AN EXTENSION OF A THEOREM OF MAMAY, WITH APPLICATION*

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SUMMARY. This paper is divided into three sections. Section 1 deals with Mamay's theorem itself and gives certain additional arguments which appear to me to be necessary for a complete proof thereof. In Section 2 is given the main result of this paper, extending Mamay's theorem in a comprehensive form to the case where the functions concerned have zeros in the domain of analyticity. Section 3 deals with an application of the (extended) theorem and establishes a 'denumerable α-decomposition theorem' for the composition of a binomial and a Poisson law having the same maximum span. An announcement by R. Cuppens (1963) contains a statement of part of Theorem 1 of the present paper (namely, that every f_j is analytic in D, in the case where D is a circle around the origin). Cuppens appeals to a theorem announced by R. G. Laha (*Bull. Amer. Math. Soc*, Vol. 67, 1961, pp. 148—149) concerning properties of absolutely monotonic functions, while our aproach makes direct use of properties of characteristic functions as in Mamay (1960).

1. Completion of the proof of Mamay's theorem

Let f_j, $j = 1, 2, \ldots$ be a sequence of characteristic functions (c.f.) of one-dimensional probability distribution functions (d.f.), and let $\{\alpha_j\}$ be a sequence of positive constants bounded below by a positive constant. Throughout what follows, let D denote *either* $|t| < R$ *or* $|\text{Im}\, t| < R$. Also, for convenience, we shall merely write Σ or π to denote an infinite sum or product respectively, and specify the range of the index only in the case of finite sums and products. Suppose f is analytic and non-vanishing in D and that there exists a real neighbourhood of $t = 0$ in which every f_j is non-vanishing and in which the relation

$$\pi f_j^{\alpha_j}(t) = f(t) \qquad \ldots \quad (1.1)$$

holds. (By the logarithm of a c.f. in any neighbourhood of the origin in which the c.f. does not vanish, we shall always mean that branch defined by continuity and by the condition that it vanishes at the origin. $f_j^{\alpha_j}(t)$ shall be taken to mean $\exp[\alpha_j \ln f_j(t)]$.) Setting $\phi_j(t) = f_j(t) f_j(-t)$ and $\phi(t) = f(t) f(-t)$, Mamay (1960) has proved that (a) every ϕ_j is analytic and non-vanishing in D, and (b) the relation

$$\pi \phi_j^{\alpha_j}(t) = \phi(t) \qquad \ldots \quad (1.2)$$

holds there. From (a) and a theorem due to D. A. Raikov (see, for instance, Lukacs, 1960, p. 173), it follows that every f_j is also analytic and (so) non-vanishing in D, since $\phi_j(t)$ is the product of the c.f.'s $f_j(t)$ and $f_j(-t)$. In Mamay (1960), it is asserted that it then follows (presumably without any further argument being needed) that (1.1) is valid throughout D. I have not been able to convince myself that this assertion is an immediate consequence of the foregoing facts, but have, however, been able to supply the following proof. (Incidentally, it is of interest to note that, while

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the validity of relation (1.2) is sufficient, taken in conjunction with the Lévy-Cramér theorem on the Normal law, to establish that, if f is a normal c.f., so is every f_j, the validity of relation (1.1) appears to be indispensable for the discussion of the case where f is a Poisson c.f. : vide Ramachandran (1964).)

For $t = iy$, $0 < y < R$, (1.2) can be written in the form

$$\Sigma \alpha_j [(1/y) \ln f_j (iy) + (1/y) \ln f_j (-iy)] = (1/y) \ln \phi(iy). \qquad \dots \quad (1.3)$$

