M.Tech. (Computer Science) Dissertation Series

Reasoning About Knowledge and Probability

a dissertation submitted in partial fulfilment of the requirements for the M. Tech. (Computer Science) degree of the Indian Statistical Institute

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Certificate of Approval

This is to certify that the thesis entitled *Reasoning About Knowledge and Probability* submitted by *Rakesh Kumar Bhatia*, towards partial fulfillment of the requirement for ***M. Tech.*** in *Computer Science* degree of the *Indian Statistical Institute, Calcutta*, is an acceptable work for the award of the degree.

July 19, 2000 **(Supervisor)**

(External Examiner)

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Rakesh Kumar Bhatia

We have reviewed works done on Reasoning About Knowledge and Probability introduced by Ronald Fagin, Joseph Y. Halpern (IBM Almaden Research Centre, San Jose, California)

We allow explicit mentions of probabi-lities in formulas, so that our language has formulas that essentially says "according to agent i, formula ∅ holds with probability at least b". The language is powerful enough to allow reasoning about higher order probabilities as well as the explicit comparisons of the probabilities an agent places on distinct events. and on the basis of the results we have tried to give probability assign-ments to Linear Time Temporal Logic formulas.

Reasoning About Knowledge and Probability

Introduction

In many of the application areas for reasoning about knowledge, it is impor-tant to be able to reason about the probability of certain events as well as the knowledge of agents.

In the standard possible worlds model of knowledge, agent i knows a fact ϕ, written $K_i\phi$ in a world or state s, if ϕ is true in all the worlds the agent considers possible in world s.

We want to reason not only about agents knowledge, but also about the probability he places on certain events. In order to do this we extend the language. Typical formulas in the logic of Fagin et al. includes $w(\phi) \geq 2w(\psi)$ and $w(\phi) < \frac{1}{3}$, where ϕ and ψ are propositional formulas. These formulas can be viewed as saying,

" ϕ is twice as probable as ψ " and " ϕ has probability less than $\frac{1}{3}$ ".

In order to give semantics to such a language in the possible world framework, we assume that roughly speaking, at each state each agent has a probability on the worlds he considers possible. Then a formula such as $w_i(\phi) \geq 2w_i(\psi)$ is true at each state s if, according to agent i's probability assignment at state s, the event ϕ is twice as probable as ψ.

The Standard Kripke structure for Knowledge and Probability

Syntax

P : non-empty set of propositional letters.

K_i : $1 \leq i \leq n$ Knowledge modalities

W_i : probability modalities (behaviour distinct from K_i) $1 \leq i \leq n$.

Then ϕ, the set of formulas is defined inductively as follows.

- Every member of P belongs to ϕ.
- If ϕ_1 and ϕ_2 are formulas in ϕ, then so are $\sim\phi_1$, $\phi_1 \vee \phi_2$, $K_i(\phi_1)$
- If $\phi_1, \phi_2, \dots, \phi_k$ are formulas in ϕ. then so is

$$a_1 W_i(\phi_1) + \dots + a_k W_i(\phi_k) \geq b.$$

Semantics

$v: P \longrightarrow \{T, F\}$ is a valuation

V = set of all valuations

$(S, \Pi, K_1, K_2, ---, K_n)$ is a Kripke structure.

where $S = \{s_1, s_2, ---, s_m\}$ are the set of worlds.

$\Pi : S \longrightarrow V$, $\Pi(s)(p) = T$ or F

and K_i is an equivalence relation on S.

for $s_i \in S$, $K_i(s_j)$ = equivalence class of s_j under K_i.

Probability Space

Probability space is a tuple $(\Omega, \mathcal{K}, \mu)$ where Ω is a set called the sample space, $\mathcal{K}$ is a σ-algebra of subsets of Ω (i.e. a set of subsets containing Ω and closed under complementation and countable union) whose elements are called the measurable sets and a probability measure μ defined on the elements of Ω.

μ does not assign probability to all subsets of Ω but only to the measurable sets. One natural way of attaching weight to every subsets of Ω is considering the inner measure μ_*, if $A \subseteq \Omega$, then

$$\mu_*(A) = \sup\{\mu(B) / B \subseteq A, B \in \mathcal{K}\}.$$

Kripke Structure for Knowledge and Probability

Model ; $M = (S, \pi, \mathcal{K}_1, \mathcal{K}_2, ---, \mathcal{K}_n, \mathcal{P})$
where

$\mathcal{P} : \{1, ---, n\} \times S \rightarrow$ probability space

$\mathcal{P}_{i,s} = (S_{i,s}, \mathcal{X}_{i,s}, \mu_{i,s})$

n: number of agents.

Interpretation

① $(M, s) \models p$ iff $\pi(s)(p) = T$

② $(M, s) \models \sim\phi$ iff $(M, s) \not\models \phi$

③ $(M, s) \models \phi_1 \vee \phi_2$ iff

$(M, s) \models \phi_1$ or $(M, s) \models \phi_2$

④ $(M, s) \models K_i \phi$ iff

$(M, t) \models \phi \quad \forall \; t \in \mathcal{K}_i(s)$.

⑤ $(M, s) \models w_i(\phi) \geq b$ iff

$c \geq b$ where

$c = \mu_{i,s}(\{t \in S_{i,s} \mid (M, t) \models \phi\})$

(6) $(M, s) \models a_1 w_i(\phi_1) + a_2 w_i(\phi_2) + \cdots + a_k w_i(\phi_k) \geq b$

iff $c \geq b$ where

$$c = a_1 (\mu_{i,s})_* (\{t \in S_{i,s} \mid (M, t) \models \phi_1\}) + \cdots + a_k (\mu_{i,s})_* (\{t \in S_{i,s} \mid (M, t) \models \phi_k\}).$$

Example.

