathbb{R}^+$.*

It should be specified that also in this case, by a CP-H-extendable map T on E we mean that T is a CPH-map into E_T. Likewise, in the semigroup version it is required that the φ_t turning T_t into φ_t-maps, form a semigroup. Note that E is not required full. So the $\varphi^\odot = \{\varphi_t\}_{t\in\mathbb{R}^+}$ may not be unique. If we wish to emphasize a fixed CP-semigroup $\varphi^\odot$, we say $T^\odot$ is a CP-H-extendable semigroup *associated* with $\varphi^\odot$.

Definition 4.4.3. A semigroup $T^\odot = \{T_t\}_{t\in\mathbb{R}^+}$ of maps $T_t : E \to E$ on a Hilbert $\mathcal{B}$-module E is

(i) a *(strictly) CP-semigroup* on E if it extends to a CP-semigroup $\mathfrak{T}^\odot = \{\mathfrak{T}_t\}_{t\in\mathbb{R}^+}$ of maps $\mathfrak{T}_t = \begin{bmatrix} \varphi_t & T_t^* \\ T_t & \vartheta_t \end{bmatrix}$ acting blockwise on the extended linking algebra of E (with strict ϑ_t);

(ii) a *(strictly) CPH$_0$-semigroup* on E if it is a (strictly) CP-semigroup where the ϑ_t can be chosen to form an $E(_0)$-semigroup and where the φ_t can be chosen such that each T_t is a φ_t-map.

Recall that, by the discussion preceding Theorem 4.3.6, the case when $\mathcal{B}$ is a unital C^*-algebra $T^\odot$ being strictly CP-semigroup (and so forth) on a Hilbert $\mathcal{B}$-module, simply means that each $\mathfrak{T}_t$ is strict. In that case, we will just say, T is a strict CP-semigroup (and so forth).

Theorem 4.4.4. *Let $\mathcal{B}$ be a unital C^*-algebra. Then for a semigroup $T^\odot = \{T_t\}_{t\in\mathbb{R}^+}$ of maps on a Hilbert $\mathcal{B}$-module E the following are equivalent:*

(i) *$T^\odot$ is a strict CP-semigroup.*

(ii) *There exists a product system $E^\odot = \{E_t\}_{t\in\mathbb{R}^+}$ of $\mathcal{B}$-correspondences, a unit $\xi^\odot$ for $E^\odot$, and a family $\{v_t\}_{t\in\mathbb{R}^+}$ of maps $v_t \in \mathcal{B}^a(E\odot E_t, E)$ fulfilling $(xy_s)z_t = x(y_sz_t)$, such that $T_t(x) = v_t(x\odot\xi_t)$ for all $x\in E,\ t\in\mathbb{R}^+$.*

Theorem 4.4.5. *Let $\mathcal{B}$ be a unital C^*-algebra and let $T^\odot = \{T_t\}_{t\in\mathbb{R}^+}$ be a family of maps on a Hilbert $\mathcal{B}$-module E. Then the following are equivalent:*

(i) *$T^\odot$ is a strict CPH-semigroup (CPH$_0$-semigroup).*

(ii) *There exists a product system $E^\odot = \{E_t\}_{t\in\mathbb{R}^+}$ of $\mathcal{B}$-correspondences, a unit $\xi^\odot$ for $E^\odot$, and a left quasi-semidilation (a left quasi-dilation) $\{v_t\}_{t\in\mathbb{R}^+}$ of $E^\odot$ to E, such that $T_t(x) = v_t(x\odot\xi_t)$ for all $x\in E,\ t\in\mathbb{R}^+$.*

Corollary 4.4.6. *Let $\varphi^\odot$ be a (strongly continuous) CP-semigroup (of contractions) on the unital C^*-algebra $\mathcal{B}$. Then there exists a (strongly continuous) CPH-semigroup $T^\odot$ on a full Hilbert $\mathcal{B}$-module associated with $\varphi^\odot$.*

Definition 4.4.7. A CP-H-extendable semigroup $T^\odot = \{T_t\}_{t\in\mathbb{R}^+}$ on a Hilbert $\mathcal{B}$-module E (E full or not, $\mathcal{B}$ unital or not) is *minimal* if $T^\odot$ fulfills

$$E = \overline{\mathsf{span}}\left\{T_{t_1}(T_{t_2}(\dots T_{t_n}(x)b_{n-1}\dots)b_1)b_0 : b_i\in\mathcal{B},\ x\in E,\ t_1+\dots+t_n = t,\ n\in\mathbb{N}\right\}$$

for some $t>0$.

Theorem 4.4.8. *Let $\varphi^\odot$ be a CP-semigroup on a unital C^*-algebra $\mathcal{B}$, and denote by $(E^\odot,\xi^\odot)$ its GNS-system and cyclic unit. Let E be a full Hilbert $\mathcal{B}$-module. Then* *the formula* $T_t(\cdot) = v_t\big((\cdot) \odot \xi_t\big)$ *establishes a one-to-one correspondence between:*

(i) *Left dilations* $v_t : E \odot E_t \to E$ *of* $E^\odot$ *to* E*.*

(ii) *Minimal CP-H-extendable semigroups* $T^\odot$ *on* E *associated with* $\varphi^\odot$*.*

In either case, $\vartheta^\odot = \{\vartheta_t\}_{t\in\mathbb{R}^+}$ *with* $\vartheta_t(\cdot) = v_t\big((\cdot) \odot id_t\big)v_t^*$ *is the unique strict* E_0*-semigroup on* $\mathcal{B}^a(E)$ *making* $\mathfrak{T}^\odot = \left\{\begin{bmatrix}\varphi_t & T_t^* \\ T_t & \vartheta_t\end{bmatrix}\right\}_{t\in\mathbb{R}^+}$ *a* CPH_0*-extension of* $T^\odot$*.*

Corollary 4.4.9. *Let* $T^\odot$ *and* $T'^\odot$ *be two minimal CP-H-extendable semigroups on the same (necessarily full) Hilbert* C^**-module* E *over the unital* C^**-algebra* $\mathcal{B}$*. Then* $T^\odot$ *and* $T'^\odot$ *are associated with the same CP-semigroup* $\varphi^\odot$ *on* $\mathcal{B}$ *if and only if there is a unitary left cocycle* $\mathfrak{u}^\odot = \{\mathfrak{u}_t\}_{t\in\mathbb{R}^+}$ *for* $\vartheta^\odot$ *satisfying* $\mathfrak{u}_t : T_t(x) \mapsto T'_t(x)$*. Moreover, if* $\mathfrak{u}_t$ *exists, then it is determined uniquely and* $\vartheta'_t(\cdot) = \mathfrak{u}_t\vartheta_t(\cdot)\mathfrak{u}_t^*$*.*

It might be worth to compare the results in this section with [HJ11], who investigated semigroups that, in our terminology, are CP-H-extendable, but who call them CP-semigroups.

4.4.2 An application: CPH-dilations

As a first attempt to give some application of φ-maps, Skeide ([SS14]) interprets φ-maps as a notion that generalizes the notion of dilation of a CP-map $\varphi : \mathcal{B} \to \mathcal{C}$ to a $*$-homomorphism $\vartheta : \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ to the notion of *CPH-dilation*. In the situation of semigroups, this dilation allows for new features: While CP-semigroups that allow weak dilations to an E_0-semigroup, are necessarily Markov (i.e., unital CP-semigroup), results from [SS14, Section 4] allow us to show that many nonunital CP-semigroups allow CPH-dilations to E_0-semigroups, which are called *CPH*$_0$*-dilations*.

Definition 4.4.10. Suppose E, F are Hilbert C^*-modules over C^*-algebras (do not require unital) $\mathcal{B}, \mathcal{C}$, respectively and let $\varphi : \mathcal{B} \to \mathcal{C}$ be a CP-map. A $*$-homomorphism $\vartheta : \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ is a *CPH-dilation* of φ if E is full and if there exists a map

$T : E \to F$ such that the diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\varphi} & \mathcal{B} \\ \uparrow{\scriptstyle \langle x,(\cdot)x'\rangle} & & \uparrow{\scriptstyle \langle T(x),(\cdot)T(x')\rangle} \\ \mathcal{B}^a(E) & \xrightarrow[\vartheta]{} & \mathcal{B}^a(E) \end{array}$$

commutes for all $x, x' \in E$. If E is not necessarily full, then we speak of a *CPH-quasi-dilation*. A CPH-(quasi-)dilation is *strict* if ϑ is strict. A CPH-(quasi-)dilation is a *CPH*$_0$*-(quasi-)dilation* if ϑ is unital.

Proposition 4.4.11. *If ϑ is a CPH$_0$-quasi-dilation of a CP-map φ, then every map T making the diagram commute is a φ-map fulfilling $T(ax) = \vartheta(a)T(x)$.*

From now on we shall assume that $\mathcal{B}$ is unital.

