CYCLE PACKING IN COMPLETE UNDIRECTED GRAPH

A dissertation submitted in partial fulfilment of the requirements for the M. Tech. (Computer Science) degree of the Indian Statistical Institute

By

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Certificate of Approval

This is to certify that the thesis titled, **CYCLE PACKING IN COMPLETE UNDIRECTED GRAPH** submitted by **Tapan Kumar Adak**, towards partial fulfilment of the requirements for the degree of M. Tech. in Computer Science at the Indian Statistical Institute, Calcutta, embodies the work done under my supervision.

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Abstract

Packing of uniform 2–factors, (existing of 4–cycles) in the complete undirected graph with n vertex considered. For n = 4t, an algorithm is given which is considerably simpler than the existing algorithm. This case is modified to get result for n = 4t + 2. For the cases n = 4t + 1 and n = 4t + 3, empirical studies are done.

1 Introduction

A *Steiner triple system* (more simply, triple simple) is a pair (S,T), where S is the vertex set of the complete undirected graph K_n of n vertices and T is a collection of edge disjoint triangles (or triples) which partition the K_n in to the vertex set S. The number $n = |S|$ is called the *order* of the triple system (S,T). It has been known that a triple system of order n exists if and only if $n \equiv 1$ *or* 3 $(mod 6)$[1]. It is trivial to see that if (S,T) is a triple system of order n then $|T| = n(n-1)/6$.

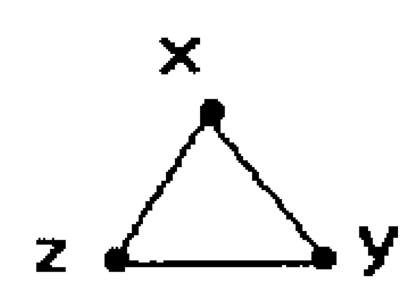
Now a triangle is also a 3-cycle and so a Steiner triple system (S,T) can be describe as an edge disjoint collection of 3-cycles which partition K_n (based on S). Since there is nothing particularly sacred about the number 3, every single question raised for triple systems can also be raised for m-cycle systems for $m \geq 4$. An obvious definition here: an *m-cycle system* of *order n* is a pair (S,C), where S is the vertex set of complete undirected graph K_n and C is an edge disjoint collection of m-cycles which partition K_n based on S. Of course $|C| = n(n-1)/2m$. Roughly twenty-five years ago, serious work began on attacking a wide range of m-cycle system problems[1].

2 Existence of $m-$cycle systems

Certainly the place to start any survey on $m-$cycle systems is with the existence problem; i.e., the determination for each $m \geq 3$ of the set of all n such that an $m-$cycle system (S,C) of order n exists.

2.1 Steiner triple systems

Consider some example. In what follows we will denote the triangle



by $\{x,y,z\}$ or simply xyz in any order.

Example 1 . *The unique triple system of order 3.*

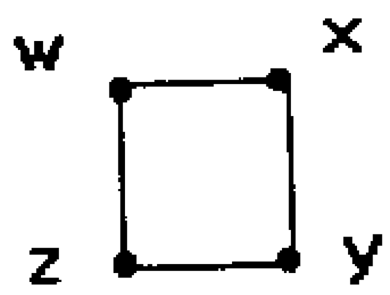
$$1,2,3 \text{ or } 123$$

Example 2 . *The unique (to within isomorphism) triple system of order* 7.

$$(1,2,4)\ (3,4,6)\ (5,6,1)\ (7,1,3)\ (2,3,5)\ (4,5,7)\ (6,7,2).$$

A new idea has been proposed in this paper using cycles of the complete undirected graph K_n which is as follows.

Note. In what follows, denote the 4 -cycle



by any cyclic shift of (x,y,z,w) *or* (y,x,w,z) or simply $(y\ x\ w\ z\ )$.

3 Packing system

Definition 1 : $2 \times l-uniform$ $k-$ $near\,factor$: *Given a complete undirected graph* K_n, *with n vertices,* $n = lt + k$, $l \geq 3$, $k = 0, 1, ..., l-1$, *for some* $t \geq 1$, (*all* n, l *and t are +ve integers, and k is non -ve integer.*). *By* $2 \times l$ *(read as 2 by l)-uniform k-near factor of* K_n, *we shall mean a collection of vertex disjoint t-number of cycles each of length l with k vertices not occurring in that collection.*

Example 3 . *Consider* K_5, $t = 1, l = 4$ *and* $k = 1$, *then* $\{(1,2,3,4)\}$ *is a* 2×4*-uniform* 1*-near factor of* K_5 *with one vertex viz, '5' dose not occur in this collection. Also* $\{(2,3,4,5)\}$ *is another* 2×4*-uniform* 1*-near factor of* K_5 *with one vertex viz '1' dose not occur in this collection.*

Example 4 . *Consider* K_9, $t = 2, l = 4$ *and* $k = 1$, *then* $\{(1,2,3,4),(5,6,7,8)\}$ *is a* 2×4*-uniform* 1*-near factor of* K_9 *with one vertex viz '9' not in this collection. Also there are other such factors exists.*

Example 5 . *Consider* K_8, $t = 2, l = 4$ *and* $k = 0$, *then* $\{(1,2,3,4),(5,6,7,8)\}$ *is a* 2×4*-uniform* 0*-near factor of* K_8.