Suppose there are two agents. Agent 2 has an input bit either 0 or 1. He then tosses a fair coin, and perform an action a if the coin toss agrees with the input bit. i.e. if the coin toss lands heads and the input bit is 1 or if the coin lands tails and the input bit is 0. It is assumed that agent 1 never learns agent 2's input bit or the outcome of his coin toss.

An easy argument shows that according to Agent 2 who knows the input bit, the probability (before he tosses the coin) of performing action a is $\frac{1}{2}$. There is also a reasonable argument to show that, even according to agent 1 (who does not know the input bit), the probability that action will be performed is $\frac{1}{2}$.

Now we try to capture this argument in our formal system.

$b \in \{0, 1\}$

coin toss $(t) \in \{H, T\}$

Action a will be performed if

$b = 1,\ t = H$ or

$b = 0,\ t = T$.

$S = \{s_1, s_2, s_3, s_4\}$

where $s_1 = (1, H)$

$s_2 = (1, T)$

$s_3 = (0, H)$

$s_4 = (0, T)$

A: Action a is performed by agent 2.

H: coin landed head

T: coin landed tail

B_0: bit is 0

B_1: bit is 1

$M = (S, \pi, \mathcal{K}_1, \mathcal{K}_2, \mathcal{P})$

where $\mathcal{K}_1 = \{s_1, s_2, s_3, s_4\}$

$\mathcal{K}_2 = \{\{s_1\}, \{s_2\}, \{s_3\}, \{s_4\}\}$.

	A	H	T	B_0	B_1
$\pi(s_1)$	t	t	f	f	t
$\pi(s_2)$	f	f	t	f	t
$\pi(s_3)$	f	t	f	t	f
$\pi(s_4)$	t	f	t	t	f

What should agent 1's probability assignment be?
There are three possible answers to this question.

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① We can associate with each state the sample space consisting of all four states.

i.e. $S_{1,s_i} = \{s_1, s_2, s_3, s_4\}$
for $1 \leq i \leq 4$.

Because there is no probability "the input bit is 0" ("respectively input bit is 1"), the candidates for measurable sets are $\{s_1, s_3\}$ (which corresponds to the event "the coin landed head") and $\{s_2, s_4\}$ ("the coin landed tail")
i.e if we take the Kripke structure M_0, then we have,

$$P_{1,s_1} = P_{1,s_2} = P_{1,s_3} = P_{1,s_4}$$

and $\mathcal{X}_{1,s_i} = \{ \{s_1, s_3\}, \{s_2, s_4\}, \phi, \{s_1, s_2, s_3, s_4\} \}$.

where $1 \leq i \leq 4$

$$\left.\begin{aligned} \mu_{1,s_i}(\{s_1, s_3\}) &= \tfrac{1}{2} \\ \mu_{1,s_i}(\{s_2, s_4\}) &= \tfrac{1}{2} \end{aligned}\right\} \forall i$$

Here the events $\{s_1\}$ and $\{s_2\}$ cannot both be measurable, for then the event $\{s_1, s_2\}$ which corresponds to "input bit is 1" would also be measurable.

② $P_{1,s_1} = P_{1,s_2}$ (input bit is 1)
where

$$S_{1,s_i} = \{s_1, s_2\}$$

$$\mathcal{N}_{1,s_i} = \{\{s_1\}, \{s_2\}, \phi, \{s_1, s_2\}\}$$

$$\mu_{1,s_i}(\{s_1\}) = \mu_{1,s_i}(\{s_2\}) = \frac{1}{2}$$

$$1 \leq i \leq 2$$

$P_{1,s_3} = P_{1,s_4}$ (input bit is 0)
where

$$S_{1,s_i} = \{s_3, s_4\}$$

$$\mathcal{N}_{1,s_i} = \{\{s_3\}, \{s_4\}, \phi, \{s_3, s_4\}\}$$

$$\mu_{1,s_i}(\{s_3\}) = \mu_{1,s_i}(\{s_4\}) = \frac{1}{2}$$

$$3 \leq i \leq 4$$

We call the above structure M_1.

3. $S_{1,s_1} = \{s_1\}$

$S_{1,s_2} = \{s_2\}$

$S_{1,s_3} = \{s_3\}$

$S_{1,s_4} = \{s_4\}$

$$\mu_{1,s_1}(\{s_1\}) = \mu_{1,s_2}(\{s_2\}) = \mu_{1,s_3}(\{s_3\}) = \mu_{1,s_4}(\{s_4\}) = 1$$

The above structure is M_2.

Now it is easy to check that for each s in S.

$$(M_1, s) \models K_1(w_1(A) \geq \tfrac{1}{2})$$

First we show that

$$(M_0, s) \not\models K_1(w_1(A) \geq \tfrac{1}{2})$$

$$(M_0, s) \models K_1(w_1(A) \geq \tfrac{1}{2})$$

iff

$$(M_0, t) \models w_1(A) \geq \tfrac{1}{2} \quad \forall t \in \mathcal{K}_1(s)$$

Now

$$(M_0, s_1) \models w_1(A) \geq \tfrac{1}{2}$$

iff

$$\mu_{1,s_1*}(\{t \in S_{1,s_1} \mid (M_0, t) \models A\}) \geq \tfrac{1}{2}.$$

i.e. $\mu_{1,s_1*}(\{s_1, s_4\}) \geq \frac{1}{2}$.

since A is true in s_1 and s_4 only.

$$\mu_{1,s_1*}(\{s_1, s_4\}) = \mu_{1,s_1}(\emptyset) = 0.$$

since the only measurable set contained in $\{s_1, s_4\}$ is $\emptyset$.