Theorem 4.4.12. *If ϑ is a strict CPH$_0$-dilation of a CP-map, then every map T making the diagram commutes is a strict CPH$_0$-map.*

Definition 4.4.13. An E_0-semigroup $\vartheta^\odot$ on $\mathcal{B}^a(E)$ for a full Hilbert $\mathcal{B}$-module E is a *CPH*$_0$*-dilation* of a CP-semigroup $\varphi^\odot$ if there exists a CPH$_0$-semigroup $T^\odot$ on E making the diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\varphi_t} & \mathcal{B} \\ \uparrow{\scriptstyle \langle x,(\cdot)x'\rangle} & & \uparrow{\scriptstyle \langle T_t(x),(\cdot)T_t(x')\rangle} \\ \mathcal{B}^a(E) & \xrightarrow[\vartheta_t]{} & \mathcal{B}^a(E) \end{array}$$

commutes for all $x, x' \in E$ and all $t \in \mathbb{R}^+$.

If $\varphi^\odot$ is not Markov, then [BS00] provide a weak dilation to an E-semigroup. But $\varphi^\odot$ cannot posses a weak dilation to an E_0-semigroup. On the contrary, we can see that $\varphi^\odot$ can possess a CPH$_0$-dilation:

Observation 4.4.14. Finding a strict CPH$(_0)$-dilation for a CP-semigroup $\varphi^\odot$, is the same as finding a CPH$(_0)$-semigroup $T^\odot$ associated with that $\varphi^\odot$. So, all results from Section 4.4.1 are applicable.

1. From Corollary 4.4.6, we recover existence of a strict CPH-dilation. (As said, we knew this from the stronger existence of a weak dilation in [BS00].)
2. From existence of E_0-semigroups for full product systems, we infer that every CP-semigroup, Markov or not, with full product system admits a strict CPH$_0$-dilation.
3. In the case of CPH$_0$-dilations, also the notion of minimality and the results about uniqueness up to cocycle conjugacy remain intact. It is noteworthy that for a weak E_0-dilation of a (necessarily) Markov semigroup, minimality of the weak dilation coincides with minimality of the associated CPH$_0$-semigroup.

In the end, Skeide comments on some relations with (completely positive definite) *CPD-kernels* and with *Morita equivalence*. If CPH-dilations can be considered an interesting concept, and if, as demonstrated, understanding CPH-dilations is the same as understanding CPH-maps and CPH-semigroups, then [SS14, Section 5] shows the road to what might be the first application of CPH-maps.

Appendix A

Basic operator algebra theory

A.1 Banach algebras and C^*-algebras

An *algebra* is a complex vector space $\mathcal{A}$ with a bilinear map, called *multiplication*, $\mathcal{A} \times \mathcal{A} \ni (a,b) \mapsto ab \in \mathcal{A}$ such that $(ab)c = a(bc)$ and $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for all $a, b, c \in \mathcal{A}$, $\lambda \in \mathbb{C}$. The algebra $\mathcal{A}$ is said to be *commutative* (or *abelian*) if $ab = ba$ for all $a, b \in \mathcal{A}$, and $\mathcal{A}$ is said to be *unital* if it has a multiplicative identity, denoted by $1_{\mathcal{A}}$ or simply 1.

Definition A.1.1. An algebra $\mathcal{A}$ is said to be a *normed algebra* if it has a norm that makes it into a normed linear space and if $\|ab\| \leq \|a\| \|b\|$ for all $a, b \in \mathcal{A}$. A complete normed algebra is called a *Banach algebra*.

Note that if a Banach algebra $\mathcal{A}$ has an multiplicative identity, then it is unique. Also since $1 = 1^2$ we have $\|1\| \leq \|1\| \|1\|$, which implies that $\|1\| \geq 1$. It is well-known that if $\mathcal{A}$ is a Banach algebra with identity 1, then there is a norm $\|\cdot\|'$ on $\mathcal{A}$, equivalent to the original norm, such that $(\mathcal{A}, \|\cdot\|')$ is a unital Banach algebra with $\|1\|' = 1$. So we always assume that the multiplicative unit of a unital Banach algebra has norm 1. In fact, this is often taken as part of the definition of a unital Banach algebra.

Example A.1.2. Let Ω be a topological space.

(i) If Ω is compact, then the set $C(\Omega)$ of all complex-valued continuous functions on Ω is a unital Banach algebra with point-wise operations and sup-norm.

(ii) The set $C_b(\Omega)$ of all bounded continuous complex-valued functions on Ω is a unital Banach algebra. If Ω is compact, then $C_b(\Omega) = C(\Omega)$.

(iii) If Ω is a locally compact Hausdorff space, then the set $C_0(\Omega)$ of all complex-valued continuous functions vanishing at infinity is a closed subalgebra of $C_b(\Omega)$, and therefore, a Banach algebra. It is unital if and only if Ω is compact, and in that case $C_0(\Omega) = C(\Omega)$.

(iv) If (Ω, μ) is a measure space, then the set $L^\infty(\Omega, \mu)$ of (classes) of essentially bounded complex-valued measurable functions on Ω is a unital Banach algebra with usual point-wise operations and essential supremum norm.

(v) If X is a normed vector space, then the set $\mathcal{B}(X)$ of all bounded linear maps from X to itself is a unital normed algebra with point-wise operations for addition and scalar multiplication, multiplication given by $(T, S) \mapsto T \circ S$, and norm the operator norm. If X is a Banach algebra, then $\mathcal{B}(X)$ is a unital Banach algebra.

Definition A.1.3. A normed algebra (Banach algebra) $(\mathcal{A}, \|\cdot\|)$ with an *involution* $^* : \mathcal{A} \to \mathcal{A}(a \mapsto a^*)$ satisfying

(i) $a^{**} := (a^*)^* = a$,

(ii) $(a + \lambda b)^* = a^* + \overline{\lambda} b^*$,

(iii) $(ab)^* = b^* a^*$,

(iv) $\|a^* a\| = \|a\|^2$

for all $\lambda \in \mathbb{C},\ a, b \in \mathcal{A}$ is called a pre-C^*-algebra (C^*-algebra).

An (Banach) algebra with an involution satisfying conditions $(i) - (iii)$ is called a *(Banach) $*$-algebra*. It is well-known that norm on a $*$-algebra which makes it a C^*-algebra is unique. If $\mathcal{A}$ is a C^*-algebra, then $\|a^*\| = \|a\|$.

Example A.1.4. Suppose Ω is a topological space. The following algebras are C^*-algebras with involution $f \mapsto \overline{f}$.

(i) $C_b(\Omega)$ is a unital C^*-algebra.

(ii) If Ω is locally compact Hausdorff space, then $C_0(\Omega)$ is a C^*-algebra. It is unital if Ω is compact.

(iii) If H is a Hilbert space, then $\mathcal{B}(H)$ is a unital C^*-algebra with adjoint as the involution.

Unitalization

If A is a nonunital algebra we set $\widetilde{\mathcal{A}} := \mathcal{A} \oplus \mathbb{C}$ as a vector space. Define multiplication on $\widetilde{\mathcal{A}}$ by

$$(a_1, \lambda_1)(a_2, \lambda_2) := (a_1 a_2 + \lambda_1 a_2 + \lambda_2 a_1, \lambda_1 \lambda_2).$$

Then $\widetilde{\mathcal{A}}$ is an algebra with unit $(0,1)$, and is called the *unitalization* of $\mathcal{A}$. The map $\mathcal{A} \ni a \mapsto (a,0) \in \widetilde{\mathcal{A}}$ is an injective homomorphism, which we used to identify $\mathcal{A}$ as a two-sided ideal of $\widetilde{\mathcal{A}}$. If $\mathcal{A}$ is a normed (Banach) algebra, then $\widetilde{\mathcal{A}}$ is a normed (Banach) algebra with norm

$$\|(a,\lambda)\| := \|a\| + |\lambda| \,. \tag{A.1.1}$$

If $\mathcal{A}$ is a $*$-algebra, then $\widetilde{\mathcal{A}}$ is a $*$-algebra with involution $(a,\lambda)^* := (a^*, \overline{\lambda})$. But $\widetilde{\mathcal{A}}$ may not be a C^*-algebra with the norm given by (A.1.1). To make it a C^*-algebra, given $(a,\lambda) \in \widetilde{\mathcal{A}}$, we define $L_{(a,\lambda)} \in \mathcal{B}(\mathcal{A})$ by $a' \mapsto aa' + \lambda a'$. Then $\|(a,\lambda)\| := \left\|L_{(a,\lambda)}\right\| = \sup\{\|aa' + \lambda a'\| : a' \in \mathcal{A}, \|a'\| \leq 1\}$ makes $\widetilde{\mathcal{A}}$ a unital C^*-algebra. Since $\|a\| = \left\|L_{(a,0)}\right\|$ for all $a \in \mathcal{A}$, the embedding of $\mathcal{A}$ into $\widetilde{\mathcal{A}}$ is an isometry. If $\mathcal{A}$ already has a unit, then the mapping $(a,\lambda) \mapsto (a+\lambda, \lambda)$ identifies $\widetilde{A} = \mathcal{A} \oplus \mathbb{C}$ as algebras.

Suppose $\mathcal{A}, \mathcal{B}$ are $*$-algebras. A $*$-preserving algebraic homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ is called a *$*$-homomorphism.*

- A $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ extends uniquely to a unital $*$-homomorphism $\widetilde{\pi} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$.
- A $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ from a Banach $*$-algebra $\mathcal{A}$ to a C^*-algebra $\mathcal{B}$ is necessarily norm-decreasing.
- If $\pi : \mathcal{A} \to \mathcal{B}$ is an injective $*$-homomorphism between C^*-algebras, then π is necessarily isometric.