Remark.

1 . It is noted that, if $k = 0$, then a 2×4-uniform 0-near factor of K_n is nothing but a 2-uniform factor of K_n. So in such a case for simplicity we use only the terminology 'factor'.

2 . For $k \neq 0$, the k vertices of K_n which dose not occur in a $2 \times l$-uniform k-near factor f of K_n said to be *left vertices of K_n with respect to f.*

3 . Since each vertex occurs in a collection of above type contribute 2 degree and each cycle is of same length, that is the reason for useing the terminology '2-uniform'.

Definition 2 : *Packing system: Given a complete graph K_n with $n = lt + k$ for $t \geq 1$, $l \geq 3$ and $k = 0, 1, ..., l-1$. By a packing of K_n we mean a collection $\mathcal{P}_{l,t}$ of all $2 \times l$-uniform k-near factor of K_n such that any two $2 \times l$-uniform k-near factor of that collection has no common edge and it is denoted by $(\mathcal{P}_{l,t}, n)$. The number of $2 \times l$-uniform k-near factor of K_n in $\mathcal{P}_{l,t}$ is said to be size (or order) of $\mathcal{P}_{l,t}$ and denoted by $O(\mathcal{P}_{l,t}, n)$.*

Problem:

This paper exclusively studies for a complete undirected graph K_n for $n = lt + k$ for $l = 4$. More precisely it provides an algorithm for computing $\mathcal{P}_{4,t}$ when $k = 0$ *and* 2 and compute the size of $\mathcal{P}_{4,t}$.

4 Existing Results.

The reader is referred to [2] for this section.

Let v be an even integer and H_v be the complete undirected graph on v vertices with the edge of a 1 -factor deleted, throughout this section. Let us denote $D(m)$ to be the set of all integers such that H_v decomposed into isomorphic 2 uniform -factors consisting entirely cycles of length m.

Definition 3 . *An (A, k) -system is a set of k disjoint pairs (p_r, q_r) covering the elements of $\{1, 2, ..., 2k\}$ exactly once and such that $q_r - p_r = r$ for $r = 1, 2, ..., k$. Similarly, a (B, k) -system is a set of disjoint pairs (p_r, q_r) covering the elements* *of* $\{1, 2, ..., 2k-1, 2k+1\}$ *exactly once and such that* $q_r - p_r = r$ *for* $r = 1, 2, ..., k$.

Lemma 1 . *An* (A, k) *system exists if and only if* $k \equiv 0$ *or* 1 $(mod4)$ *and an* (B, k) *system exists if and only if* $k \equiv 2$ *or* 3 $(mod4)$.

Definition 4 *:Design. Let* X *be a finite set of points and let* $\mathcal{B} = \{B_i : i \in I\}$ *where* I *is an index set, be a family of subsets* B_i *(not necessarily disjoint) -said to be blocks of* X. *The pair* $(X, \mathcal{B})$ *is called a design.*
The order of a design $(X, \mathcal{B})$ *is* $|X|$ *(the cardinality of* X*) and the set* $\{|B_i| : B_i \in \mathcal{B}\}$ *is the set of block-size of the design.*

Definition 5 *:Parrallel design. Let a design* $(X, \mathcal{B})$ *be given. A parallel class of blocks is a subfamily* $\mathcal{G} \subset \mathcal{B}$ *of disjoint blocks, the union of which equals* X.

Definition 6 *: We shall consider of length design of the form* $(X, \mathcal{G}, \mathcal{P})$, *where* X *is a finite set of points,* $\mathcal{G}$ *is a parallel class of subsets of* X *called groups and* $\mathcal{P}$ *is a family of subsets of* X *called proper blocks or simply blocks . Let* s, r *and* λ *be positive integers. A design* $(X, \mathcal{G}, \mathcal{P})$ *is a transversal design* $T[s, \lambda; r]$ *if*
(i) $|G_i| = r$ *for every* $G_i \in \mathcal{G}$,
(ii) $|\mathcal{G}| = s$,
(iii) $|G_i \bigcap B_j| = 1$ *for every* $G_i \in \mathcal{G}$ *and every* $B_j \in \mathcal{P}$ *and*
(iv) every pair set $\{x, y\} \subset X$, *such that* x *and* y *belongs to disjoint groups is contained in exactly* λ *blocks of* $\mathcal{P}$.

Obviously in this case, $|X| = sr, |B_j| = s$, for all $B_j \in \mathcal{P}$ and $|\mathcal{P}| = \lambda^2 r$.