$$\therefore (M_0, s_1) \not\models K_1(w_1(A) \geq \tfrac{1}{2})$$

Now since,

$$P_{1,s_1} = P_{1,s_2} = P_{1,s_3} = P_{1,s_4}.$$

There does not exist any state s, such that

$$(M_0, s) \models K_1(w_1(A) \geq \tfrac{1}{2})$$

Now we show that,

$$(M_1, s) \models K_1(w_1(A) \geq \tfrac{1}{2}) \quad \text{——} \ (1)$$

(1) holds iff

$$(M_1, t) \models w_1(A) \geq \tfrac{1}{2} \quad \forall\, t \text{ in } \mathcal{K}_1(s)$$

$$(M_1, s_1) \models w_1(A) \geq \tfrac{1}{2}.$$

iff

$$\mu_{1,s_1*}(\{y \in S_{1,s_1} \mid (M_1, y) \models A\}) \geq \tfrac{1}{2}.$$

i.e

$$\mu_{1,s_1*}(\{s_1, s_4\}) \geq \tfrac{1}{2}.$$

Now $\mu_{1,s_1*}(\{s_1, s_4\}) = \mu_{1,s_1}(\{s_1\})$

$$= \tfrac{1}{2}$$

(since s_1 is measurable)

$$\therefore (M_1, s_1) \models w_1(A) \geq \tfrac{1}{2}.$$

Since $P_{1,s_1} = P_{1,s_2}$

$(M_1, s_2) \models w_1(A) \geq \frac{1}{2}$.

Similarly we can show that

$(M_1, s_3) \models w_1(A) \geq \frac{1}{2}$.

$(M_1, s_4) \models w_1(A) \geq \frac{1}{2}$.

$\therefore\ (M_1, s_i) \models K_1(\,w_1(A) \geq \frac{1}{2})$

where $1 \leq i \leq 4$.

because

$(M_1, s_i) \models K_1(\,w_1(A) \geq \frac{1}{2})$

iff

$(M_1, t) \models w_1(A) \geq \frac{1}{2} \quad \forall\ t \text{ in } \mathcal{K}_1(s_i)$

Now $\mathcal{K}_1(s_1) = \mathcal{K}_1(s_2) = \mathcal{K}_1(s_3) = \mathcal{K}_1(s_4)$

$= \{s_1, s_2, s_3, s_4\}$.

The above argument holds in any state.

Now we show that

$$(M_2, s) \not\models K_1(w_1(A) \geq \tfrac{1}{2})$$

$$(M_2, s_1) \models K_1(w_1(A) \geq \tfrac{1}{2})$$

iff

$$(M_2, t) \models w_1(A) \geq \tfrac{1}{2} \quad \forall t \in \mathcal{K}_1(s_1).$$

Now $\mathcal{K}_1(s_1) = \{s_1, s_2, s_3, s_4\}$

$$(M_2, s_1) \models w_1(A) \geq \tfrac{1}{2}$$

iff $\mu_{1,s_1}{}^*(\{t \in S_{1,s_1} \mid (M_2, t) \models A\}) \geq \frac{1}{2}$.

i.e $\mu_{1,s_1}{}^*(\{s_1\}) \geq \frac{1}{2}$.

But $\mu_{1,s_1}(\{s_1\}) = 1$

$\therefore$ $(M_2, s_1) \models w_1(A) \geq \frac{1}{2}$. holds.

Now

$$(M_2, s_2) \models w_1(A) \geq \tfrac{1}{2}.$$

iff

$$\mu_{1,s_2}{}^*(\{t \in S_{1,s_2} \mid (M_2, t) \models A\}) \geq \tfrac{1}{2}.$$

i.e $\mu_{1,s_2}{}^*(\{t \in s_2 \mid (M_2, t) \models A\}) \geq \frac{1}{2}$.

i.e $\mu_{1,s_2}{}^*(\phi) \geq \frac{1}{2}$. [since A is true in s_2 and s_4 only].

But

$\mu_{1,s_2}{}^*(\phi) = 0$.

$\therefore$ $(M_2, s_2) \not\models w_1(A) \geq \frac{1}{2}$.

Hence

$$(M_2, s_1) \nvDash K_1(w_1(A) \geq \tfrac{1}{2}).$$

Since $\mathcal{K}_1(s_1) = \mathcal{K}_1(s_2) = \mathcal{K}_1(s_3) = \mathcal{K}_1(s_4)$
$= \{s_1, s_2, s_3, s_4\}$.

There does not exist any state s in S such that

$$(M_2, s) \vDash K_1(w_1(A) \geq \tfrac{1}{2})$$

Further, we can show.
for every state $s \in S$, we have

$$(M_2, s) \vDash K_1((w_1(A) = 1) \vee (w_1(A) = 0))$$

The above statement holds iff

$$(M_2, t) \vDash (w_1(A) = 1 \vee w_1(A) = 0) \quad \forall t \in \mathcal{K}_1(s)$$

$$(M_2, s_1) \vDash (w_1(A) = 1 \vee w_1(A) = 0)$$

iff

$$(M_2, s_1) \vDash w_1(A) = 1$$

$$\vee \quad (M_2, s_1) \vDash w_1(A) = 0.$$

Now

$$(M_2, s_1) \vDash w_1(A) = 1.$$

iff

$$\mu_{1, s_1 *}(\{t \in S_{1, s_1} \mid (M_2, t) \vDash A\}) = 1.$$

i.e.

$$\mu_{1,s_1*}(\{s_1\}) = 1.$$

But

$$\mu_{1,s_1*}(\{s_1\}) = 1$$

[since $\{s_1\}$ is the only measurable set in S_{1,s_1}].