Commutative C^*-algebras

Suppose $\mathcal{A}$ is a unital Banach algebra. We say $a \in \mathcal{A}$ is *invertible* if there exists $b \in \mathcal{A}$ such that $ab = 1 = ba$. In this case b is unique and is denoted by a^{-1}. We define the *spectrum* of a to be the set

$$\sigma_{\mathcal{A}}(a) := \{\lambda \in \mathbb{C} : \lambda 1 - a \text{ is not invertible in } \mathcal{A}\}.$$

It is well-known that $\sigma_{\mathcal{A}}(a)$ is a nonempty compact set. If $\mathcal{A}$ is nonunital Banach algebra, then for any $a \in \mathcal{A}$, we set $\sigma_{\mathcal{A}}(a) := \sigma_{\widetilde{\mathcal{A}}}(a)$.

Theorem A.1.5. *Suppose $\mathcal{A}$ is a commutative C^*-algebra. Then there exists a locally compact Hausdorff space Ω such that $\mathcal{A}$ is isometrically $*$-isomorphic to $C_0(\Omega)$.*

Further, Ω is compact if and only if $\mathcal{A}$ is unital, and in that case $\mathcal{A} \cong C(\Omega)$.

Let $\mathcal{A}$ be a C^*-algebra. Then $a \in \mathcal{A}$ is said to be *projection* if $a = a^* = a^2$, *self-adjoint* if $a = a^*$, *normal* if $aa^* = a^*a$ and *positive* if $a = b^*b$ for some $b \in \mathcal{A}$. In addition if $\mathcal{A}$ is unital, then a is said to be *isometry* if $a^*a = 1$, *unitary* if $a^*a = 1 = aa^*$.

The set of positive elements in a C^*-algebra $\mathcal{A}$ is denoted by $\mathcal{A}^+$. If $a \in \mathcal{A}^+$ we write $a \geq 0$ (or $0 \leq a$). For $a, b \in \mathcal{A}$ by $a \geq b$ we mean $a - b \in \mathcal{A}^+$. Given $a \in \mathcal{A}^+$ there exists a unique element, denoted by $a^{\frac{1}{2}}$, in $\mathcal{A}^+$ such that $a = (a^{\frac{1}{2}})^2$. Given $a \in \mathcal{A}$ we have $a^*a \geq 0$, and we set $|a| = (a^*a)^{\frac{1}{2}}$. If $a \leq b$, then $c^*ac \leq c^*bc$ for all $c \in \mathcal{A}$. Also for a unital C^*-algebra $\mathcal{A}$ we have $0 \leq a \leq \|a\| \, 1$ for all $a \in \mathcal{A}^+$.

An *approximate unit* for a C^*-algebra $\mathcal{A}$ is an increasing net $\{e_\alpha\}_{\alpha \in \Lambda}$ of positive elements in the closed unit ball of $\mathcal{A}$ such that $a = \lim ae_\alpha$ (equivalently, $a = \lim e_\alpha a$) for all $a \in \mathcal{A}$. Note that in that case $a = \lim e_\alpha a e_\alpha$. Every C^*-algebra admits an approximate unit. A C^*-algebra is called *σ-unital* if it has a countable approximate unit.

Theorem A.1.6. *Let a be a normal element of a unital C^*-algebra $\mathcal{A}$, and suppose that f_1 is the inclusion map of $\sigma(a)$ in $\mathbb{C}$. Then there exists a unique unital $*$-homomorphism $\pi : C(\sigma(a)) \to \mathcal{A}$ such that $\pi(a) = f_1$. Moreover, π is isometric and $\mathsf{ran}(\pi)$ is the C^*-subalgebra of $\mathcal{A}$ generated by 1 and a (i.e., the smallest C^*-subalgebra containing 1 and a).*

GNS representation

A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras is said to be *positive* if $\varphi(a) \geq 0$ for all $a \geq 0$. Clearly $*$-homomorphisms are positive maps. All positive linear functionals are bounded.

Proposition A.1.7. *Let $\phi : \mathcal{A} \to \mathbb{C}$ be a bounded linear functional. The following conditions are equivalent:*

(i) *ϕ is positive.*

(ii) *For each approximate unit $\{e_\alpha\}_{\alpha \in \Lambda}$ of $\mathcal{A}$, $\|\phi\| = \lim \phi(e_\alpha)$.*

(iii) *For some approximate unit $\{e_\alpha\}_{\alpha \in \Lambda}$ of $\mathcal{A}$, $\|\phi\| = \lim \phi(e_\alpha)$.*

A positive linear functional $\phi : \mathcal{A} \to \mathbb{C}$ of norm one is known as a *state* on $\mathcal{A}$. We let $\mathcal{S}(\mathcal{A})$ denote the space of all states on $\mathcal{A}$. The *state space* $\mathcal{S}(\mathcal{A})$ is a convex, compact and Hausdorff space. If $a \in \mathcal{A}$, then

- $a = 0$ if and only if $\phi(a) = 0$ for all $\phi \in \mathcal{S}(\mathcal{A})$,
- $a = a^*$ if and only if $\phi(a) \in \mathbb{R}$ for all $\phi \in \mathcal{S}(\mathcal{A})$,
- $a \geq 0$ if and only if $\phi(a) \geq 0$ for all $\phi \in \mathcal{S}(\mathcal{A})$,
- If a is normal, then $\|a\| = |\phi(a)|$ for some $\phi \in \mathcal{S}(\mathcal{A})$.

A positive element a of a C^*-algebra $\mathcal{A}$ is called *strictly positive* if $\phi(a) > 0$ for all $\phi \in \mathcal{S}(\mathcal{A})$. A positive element $a \in \mathcal{A}$ is strictly positive if and only if the closed right ideal generated by a is the whole of $\mathcal{A}$. A C^*-algebra is σ-unital if and only if it has a strictly positive element.

Suppose $\mathcal{A}_0$ is a C^*-subalgebra of $\mathcal{A}$ and ϕ is a positive linear functional on $\mathcal{A}_0$. Then there exists a positive linear functional ϕ' on $\mathcal{A}$ extending ϕ such that $\|\phi\| = \|\phi'\|$.

Theorem A.1.8. *Let ϕ be a state on a unital C^*-algebra $\mathcal{A}$. Then there exists a Hilbert space H, a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(H)$ and a unit vector $x \in H$ such that $\phi(a) = \langle x, \pi(a)x \rangle$ for all $a \in \mathcal{A}$.*

The triple (H, π, x) is called a *GNS-construction* for ϕ. It is said to be *minimal* if $H = \overline{\mathsf{span}}\, \pi(\mathcal{A})x$. In that case x is called a *cyclic vector*. Minimal GNS-constructions are unique up to isomorphism.

A *representation* of a C^*-algebra $\mathcal{A}$ is a pair (H, π) where H is a Hilbert space and $\pi : \mathcal{A} \to \mathcal{B}(H)$ is a $*$-homomorphism. If both $\mathcal{A}$ and π are unital, then we say the representation is unital. We say (H, π) is *faithful* if π is injective. The *direct sum* of a family of representations $\{(H_\alpha, \pi_\alpha)\}_{\alpha \in \Lambda}$ of $\mathcal{A}$ is the representation (H, π) obtained by setting $H = \oplus H_\alpha$, and defining $\pi(a)(\oplus_\alpha x_\alpha) := \oplus_\alpha \pi(a) x_\alpha$ for all $a \in \mathcal{A}$ and all $\oplus_\alpha x_\alpha \in H$. Then (H, π) is indeed a representation of $\mathcal{A}$.

Theorem A.1.9 (Gelfand-Naimark). *If $\mathcal{A}$ is a (unital) C^*-algebra, then it has a faithful (unital) representation.*

As a consequence, given a C^*-algebra $\mathcal{A}$ there exists a unique norm on $M_n(\mathcal{A})$ making it a C^*-algebra.

A.2 von Neumann algebras

Let H be a Hilbert space and $X \subseteq \mathcal{B}(H)$ be a subset. The *commutant* of X is defined by

$$X' := \{T \in \mathcal{B}(H) : TS = ST \text{ for all } S \in X\}.$$

The *double commutant* of X, denoted by X'', is the commutant of X'. If X is convex subset, then the SOT closure of X in $\mathcal{B}(H)$ coincides with the WOT closure of X.

Definition A.2.1. A $*$-subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ is called a *von Neumann algebra* if $\mathcal{A}$ is SOT (equivalently WOT) closed in $\mathcal{B}(H)$.

Since the *SOT* is weaker than norm topology, a von Neumann algebra is necessarily a C^*-algebra. If $\mathcal{A}$ is a nonzero von Neumann algebra, then it is unital. But the unit may not be the identity map of the underlying Hilbert space.

Theorem A.2.2 (Double commutant theorem). *Suppose $\mathcal{A}$ is a unital $*$-subalgebra of $\mathcal{B}(H)$. Then $\mathcal{A}$ is a von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''$.*

If $\mathcal{A} \subseteq \mathcal{B}(H)$ is a $*$-algebra, then its commutant $\mathcal{A}'$ is a von Neumann algebra on H. If $\mathcal{A}$ is unital also, then $\mathcal{A}$ is SOT (as well as WOT) dense in $\mathcal{A}''$, that is, $\mathcal{A}''$ is the SOT (as well as WOT) closure of $\mathcal{A}$. Thus, $\mathcal{A}''$ is the smallest von Neumann algebra containing $\mathcal{A}$.

