WEAK CONVERGENCE OF MEASURES ON SEPARABLE METRIC SPACES

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1. Introduction

This paper studies the weak convergence of measures over a given separable metric space X. A sequence $\{\mu_n\}$ of measures on X is said to converge weakly to a measure μ if $\int_X g d\mu_n \rightarrow \int_X g d\mu$ for each bounded continuous function g on X. The following problems are discussed: (i) whether this notion of convergence arises from a topology on the space of measures, and if so, (ii) whether the corresponding topology is metrizable or metrizable as a complete metric space. These problems are studied on $\mathfrak{M}$, the space of finite signed measures; on $\mathfrak{M}^+$, the space of finite measures; and on $\mathfrak{M}_p$, the space of probability measures.

The main conclusions of the paper may be summarized as follows :

(i) Weak convergence in $\mathfrak{M}$ does correspond to a certain topology on $\mathfrak{M}$ referred to as the *weak topology* on $\mathfrak{M}$.

(ii) In this topology, $\mathfrak{M}^+$ (and also $\mathfrak{M}_p$ is a closed set and is metrizable as a separable metric space.

(iii) $\mathfrak{M}^+$ (and also $\mathfrak{M}_p$) is metrizable as a complete metric space if the basic space is complete.

(iv) $\mathfrak{M}$ is metrizable if and only if the weak and norm topologies on $\mathfrak{M}$ are identical.

In the particular case when X is the real line, the above results for $\mathfrak{M}_p$ reduce to the well known results of P. Levy (see for instance Gnedenko and Kolmogorov, 1949).

2. Preliminary notions

Before we proceed, we discuss some preliminary notions. The entire paper will centre round convergence rather than topology. Consequently, we rely heavily on the machinery of Moore-Smith convergence of nets or directed sets (Kelley, 1955). A directed set is a pair $(L, >)$ such that $>$ directs L. A net is a pair $(S, >)$ where S is a function and $>$ directs the domain L of S. The range of the function is usually a topological space. It is also written as $\{S_n : n \epsilon L, >\}$; when no confusion is likely to arise, we simply say the net $\{S_n\}$. We assume as known the theory of limits in the sense of Moore-Smith as applied to nets. If a net $\{S_n\}$ tends to S, we write $\lim_{n \to \infty} S_n = S$. Here of course it is to be noted that the notation $n \rightarrow \infty$ is a pure symbolism. When L is the set of integers > 0 and $>$ is the usual ordering, the above definitions reduce to the classical notion of limits of sequences. Nets are important in the sense that they will describe the topological structure of a space completely (Kelley, 1955).

Let B be a Banach space and B^* its adjoint space. B^* is the space of all real-valued bounded linear functionals on B. In B^* we can introduce a topology by defining at any $\Lambda_0 \epsilon B^*$, the neighbourhood system as the family of sets $\{N(\Lambda_0; x_1, \ldots x_n, \epsilon)\}$ for all possible choices of $\epsilon > 0$, integer n and points $x_1, \ldots x_n \epsilon B$. Here $N(\Lambda_0; x_1 \ldots x_n, \epsilon) = \{\Lambda : \Lambda \epsilon B^*, |\Lambda(x_i) - \Lambda_0(x_i)| < \epsilon \text{ for } i = 1, 2, \ldots n\}$. It is easy to verify that these sets satisfy the Hausdorff postulates for neighbourhoods. The topology derived from this neighbourhood system is called the weak topology on B^* A net $\{\Lambda_n\}$ in B^* converges to Λ in B^* in this topology if and only if $\Lambda_n(x) \to \Lambda(x)$ for all $x \epsilon B$. We say that Λ_n converges weakly to Λ and write $\Lambda_n \Longrightarrow \Lambda$ in symbols.

It is obvious that these remarks apply not only to B^* but to its subsets as well. The convergence and topology will then be the corresponding relativized notions.

Suppose now X is a metric space. For any finite measure μ on X $\int_X g d\mu$ is a bounded linear functional on the Banach space $C(X)$ of bounded real-valued continuous functions on X and consequently the space of finite measures on X can be considered as a subset of the adjoint space of $C(X)$. We discuss in this paper the weak topology of this subset. The precise definition will be given at the end of this section.

Notation and Terminology. Throughout most of the rest of the paper, X denotes a separable metric space with distance function d. $\mathcal{G}$ and $\mathcal{F}$ are respectively the classes of open and closed subsets of X. $\mathcal{S}$ denotes the smallest σ-field containing $\mathcal{G}$. Sets in $\mathcal{S}$ will be called the Borel sets of X. A signed measure is a finite, real-valued and countably additive set function on $\mathcal{S}$. A measure is a non-negative signed measure. A probability measure is a measure m with $m(X) = 1$. $\mathfrak{M}$ is the space of all signed measures on $\mathcal{S}$; $\mathfrak{M}^+$ is the space of all measures on $\mathcal{S}$; and $\mathfrak{M}_p$ is the space of all probability measures on $\mathcal{S}$; $\mathfrak{M}_p \subset \mathfrak{M}^+ \subset \mathfrak{M}$. For $\phi \epsilon \mathfrak{M}$, $|\phi|$ is its total variation and ϕ^+ and ϕ^- are its positive and negative parts. $|\phi|$, ϕ^+ and ϕ^- are all in $\mathfrak{M}^+$ and $\phi = \phi^+ - \phi^-$, $|\phi| = \phi^+ + \phi^-$. (Notation and terminology as in Halmos, 1950).

We give below some lemmas. These are known and proofs are sketched here only for the sake of completeness.