Now since A is false in state s_2 and s_3 we have,

$(M_2, s_2) \models w_1(A)=0$

and $(M_2, s_3) \models w_1(A)=0$.

Again since A is true in s_4.

$(M_2, s_4) \models w_1(A)=1$.

Hence

$(M_2, s_1) \models K_1((w_1(A)=1) \vee (w_1(A)=0))$.

Now since

$\mathcal{K}_1(s_1) = \mathcal{K}_1(s_2) = \mathcal{K}_1(s_3) = \mathcal{K}_1(s_4)$.

we have

$(M_2, s) \models K_1((w_1(A)=1) \vee (w_1(A)=0))$

for each s in S.

Finally in M_0, the event A is not measurable, nor does it contain any non-empty measurable sets.
Thus we have,

$$(M_0, s) \models K_1(w_1(A)=0)$$

[The proof is similar to the above proofs].

FEW PROPERTIES

① We term the following condition consistent (CONS)

For all i and s if

$\mathcal{P}_{i,s} = (S_{i,s}, \mathcal{X}_{i,s}, \mu_{i,s})$

then $S_{i,s} \subseteq \mathcal{K}_i(s)$.

Consistency does not imply that $s \in S_{i,s}$; an agent is not required to view the state that he is in as one of the set of states in his probability space.

② In the content of Kripke structure for knowledge and probability, having an objective probability assignment corresponds to the following condition

OBJ: $\mathcal{P}_{i,s} = \mathcal{P}_{j,s} \quad \forall\ i, j$ and s.

The above property can also be termed as.

AGENT INDEPENDENT PROBABILITY DISTRIBUTION.

③ The following property is called SDP (state determined property)

For all i and s, and t if $t \in \mathcal{K}_i(s)$ then $\mathcal{P}_{i,s} = \mathcal{P}_{i,t}$.

④ Formally we say uniformity holds if

For all i, s and t if

$\mathcal{P}_{i,s} = (S_{i,s}, \mathcal{X}_{i,s}, \mu_{i,s})$

and $t \in S_{i,s}$, then

$\mathcal{P}_{i,t} = \mathcal{P}_{i,s}$

⑤ We say that formulas define measurable sets in M if

MEAS: For all i and s and for every formula ϕ, the set

$$S_{i,s}(\phi) \in \mathcal{X}_{i,s}$$

where $S_{i,s}(\phi) = \{ t \in S_{i,s} \mid (M,t) \models \phi \}$

Let PMEAS be the property which says that all primitive propositions define measurable sets.

<u>Lemma</u> If M is a structure satisfying CONS, OBJ, UNIF and PMEAS, then M satisfies MEAS.

PROOF: The proof is by induction on the structure of formulae.

<u>Base</u>: Since PMEAS holds, all primitive propositions define measurable sets.

Suppose $S_{i,s}(\phi_1) \in \mathcal{X}_{i,s}$ and

$$S_{i,s}(\phi_2) \in \mathcal{X}_{i,s}$$

then $S_{i,s}(\phi_1 \vee \phi_2) \in \mathcal{X}_{i,s}$

and $S_{i,s}(\sim\phi_1) \in \mathcal{X}_{i,s}$.

[since the set of measurable sets are closed countable union and complementation]

Now we prove that

$S_{i,s}(w_i(\phi) \geq b) \in \mathcal{X}_{i,s}$

$S_{i,s}(w_i(\phi) \geq b)$

$= (\{t \in S_{i,s} / (M,t) \models w_i(\phi) \geq b\})$

suppose there is a state t in $S_{i,s}$ s.t.

$(M,t) \models w_i(\phi) \geq b$.

i.e.

$\mu_{i,t^*}(\{y \in S_{i,t} / (M,y) \models \phi\}) \geq b$.

Since $t \in S_{i,s}$. $P_{i,t} = P_{i,s}$ by UNIF.

Hence

$\mu_{i,t^*}(\{y \in S_{i,t} / (M,y) \models \phi\}) \geq b$.

can be written as

$\mu_{i,s^*}(\{y \in S_{i,s} / (M,y) \models \phi\}) \geq b$.

and so is either identically true or false for all $t \in S_{i,s}$

i.e. either $S_{i,s}(w_i(\phi) \geq b)$ is ϕ or $S_{i,s}$ and hence in both the case $S_{i,s}(w_i(\phi) \geq b)$ is measureable.

Now we have to prove that
$S_{i,s}(K_j \phi) \in \mathcal{H}_{i,s}$.

$S_{i,s}(K_j \phi) = (\{ y \in S_{i,s} / (M, y) \models K_j \phi \})$

Let $y_1 \in S_{i,s}$ be such that
$(M, y_1) \models K_j \phi$

i.e $(M, t) \models \phi \quad \forall\, t \in \mathcal{K}_j(y_1)$

Now since <u>objectivity</u> and <u>uniformity</u> holds all the agents will have the same probability distribution in a particular state, and in all the states of $S_{i,s}$, agent i will have the same probability distribution.

By combining these two facts we can say that in all the states of $S_{i,s}$ agent i and agent j will have the same probability distribution.

Now since

$(M, y_1) \models K_j \phi$

i.e $(M, t) \models \phi \ \forall \ t \in \mathcal{K}_j(y_1)$.

this does not imply.

$(M, y_2) \models K_j \phi$

i.e $(M, t) \models \phi \ \forall \ t \in \mathcal{K}_j(y_2)$

where $y_2 \in S_{i,s}$.

i.e if $S_{j,y_1} = S_{j,y_2}$ [since $P_{j,y_1} = P_{j,y_2}$].

this does not imply

if $(M, y_1) \models K_j \phi$

Then also $(M, y_2) \models K_j \phi$.