If $\{H_\alpha\}_{\alpha\in\Lambda}$ is a family of Hilbert spaces and $\mathcal{A}_\alpha$ is a von Neumann algebra on H_α, then the direct sum $\oplus\mathcal{A}_\alpha$ is a von Neumann algebra on $\oplus H_\alpha$.

Suppose $\mathcal{A}$ is a von Neumann algebra on a Hilbert space H. Then

- $\mathcal{A}$ contains projections, and $\mathcal{A}$ is the closed linear span of its projections.
- If $a \in \mathcal{A}$ is with polar decomposition $a = v\,|a|$, then $v \in \mathcal{A}$.
- If $id_H \in \mathcal{A}$ and $T \in \mathcal{B}(H)$, then $T \in \mathcal{A}$ if and only if T commutes with all the projections of $\mathcal{A}'$.
- $M_n(\mathcal{A})$ is a von Neumann algebra on H^n.

Theorem A.2.3 (Kaplansky density theorem). *Suppose $\mathcal{A}_0$ is a C^*-subalgebra of $\mathcal{B}(H)$ with SOT closure $\mathcal{A}$ in $\mathcal{B}(H)$.*

(i) *The set $\mathcal{A}_0^{sa}$ of all self-adjoint operators in $\mathcal{A}_0$ is strongly dense in the set $\mathcal{A}^{sa}$*

of all self-adjoint operators in $\mathcal{A}$.

(ii) *The closed unit ball of* $\mathcal{A}_0^{sa}$ *is strongly dense in the closed unit ball of* $\mathcal{A}^{sa}$.

(iii) *The closed unit ball of* $\mathcal{A}_0$ *is strongly dense in the closed unit ball of* $\mathcal{A}$.

(iv) *If* $id_H \in \mathcal{A}$, *then the unitaries of* $\mathcal{A}_0$ *are strongly dense in the unitaries of* $\mathcal{A}$.

Corollary A.2.4. *Suppose* $\mathcal{A}$ *is a* C^**-subalgebra of* $\mathcal{B}(H)$. *Then the following are equivalent:*

(i) $\mathcal{A}$ *is a von Neumann algebra.*

(ii) *The closed unit ball of* $\mathcal{A}$ *is SOT-closed.*

(iii) *The closed unit ball of* $\mathcal{A}$ *is WOT-closed.*

Normal maps

Suppose H is a Hilbert space and $\{T_\alpha\}$ is an increasing net of hermitian operators on H such that $\sup \|T_\alpha\| < \infty$. Then there is an operator $T \in \mathcal{B}(H)$ such that the following holds:

- $T = \sup T_\alpha$, i.e., if $T_\alpha \leq T$ for all α and if S is any other hermitian operator satisfying $T_\alpha \leq S$ for all α, then $T \leq S$.
- $T_\alpha \longrightarrow T$ in WOT.
- $T_\alpha \longrightarrow T$ in SOT.
- $T_\alpha \longrightarrow T$ in σ-weak topology.

If $\mathcal{A}$ is a C^*-algebra contained in $\mathcal{B}(H)$, then $\mathcal{A}$ is weak* closed if and only if it contains the supremum of every bounded increasing net of hermitian operators in the algebra.

Definition A.2.5. Let $\mathcal{A}, \mathcal{B}$ be von Neumann algebras. A positive linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is said to be *normal* if $\varphi(a_\alpha) \xrightarrow{SOT} \varphi(a)$ for any increasing net $\{a_\alpha\}_{\alpha\in\Lambda}$ in $\mathcal{A}$ that converges strongly to a.

Note that, von Neumann algebras are *order complete*, i.e., any bounded increasing net of positive elements in a von Neumann algebra converges in the strong operator topology to its unique least upper bound. Normal maps are order continuous, i.e., $\limsup_\alpha \varphi(a_\alpha) = \varphi(\limsup_\alpha a_\alpha)$ for each bounded increasing net $\{a_\alpha\}_{\alpha\in\Lambda}$.

Proposition A.2.6. *Every $*$-isomorphism between von Neumann algebras is normal.*

Theorem A.2.7. *Let $\mathcal{A} \subseteq \mathcal{B}(H), \mathcal{B} \subseteq \mathcal{B}(G)$ be von-Neumann algebras and $\varphi : \mathcal{A} \to \mathcal{B}$ be a positive linear map. Then the following are equivalent:*

(i) *φ is normal.*

(ii) *φ is σ-weakly (weak*) continuous.*

(iii) *For every increasing net $\{a_\alpha\}_{\alpha\in\Lambda} \subseteq \mathcal{A}^+$ with least upper bound $a \in \mathcal{A}^+$ the increasing net $\{\varphi(a_\alpha)\}_{\alpha\in\Lambda} \subseteq \mathcal{B}^+$ converges σ-weakly to $\varphi(a)$.*

(iv) *For every increasing net $\{a_\alpha\}_{\alpha\in\Lambda} \subseteq \mathcal{A}^+$ with least upper bound $a \in \mathcal{A}^+$ we have*

$$\lim_\alpha \langle g, \varphi(a_\alpha)g\rangle = \sup_\alpha \langle g, \varphi(a_\alpha)g\rangle = \langle g, \varphi(a)g\rangle$$

for each g in a norm-dense linear submanifold of G.

(v) *For every increasing net $\{a_\alpha\}_{\alpha\in\Lambda} \subseteq \mathcal{A}^+$ with least upper bound $a \in \mathcal{A}^+$ we have*

$$\lim_\alpha \langle g_1, \varphi(a_\alpha)g_2\rangle = \langle g_1, \varphi(a)g_2\rangle$$

for all g_1, g_2 in a total subset of G.

(vi) *Restriction of φ to bounded sets is strongly continuous.*

Any positive linear map between von Neumann algebras that is strongly continuous is normal. The converse is not necessarily true.

A.3 Completely positive maps

Definition A.3.1. Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is said to be *completely positive* (*CP-*) map, if $\sum_{i,j=1}^n b_i^* \varphi(a_i^* a_j) b_j \geq 0$ for all $a_i \in \mathcal{A},\ b_i \in \mathcal{B}$.

Proposition A.3.2. *For a map $\varphi \in \mathcal{B}(\mathcal{A}, \mathcal{B})$ the following conditions are equivalent:*

(i) *φ is a CP-map.*

(ii) *The maps $\varphi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$, defined by $\varphi_n([a_{i,j}]) := [\varphi(a_{i,j})]$ is positive for all $n \in \mathbb{N}$.*

(iii) *φ_n is CP-map for all $n \in \mathbb{N}$.*

If either $\mathcal{A}$ or $\mathcal{B}$ is a commutative C^*-algebra, then any positive linear map from

$\mathcal{A}$ to $\mathcal{B}$ is a CP-map. In particular, positive linear functionals on a C^*-algebra are CP-maps.

Suppose $\varphi : \mathcal{A} \to \mathcal{B}$ is a CP-map. Then φ is bounded. If $\{e_\alpha\}_{\alpha\in\Lambda}$ is an approximate unit for $\mathcal{A}$, then $\|\varphi\| = \sup \|\varphi(e_\alpha)\|$. (If $\mathcal{A}$ is unital, then $\|\varphi\| = \|\varphi(1)\|$.) Also for all $a, a_1, \cdots, a_n \in \mathcal{A}$,

- $\varphi(a^*) = \varphi(a)^*$,
- $\varphi(a^*a')\varphi(a'^*a) \leq \|\varphi(a'^*a')\|\,\varphi(a^*a)$,
- $\varphi(a^*)\varphi(a) \leq \|\varphi\|\,\varphi(a^*a)$,
- $\left[\varphi(a_i^*)\varphi(a_j)\right] \leq \|\varphi\|\left[\varphi(a_i^*a_j)\right]$ in $M_n(\mathcal{B})$.

Theorem A.3.3 ([Sti55]). *Let $\mathcal{A}$ be a unital C^*-algebra and H be a Hilbert space. Suppose $\varphi : \mathcal{A} \to \mathcal{B}(H)$ is a CP-map. Then there exists a Hilbert space K, a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(K)$ and $V \in \mathcal{B}(H, K)$ with $\|\varphi(1)\|^2 = \|V\|$ such that $\varphi(a) = V^*\pi(a)V$.*

The triple (K, π, V) is called a *Stinespring representation* for φ. It is said to be minimal if $\overline{\operatorname{span}}\,\pi(\mathcal{A})VH = K$. Minimal representation is unique up to unitary isomorphism.

Suppose $\mathcal{A}$ is a unital C^*-algebra and X is a subset of $\mathcal{A}$ containing $1_{\mathcal{A}}$.

- If $X = \{a \in \mathcal{A} : a^* \in X\}$, then X is called an *operator system*.
- If X is a subspace of $\mathcal{A} = \mathcal{B}(H)$ we call X an *operator space*. (See Appendix A.6.)
- If X is a subalgebra (not necessarily $*$-closed) we call X an *operator algebra*.