Definition 7 *: A resolvable transversal design* $RT[s, \lambda; r]$ *is a transversal design* $T[s, \lambda; r]$ *in which the family* $\mathcal{P}$ *of blocks can be partitioned into* λr *parallel classes. Denote* $RT(s, \lambda)$, *the set of integers* r *for which resolvable transversal design* $RT[s, \lambda; r]$ *exists.*

Theorem 1 . *An resolvable transversal design* $RT(s, \lambda)$ *is same as a decomposition of the complete s-partite graph* $K_{\lambda, \lambda, ..., \lambda}$ *into isomorphic* $s - 1$ *factors with each component of each factor being the complete graph* K_s.

Lemma 2 . *Let* k *be an even integer. If there exist* $v \in D(k)$ *and a resolvable decomposition of* $K_{v,v}$ *into k-cycle then* $2v \in D(k)$.

Theorem 2 . *$v \in D(4)$ if and only if $v \equiv 0$ (mod4).*

Proof. The method is essentially Bose's method of "symmetrically repeated difference" from the theory of BIBD's. It turns out that for the problem in question it is more often then not convenient to take as the vertices of the complete graph the elements of the additive group Z_n of residues modulo n, with one or two additional elements ∞_1 and ∞_2 and assume our problem to have an automorphism having a cycle of length n and one or two fixed points. The problem of obtain a solution then reduces to constructing a "base" 2-factor R; the remaining 2-factor are obtained by developing R with respect to the group in question.

The necessity of the theorem is obvious. Let $v = 4t, t \geq 1$. First we show that there exists a resolvable decomposition of the complete bipartite graph $K_{v,v}$ into 4-cycles. If $V = V_1 \bigcup V_2$ is the bipartition of vertex set of $K_{v,v}$, partition each V_i arbitrarily into 4-subsets S_{ij} $(i = 1,2; j = 1,2,...,t)$. Consider a new complete bipartite graph $K_{t,t}$ whose vertices are the sets S_{ij} and decompose it into 1-factors. From each 1-factor of this $K_{t,t}$ we get two 2-factors has edges S_{1j} and S_{2j}, $j = 1,2,...,t$ and if $S_{ij} = (a_{ij}, b_{ij}, c_{ij}, d_{ij})$ then the 4-cycles $(a_{1j}, a_{2j}, b_{1j}, b_{2j})$, and $(c_{1j}, c_{2j}, d_{1j}, d_{2j})$ belongs to one 2-factor while the cycles $(a_{1j}, c_{2j}, b_{1j}, d_{2j})$, and $(a_{2j}, d_{1j}, b_{2j}, c_{1j})$ belongs to the other 2-factor.

Now by the previous lemma, the existence of $v \in D(4)$ if and only if $2v \in D(4)$. Hence we only need to construct decomposition of H_v for $v \equiv 4$ (*mod* 8). Thus let $v = 8t+4$, and let the vertex set of our complete graph be $V = Z_{4t+1} \times \{1,2\} \bigcup \{\infty_1, \infty_2\}$. An element (a,i) of $Z \times \{i\}$ will simply be denoted by a_i. Let $\{(p_r, q_r) : r = 1,2,...,2t\}$ be an $(A, 2t)$–system or a $(B, 2t)$– system depending on whether t is even or odd. That is $q_r - p_r = r$ *forall* $r = 1,2,...,2t$ *and* $\bigcup_{r=1}^{2t} \{p_r, q_r\} \subset \{1,2,...,4t+1\}$. Let x and y be elements defined by

$$\{x\} = \{1,2,...,4t+1\} \setminus \bigcup_{r=1}^{2t} \{p_r, q_r\}$$

$$\{y\} = \{1,2,...,4t+1\} \setminus \bigcup_{r=1}^{2t} \{2p_r, 2q_r\}$$

Define a 2-factor R whose components are 4-cycle as follows:-

One component of R is

$$(\infty_1, x_1, \infty_2, y_2)$$

and the remaining $2t$ components are

$$((p_r)_1, (q_r)_1, (2q_r)_2, (2p_r)_2), \text{ where } r = 1,2,...,2t.$$

Then R is a base 2-factor with respect to

$$\alpha = (\infty_1)\ (\infty_2)\ (0_1, 1_1, ..., (4t)_1)\ (0_2, 1_2, ..., (4t)_2).$$

The unused edges form a 1-factor

$$F = \{[\infty_1, \infty_2], [i_1, (i+z)_2] : i = 0, 1, ..., 4t\}$$

where $z = 0$ *or* 4 depending on whether t is even or odd.

5 Improvement over the existing algorithm.

From the existing result for computing all $2 \times\ 4-$ *uniform* 0−near factors of a K_n for $n = 4t$ it is clear that if a perfect matching deleted from K_n, then it is entirely 2− uniform factorizable. This paper supports a simple algorithm for finding all $2 \times 4-$uniform factor. Also this algorithm can be applied for K_n, $n = 4t + 2$ by slightly modifying it.