~~Hence I, think the above lemma is not correct~~

[The above lemma was given by Ronald Fagin, Joseph Y Halpern in their paper named Reasoning about Knowledge and Probability]

Suppose
$y_1 \in S_{i,s}$ be such that
$(M, y_1) \models K_j \phi$
i.e $(M, t) \models \phi \ \forall \ \phi \in \mathcal{K}_j(y_1)$

Then we have to prove that if
$y_2 \in S_{i,s}$
Then $(M, y_2) \models K_j \phi$
i.e $(M, t) \models \phi \ \forall \ \phi \in \mathcal{K}_j(y_2)$

Now since uniformity holds
$P_{i,s} = P_{i,y_1} = P_{i,y_2}$
i.e $S_{i,y_1} = S_{i,y_2}$
Since objectivity holds.
$P_{i,y_1} = P_{j,y_1}$ and $P_{i,y_2} = P_{j,y_2}$
i.e $S_{i,y_1} = S_{j,y_1}$ and $S_{i,y_2} = S_{j,y_2}$
i.e $S_{j,y_1} = S_{j,y_2}$
Now if $S_{j,y_1} = \mathcal{K}_j(y_1)$ and
$S_{j,y_2} = \mathcal{K}_j(y_2)$
then $\mathcal{K}_j(y_1) = \mathcal{K}_j(y_2)$

Hence
The above lemma holds if
$S_{i,s} = \mathcal{K}_i(s)$ for each agent i
and each state s.

Millers Principle

$w_i(\phi) \geq b \; w_i(w_i(\phi) \geq b)$.

uniformity implies above axiom.

proof. $w_i(\phi) \geq b$ is either true in state s or false in state s.

Suppose it is false in state s.

i.e

$$(M, s) \nvDash w_i(\phi) \geq b, \quad \text{— (I)}$$

Then we have

$$(M, s) \models w_i(w_i(\phi) \geq b) = 0 \quad \text{— (II)}$$

Let us assume that the statement (II) is false,

i.e There exist a state $y \in S_{i,s}$

s.t

$$(M, y) \models w_i(\phi) \geq b$$

i.e.

$$\mu_{i,y*}(\{t \in S_{i,y} \mid (M, t) \models \phi\}) \geq b.$$

Now since uniformity holds, agent i will have the same probability distribution in all the states of $S_{i,s}$.

$\therefore$

$\mu_{i,s^*}(\{t \in S_{i,s} \mid (M,t) \models \phi\}) \geq b$

i.e.

$(M,s) \models w_i(\phi) \geq b$

Which is a contradiction to the statement (1).

$\therefore$ (11) holds.

and hence

$(M,s) \models w_i(\phi) \geq 0$.

Now suppose,

$(M,s) \models w_i(\phi) \geq b$.

Then we have

$(M,s) \models w_i(w_i(\phi) \geq b) = 1$.

because

$(M,s) \models w_i(\phi) \geq b$.

iff

$\mu_{i,s^*}(\{t \in S_{i,s} / (M,t) \models \phi\}) \geq b$.

Now since <u>uniformity</u> holds

$\mu_{i,y^*}(\{t \in S_{i,y} \mid (M,t) \models \phi\}) \geq b$

$\forall y \in S_{i,s}$.

$\therefore$ $(M,s) \models w_i(w_i(\phi) \geq b) = 1$.

i.e. $(M,s) \models w_i(\phi) \geq b$. which holds.

Hence the proof.

We can also capture some of the assumptions we made about the system axiomatically.

① CONS corresponds to the axiom

$K_i\phi \Rightarrow (w_i(\phi)=1)$.

First we proof that if

$(M,s) \models K_i\phi$.

then $(M,s) \models w_i(\phi)=1$

$(M,s) \models w_i(\phi)=1$

iff

$\mu_{i,s*}(\{t \in S_{i,s} / (M,t) \models \phi\}) = 1$

Now since $S_{i,s} \subseteq \mathcal{K}_i(s)$ and

$(M,s) \models K_i\phi$

ϕ must be true in all the states of $S_{i,s}$.

i.e $\mu_{i,s*}(\{t \in S_{i,s} / (M,t) \models \phi\})$

$= \mu_{i,s}(\{S_{i,s}\}) = 1$.

$\therefore$ $(M,s) \models w_i(\phi)=1$.

Now we prove if
$(M, s) \models K_i \phi \Rightarrow (w_i(\phi) = 1)$ for any
$s \in S$, then
CONS. must hold in M.

Now here if ϕ is true in all the states of $\mathcal{K}_i(s)$, then ϕ is true in all the states of $S_{i,s}$.
i.e there does not exist any state $y \in S$ such that ϕ_y is true in all the states of $\mathcal{K}_i(y)$, but ϕ is not true at least in one state of $S_{i,y}$.
Hence $S_{i,s} \subseteq \mathcal{K}_i(s)$ $\forall$ s in S.
i.e CONS holds in M.

(2) OBJ corresponds to the axiom:

$$(a_1 w_i(\phi_1) + a_2 w_i(\phi_2) + \dots + a_k w_i(\phi_k) \geq b)$$
$$\Rightarrow (a_1 w_j(\phi_1) + \dots + a_k w_j(\phi_k) \geq b).$$

proof.

The proof goes very much like this, if objectivity holds, then all the agents will have the same probabi-lity distribution in a state $s \in S$. and hence above formula must holds for each $s \in S$.

If the above formula ~~must~~ holds for all agents i and j, then objectivity must hold.

[detail of the proof can be worked out easily].

③ uniformity corrospends to the axiom

$\phi \Rightarrow (w_i(\phi)=1)$ if ϕ is an i-probability formula or the negation of an i-probability formula.