Theorem A.3.4 (Arverson's extension theorem). *Let $\mathcal{A}$ be a C^*-algebra, X be an operator system and $\varphi : X \to \mathcal{B}(H)$ be a CP-map. Then there exists a CP-map, $\widehat{\varphi} : \mathcal{A} \to \mathcal{B}(H)$, extending φ.*

A C^*-algebra $\mathcal{B}$ is called *injective* if for every C^*-algebra $\mathcal{A}$ and operator system $X \subseteq \mathcal{A}$, every CP-map $\varphi : X \to \mathcal{B}$ can be extended to a CP-map on all of $\mathcal{A}$.

Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras, $X \subseteq \mathcal{A}$ an operator space, and let $\psi : X \to \mathcal{B}$ be a linear map. If ψ is bounded, then ψ_n is also bounded with $\|\psi_n\| \leq n\,\|\psi\|$ for all $n \in \mathbb{N}$. We call ψ a *completely bounded* (CB-) map (respectively, *completely contractive*) if $\|\psi\|_{cb} := \sup_n \|\psi_n\| < \infty$ (respectively, $\|\psi\|_{cb} \leq 1$). Note that $\|\cdot\|_{cb}$

is a norm on the space $CB(\mathcal{A},\mathcal{B})$ of all CB-maps. We call ψ a *completely isometry* if each ψ_n is isometric, and a *complete isomorphism* if it is a linear isomorphism with $\|\varphi\|_{cb}, \|\varphi^{-1}\|_{cb} < \infty$. All CP-maps φ are CB-maps with $\|\varphi\|_{cb} = \|\phi\|$, which is equal to $\|\varphi(1)\|$ if X is an operator system. If $\mathcal{B}$ is commutative unital C^*-algebra, then all bounded maps $\psi : X \to \mathcal{B}$ are CB-maps with $\|\psi\|_{cb} = \|\psi\|$.

Theorem A.3.5 (Arverson). *Let $\mathcal{A}$ be a C^*-algebra, $X \subseteq \mathcal{A}$ a subspace with $1 \in X$, and let $\psi : X \to \mathcal{B}(H)$ be a unital complete contraction. Then there exists a CP-map $\varphi : \mathcal{A} \to \mathcal{B}(H)$ extending ψ.*

Theorem A.3.6. *Let $\mathcal{A}$ be a unital C^*-algebra and $\psi : \mathcal{A} \to \mathcal{B}(H)$ be a CB-map. Then there exists a Hilbert space K, a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(K)$ and $V_i \in \mathcal{B}(H,K)$ with $\|\varphi\|_{cb} = \|V_1\|\,\|V_2\|$ such that $\psi(a) = V_1^*\pi(a)V_2$ for all $a \in \mathcal{A}$. Moreover, if $\|\psi\|_{cb} = 1$, then V_i may be taken to be isometries.*

The following Wittstock's decomposition theorem says that CB-maps on a unital C^*-algebra are the linear span of CP-maps. See [Pau02, Theorem 8.5] for details.

Theorem A.3.7. *Let $\mathcal{A}$ be a unital C^*-algebra and $\psi : \mathcal{A} \to \mathcal{B}(H)$ be a CB-map. Then there exists a CP-map $\varphi : \mathcal{A} \to \mathcal{B}(H)$ with $\|\varphi\|_{cb} \leq \|\psi\|_{cb}$ such that $\varphi \pm \mathsf{Re}(\psi)$ and $\varphi \pm \mathsf{Im}\,(\psi)$ are all CP-map.*

A.4 Semigroups

Generators of semigroups

Definition A.4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras such that the former is a subalgebra of the latter, and $\mathsf{L} : \mathcal{A} \to \mathcal{B}$ be a bounded linear map with the property that L is real, that is, $\mathsf{L}(a^*) = \mathsf{L}(a)^*$ for all $a \in \mathcal{A}$. We call L *conditionally completely positive* (CCP) if $\sum_{i,j=1}^n b_i^*\mathsf{L}(a_i^*a_j)b_j \geq 0$ for all $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ satisfying $\sum_{i=1}^n a_ib_i = 0$ and for all $n \in \mathbb{N}$.

Theorem A.4.2. *A bounded linear adjoint-preserving map L from a unital C^*-algebra $\mathcal{B}$ to itself is CCP if and only if $e^{t\mathsf{L}}$ is CP for all $t \in \mathbb{R}^+$.*

Definition A.4.3. A *semigroup* on a Banach space X is a family $T^{\odot} = \{T_t\}_{t\in\mathbb{R}^+}$ of bounded operators on X with the following properties:

(i) $T_0 = id_X$.

(ii) $T_{s+t} = T_s \circ T_t$ for all $s, t \in \mathbb{R}^+$.

A semigroup is said to be *uniformly continuous* (UC) if $t \mapsto T_t$ is norm continuous (i.e., $\|T_t - I\| \longrightarrow 0$ as $t \to 0^+$).

Theorem A.4.4. *Let X be a Banach space and $T^{\odot} = \{T_t\}_{t\in\mathbb{R}^+} \subseteq \mathcal{B}(X)$ be a semigroup. Then $T^{\odot}$ is UC if and only if there exists $\mathsf{L} \in \mathcal{B}(X)$ such that $T_t = e^{t\mathsf{L}}$ for all $t \in \mathbb{R}^+$ and $\mathsf{L}(x) = \lim\limits_{t\to 0^+} \frac{T_t(x)-x}{t}$ for all $x \in X$.*

Proposition A.4.5. *Let X be a Banach space, $\mathsf{L} \in \mathcal{B}(X)$ and $T_t := e^{t\mathsf{L}}$ for all $t \in \mathbb{R}^+$. Then*

(i) $\|T_t\| \leq e^{|t|\|\mathsf{L}\|}$.

(ii) $T : [0, \infty) \to \mathcal{B}(X)$ *given by* $t \mapsto T_t$ *is continuous.*

(iii) $T : [0, \infty) \to \mathcal{B}(X)$ *is infinitely differentiable and* $\frac{d^n T_t}{dt^n} = \mathsf{L}^n T_t = T_t \mathsf{L}^n$ *as operators on* X *for* $n = 0, 1, 2, \cdots$.

Now from here onwards we assume that $X = \mathcal{B}$ is a C^*-algebra.

Definition A.4.6. Let $T^{\odot} = \{T_t\}_{t\in\mathbb{R}^+}$ is a UC-semigroup on $\mathcal{B}$. Then the operator $\mathsf{L} \in \mathcal{B}(\mathcal{B})$ defined by $\mathsf{L}(b) = \lim\limits_{t\to 0^+} \frac{T_t(b)-b}{t}$ is called the *(infinitesimal) generator* of $T^{\odot}$.

Proposition A.4.7. *Let $T^{\odot} = \{T_t\}_{t\in\mathbb{R}^+}$ be a UC-semigroup on $\mathcal{B}$ with generator $\mathsf{L} \in \mathcal{B}(\mathcal{B})$. Then T_t is CP for all $t \in \mathbb{R}^+$ if and only if L is CCP and $\mathsf{L}(b^*) = \mathsf{L}(b)^*$ for all $b \in \mathcal{B}$.*

CP-semigroups

Definition A.4.8. A CP-semigroup on a C^*-algebra $\mathcal{B}$ is a semigroup $\varphi^{\odot} = \{\varphi_t\}_{t\in\mathbb{R}_+}$ of CP-maps $\varphi_t : \mathcal{B} \to \mathcal{B}$. If $\mathcal{B}$ is unital, then $\varphi^{\odot}$ is said to be *unital* if all φ_t are unital.

Theorem A.4.9. *The formula $\varphi_t = e^{t\mathsf{L}}$ establish a one-one correspondence between*

UC-CP-semigroup on $\mathcal{B}$ and hermitian CCP mappings $\mathsf{L} \in \mathfrak{B}(\mathcal{B})$.

Proposition A.4.10. *Let $\varphi^{\odot} = \{\varphi_t\}_{t\in\mathbb{R}^+}$ be a UC-contractive semigroup on a von Neumann algebra $\mathcal{B}$ with generator $\mathsf{L} \in \mathfrak{B}(\mathcal{B})$. Then φ_t is normal for all $t \in \mathbb{R}^+$ if and only if L is σ-weakly (and hence σ-strongly) continuous on any norm-bounded subset of $\mathcal{B}$.*

Proposition A.4.11. *Let $\mathcal{B}$ be a unital C^*-algebra, let y be an element in a pre-Hilbert $\mathcal{B}$-$\mathcal{B}$ module F and let $\mathfrak{p} \in \mathcal{B}$. Then $\mathsf{L}(b) := b\mathfrak{p} + \mathfrak{p}^*b + \langle y, by\rangle$ is CCP and hermitian so that $\varphi^{\odot} = \{e^{t\mathsf{L}}\}_{t\in\mathbb{R}^+}$ is a UC-CP-semigroup.*

Theorem A.4.12. *Let $\varphi^{\odot} = \{\varphi_t\}_{t\geq 0}$ be a normal uniformly continuous CP-semigroup on a von Neumann algebra $\mathcal{B}$ with generator L. Then there exists a two-sided von Neumann $\mathcal{B}$-$\mathcal{B}$-module F, an element $y \in F$ and an element $\mathfrak{p} \in \mathcal{B}$ such that $\mathsf{L}(b) = b\mathfrak{p} + \mathfrak{p}^*b + \langle y, by\rangle$ and such that F is the strongly closed submodule of F generated by the derivation $d(b) := by - yb$. Moreover, F is determined by L up to (two-sided) isomorphism.*

Let $\mathsf{L} \in \mathfrak{B}(\mathcal{B})$ be a hermitian CCP map which is σ-weakly continuous on all norm-bounded subsets of $\mathcal{B}$. Then $\varphi^{\odot} = \{e^{t\mathsf{L}}\}_{t\in\mathbb{R}^+}$ is a normal UC-CP-semigroup on $\mathcal{B}$ with generator L. Then a triple $(F, y, \mathfrak{p})$ obtained as in above theorem is known as a *dilation* for L and it is said to be *minimal* if F is the strongly closed submodule of F generated by the derivation $d(b) = by - yb$.