Since both $f_j(t)$ and $f_j(-t)$ are c.f.'s analytic in D, it follows that the functions $(1/y) \ln f(\pm iy)$ are both non-decreasing functions of y in $0 < y < R$ (cf. Ramachandran, 1964, pp. 15-16). It then follows from (1.3) that, for $0 < y' < y < R$ and for all positive integers N,

$$\sum_{j=1}^{N} \alpha_j [(1/y) \ln f_j (iy) - (1/y') \ln f_j (iy')]$$

is a series of non-negative terms with sum less than or equal to

$$(1/y) \ln \phi(iy) - (1/y') \ln \phi(iy').$$

Letting y' tend to zero from above in this inequality, we obtain

$$\sum_{j=1}^{N} \alpha_j \; [(1/y) \ln f_j (iy) + \mu_j] \leqslant (1/y) \ln \phi(iy),$$

or

$$\sum_{j=1}^{N} \alpha_j [\ln f_j (iy) + \mu_j y] \leqslant \ln \phi(iy),$$

where μ_j is the first moment of the d.f. corresponding to the c.f. f_j (note that, f_j being analytic, all the moments exist for every one of these d.f.'s). We note that every term of the series on the left in the above inequality is non-negative.

A similar argument yields the dual relation

$$\sum_{j=1}^{N} \alpha_j [\ln f_j (-iy) - \mu_j y] \leqslant \ln \phi(iy),$$

valid for all positive integers N for $0 < y < R$. Here again, every term of the series on the left is non-negative.

It follows that the sequence

$$\exp \left\{ \sum_{j=1}^{N} \alpha_j [\ln f_j (t) - i\mu_j t] \right\} = \prod_{j=1}^{N} \; [f_j (t) \exp(-i\mu_j t)]^{\alpha_j} \qquad \dots \quad (1.4)$$

of functions analytic in D is, for every $0 < r < R$, bounded uniformly by $\max_{0 < y < r} \phi(iy) = \phi(ir)$ for all N and for all t in $|t| < r$ or in $|\text{Im } t| < r$ respectively according as D denotes $|t| < R$ or $|\text{Im } t| < R$. Also, for every $t = iy$ with $-R < y < R$, it is a non-decreasing sequence and consequently has a limit (which is finite). Hence, by Vitali's theorem (see, for instance, Titchmarsh (1939), p. 168), the sequence (1.4) has a limit function g in D which is analytic there. Since $g(0) = 1$, there exists a neighbourhood of the origin in the t-plane in which g is non-vanishing, and consequently the series $\Sigma \alpha_j [\ln f_j(t) - i\mu_j t]$ converges (indeed, represents an analytic function, namely, $\ln g(t)$) there. Since $\Sigma \alpha_j \ln f_j(t)$ itself converges in a (real) neighbourhood of $t = 0$ by virtue of our basic assumption, it follows (on considering any point $\neq 0$ common to both neighbourhoods) that $\Sigma \alpha_j \mu_j$ converges and consequently $\pi f_j^{\alpha_j}(t)$ represents in D an analytic function, which, by analytic continuation, can only be f. Hence (1.1) holds throughout D, as desired to prove.

2. Extension of Mamay's theorem where the functions concerned have zeros in the domain D

We shall now establish the following extension of Mamay's theorem.

Theorem 1: *Let f_j, $j = 1, 2, \ldots$, be c.f.'s, and let $\{\alpha_j\}$ be a sequence of positive constants bounded below by a positive constant. Let f be analytic in D and the relation* (1.1) *hold in some real neighbourhood of $t = 0$. Then, every f_j is analytic in D, non-vanishing at every point of D where f is and relation* (1.1) *holds at all such points.*

Proof: We shall say that a t-domain contained in D has the property (P) if the ϕ_j and ϕ are analytic and non-vanishing there and relation (1.2) holds there.

First, we shall consider the case where D is the circle $|t| < R$.

Here again, let us first examine the case where ϕ does not vanish on the imaginary axis in D. Let $\varepsilon > 0$ be arbitrary but fixed. Then ϕ has only a finite number of zeros in $|t| < R-\varepsilon$, and, by our assumption, none of them lies on the imaginary axis. Hence there exists an $h = h(\varepsilon) > 0$ such that $\{|t| < R-\varepsilon, |\text{Re } t| < 2h\}$ is free of the zeros of ϕ. Then, by Mamay's theorem, $|t| < 2h$ has the property (P); the Fourier-Stieltjes transform representation is valid for every ϕ_j in $|\text{Im } t| < 2h$, since it is analytic there; and consequently, the functions $\psi_j(u) = \phi_j(ih+u)/\phi_j(ih)$ are c.f.'s related to the function $\psi(u) = \phi(ih+u)/\phi(ih)$ according to