<u>Proof</u>. First we will proof that

$(M,s) \models \phi \Rightarrow (w_i(\phi)=1)$ if uniformity holds in M.

i.e $(M,s) \models \phi$

Then

$(M,s) \models w_i(\phi)=1$.

i.e $\mu_{i,s^*}(\{t \in S_{i,s} / (M,t) \models \phi\}) = 1$.

i.e ϕ is true in all the states of $S_{i,s}$.

Now since uniformity holds in M, agent i will have the same probability distribution in all the states of $S_{i,s}$.

and since ϕ is true in s, ϕ must be true in all the states of $S_{i,s}$.

$\therefore$ $(M,s) \models w_i(\phi)=1$.

Now we try to prove that
if
$(M, s) \models \phi \Rightarrow (w_i(\phi) = 1)$,
then
<u>uniformity</u> holds in M.

$(M, s) \models \phi \Rightarrow (w_i(\phi) = 1)$

i.e

if $(M, s) \models \phi$, then
ϕ is true in all the states of $S_{i,s}$.

i.e agent i will have the same probability distribution in all the states of $S_{i,s}$.

Now since s and i are arbitrary
<u>uniformity</u> holds in M.

(4) SDP corresponds to the axiom:
$\phi \Rightarrow K_i\phi$ if ϕ is an i-probability formula or the negation of an i-probability formula.

proof. First we try to prove if SDP holds in M and if

$(M, s) \models \phi$.

Then $(M, s) \models K_i\phi$.

Now since SDP holds, agent i will have the same probability distribution in all the states of $\mathcal{K}_i(s)$.

Since ϕ is true in s, it must be true in all the states of $\mathcal{K}_i(s)$.

i.e $(M, s) \models K_i\phi$.

Now we try to proof if
$(M, s) \models \phi \Rightarrow K_i\phi$,
Then $\therefore$ SDP holds in M.

[The above proof is very much similar to proof (3) and can be worked out easily]

Complete Axiomatizations

We now describe a natural complete axiomatizations for the logic of probability and knowledge.

I. Axioms and rule for reasoning about knowledge.

K1. $(K_i\phi \wedge K_i(\phi \Rightarrow \psi)) \Rightarrow K_i\psi$

K2. $K_i\phi \Rightarrow \phi$

K3. $K_i\phi \Rightarrow K_iK_i\phi$

K4. $\sim K_i\phi \Rightarrow K_i \sim K_i\phi$

R1. From ϕ infer $K_i\phi$ (Knowledge generalization)

II Axioms for reasoning about probabilities.

W1. $w_i(\phi) \geq 0$ (nonnegativity)

W2. $w_i(\text{true}) = 1$ (The probability of the event true is 1)

W3. $w_i(\phi \wedge \psi) + w_i(\phi \wedge \sim\psi) = w_i(\phi)$ (additivity)

W4. $w_i(\phi) = w_i(\psi)$ if $\phi \Leftrightarrow \psi$ is a propositional tautology (distributivity)

W5. $w_i(\text{false}) = 0$. (The probability of the event false is 0).

<u>Theorm</u> AX_{MEAS} is a sound and complete axiomatization for the logic of knowledge and probability.

Before proving the above theorem we give one difinition and prove one lemma.

An <u>n-atom</u> is a formula of the form $p_1' \wedge p_2' --- \wedge p_n$ where p_i' is either p_i or $\sim p_i$ for each i.

<u>Lemma</u>. Let ϕ be a propositional formula. Assume that $\{p_1, p_2, --- p_n\}$ includes all of the primitive propositions that appear in ϕ. Let $At_n(\phi)$ consist of all the n-atoms δ such that $\delta \Rightarrow \phi$ is a propositional tautology. Then

$$W(\phi) = \sum_{\delta \in At_n(\phi)} W(\delta) \quad \text{is provable.}$$

<u>proof</u>. We show by induction on $j \geq 1$ that if $\psi_1, \psi_2, \ldots, \psi_{2^j}$ are all of the j-atoms, then

$W(\phi) = W(\phi \wedge \psi_1) + \cdots + W(\phi \wedge \psi_{2^j})$ is provable.

If $j = 1$, this follows by finite additivity (axiom W3).

Assume inductively, we have shown that

$W(\phi) = W(\phi \wedge \psi_1) + \cdots + W(\phi \wedge \psi_{2^j})$.

is provable.

By W3.

$$W(\phi \wedge \psi_1 \wedge p_{j+1}) + W(\phi \wedge \psi_1 \wedge \sim p_{j+1})$$
$$= W(\phi \wedge \psi_1) \text{ is provable.}$$

Therefore we can replace each $W(\phi \wedge \psi_r)$ by $W(\phi \wedge \psi_r \wedge p_{j+1}) \wedge W(\phi \wedge \psi_r \wedge \sim p_{j+1}]$

This proves the inductive step.

In particular

$W(\phi) = W(\phi \wedge \sigma_1) + \cdots + W(\phi \wedge \sigma_{2^n})$ is provable.

since $\{p_1, \ldots, p_n\}$ includes all of the primitive propositions that appear in ϕ. if $\delta_r \in Atn(\phi)$.
Then $\phi \wedge \delta_r \equiv \delta_r$.
because
if $\phi \wedge \delta_r$ is true; then δ_r must be true and if δ_r is true then ϕ must be true, since $\delta_r \Rightarrow \phi$ is a propositional tautology.

If $\delta_r \notin Atn(\phi)$, then

$$\phi \wedge \delta_r \equiv false.$$

So by W4, we see that if $\delta_r \in Atn(\phi)$
Then $W(\phi \wedge \delta_r) = W(\delta_r)$ is provable.
and if $\delta_r \notin Atn(\phi)$ then
$W(\phi \wedge \delta_r) = W(false)$ is provable.