E_0-semigroups

Definition A.4.13. An *E-semigroup* on a C^*-algebra $\mathcal{B}$ is a semigroup $\vartheta^{\odot} = \{\vartheta_t\}_{t\in\mathbb{R}^+}$ of endomorphisms $\vartheta_t : \mathcal{B} \to \mathcal{B}$. If $\mathcal{B}$ is unital and all ϑ_t are unital, then we call $\vartheta^{\odot}$ a *E_0-semigroup.*

Definition A.4.14. Suppose $\vartheta^{\odot} = \{\vartheta_t\}_{t\in\mathbb{R}^+}$ is a E_0-semigroup on a unital C^*-algebra $\mathcal{A}$. A *left (right) cocycle* in $\mathcal{A}$ with respect to $\vartheta^{\odot}$ is a family $\mathfrak{u}^{\odot} = \{\mathfrak{u}_t\}_{t\in\mathbb{R}^+}$ of elements $\mathfrak{u}_t \in \mathcal{A}$ satisfying

$$\mathfrak{u}_{s+t} = \mathfrak{u}_t \vartheta_t(\mathfrak{u}_s) \qquad \left(\mathfrak{u}_{s+t} = \vartheta_t(\mathfrak{u}_s)\mathfrak{u}_t\right)$$

and $\mathfrak{u}_0 = 1$. A cocycle is *positive, contractive, isometric, unitary* if so is $\mathfrak{u}_t$ for all t.

Proposition A.4.15. *$\mathfrak{u}^\odot$ is a left cocycle in $\mathcal{A}$ if and only if $\mathfrak{u}^{\odot *} := \{\mathfrak{u}_t^*\}$ is a right cocycle. In this case $\vartheta^{\odot \mathfrak{u}} = \{\vartheta_t^{\mathfrak{u}}\}_{t\in\mathbb{R}^+}$ with $\vartheta_t^{\mathfrak{u}}(\cdot) := \mathfrak{u}_t \vartheta_t(\cdot)\mathfrak{u}_t^*$ is a CP-semigroup on $\mathcal{A}$. This semigroup is unital, an E-semigroup, an E_0-semigroup if and only if $\mathfrak{u}^\odot$ is co-isometric, isometric, unitary, respectively.*

Definition A.4.16. We say the semigroup $\vartheta^{\odot \mathfrak{u}}$ is *conjugate* to the semigroup $\vartheta^\odot$ via the cocycle $\mathfrak{u}^\odot$. We say two E_0-semigroups $\vartheta^\odot, \vartheta'^\odot$ on $\mathcal{A}$ are *outer conjugate*, if $\vartheta'^\odot$ is conjugate to $\vartheta^\odot$ via a unitary cocylce $\mathfrak{u}^\odot$.

Remark A.4.17. Outer conjugacy is an equivalence relation among E_0-semigroups on $\mathcal{A}$.

A.5 Dilations of semigroups

Definition A.5.1. Let $\varphi^\odot = \{\varphi_t\}_{t\in\mathbb{R}^+}$ be a unital CP-semigroup on a unital C^*-algebra $\mathcal{B}$. A *dilation* of $\varphi^\odot$ is a quadruple $(\mathcal{A}, \vartheta^\odot, \mathfrak{i}, \mathfrak{p})$ consisting of a unital C^*-algebra $\mathcal{A}$, an E_0-semigroup $\vartheta^\odot = \{\vartheta_t\}_{t\in\mathbb{R}^+}$ on $\mathcal{A}$, a canonical injection (i.e., an injective $*$-homomorphism) $\mathfrak{i} : \mathcal{B} \to \mathcal{A}$, and an expectation $\mathfrak{p} : \mathcal{A} \to \mathcal{B}$ (i.e., a unital CP-map such that $\mathfrak{i}\circ\mathfrak{p}$ is a conditional expectation onto $\mathfrak{i}(\mathcal{B})$) such that the following diagram is commutative (i.e., $\mathfrak{p} \circ \vartheta_t \circ \mathfrak{i} = \varphi_t$ for all $t \in \mathbb{R}^+$).

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\ \varphi_t\ } & \mathcal{B} \\ {\scriptstyle \mathfrak{i}}\downarrow & & \uparrow{\scriptstyle \mathfrak{p}} \\ \mathcal{A} & \xrightarrow[\ \vartheta_t\ ]{} & \mathcal{A} \end{array}$$

A dilation $(\mathcal{A}, \vartheta^\odot, \mathfrak{i}, \mathfrak{p})$ of $\varphi^\odot$ is a *weak dilation*, if $\mathfrak{i} \circ \mathfrak{p}(\cdot) = \mathfrak{i}(1)(\cdot)\mathfrak{i}(1)$.

Definition A.5.2. A pair $(\mathcal{A}, j^\odot)$ consisting of a unital C^*-algebra $\mathcal{A}$ and a family

$j^{\odot} = \{j_t\}_{t\in\mathbb{R}^+}$ of $*$-homomorphisms $j_t : \mathcal{B} \to \mathcal{A}$ is a *weak Markov flow* for the CP-semigroup $\varphi^{\odot}$, if

$$j_t(1)j_{s+t}(b)j_t(1) = j_t \circ \varphi_s(b) \quad \text{for all } s,t \in \mathbb{R}^+, \text{ and } b \in \mathcal{B}.$$

A *weak Markov quasiflow* is a weak Markov flow $(\mathcal{A}, j^{\odot})$ except that j_0 need not be injective and $\mathcal{A}$ need not be unital.

If $(\mathcal{A}, \vartheta^{\odot}, \mathfrak{i}, \mathfrak{p})$ is a weak dilation, then the $*$-homomorphisms $j_t := \vartheta_t \circ \mathfrak{i}$ form a weak Markov flow. Thus a weak dilation gives rise to a weak Markov flow. In [Bha99] Bhat proved that the converse is true under certain minimality condition on a weak Markov flow.

A.6 Operator spaces

Definition A.6.1. A *matrix norm* $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ on a vector space X is an assignment of a norm $\|\cdot\|_n$ on the matrix space $M_n(X)$ for each $n \in \mathbb{N}$. An *operator space* is a pair $(X, \{\|\cdot\|_n\}_{n\in\mathbb{N}})$ consisting of a vector space X and a matrix norm $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ satisfying:

(R1) $\|\lambda x \lambda'\|_n \leq \|\lambda\| \, \|x\|_n \, \|\lambda'\|$ for all $\lambda, \lambda' \in M_n(\mathbb{C})$, $x \in M_n(X)$;

(R2) $\|x \oplus y\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$ for all $x \in M_n(X)$, $y \in M_m(X)$.

We say that a matrix norm is an *operator space matrix norm* if it satisfies the above two conditions (called *Ruan axioms*).

Example A.6.2. (i) Given a Hilbert space H, the operator norms on $\mathcal{B}(H^n)$ defines an operator space matrix norm on $\mathcal{B}(H)$, and so $\mathcal{B}(H)$ is an operator space.

(ii) Given Hilbert spaces H and K, $\mathcal{B}(H,K)$ is an operator space. We use the identifications $M_n(\mathcal{B}(H,K)) \cong \mathcal{B}(H^n, K^n)$ to determine a matrix norm on $\mathcal{B}(H,K)$. Alternatively, we may consider $\mathcal{B}(H,K)$ as a subspace of $\mathcal{B}(H \oplus K)$.

(iii) If $\mathcal{A}$ is a C^*-algebra, by fixing a faithful representation of $\mathcal{A}$ on a Hilbert space H we may regard $M_n(\mathcal{A})$ as a C^*-subalgebra of $\mathcal{B}(H^n)$. Then $\mathcal{A}$ has a canonical operator space structure, namely by assigning to each $M_n(\mathcal{A})$ the unique norm that makes it a C^*-algebra. Note that the matrix norm does not depend on the representation.

(iv) Suppose X, Y are operator spaces and $\psi : X \to Y$ bounded linear map. Consider the dual spaces $X^* = \mathcal{B}(X, \mathbb{C}) = CB(X, \mathbb{C})$ and $Y^* = \mathcal{B}(Y, \mathbb{C}) = CB(Y, \mathbb{C})$, and define $\psi^* : Y^* \to X^*$ by $\psi^*(\phi)(x) = \phi(\psi(x))$. From Hahn-Banach theorem, $\|\psi^*\| = \|\psi\|$. Now

$$M_n(X^*) \ni f = [f_{ij}] \mapsto \big(x \mapsto [f_{ij}(x)]\big) \in CB(X, M_n(\mathbb{C}))$$

defines a linear isomorphism from $M_n(X^*) \to CB(X, M_n(\mathbb{C}))$, which we use to determine the norm on $M_n(X^*)$. Thus we have the isometric identification $M_n(X^*) = CB(X, M_n(\mathbb{C}))$. The matrix norms on X^* determine an operator space space. If $\psi : X \to Y$ is a CB-map, then $\|\psi_n^*\| = \|\psi_n\|$ for all $n \in \mathbb{N}$ and $\|\psi^*\|_{cb} = \|\psi\|_{cb}$.