$$\pi \psi_j^{\alpha_j}(u) = \psi(u) \qquad \ldots \quad (2.1)$$

valid for $|u| < h$, u real. Then Mamay's theorem shows that every ψ_j is analytic and non-vanishing in $|u| < \min(2h, R-\varepsilon-h)$ and (2.1) holds there, and hence that $|t-ih| < \min(2h, R-\varepsilon-h)$ has the property (P). We may proceed in this manner, 'shifting the origin' by steps of h upwards along the imaginary axis as far as necessary, concluding that all the circles $|t-ikh| < 2h$ for $0 \leqslant k \leqslant n-1$ and the circle $|t-inh| < R-\varepsilon-nh$ have the property (P), where $n+1$ is the largest integer not exceeding $(R-\varepsilon)/h$. Hence none of the ϕ_j has a singularity on the upper imaginary axis in $|t| < R-\varepsilon$. Since $\varepsilon > 0$ is arbitrary, this assertion is true of the upper imaginary axis in $|t| < R$ itself. But, from the theory of analytic c.f.'s, we know that the singularities nearest to the origin of any of the ϕ_j's (which are symmetric c.f.'s) must lie (symmetrically about the origin) on the imaginary axis. Hence every ϕ_j is analytic in $|t| < R$ — in the case where ϕ does not vanish on the imaginary axis in $|t| < R$.

We now dispose of the possibility that ϕ can vanish on the imaginary axis in $|t| < R$. Suppose ϕ does, and let $\pm ir$ be the zeros nearest to the origin so that $r < R$. Then, our above analysis applies in $|t| < r$, and so (1.2) holds at all points iy with $0 < y < r$. Since ϕ_j is a symmetric c.f., analytic in $|t| < r$, every $\phi_j(iy)$ is a non-decreasing function of y in $(0, r)$ and hence the same is true of $\phi(iy)$ in that interval. This implies, by virtue of the continuity of ϕ, that $\phi(ir) \neq 0$, contrary to our assumption. Hence ϕ cannot vanish on the imaginary axis in $|t| < R$, the earlier analysis applies, and so every ϕ_j is analytic in $|t| < R$. It then follows from Raikov's theorem that so is every f_j.

Remark : We note that it has been incidentally proved that every point on the imaginary axis in $|t| < R$ has a neighbourhood with property (P), and that in particular (1.2) is valid at all points on the imaginary axis in $|t| < R$.

We turn to a proof of the second assertion of the theorem. Let $\varepsilon > 0$ be arbitrary. Then f has only a finite number of zeros in $|t| < R-\varepsilon$. Let $h = h(\varepsilon)$ be such that the minimum of the absolute values of the imaginary parts of those zeros in $|t| < R-\varepsilon$ which are not real is $2h$; obviously $h > 0$. Considering the c.f.'s $f_j(ih+u)/f_j(ih)$, and the function $f(ih+u)/f(ih)$ which is analytic and non-vanishing in $\{|ih+u| < R-\varepsilon, |\text{Im } u| < h\}$, it is easy to see that, as a consequence of (the argument leading to) Mamay's theorem, relation (1.1) holds in the domain $\{|t| < R-\varepsilon, 0 < \text{Im } t < 2h\}$. Similarly, considering the c.f.'s $f_j(-ih+u)/f_j(-ih)$, and the function $f(-ih+u)/f(-ih)$ which is analytic and non-vanishing in $\{|-ih+u| < R-\varepsilon, |\text{Im } u| < h\}$, we see that (1.1) holds in the domain $\{|t| < R-\varepsilon, -2h < \text{Im } t < 0\}$ as well, so that (1.2) holds in the domain $\{0 < \text{Im } t < 2h, |t| < R-\varepsilon\}$.