$\therefore$ we can replace each $W(\phi \wedge \delta_r)$ by either $W(\delta_r)$ or $W(false)$. Also we can drop the $W(false)$ terms, since $W(false) = 0$ is provable.

The lemma now follows.

Proof of The main theorem:

Soundness is straighforward, so we focus on completeness.

In order to show completeness we need to show that if the formula ϕ is consistent, then ϕ is satisfiable in a Kripke structure for knowledge and probability satisfying MEAS.

Let sub(ϕ) be the set of all subformulas of ϕ and sub+(ϕ) be the set of all subformulas of ϕ and their negations.

Let s be the finite set of formulas and let ϕ_s be the conjunctions of the formulas in s. We say that s is consistent if it is not the case that $AX_{MEAS} \vdash \sim \phi_s$

The set s is a maximal consistent subset of Sub+(ϕ) if it is consistent, a subset of sub+(ϕ) and for every subformula $\cancel{\phi}$ ψ of ϕ includes one of ψ or $\sim\psi$.

We now construct a Kripke structure for knowledge (but not probability) $(S, \pi, \mathcal{K}_1, \mathcal{K}_2, \ldots \mathcal{K}_n)$ as follows.

We take S, the set of states to consist of all maximal consistent subsets of $\mathrm{sub}^+(\phi)$. If s and t are states ~~that~~ than

$(s, t) \in \mathcal{K}_i$ iff s and t contains the same formula of the type $K_i \psi$

We define π so that for a primitive proposition p, we have $\pi(s)(p) =$ true iff p is one of the formulas in the set s.

Our goal is to define a probability assignment $\mathcal{P}$ such that if we consider the Kripke structure for knowledge and probability $M = (S, \pi, \mathcal{K}_1, \ldots \mathcal{K}_n, \mathcal{P})$, then for every state $s \in S$ and every formula $\psi \in \mathrm{Sub}^+(\phi)$

$(M, s) \models \psi$ iff $\psi \in s$.

Now we have

$w_i(\psi) = \sum_{s \in S / \psi \in s} w_i(\phi_s)$ is provable in AX_{MEAS}.

(This follows from the above lemma)

Hence we can show that any i-probability formula $\psi \in Sub^+(\phi)$ is provably equivalent to a formula of the form

$\sum_{s \in S} c_s w_i(\phi_s) \geq b$ for some appropriate coefficient c_s.

Fix an agent i and state $s \in S$.

Assume that ψ is equivalent to

$\sum_{s' \in S} c_{s'} w_i(\phi_{s'}) \geq b$.

Since s is maximal consistent set either ψ or $\sim\psi \in s$.

If $\psi \in s$. Then the corrospending inequality is

$\sum_{s' \in S} c_{s'} \mu_{i,s}\{ t \in S_{i,s} / (M,t) \models \phi_{s'} \} \geq b$

But since s' is the maximal consistent set. $\phi_{s'}$ is true in s' only.

$$\sum_{s' \in S} c_{s'}\, \mu_{i,s}(s') \geq b.$$

Here the summation is for all states $s' \in S$. but I think it should not be for all the states.

(But here how we know that either s' is present in $S_{i,s}$ or not If $S_{i,s} = S$, then it is O.K.)

Example

We take an i-probability formula

$$w_i(\psi_1) + w_i(\psi_2) + \cdots + w_i(\psi_n) \geq b.$$

But how do we confirm that in each state $s' \in S$ one of $\psi_1, \psi_2, \ldots \psi_n$ is present.

Finally we have the inequality

$$\sum_{s' \in S} \mu_{i,s}(s') = 1.$$

This set of linear equality and inequalities has a solution.

For each i and s, we solve the corrosponding set of inequalities seperately.

We know define $\mathcal{P}$ so that $\mathcal{P}_{i,s} = (S, 2^S, \mu_{i,s})$ where if $A \subseteq S$, then

$$\mu_{i,s}(A) = \sum_{s' \in A} \mu_{i,s}(s').$$

Now we want to show that for every subformula ψ of $Sub^+(\phi)$ and every state in s, we have

$(M, s) \models \psi$ iff $\psi \in s$.

The proof proceeds by induction on ψ. If ψ is primitive proposition, the result is immediate from the definition of π. The case where ψ is a negation or conjunction are straightforward.

Now if ψ is an i-probability formula then suppose

$(M, s) \models \psi$ suppose ψ is of the form

Then $w_i(\psi_1) + w_i(\psi_2) \geq b$.

$(M, s) \models w_i(\psi_1) + w_i(\psi_2) \geq b$.

iff

$\mu_{i,s}(\{t \in S_{i,s} \mid (M, t) \models \psi_1\})$

$+ \mu_{i,s}(\{t \in S_{i,s} \mid (M, t) \models \psi_2\}) \geq b$.

$$\equiv \sum_{\substack{t \in S_{i,s} \\ (M,t) \models \psi_1}} \mu_{i,s}(t) + \sum_{\substack{t \in S_{i,s} \\ (M,t) \models \psi_2}} \mu_{i,s}(t) \geq b.$$

which confirms that $\psi \in s$.

The proof of the opposite side is same as above.

Now suppose ψ is of the form

$K_i \psi'$.

$K_i \psi' \in s$. Then for all $t \in \mathcal{K}_i(s)$

$K_i \psi' \in t$.

Now since t is maximal consistent set, either $\psi' \in t$ or $\sim \psi' \in t$.

Now from K3

$\psi' \in t$

By the induction hypothesis

$(M, t) \models \psi'$

i.e $(M, s) \models K_i \psi'$.