Theorem A.6.3 ([Rua88]). *Suppose that X is a vector space and $\|\cdot\|_n$ is a norm on $M_n(X)$ for each $n \in N$. Then X is completely isometrically isomorphic to a linear subspace of $\mathcal{B}(H)$, for some Hilbert space H, if and only if the conditions $(R1)$ and $(R2)$ hold. In other words, if X is an operator space, then there exists a Hilbert space H, a subspace $Y \subseteq \mathcal{B}(H)$, and a complete isometry $\psi : X \to Y$.*

For more details on operator spaces see [BLM04, ER88, Rua88].

Bibliography

[ACS99] E. Andruchow, G. Corach, and D. Stojanoff. Geometry of the sphere of a Hilbert module. *Math. Proc. Cambridge Philos. Soc.*, 127(2):295–315, 1999.

[Akh07] S. J. Akhtarshenas. An explicit computation of the Bures metric over the space of N-dimensional density matrices. *J. Phys. A*, 40(37):11333–11341, 2007.

[AP00] P. M. Alberti and G. Peltri. On bures distance over standard form vN-algebras. *Preprint, arXiv:math/0008164*, 2000.

[Ara72] H. Araki. Bures distance function and a generalization of Sakai's non-commutative Radon-Nikodym theorem. *Publ. Res. Inst. Math. Sci.*, 8:335–362, 1972.

[Ara05] L. Arambašić. Irreducible representations of Hilbert C^*-modules. *Math. Proc. R. Ir. Acad.*, 105A(2):11–24 (electronic), 2005.

[Ara08] L. Arambašić. Another characterization of Hilbert C^*-modules over compact operators. *J. Math. Anal. Appl.*, 344(2):735–740, 2008.

[Arv69a] W. B. Arveson. On subalgebras of C^*-algebras. *Bull. Amer. Math. Soc.*, 75:790–794, 1969.

[Arv69b] W. B. Arveson. Subalgebras of C^*-algebras. *Acta Math.*, 123:141–224, 1969.

[Arv76] W. Arveson. *An invitation to C^*-algebras.* Springer-Verlag, New York, 1976. Graduate Texts in Mathematics, No. 39.

[Arv89] W. Arveson. Continuous analogues of Fock space. *Mem. Amer. Math. Soc.*, 80(409):iv+66, 1989.

[Asa09] M. B. Asadi. Stinespring's theorem for Hilbert C^*-modules. *J. Operator Theory*, 62(2):235–238, 2009.

[AZ09] D. F. Abasto and P. Zanardi. Thermal states of the Kitaev honeycomb model: Bures metric analysis. *Phys. Rev. A (3)*, 79(1):012321, 6, 2009.

[BG02a] D. Bakić and B. Guljaš. Hilbert C^*-modules over C^*-algebras of compact operators. *Acta Sci. Math. (Szeged)*, 68(1-2):249–269, 2002.

[BG02b] D. Bakić and B. Guljaš. On a class of module maps of Hilbert C^*-modules. *Math. Commun.*, 7(2):177–192, 2002.

[BG03] D. Bakić and B. Guljaš. Extensions of Hilbert C^*-modules. II. *Glas. Mat. Ser. III*, 38(58)(2):341–357, 2003.

[Bha99] B. V. Rajarama Bhat. Minimal dilations of quantum dynamical semigroups to semigroups of endomorphisms of C^*-algebras. *J. Ramanujan Math. Soc.*, 14(2):109–124, 1999.

[Ble97a] D. P. Blecher. A new approach to Hilbert C^*-modules. *Math. Ann.*, 307(2):253–290, 1997.

[Ble97b] D. P. Blecher. On selfdual Hilbert modules. In *Operator algebras and their applications (Waterloo, ON, 1994/1995)*, volume 13 of *Fields Inst. Commun.*, pages 65–80. Amer. Math. Soc., Providence, RI, 1997.

[BLM04] D.P. Blecher and C. Le Merdy. *Operator algebras and their modules– an operator space approach*, volume 30 of *London Mathematical Society Monographs. New Series.* The Clarendon Press Oxford University Press, Oxford, 2004. Oxford Science Publications.

[BLS08] B. V. Rajaram Bhat, V. Liebscher, and M. Skeide. A problem of powers and the product of spatial product systems. In *Quantum probability and related topics*, volume 23 of *QP–PQ: Quantum Probab. White Noise Anal.*, pages 93–106. World Sci. Publ., Hackensack, NJ, 2008.

[BMP00] D. P. Blecher, P. S. Muhly, and V. I. Paulsen. Categories of operator modules (Morita equivalence and projective modules). *Mem. Amer. Math. Soc.*, 143(681):viii+94, 2000.

[BP91] D. P. Blecher and V. I. Paulsen. Tensor products of operator spaces. *J. Funct. Anal.*, 99(2):262–292, 1991.

[Bro85] L. G Brown. Close hereditary C^*-subalgebras and the structure of quasi-multipliers. *MSRI Preprint,11211-85*, 1985.

[BRS12] B. V. Rajarama Bhat, G. Ramesh, and K. Sumesh. Stinespring's theorem for maps on Hilbert C^*-modules. *J. Operator Theory*, 68(1):173–178, 2012.

[Brü99] F. M. Brückler. Tensor products of C^*-algebras, operator spaces and Hilbert C^*-modules. *Math. Commun.*, 4(2):257–268, 1999.

[Brü04] F. M. Brückler. A note on extensions of Hilbert C^*-modules and their morphisms. *Glas. Mat. Ser. III*, 39(59)(2):313–326, 2004.

[BS00] B. V. Rajarama Bhat and M. Skeide. Tensor product systems of Hilbert modules and dilations of completely positive semigroups. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 3(4):519–575, 2000.

[BS13] B. V. Rajarama Bhat and K. Sumesh. Bures distance for completely positive maps. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 16(4):1350031, 2013.

[Bur69] D. Bures. An extension of Kakutani's theorem on infinite product measures to the tensor product of semifinite w^*-algebras. *Trans. Amer. Math. Soc.*, 135:199–212, 1969.

[Coh59] P. J. Cohen. Factorization in group algebras. *Duke Math. J*, 26:199–205, 1959.

[Dit98] J. Dittmann. Yang-Mills equation and Bures metric. *Lett. Math. Phys.*, 46(4):281–287, 1998.

[Dit99] J. Dittmann. Explicit formulae for the Bures metric. *J. Phys. A*, 32(14):2663–2670, 1999.

[EN06] K.J. Engel and R. Nagel. *A short course on operator semigroups.* Universitext. Springer, New York, 2006.

[ER88] E.G. Effros and Z.J. Ruan. On matricially normed spaces. *Pacific J. Math.*, 132(2):243–264, 1988.

[ER91] E. G. Effros and Z.J. Ruan. Self-duality for the Haagerup tensor product and Hilbert space factorizations. *J. Funct. Anal.*, 100(2):257–284, 1991.

[Fra90] M. Frank. Self-duality and C^*-reflexivity of Hilbert C^*-moduli. *Z. Anal. Anwendungen*, 9(2):165–176, 1990.

[Fra97a] M. Frank. Isomorphisms of Hilbert C^*-modules and $*$-isomorphisms of related operator C^*-algebras. *Math. Scand.*, 80(2):313–319, 1997.

[Fra97b] M. Frank. A multiplier approach to the Lance-Blecher theorem. *Z. Anal. Anwendungen*, 16(3):565–573, 1997.

[Fra99] M. Frank. Geometrical aspects of Hilbert C^*-modules. *Positivity*, 3(3):215–243, 1999.

[Fra02] M. Frank. On Hahn-Banach type theorems for Hilbert C^*-modules. *Internat. J. Math.*, 13(7):675–693, 2002.

[Fra08] M. Frank. Characterizing C^*-algebras of compact operators by generic categorical properties of Hilbert C^*-modules. *J. K-Theory*, 2(3):453–462, 2008.

[FS10] M. Frank and K. Sharifi. Adjointability of densely defined closed operators and the Magajna-Schweizer theorem. *J. Operator Theory*, 63(2):271–282, 2010.

[Heo99] J. Heo. Hilbert C^*-module representation on Haagerup tensor products and group systems. *Publ. Res. Inst. Math. Sci.*, 35(5):757–768, 1999.

[HJ11] J. Heo and Un Cig Ji. Quantum stochastic processes for maps on Hilbert C^*-modules. *J. Math. Phys.*, 52(5):053501, 16, 2011.

[Hüb92] M. Hübner. Explicit computation of the Bures distance for density matrices. *Phys. Lett. A*, 163(4):239–242, 1992.

[Joi11] M. Joita. Covariant version of the Stinespring type theorem for Hilbert C^*-modules. *Cent. Eur. J. Math.*, 9(4):803–813, 2011.

[Joi12] M. Joita. Comparison of completely positive maps on Hilbert C^*-modules. *J. Math. Anal. Appl.*, 393(2):644–650, 2012.

[JT91] K. K. Jensen and K. Thomsen. *Elements of KK-theory*. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1991.