Let now a be any point on the real line in $|t| < R-\varepsilon$ at which f does not vanish. Since $\phi(t) = |f(t)|^2$ on the real axis, $\phi(a) \neq 0$ also. Hence there exists a circular neighbourhood of a of radius 2δ, where $\delta = \delta(a) > 0$, in which ϕ does not vanish. Choose and fix a y belonging to $(0, \delta)$, and let $z = a+iy$. Then (1.2) holds for $t = z$ by what we have just proved, and for $t = iy+\xi$ for all ξ with $|\xi|$ sufficiently small, by the 'Remark' made earlier. Hence, setting $v_j(t) = \phi_j(it)$ and $v(t) = \phi(it)$, we have for all ξ, with $|\xi|$ sufficiently small,

$$\pi v_j^{\alpha_j}(y+\xi) = v(y+\xi).$$

Let p be a positive integer such that $\beta_j = p\alpha_j \geqslant 1$ for all j. (Such a p exists since the α_j have been assumed to be bounded below by a positive number.) Hence we have, for any positive integer n,

$$\pi v_j^{n\beta_j}(y+\xi) = v^{np}(y+\xi).$$

The power-series expansion for v_j about the origin has all its odd coefficients zero and even coefficients real and non-negative; hence the derivatives of v_j at y (which may be obtained by successive termwise differentiations of the power-series) are all non-negative. Therefore, if we raise both sides of the last relation above to the q-th power, differentiate q times with respect to ξ, set $\xi = 0$, and omit in the resulting expression on the left all the terms involving derivatives of orders less than q, we obtain, since all the omitted terms are non-negative (c.f. Mamay, 1960, p. 95),

$$\Sigma nq\beta_j \left[v_j^{(q)}(y)/v_j(y)\right] \left[\pi v_k^{nq\beta_k}(y)\right] \leqslant \frac{d^q}{d\xi^q} v^{npq}(y+\xi)\Big|_{\xi=0}$$

Dividing through by $\qquad \pi \mid \phi_k^{nq\beta_k}(z) \mid = \mid \Phi(z) \mid^{npq},$

and noting that

$$\left[\pi v_k^{nq\beta_k}(y)\right] \Big/ \left[v_j(y) \left\{ \pi \left| \phi_k^{nq\beta_k}(z) \right| \right\} \right]$$

$$= \prod_{k \neq j} (v_k(y)/|\phi_k(z)|)^{nq\beta_k} \cdot (v_j(y)/|\phi_j(z)|)^{n(q\beta_j-1)} \cdot [v_j(y)]^{n-1} |\phi_j(z)|^{-n} > |\phi_j(z)|^{-n},$$

since, for all $\qquad k, \ |\phi_k(z)| \leqslant v_k(y)$, and $1 \leqslant v_j(y)$,

we have

$$\Sigma \, nq\beta_j \{ v_j^{(q)}(y) / |\phi_j(z)|^n \} \leqslant \frac{d^q}{d\xi^q} \{ v^{npq}(y+\xi) / |\phi(z)|^{npq} \} |_{\xi=0}$$

$$= [q!/2\pi i |\phi(z)|^{npq}] \cdot \int_{|\xi|=\delta} [v^{npq}(y+\xi)/\xi^{q+1}] \, d\xi,$$

by Cauchy's integral formula,

$$\leqslant q!\,(M/m)^{npq} \delta^{-q},$$

where $\quad M = \max [\, |\phi(t)| : |t| \leqslant R-\epsilon]$ and $m = \min [\, |\phi(t)| : |t-a| < \delta] > 0$.
Hence we have for all j and n,

$$|\phi_j^{(q)}(iy)| = v_j^{(q)}(y) \leqslant q!\,(M/m)^{npq} \, \delta^{-q} \, |\phi_j(z)|^n. \qquad \ldots \quad (2.2)$$

We now note the following easily-proved facts (recalling that $z = a+iy$) :

(i) $\quad |\phi_j^{(2q)}(z)| \leqslant |\phi_j^{(2q)}(iy)|$

(ii) $\quad 2|\phi_j^{(2q-1)}(z)| \leqslant |\phi_j^{(2q)}(iy)| + |\phi_j^{(2q-2)}(iy)|$