<u>Case 1. $n = 4t$</u>

Algorithm 1 .

Input: A complete undirected graph K_{4t} with vertex set (say) $\{1, 2, ..., 4t\}$.

Output: Set of all $2 \times\ 4-$uniform factor of K_n.

Step 1.

Find any perfect matching of K_n, (call it as initial perfect matching) say

$$\mu = \{(1, 2),\ (3, 4),\ ...,\ (4t-1, 4t)\}.$$

there are 2t edges of K_n in μ; denote those as t_i, for $i = 1,\ 2,\ ...,\ 2t$.

Step 2.

Find all disjoint perfect matching of K_{2t}, the complete undirected graph of order 2t. Since a complete undirected graph of order 2k has $2k - 1$ disjoint perfect matching, so there $2t - 1$ disjoint perfect matching of K_{2t}, let those be

$$\{\mu_1, \mu_2, ..., \mu_{2t-1}\}.$$

Step 3.

For each edge ab of $K_{2t}\ \in\ \mu_i$, for some $i = 1, 2, ..., 2t-1$, if $t_a = (x, y)$ and $t_b = (u, v)$, form the four cycle

$$\{x, u, y, v\}\ or\ \{u, x, v, y\}.$$

Since there are t edges in any μ_i, there are t number of 4−cycles for each μ_i, and they are vertex disjoint. Since there are $2t - 1$ disjoint perfect matching of K_{2t}, there are $2t - 1$, 2×4 -uniform factor of K_n.

Let us look at an illustration before a formal proof of the algorithm.

5.1 Illustration of the algorithm.

Consider the complete graph K_{12}, of order 12, and let its vertex set be $\{0, 1, ..., 11\}$. One of its perfect matching is

$$\mu = \{(0,1),\ (2,3),\ ...,\ (10,11)\ \}$$

taken as initial perfect matching and let

$$t_i \ = \ (2i, 2i+1),\ \ for\ i = 0, 1, ..., 5.$$

Then all disjoint perfect matching of K_6, with vertex set

$$\{0, 1, ..., 5\}\ are:$$

$$\mu_1 \ = \ \{(0,5),\ (1,4),\ (2,3)\ \}$$

$$\mu_2 \ = \ \{(1,5),\ (2,0),\ (3,4)\ \}$$

$$\mu_3 \ = \ \{(2,5),\ (3,1),\ (4,0)\ \}$$

$$\mu_4 \ = \ \{(3,5),\ (4,2),\ (0,1)\ \}$$

$$\mu_5 \ = \ \{(4,5),\ (0,3),\ (1,2)\ \}.$$

Hence the $2 \times$ 4-uniform factors of K_{12} are:

$$\{(0,10,1,11),\ (2,8,3,9),\ (4,6,5,8)\ \}$$

$$\{(2,10,3,11),\ (0,4,1,5),\ (6,8,7,9)\ \}$$

$$\{(4,10,5,11),\ (0,6,1,7),\ (2,8,3,9),\ \}$$

$$\{(6,10,7,11),\ (4,8,5,9),\ (0,2,1,3)\ \}$$

$$\{(8,10,9,11),\ (0,6,1,7),\ (2,4,3,5)\ \}.$$

Now it remains to prove the correctness of the above algorithm. By correctness of the above algorithm, we shall mean that, each edge of the K_n not in the initial perfect matching μ used in the $2\times4-$ uniform factor one and only once and edges in the initial perfect matching are not used in any of the $2\times4-$uniform factors.

Lemma 3 . *Each edge of K_n not in the initial perfect matching μ of K_n occurs in the 2×4-uniform factors of K_n exactly once and edges in the initial perfect matching does not occurs in an of the $2\times4-$ uniform factors.*

Proof. To prove this, recall that [3] for a complete undirected graph K_{2t} of even order, the set of all perfect matchings of K_{2t} are given as follows:

Let the vertex set of K_{2t} be

$$V = Z_{2t-1} \bigcup \{\infty\}$$

Let $F_0 = \{(j,-j) : j \in Z_t \setminus \{0\}\} \bigcup\{(0,\infty)\}$. Then F_0 is a perfect matching of K_{2t}. The other perfect matching of K_{2t} are

$$F_i = F_0 + i = \{(i+j, i-j) : j \in Z_t \setminus \{0\}\} \bigcup \{(i,\infty)\}, i = 1,2,...,2t-2.$$

Now let (i,j) be any edge of K_n not in the initial perfect matching of K_n. Then there exist integers a and b such that $a \neq b$ and i *and* j are the end vertices of edge t_a and t_b respectively. Since the edge (a,b) of K_{2t} occurs exactly once in the set of all perfect matching of K_{2t} it follows that, the edge (i,j) of K_n occurs exactly once in the set of all 2×4-uniform factor of K_n.
Now it remains to prove that any edge of initial perfect matching does not occurs in any of the $2\times4-$uniform factors of K_n. This follows trivially from the formation of $4-$cycle from any edge ab of K_{2t}. ∎

On the basis of above lemma we can state the following theorem.