[Kap53] I. Kaplansky. Modules over operator algebras. *Amer. J. Math.*, 75:839–858, 1953.

[Kas80] G. G. Kasparov. Hilbert C^*-modules: theorems of Stinespring and Voiculescu. *J. Operator Theory*, 4(1):133–150, 1980.

[Kos83] H. Kosaki. On the Bures distance and the Uhlmann transition probability of states on a von Neumann algebra. *Proc. Amer. Math. Soc.*, 89(2):285–288, 1983.

[KR83] R. V. Kadison and J. R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. I*, volume 100 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983. Elementary theory.

[KR86] R. V. Kadison and J. R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. II*, volume 100 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986. Advanced theory.

[KSW08a] D. Kretschmann, D. Schlingemann, and R. F. Werner. A continuity theorem for Stinespring's dilation. *J. Funct. Anal.*, 255(8):1889–1904, 2008.

[KSW08b] D. Kretschmann, D. Schlingemann, and R. F. Werner. The information-disturbance tradeoff and the continuity of Stinespring's representation. *IEEE Trans. Inform. Theory*, 54(4):1708–1717, 2008.

[Kus05] M. Kusuda. Discrete spectra of C^*-algebras and orthogonally closed submodules in Hilbert C^*-modules. *Proc. Amer. Math. Soc.*, 133(11):3341–3344 (electronic), 2005.

[Lan94] E. C. Lance. Unitary operators on Hilbert C^*-modules. *Bull. London Math. Soc.*, 26(4):363–366, 1994.

[Lan95] E. C. Lance. *Hilbert C^*-modules*, volume 210 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1995. A toolkit for operator algebraists.

[Lin92] Hua Xin Lin. Injective Hilbert C^*-modules. *Pacific J. Math.*, 154(1):131–164, 1992.

[Mag97a] B. Magajna. Hilbert C^*-modules in which all closed submodules are complemented. *Proc. Amer. Math. Soc.*, 125(3):849–852, 1997.

[Mag97b] B. Magajna. Hilbert modules and tensor products of operator spaces. In *Linear operators (Warsaw, 1994)*, volume 38 of *Banach Center Publ.*, pages 227–246. Polish Acad. Sci., Warsaw, 1997.

[Man96a] V. M. Manuilov. Adjointability of operators on Hilbert C^*-modules. *Acta Math. Univ. Comenian. (N.S.)*, 65(2):161–169, 1996.

[Man96b] V. M. Manuilov. Representability of functionals in Hilbert C^*-modules. *Funktsional. Anal. i Prilozhen.*, 30(4):83–86, 1996.

[McB87] A. C. McBride. *Semigroups of linear operators: an introduction*, volume 156 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1987.

[MS00] P. S. Muhly and B. Solel. On the Morita equivalence of tensor algebras. *Proc. London Math. Soc. (3)*, 81(1):113–168, 2000.

[MSS06] P. S. Muhly, M. Skeide, and B. Solel. Representations of $\mathcal{B}^a(E)$. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 9(1):47–66, 2006.

[Mur90] G. J. Murphy. *C^*-algebras and operator theory*. Academic Press Inc., Boston, MA, 1990.

[Mur97] G. J. Murphy. Positive definite kernels and Hilbert C^*-modules. *Proc. Edinburgh Math. Soc. (2)*, 40(2):367–374, 1997.

[Pas73] W. L. Paschke. Inner product modules over B^*-algebras. *Trans. Amer. Math. Soc.*, 182:443–468, 1973.

[Pau86] V. I. Paulsen. *Completely bounded maps and dilations*, volume 146 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1986.

[Pau02] V. Paulsen. *Completely bounded maps and operator algebras*, volume 78 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002.

[Ped79] G. K. Pedersen. *C^*-algebras and their automorphism groups*, volume 14 of *London Mathematical Society Monographs*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1979.

[Pli12] M. Pliev. Stinespring type theorem for a finite family of maps on Hilbert C^*-modules. *Preprint, arXiv:1210.5716*, 2012.

[PS87] V. I. Paulsen and R. R. Smith. Multilinear maps and tensor norms on operator systems. *J. Funct. Anal.*, 73(2):258–276, 1987.

[Rie67] M. A. Rieffel. Induced Banach representations of Banach algebras and locally compact groups. *J. Functional Analysis*, 1:443–491, 1967.

[Rie74a] M. A. Rieffel. Induced representations of C^*-algebras. *Advances in Math.*, 13:176–257, 1974.

[Rie74b] M. A. Rieffel. Morita equivalence for C^*-algebras and W^*-algebras. *J. Pure Appl. Algebra*, 5:51–96, 1974.

[Rua88] Z.J. Ruan. Subspaces of C^*-algebras. *J. Funct. Anal.*, 76(1):217–230, 1988.

[Sak71] S. Sakai. *C^*-algebras and W^*-algebras*. Springer-Verlag, New York, 1971. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 60.

[Sch99] J. Schweizer. A description of Hilbert C^*-modules in which all closed submodules are orthogonally closed. *Proc. Amer. Math. Soc.*, 127(7):2123–2125, 1999.

[Ske00] M. Skeide. Generalised matrix C^*-algebras and representations of Hilbert modules. *Math. Proc. R. Ir. Acad.*, 100A(1):11–38, 2000.

[Ske01] M. Skeide. *Hilbert modules and applications in quantum probability*. Habilitationsschrift, Cottbus, 2001.

[Ske02] M. Skeide. Dilations, product systems, and weak dilations. *Mat. Zametki*, 71(6):914–923, 2002.

[Ske03a] M. Skeide. Commutants of von Neumann modules, representations of $\mathcal{B}^a(E)$ and other topics related to product systems of Hilbert modules. In *Advances in quantum dynamics (South Hadley, MA, 2002)*, volume 335 of *Contemp. Math.*, pages 253–262. Amer. Math. Soc., Providence, RI, 2003.

[Ske03b] M. Skeide. Dilation theory and continuous tensor product systems of Hilbert modules. In *Quantum probability and infinite dimensional analysis (Burg, 2001)*, volume 15 of *QP–PQ: Quantum Probab. White Noise Anal.*, pages 215–242. World Sci. Publ., River Edge, NJ, 2003.

[Ske05a] M. Skeide. Three ways to representations of $\mathcal{B}^a(E)$. In *Quantum probability and infinite dimensional analysis*, volume 18 of *QP–PQ: Quantum Probab. White Noise Anal.*, pages 504–517. World Sci. Publ., Hackensack, NJ, 2005.

[Ske05b] M. Skeide. von Neumann modules, intertwiners and self-duality. *J. Operator Theory*, 54(1):119–124, 2005.

[Ske06a] M. Skeide. Commutants of von Neumann correspondences and duality of Eilenberg-Watts theorems by Rieffel and by Blecher. In *Quantum probability*, volume 73 of *Banach Center Publ.*, pages 391–408. Polish Acad. Sci. Inst. Math., Warsaw, 2006.

[Ske06b] M. Skeide. Generalized unitaries and the Picard group. *Proc. Indian Acad. Sci. Math. Sci.*, 116(4):429–442, 2006.

[Ske07] M. Skeide. E_0-semigroups for continuous product systems. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 10(3):381–395, 2007.

[Ske08] M. Skeide. Product systems; a survey with commutants in view. In *Quantum stochastics and information*, pages 47–86. World Sci. Publ., Hackensack, NJ, 2008.

[Ske09a] M. Skeide. Classification of e_0-semigroups by product systems. *Preprint, arXiv:0901.17983v3, 2009*, 2009.

[Ske09b] M. Skeide. E_0-semigroups for continuous product systems: the nonunital case. *Banach J. Math. Anal.*, 3(2):16–27, 2009.

[Ske09c] M. Skeide. Unit vectors, Morita equivalence and endomorphisms. *Publ. Res. Inst. Math. Sci.*, 45(2):475–518, 2009.

[Ske11] M. Skeide. Hilbert modules—square roots of positive maps. In *Quantum probability and related topics*, volume 27 of *QP–PQ: Quantum Probab. White Noise Anal.*, pages 296–322. World Sci. Publ., Hackensack, NJ, 2011.

[Ske12] M. Skeide. A factorization theorem for φ-maps. *J. Operator Theory*, 68(2):543–547, 2012.

[Sol01] B. Solel. Isometries of Hilbert C^*-modules. *Trans. Amer. Math. Soc.*, 353(11):4637–4660 (electronic), 2001.

[SS14] M. Skeide and K. Sumesh. CP-H-extendable maps between Hilbert modules and CPH-semigroups. *J. Math. Anal. Appl.*, 414(2):886–913, 2014.

[Sti55] W. F. Stinespring. Positive functions on C^*-algebras. *Proc. Amer. Math. Soc.*, 6:211–216, 1955.

[Tak03] M. Takesaki. *Theory of operator algebras. III*, volume 127 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2003. Operator Algebras and Non-commutative Geometry, 8.

[TS07] G. A. Tabadkan and M. Skeide. Generators of dynamical systems on Hilbert modules. *Commun. Stoch. Anal.*, 1(2):193–207, 2007.

[Zet94] H. Zettl. Some aspects of geometry of operators in Hilbert modules. *Preprint*, N. 173, 1994.

[Zhu93] Ke He Zhu. *An introduction to operator algebras*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1993.

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