(iii) $\quad |\phi_j'(z)|^2 \leqslant \phi_j(iy) \, . \, |\phi_j''(iy)|$

and $\qquad$ (iv) $\quad 1 \leqslant \phi_j(iy) \leqslant \phi_j(i\delta) \leqslant [\phi(i\delta)]^{(1/\alpha_0)} \leqslant M^{(1/\alpha_0)}$,

where α_0 is any positive lower bound to the α_j. Using relation (2.2) for $n = 2$ together with (iii) and (iv) to estimate $|\phi_j'(z)|$, and relation (2.2) for $n = 1$ together with (i) and (ii) to estimate the derivatives of all higher orders at z of the ϕ_j, we see that there exists an $r > 0$ depending only on M, m and δ, such that for $|u| < r$,

$$|\phi_j(z+u) - \phi_j(z)| = | \sum_{n=1}^{\infty} u^n \phi_j^{(n)}(z)/n! \, | < | \, \phi_j(z)/2 |,$$

so that $\phi_j(z+u)$ cannot vanish for $|u| < r$. Since r is independent of y in $(0, \delta)$, it follows that $\phi_j(a) \neq 0$ and consequently $f_j(a) \neq 0$. Hence every f_j is non-zero at every point of the real axis in $|t| < R-\epsilon$ at which f is non-zero.

Let now $a+iy$ be any point in $|t| < R-\epsilon$ at which f does not vanish. Since ϕ does not vanish on the imaginary axis in $|t| < R$ and has only a finite number of zeros in $|t| < R-\epsilon$, there exists a $\delta = \delta(\epsilon) > 0$ such that ϕ has no zeros in $\{|t| < R-\epsilon$, $|\text{Re}\, t| < \delta\}$. It can be verified as usual that every ϕ_j is non-vanishing in this domain, relation (1.2) holds there and consequently (as in Section 1 of this paper), relation (1.1) holds there as well. Hence the c.f.'s $h_j(u) = f_j(iy+u)/f_j(iy)$ are related to the function $h(u) = f(iy+u)/f(iy)$ according to a relation of the same form as (1.1), for $|u| < \delta$, u real. The function f being analytic in $|t| < R$ and non-zero at $t = iy+a$, it follows by the preceding discussion that every h_j is non-zero at $u = a$. Thus, no f_j vanishes at a point in $|t| < R-\epsilon$ at which f does not; since $\epsilon > 0$ is arbitrary, this assertion is true of $|t| < R$ itself.

It is then easy to convince ourselves, using the arguments of Section 1, that the relation (1.1) holds at every point in $|t| < R$ at which f does not vanish.

We turn to the case where D is the strip $|\text{Im}\, t| < R$.

In this case, f is analytic in $|t| < R$ *a fortiori*, and so every f_j is analytic in $|t| < R$ by the preceding discussion, and so in $|\text{Im}\, t| < R$ by the well-known property of analytic c.f.'s. In this case, we then consider an arbitrary rectangle contained in D, say $\{|\text{Im}\, t| < R-\epsilon, |\text{Re}\, t| < X\}$ where $(R >)\, \epsilon > 0$ and $X > 0$ are arbitrarily chosen and fixed, and convince ourselves as above that any f_j can vanish at a point in this rectangle only if f does (note that there are only a finite number of zeros of f in any such rectangle), and that relation (1.1) holds at all points of this rectangle at which f does not vanish. This completes the proof of our theorem.

Additional remarks: An immediate consequence of our theorem is :

If f is an entire function, so is every f_j.