Theorem 3 . *Size of packing $(\mathcal{P}_{4,t}, n)$, where $n = 4t$ is $2t-1$.*

Proof. It follows from the above lemma that, the number of disjoint perfect matching of K_{2t} is $2t-1$ and each such perfect matching determine a disjoint $2\times4-$uniform factor of K_{4t}, hence the size of packing $(\mathcal{P}_{4,t}, n)$, *where* $n = 4t$ is $2t-1$. ∎

Case 2 : $n = 4t + 2$.

In this case, if we delete a perfect matching F of K_n from K_n, then any edge of resulting graph occurs exactly once in the set of all 2×4-unform 2−near factors of K_n and the size of packing $(\mathcal{P}_{4,t}, n)$ is $2t + 1$.

The algorithm for finding the set of all 2×4-uniform 2− near factors are exactly same, only the necessary changes takes place in the step 2. Instead of finding all the perfect matching of K_{2t}, find out all the 1−near factor (or maximum matching) of K_{2t+1}. To find out all the 1−near factor of K_l, where l is odd integer, add one auxiliary vertex ∞ with the vertex set of K_l, find all the perfect matching of K_{l+1} and then delete the edge from each perfect matching of K_{l+1} having one vertex as ∞. The resulting matching is a 1−near factor of K_l.

Now it easy to obtain a 2×4-uniform 2−near factor of K_n from each 1−near factor of K_{2t+1}. Since each 1−near factor of K_{2t+1} contains t edges of K_{2t+1}, so from each 1−near factor of K_{2t+1}, it is possible to obtain a 2×4−uniform 2− near factor of K_{4t+2}. Since there are l number of disjoint 1−near factor for K_l, when l is odd, it follows that the size of packing is $(\mathcal{P}_{4,t}, n)$ for $n = 4t + 2$ is $2t + 1$.

Hence we can state the following theorem.

Theorem 4 . *Size of packing $(\mathcal{P}_{4,t}, n)$ when $n = 4t + 2$ is $2t + 1$.*

Remark.

It is noted that, the size of packing for K_n for $n = 4t$ *and* $n = 4t + 2$ whatever obtained in above are maximum. Since in this cases the degree of each vertex is odd, so each vertex has degree at least one after constructing the packing of maximum size. Now the algorithm describe above construct a packing for K_n for $n = 4t$ *and* $n = 4t + 2$ and after that each vertex of K_n in this case has degree one. Hence the size of packing in this cases whatever obtained by algorithm is maximum for the graph K_n for $n = 4t$ *and* $n = 4t + 2$.

6 Computation of Size of packing for $n = 4t + 1$ *and* $n = 4t + 3$.

It is noted that, there is a theoretical upper bound of packing size and that is $\lfloor \frac{\binom{n}{2}}{4t} \rfloor$ for K_n, where $n = 4t + k$, $k = 0, 1, 2$ *and* 3 . For $k = 1$ it becomes

$\lfloor \frac{4t \times (4t+1)}{2 \times 4t} \rfloor$ that is $\lfloor \frac{4t+1}{2} \rfloor$ that is $2t$. But empirically it has been seen that this is not acheivable for $n = 5,\ 9,\ 13,\ 17\ and\ 21$. In this case maximum size of the packing is $2t - 1$.

In the case of $n = 4t + 3$ the theoretical upper bound is $\lfloor \frac{(4t+2) \times (4t+3)}{2 \times 4t} \rfloor$ that is $(2t + 1) + \lfloor \frac{3}{2} + \frac{3}{4t} \rfloor$ that is 5 for $t = 1$ and $2t + 2$ for $t \geq 2$ i,e. $\lceil \frac{n}{2} \rceil$ for $t \geq 2$.

But for K_7 i,e. for $t = 1$ the size of packing is 4. Since in this case degree of each vertices is even, so after constructing packing for K_7, each vertex has degree either zero or even. Now K_7 has 21 edges and so if packing size of K_7 is 5 then one edge is remain which is not in the packing. This edge contributes one degree to two vertex of the K_7, which is not possible as each vertex is either of degree zero or even degree. Therefore for K_7 the size of packing is 4. One of such packing is as follows:

$$\{\{(1,2,3,4)\},\ \{(1,3,5,6)\},\ \{(1,5,2,7)\},\ \{(4,2,6,7)\}\}$$

For K_{11}, the theoretical upper bound is acheivable. One of the packing for K_{11} is

$$\{\{(7,1,2,3),\ (8,4,5,6)\},$$
$$\{(7,4,2,6),\ (9,1,3,5)\},$$
$$\{(9,2,5,7),\ (10,1,4,3)\},$$
$$\{(9,3,6,10),\ (11,1,5,8)\},$$
$$\{(4,7,8,9),\ (5,10,2,11)\},$$
$$\{(4,10,7,11),\ (6,1,8,2)\}\}$$

For $n = 15\ and\ 19$ this upper bound is not acheivable which has been seen manually. Instead, the packing size in this cases is $2t + 1$.