Now suppose that

$(M, s) \models K_i \psi'$

Let s_i be the subset of s consisting of all the formulas in s of the form $K_i \psi''$ or $\sim K_i \psi''$.

s_i includes one of $K_i \psi'$ or $\neg K_i \psi'$

We plan to show that in fact it must include $K_i \psi'$.

We claim that

$AX_{MEAS} \vdash \phi_{s_i} \Rightarrow \psi'$ —— ①

For suppose hat,

then $\phi_{s_i} \wedge \sim \psi'$ is consistent, then there is a maximal consistent set t which includes $s_i \cup \sim \psi'$.

Then we have $(s, t) \in \mathcal{K}_i$ and by induction hypothesis $(M, t) \models \sim \psi'$ which contradicts our assumption $(M, s) \models K_i \psi'$.

Thus ① holds.

By R1 from ①, we have

$AX_{MEAS} \vdash K_i(\phi_{si} \Rightarrow \psi')$ —— ②

Using K1 we get

$AX_{MEAS} \vdash K_i\phi_{si} \Rightarrow K_i\psi'$ —— ③

Every conjunct of ϕ_{si} is of the form $K_i\psi''$ or $\sim K_i\psi''$. Thus if σ is one of the conjuncts of ϕ_{si} using either K3 or K4

$AX_{MEAS} \vdash \sigma \Rightarrow K_i\sigma$. —— ④

It is well known that for any formulas σ_1 and σ_2

$AX_{MEAS} \vdash K_i(\sigma_1 \wedge \sigma_2) \Leftrightarrow K_i\sigma_1 \wedge K_i\sigma_2$.

Thus from ④ it follows that

$AX_{MEAS} \vdash \phi_{si} \Rightarrow K_i\phi_{si}$ —— ⑤

from ③ and ⑤ it follows that

$AX_{MEAS} \vdash \phi_{si} \Rightarrow K_i\psi'$.

since ϕ_s and hence ϕ_{si} is consistent it now follows that $\neg K_i\psi'$ cannot be one of the conjuncts of ϕ_{si}, hence $K_i\psi' \in S$.

If ϕ is consistent it must be in one of the maximal consistent subsets of $sub^+(\phi)$.

Thus it follows that if ϕ is consistent then it must be satisfiable in the structure M.

This completes the proof.

Probability Assignment in Linear Time Temporal Logic.

In order to give semantics to such formulas, we first need to review briefly some probability theory.

A probability space is a tuple $(S, \mathcal{X}, \mu)$ where S is a set of states, $\mathcal{X}$ is a σ-algebra of subsets of S, and μ is a probability measure defined on the measurable sets.

A probability structure is a tuple $M = (S, \mathcal{X}, \mu, \pi)$ where $(S, \mathcal{X}, \mu)$ is a probability space and π associates with each state in S a truth assignment on the primitive propositions in ϕ.

Syntax

We fix a countable set of atomic propositions $P = \{p_0, p_1, \dots\}$. Then ϕ, the set of formulas is defined inductively as follows

- Every member of P belongs to ϕ.
- If α and β are formulas in ϕ, then so are $\neg\alpha$, $\alpha \vee \beta$, $O\alpha$, $\alpha \mathcal{U} \beta$, $w(\alpha)$.

Semantics

A model is a function $M : N_0 \longrightarrow 2^S$. In other words, a model is an infinite sequence ~~$P_0 P_1$~~ $S_0 S_1 \dots$ of subsets of S.

We write

$\mathcal{P}, M, i \models \alpha$ to denote that "α is true at time instant i in the model M for probability structure $\mathcal{P}$.

This notion is defined inductively according to the structure of α.

- $\mathcal{P}, M, i \models p$, where $p \in P$ iff $\forall s \in M(i) \quad \pi(s)(p) = T$.
- $\mathcal{P}, M, i \models \sim\alpha$ iff $\mathcal{P}, M, i \not\models \alpha$.
- $\mathcal{P}, M, i \models \alpha \vee \beta$ iff $\mathcal{P}, M, i \models \alpha$ or $\mathcal{P}, M, i \models \beta$.
- $\mathcal{P}, M, i \models O\alpha$ iff $\mathcal{P}, M, i+1 \models \alpha$.
- $\mathcal{P}, M, i \models \alpha \mathcal{U} \beta$ iff there exist $k \geq i$ such that $\mathcal{P}, M, k \models \beta$ and for all j such that $i \leq j < k$ $\mathcal{P}, M, j \models \alpha$.

$\mathcal{P}, M, i \models w(\alpha) \geq b$

iff

$\mu(\{t \in M(i) / (M, t) \models \alpha\}) \geq b$.

$\mathcal{P}, M, i \models \Diamond\alpha$ iff there exist $k \geq i$ such that

$\mathcal{P}, M, k \models \alpha$.

$\mathcal{P}, M, i \models \Box\alpha$ iff $\forall\, k \geq i$

$\mathcal{P}, M, k \models \alpha$.

Now we take certain examples so that it becomes more clear.

When we say

$\mathcal{P}, M, i \models \cancel{w(}\ w(\Box\alpha) \geq b$.

$\mathcal{P}, M, i \models w(\Box\alpha) \geq b$

iff

$\mu(\{t \in M(i) \mid (M, t) \models \Box\alpha\}) \geq b$.

Now ~~when~~ we say $\Box\alpha$ is true at instant i in state t in model M

iff $t \in M(i)$ and

$M, \cancel{t, i}^{k} \models \alpha \quad \forall\, k \geq i$

When we say

$\wp, M, i \models \Box(w(\alpha) \geq b)$

$\wp, M, i \models \Box(w(\alpha) \geq b)$

iff

$\wp, M, k \models w(\alpha) \geq b \quad \forall\, k \geq i$

and we already know

when

$\wp, M, k \models w(\alpha) \geq b$.

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