Further, according to what we have seen above, in case f is entire, every point on the imaginary axis has a neighbourhood with the property (P). In particular, (1.2) holds for $t = iR$ for every $R > 0$. Since $\phi_j(iR) \geqslant \phi_j(0) = 1$, it follows that, for every j,

$$[\phi_j(iR)]^{\alpha_j} \leqslant \phi(iR) \leqslant \max[|\phi(t)| : |t| \leqslant R].$$

Since the maximum modulus of ϕ_j in $|t| \leqslant R$ is $\phi_j(iR)$, it follows that the order of ϕ_j (as an entire function) is not greater than that of ϕ. Now, since the c.f. $f_j(t)$ is a 'factor' of the c.f. $\phi_j(t)$, it follows by a well-known result that the order of f_j is not greater than that of ϕ_j (see, for instance, Lukacs, 1960, p. 173). (But, since $\phi_j(t) = f_j(t)f_j(-t)$, it follows from elementary considerations that the order of ϕ_j is not greater than that of f_j. Consequently, ϕ_j and f_j have the same order.) Since $\phi(t) = f(t)f(-t)$ shows that the order of ϕ is not greater than that of f, we finally have :

The order of any f_j is not greater than that of f, if f is entire.

3. Application to the case where f is the convolution of a binomial and a Poisson law with the same maximum span

In this section, we consider an application of our theorem to the case where f is the c.f. of the convolution of a binomial and a Poisson law having the same maximum span. This generalizes the results of the author's papers (1961) and (1964) as well as the earlier results of D. A. Raikov, N. A. Sapogov-H. Teicher and H. Teicher (1958). (For brief references to the first two of the latter set of results, see Ramachandran (1961)).

Theorem 2 : *Let, in the statement of Theorem* 1, $f(t) = (q+pe^{it})^m \cdot \exp[\lambda(e^{it}-1)]$, *where* $0 < p < 1$, $q = 1-p$, $m > 0$ *is an integer and* $\lambda > 0$, *so that f is the c.f. of the convolution of a binomial and a Poisson law having the same maximum* *span (unity). Then every* $f_j(t)$ *is of the form*

$$(q+pe^{it})^{m_j} \cdot \exp[i\xi_j t + \lambda_j (e^{it}-1)]$$

where $m_j \geqslant 0$ *is an integer,* ξ_j *is real,* $\lambda_j \geqslant 0$, $\Sigma \alpha_j \lambda_j = \lambda$, $\Sigma \alpha_j \xi_j = 0$, *and* $\Sigma \alpha_j m_j = m$. *(The relation* $\Sigma \alpha_j m_j = m$, *together with the assumption that the* α_j *are bounded below by a positive number, implies that only finitely many of the* m_j *are non-zero.)*

Proof: f is entire and its only zeros are ψ at the points $\beta_k = (2k+1)\pi - i\alpha$, where $\alpha = \ln(q/p)$ and k runs through all the integers. Hence, by our theorem, every f_j is entire, its only possible zeros are the β_k's, and the relation (1.1) holds everywhere in the t-plane except possibly at these points. In particular, it holds at $t = 2\pi$, so that we have

$$\pi \; |f_j(2\pi)|^{\alpha_j} = 1.$$

This implies that every $|f_j(2\pi)| = 1$, so that every f_j corresponds to a 'lattice distribution' with unit 'span', i.e.,

$$f_j(t) = e^{it\xi_j} \Big(\sum_{n=-\infty}^{\infty} p_{jn} e^{itn} \Big)$$

where ξ_j is real, $p_{jn} \geqslant 0$ for every n, and $\sum_{n=-\infty}^{\infty} p_{jn} = 1$. (For the necessary definitions and arguments, see the author's papers (1961) and (1964)). Without loss of generality, we may asume that ξ_j is a point of increase for the d.f. F_j corresponding to the c.f. f_j, so that $p_{j0} > 0$.

We can then show that every F_j is 'bounded below' and that, if ξ_j above be taken as the 'left extremity' of F_j (as it certainly may be, being a point of increase), $\Sigma \alpha_j \xi_j = 0$. The proof follows exactly the same lines as the author's paper (1964) dealing with the Poisson law, and is therefore omitted here. Thus, $f_j(t)$ has the form

$$f_j(t) = e^{it\xi_j} \Big(\sum_{n=0}^{\infty} p_{jn} e^{itn} \Big)$$

where $p_{j0} > 0$, every $p_{jn} \geqslant 0$, and $\sum_{n=0}^{\infty} p_{jn} = 1$. Since $\Sigma \alpha_j \xi_j = 0$, (1.1) then gives, for $t \neq$ any β_k,

$$\pi \Big(\sum_{n=0}^{\infty} p_{jn} e^{itn} \Big)^{\alpha_j} = (q+pe^{it})^m . \quad \exp[\lambda(e^{it}-1)]. \qquad \dots \quad (3.1)$$