Thus on the basis of above information, the following can be conjectured.

Conjecture 1. Packing size for K_n, where $n = 4t + 1,\ t \geq 1$ is $2t - 1$ for $n \leq 21$.

Conjecture 2. Packing size of K_n, where $n = 4t + 3,\ t = 3\ and\ 4$ is $2t + 1$.

Note.

1. According to conjecture1, the packing for the case $n = 4t + 1$, can be obtained by the algorithm1 i,e. as in the case of $n = 4t$. In this case, there is a left vertex for any $2 \times 4-$uniform $1-$near factor of K_n. Let us define *standard* packing as: If left vertices of any two $2 \times 4-$uniform $1-$near factors are distinct in a packing of K_n, this packing is said to be *standard* packing for K_n. Now to obtain a packing for K_n, $n = 4t + 1$,

omit one vertex (*say*) α of K_n and then algorithm1 is applied for the resulting graph, which gives a packing for K_{4t}. Now to obtain standard packing for K_n, insert α to each of the $2 \times 4-$ uniform $0-$near factors of the packing for K_{4t} by deleting a distinct vertex each times. The following example is illustrate this.

Example 6 . *Consider K_9 with vertex set $\{1,2,3,4,5,6,7,8,9\}$. Let $\alpha = 9$. Then the packing for K_8 is*

$$\{\{(1,7,2,8),(3,5,4,6)\},$$
$$\{(3,7,4,8),\ (1,5,2,6)\},$$
$$\{(5,7,6,8),\ (1,3,2,4)\}\}.$$

Now a standard packing for K_9 is as follows:

$$\{\{(1,7,2,9),\ (3,5,4,6)\},$$
$$\{(3,7,4,8),\ (9,5,2,6)\},$$
$$\{(5,7,6,8),\ (1,3,9,4)\}\}.$$

2. According to conjecture2, the packing for the case $n = 15$ *and* 19 can be obtained as follows:

Example 7 . *To find out packing system for K_{15}, consider K_{14} with vertex set*

$$\{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14,\}.$$

*then find out the packing system for K_{14}, by applying algorithm*1*, which is as follows:*

$$\{\{(3,\ 13,\ 4,\ 14),\ (5,\ 11,\ 6,\ 12),\ (7,\ 9,\ 8,\ 10)\},$$
$$\{(5,\ 1,\ 6,\ 2),\ (7,\ 13,\ 8,\ 14),\ (9,\ 11,\ 10,\ 12)\},$$
$$\{(7,3,8,4),\ (9,1,10,2),\ (11,13,12,14)\},$$
$$\{(9,\ 5,\ 10,\ 6),\ (11,\ 3,\ 12,\ 4),\ (1,\ 13,\ 2,\ 14)\}$$
$$\{(11,\ 7,\ 12,\ 8),\ (13,\ 5,\ 14,\ 6),\ (1,\ 3,\ 2,\ 4)\}$$

$$\{(13,\ 9,\ 14,\ 10),\ (1,\ 7,\ 2,\ 8),\ (3,\ 5,\ 4,\ 6)\}$$

$$\{(1,\ 11,\ 2,\ 12),\ (3,\ 9,\ 4,\ 10),\ (5,\ 7,\ 6,\ 8)\}\}.$$

This is a packing system of K_{15}.

It is not possible to standardis the packing system of K_{15} and K_{19} in the sense as defined in Note1. It is clear that, in a $2 \times 4-$uniform $3-$near factor of K_{4t+3}, three vertices are left vertices. If packing size is of $2t+1$, then exactly $3 \times (2t+1)$ instances are left. Hence $3 \times (2t+1) \ - \ (4t+3) \ = \ 2t$ instances are occuring twice. Thus define standard system in this case as: A packing system of K_{4t+3} for $t \geq 3$ is standard packing system if exactly $2t$ vertices become left twice and remaining vertices are left once. In this sence of standard packing system, one of the standard packing system for K_{15} is as follows:

$$\{\{(15,\ 13,\ 4,\ 14),\ (5,\ 11,\ 6,\ 12),\ (7,\ 9,\ 8,\ 10)\},$$

$$\{(15,\ 1,\ 6,\ 2),\ (7,\ 13,\ 8,\ 14),\ (9,\ 11,\ 10,\ 12)\},$$

$$\{(15,\ 3,\ 8,\ 4),\ (9,\ 1,\ 10,\ 2),\ (11,\ 13,\ 12,\ 14)\},$$

$$\{(15,\ 5,\ 10,\ 6),\ (11,\ 3,\ 12,\ 4),\ (1,\ 13,\ 2,\ 14)\},$$

$$\{(15,\ 7,\ 12,\ 8),\ (13,\ 5,\ 14,\ 6),\ (1,\ 3,\ 2,\ 4)\},$$

$$\{(15,\ 9,\ 14,\ 10),\ (1,\ 7,\ 2,\ 8),\ (3,\ 5,\ 4,\ 6)\},$$

$$\{(1,\ 11,\ 2,\ 12),\ (3,\ 9,\ 4,\ 10),\ (5,\ 7,\ 6,\ 8)\}\},$$

which can be obtained from the packing system for K_{14}, by inserting vertex 15 of K_{15} to each of the 2×4-uniform 3-near factor, except last one in the packing system of K_{14} by deleting a distinct vertex each time from it.