Now $\sum_{n=0}^{\infty} p_{jn} e^{itn}$ represents an entire function which does not vanish except possibly at the β_k; since every $z \neq 0$ has a representation as e^{it} for a suitable complex t, it follows that $\sum_{n=0}^{\infty} p_{jn} z^n$ represents an entire function $g_j(z)$ which does not vanish for $z \neq -(q/p)$: note that $p_{j0} > 0$ implies that g_j does not vanish at $z = 0$. Also, for $z \neq 0$ or $-(q/p)$, we have from (3.1) that

$$\pi\, [g_j(z)]^{\alpha_j} = (q+pz)^m . \quad \exp[\lambda(z-1)]. \qquad \dots \quad (3.2)$$

Now, since the power-series for g_j around the origin has all its coefficients non-negative, the maximum modulus of g_j in any circle $|z| \leqslant R$ is attained at $z = R$; but, from (8), we see that (for all $R > 0$)

$$\pi [g_j(R)]^{\alpha_j} = (q+pR)^m . \; \exp\,[\lambda(R-1)] \qquad \dots \quad (3.3)$$

which shows (noting that every $g_j(R) \geqslant 1$ for $R \geqslant 1$) that every g_j, as an entire function, is of order at most unity. Since g_j can vanish, if at all, only at the point $-(q/p)$, it then follows that, m_j being the order of the zero at this point,

$$g_j(z) = (q+pz)^{m_j}. \quad \exp\,[\lambda_j z + \mu_j].$$

Since g_j is a probability-generating function, it is easily verified that we must have $\lambda_j \geqslant 0$ and $\mu_j = -\lambda_j$. (Cf. Lemma 2.3 of Ramachandran, 1961). Hence the theorem, the asserted relations among the parameters following from (3.2).

Perhaps the most important special case of this result is when all the α_j are equal to unity, the corresponding result being :

If the sum (in the equivalent senses of 'almost sure', 'in probability' and 'in law') of an infinite series of independent random variables (defined on some probability space) is distributed as the convolution of a binomial and a Poisson law having the same maximum span, then (except possibly for a 'lateral shift' in each case) so is each summand.

4. Concluding remarks

In all the applications that have so far been considered, the function f is itself a c.f. Since an analytic c.f. has a strip of analyticity of the form $-\alpha < \text{Im}\, t < \beta$, $(\alpha > 0,\ \beta > 0)$, it is worth noting that the main result (Section 2) remains valid if, in its statement, D is replaced by the above strip in such cases. This observation is essentially due to D. Dugué (vide the proof of Theorem II.2, "Sur le théorème de Lévy-Cramér", *Publ. Inst. Statist. Univ. Paris.* Vol. 6, p. 220) and is a consequence of the simple fact that if $g(t)$ is an analytic c.f. then $g(t+iy)/g(iy)$ is also an analytic c.f. if iy belongs to the strip of analyticity of g. The proof of the above extension of the result in Section 2 is achieved by using this fact to make successive "shifts of the origin" starting from the strip $|\text{Im}\, t| < \min(\alpha, \beta)$ — in which the result already holds, the shifts being upward or downward according as $\beta > \alpha$ or $\beta < \alpha$.

We also notice that if f (analytic) is the c.f. of an "one-sided" d.f., then the proof in Section 3 shows that every f_j is also the c.f. of an one-sided d.f. (all bounded to the left or all to the right according to how the d.f. corresponding to f is), and that the finite extremities (left or right as the case may be) ξ_j, ξ of these d.f.'s are related according to : $\Sigma \alpha_j \xi_j = \xi$. In particular, if f is the c.f. of a "finite" d.f. (i.e., one bounded on both sides)— so that f is an entire function and either $= 1$ or of order one and of finite type—then so is every f_j, and relations of the above form hold for the left as well as the right extremities of the corresponding d.f.'s.

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