7 Some empirical results of packing system for K_n with cycle of length ≥ 5.

7.1 Packing size of K_n,with cycle length 5 for $n = 5,\ 6,\ 7,\ 8$ *and* 9.

The packing system for K_{5t+k}, $k = 0,\ 1,\ 2,\ 3$ *and* 4 is the collection of all $2 \times 5-$uniform k-near factors of K_{5t+k}. Regarding the size of packing $(\mathcal{P}_{5,1}, n)$, *for* $n \ = \ 5,\ 6,\ 7,\ 8$ *and* 9 is as follows:

1. Packing size of K_5.

   The number of edge in K_5 is 10 and so the size of packing is 2 and one of the packing is

$$\{\{(1,\ 2,\ 3,\ 4,\ 5)\},\{(1,\ 3,\ 5,\ 2,\ 4)\}\}.$$

2. Packing size of K_6.

   The number of edges in K_6 is 15 and hence theoretically the packing size is 3. But degree of each vertex is 5 and hence each vertex can occuring at most two edge disjoint cycles of length 5. Now after constructing two edge disjoint 5-cycle at least 4 vertices become of degree one, hence three edge disjoint 5-cycle is not possible. Hence the theoretical bound of packing size is not acheivable. Instead the size of packing is 2 and one of such packing is

$$\{\{(1,\ 2,\ 3,\ 4,\ 5)\},\ \{(1,\ 3,\ 5,\ 6,\ 4)\}\}.$$

3. Packing size of K_7.

   The number of edges in K_7 is 21 and so the theoretical packing size is 4. But since each vertex contribute degree two for each occurrence of it in any cycle and initially each vertex has even degree so after constructing a packing for K_7, each vertex should have either degree zero or even. If packing size is 4, then one edge remains which is not in the packing and it contribute degree one to two vertices, which is impossible. Hence the size of the packing is 3 and one of such packing is

$$\{\{(1,\ 2,\ 3,\ 4,\ 5)\},$$
$$\{(1,\ 3,\ 5,\ 6,\ 7)\},$$
$$\{(2,\ 6,\ 4,\ 7,\ 5)\}\}.$$

4. Packing size of K_8.

   The number of edge in K_8 is 28 and each vertex is of degree odd and theoretical size of packing is 5. But at least one degree of each vertex remains after constructing a packing, i.e, at least 4 edges remain which are not in the packing. Hence packing size for K_8 is 4 and one of such packing is

$$\{\{1,\ 2,\ 3,\ 4,\ 5)\},$$
$$\{(1,\ 3,\ 5,\ 6,\ 7)\},$$
$$\{(2,\ 6,\ 4,\ 7,\ 5)\},$$
$$\{(4,\ 2,\ 7,\ 3,\ 8)\}\}.$$

5. Packing size of K_9.

The number of edge in K_9 is 36 and each vertex is of degree 8, the theoretcal size of packing is 7. But after constructing a packing of this size, only one edge remains and this contribute one degree to two vertices and all vertices are of degree zero except those two. This is impossible and hence practically the size of packing is 6 and one of such packing is

$$\{\{(1,\ 2,\ 3,\ 4,\ 5)\},$$
$$\{(1,\ 3,\ 9,\ 6,\ 7)\},$$
$$\{(2,\ 6,\ 4,\ 7,\ 8)\},$$
$$\{(\ 4,\ 9,\ 7,\ 3,\ 8)\},$$
$$\{(\ 2,\ 5,\ 6,\ 8,\ 9)\},$$
$$\{(3,\ 6,\ 1,\ 8,\ 5)\}\}.$$

7.2 Packing size for K_n with cycle of length 6 for $n = 6,\ 7,\ 8,\ 9,\ 10,\ 11\ and\ 12$.

Some emperical result has been prodused for this case for $n =\ 6,\ 7,\ 8,\ 9,\ 10,\ 11\ and\ 12$.

Note In the following, $\{1, 2, ..., n\}$ denote the vertex set of K_n and $(a,\ b)$ denote the edge with $a\ and\ b$ as terminal vertex.

1. Packing for K_6.

For K_6, the number of edges is 15 and so the theoretical size of packing is 2 and one of such packing is

$$\{\{(1,\ 2,\ 3,\ 4,\ 5,\ 6)\},$$
$$\{(1,\ 3,\ 5,\ 2,\ 6,\ 4)\}\}.$$

It is noted that the remaining edges form a perfect matching viz

$$\{(1\ 5),\ (2,\ 4),\ (3,\ 6)\ \}\ of\ K_6.$$

2. Packing for K_7.

For K_7, the number of edges is 21 and so the theoretical size of packing is 3 and one of such packing is:

$$\{\{(1,\ 2,\ 3,\ 4,\ 5,\ 6)\},$$
$$\{(1,\ 3,\ 7,\ 2,\ 6,\ 4)\},$$
$$\{(7,\ 4,\ 2,\ 5,\ 3,\ 6)\}\}.$$

3. Packing for K_8.

For K_8, the number of edges is 28 and so the theoretical size of packing is 4, and one such packing is

$$\{\{(1,\ 2,\ 3,\ 4,\ 5,\ 6)\},$$
$$\{(1,\ 3,\ 7,\ 2,\ 8,\ 4)\},$$
$$\{(7,\ 8,\ 1,\ 5,\ 3,\ 6)\},$$
$$\{(8,\ 5,\ 7,\ 4,\ 2,\ 6)\}\}.$$

It is noted that the remaining edges form a perfect matching viz

$$\{(1,\ 7),\ (2,\ 5),\ (4,\ 6),\ (3,\ 8)\}\ of\ K_8.$$

4. Packing for K_9.

For K_9, the number of edges is 36 and so the theoretical size of packing is 6. But manualy it has been seen that the packing size is 5 and one of such packing is

$$\{\{(1,\ 2,\ 3,\ 4,\ 5,\ 6)\},$$
$$\{(1,\ 3,\ 5,\ 7,\ 8,\ 9)\},$$
$$\{(2,\ 7,\ 9,\ 4,\ 8,\ 6)\},$$
$$\{(1,\ 4,\ 6,\ 7,\ 3,\ 8)\},$$
$$\{(2,\ 9,\ 5,\ 1,\ 7,\ 4)\}\}.$$

5. Packing for K_{10}.

For K_{10}, the number of edges is 45 and the theoretical size of packing is 7. But as degree of each vertices is odd and if the size of packing is 7, then only three edges remains which are not in the packing - it is impossible. Hence the size of packing is 6 and one of such packing is

$$\{\{(1,\ 2,\ 3,\ 4,\ 5,\ 6)\},$$
$$\{(1,\ 10,\ 5,\ 7,\ 8,\ 9)\},$$
$$\{(2,\ 7,\ 9,\ 4,\ 10,\ 6)\},$$
$$\{(1,\ 4,\ 6,\ 7,\ 3,\ 8)\}$$
$$\{(2,\ 9,\ 5,\ 1,\ 7,\ 4)\},$$
$$\{(3,\ 10,\ 9,\ 6,\ 8,\ 5)\}\}.$$

6. Packing for K_{11}.

For K_{11}, the number og edges is 55 and so the theoretical size of packing is 9. Since each vertex is of even degree and if size of packing is 9, then one edge remains which is not in the packing and so two vertices has degree one and remaining vertices are of degree zero - which is impossible. Hence the packing size is 8 and one of such packing is

$$\{\{(4,1,2,3,5,6)\},$$
$$\{(6,\ 1,\ 3,\ 4,\ 2,\ 7)\},$$
$$\{(6,\ 2,\ 5,\ 1,\ 7,\ 3)\},$$
$$\{(6,\ 8,\ 1,\ 9,\ 2,\ 10)\},$$
$$\{(6,\ 9,\ 3,\ 8,\ 2,\ 11)\},$$
$$\{(7,\ 4,\ 5,\ 8,\ 9,\ 10)\},$$
$$\{(8,\ 4,\ 9,\ 5,\ 7,\ 11)\},$$
$$\{(8,\ 7,\ 9,\ 11,\ 1,\ 10)\}\}.$$

7. Packing for K_{12}.

For K_{12}, the number of edges is 66 and so the theoretical size of packing is 5 and one of such packing is

$$\{\{(1,\ 10,\ 2,\ 11,\ 3,\ 12),\ (4,\ 7,\ 5,\ 8,\ 6,\ 9)\},$$
$$\{(4,\ 10,\ 5,\ 11,\ 6,\ 12),\ (1,\ 7,\ 2,\ 8,\ 3,\ 9)\},$$
$$\{(7,\ 10,\ 8,\ 11,\ 9,\ 12),\ (1,\ 4,\ 2,\ 5,\ 3,\ 6)\},$$
$$\{(1,\ 11,\ 10,\ 3,\ 4,\ 8),\ (2,\ 9,\ 7,\ 6,\ 5,\ 12)\},$$
$$\{(1,\ 5,\ 4,\ 6,\ 2,\ 3),\ (12,\ 11,\ 7,\ 8,\ 9,\ 10)\}\}.$$

It is noted that the remaining edges are form a perfect matching viz,

$$\{(1,\ 2),\ (3,\ 7),\ (4,\ 11),\ (5,\ 9),\ (6,\ 10),\ (8,\ 12)\}\ of\ K_{12}.$$

8 References

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