Lemma 2.1: *For any two $\phi_1, \phi_2 \epsilon \mathfrak{M}$ the following statements are equivalent:* (a) $\phi_1 = \phi_2$ *on* $\mathcal{S}$, (b) $\phi_1 = \phi_2$ *on* $\mathcal{F}$, (c) $\phi_1 = \phi_2$ *on* $\mathcal{G}$.

Proof: The relations (a) $\Longrightarrow$ (b) and (a) $\Longrightarrow$ (c) are evident. Since closed sets are complements of open sets and X is both open and closed, (b) $\Longleftrightarrow$ (c). It thus remains to show that (b) $\Longrightarrow$ (a). We observe that $X \epsilon \mathcal{F}$ and $\mathcal{F}$ is closed under finite unions and intersections. Hence the class $\mathcal{F}_1$ of finite disjoint unions of proper differences of sets of $\mathcal{F}$ is a field. Since ϕ_1 and ϕ_2 are finite and additive and agree on $\mathcal{F}$ they agree on $\mathcal{F}_1$. Consequently they agree on the minimal σ-field over $\mathcal{F}_1$, i.e., on $\mathcal{S}$.

Given X, $C(X)$ is the space of real-valued bounded continuous functions on X and $U(X) \subset C(X)$ is the subset of all bounded uniformly continuous functions on X. In general, the set $U(X)$ depends on the choice of the metric d of X while $C(X)$ does not. Both $C(X)$ and $U(X)$ are Banach spaces under the norm $\| . \|$, where $\|f\| = \sup_{x \epsilon X} |f(x)|$. $U(X)$ is a subspace of $C(X)$.

Lemma 2.2 : *If A and B are closed subsets of X such that the minimum distance between A and B, denoted by (A, B), is > 0, then there is a $g \epsilon U(X)$ such that $0 \leqslant g(x) \leqslant 1$ for all $x \epsilon X$, $g(x) = 0$ for $x \epsilon B$ and $g(x) = 1$ for $x \epsilon A$.*

Proof: If (x, C) denotes the minimum distance of any point $x \epsilon X$ from a closed subset C, the function

$$g(x) = \frac{(x, B)}{(x, B)+(x, A)}$$

satisfies the conditions stated.

Lemma 2.3 : *$\phi = 0$ if and only if $\int_X g d\phi = 0$ for all $g \epsilon U(X)$.*

Proof: The 'only if' part is obvious. For the 'if' part, let A be any closed set and let $B_n = \left\{x : (x, A) \geqslant \frac{1}{n}\right\}$. B_n is closed and $(A, B_n) \geqslant \frac{1}{n} > 0$. Let $g_n \epsilon U(X)$ be obtained from lemma 2.2 by setting $B = B_n$. Then, from $\int_X g_n d\phi^+ = \int_X g_n d\phi^-$ we deduce that $\phi^-(A) \leqslant \phi^+(X-B_n)$ and $\phi^+(A) \leqslant \phi^-(X-B_n)$. Since $X-B_n \downarrow A$, it follows that $\phi^+(A) = \phi^-(A)$. Since A is an arbitrary closed set, it follows from lemma 2.1 that $\phi^+ = \phi^-$ on $\mathcal{S}$, i.e., $\phi = 0$. This completes the proof.

If X is a compact metric space, it is known that $C(X)$ is a separable Banach space (Kelley, 1956 p. 245). If X is compact and Λ a bounded linear functional on $C(X)$, then a famous theorem of Riesz states that there exists a $\phi \epsilon \mathfrak{M}$ such that $\Lambda(h) = \int_X h d\phi$ for all $h \epsilon C(X)$ and $\| \Lambda \| = |\phi|(X)$. ϕ is unique in virtue of lemma 2.3. If Λ is a non-negative linear functional, i.e., $\Lambda(h) \geqslant 0$ whenever $h(x) \geqslant 0$ for all $x \epsilon X$, then Λ is bounded and the ϕ occuring in the representation is in $\mathfrak{M}^+$ (for these facts, see Halmos, 1950, Ch. X).

Convergence in $\mathfrak{M}$: Given a net $\{\phi_n\}$ in $\mathfrak{M}$ and a ϕ in $\mathfrak{M}$, we say that ϕ_n *converges weakly to* ϕ ($\phi_n \Longrightarrow \phi$) if $\lim_{n \to \infty} \int_X g d\phi_n = \int_X g d\phi$ for all $g \epsilon C(X)$. Identifying $\mathfrak{M}$ as usual as a subset of the adjoint space $C^*(X)$ of $C(X)$, this is seen to be the weak convergence in $C^*(X)$ relativized to $\mathfrak{M}$. In future, the topology discussed on $\mathfrak{M}$ and its subsets will be the relativized weak topology.

3. Convergence in $\mathfrak{M}^+$

Lemma 3.1: *If $\{\mu_n\}$ is a net in $\mathfrak{M}^+$ and $\mu\epsilon\mathfrak{M}^+$, then the following statements are mutually equivalent :*

(a) $\mu_n \Longrightarrow \mu$

(b) $\lim\limits_{n\to\infty} \int\limits_X g d\mu_n = \int\limits_X g d\mu$ for all $g\epsilon U(X)$

(c) $\liminf\limits_{n\to\infty} \mu_n(G) \geqslant \mu(G)$ for all $G\epsilon\mathcal{G}$ and $\mu_n(X) \to \mu(X)$

(d) $\limsup\limits_{n\to\infty} \mu_n(C) \leqslant \mu(C)$ for all $C\epsilon\mathcal{F}$ and $\mu_n(X) \to \mu(X)$

(e) $\lim\limits_{n\to\infty} \mu_n(A) = \mu(A)$ for every $A\epsilon\mathcal{S}$ with μ-null boundary.

This is known. For a proof see Billingsley (1956).

For each $x \epsilon X$, let p_x denote the probability measure with total mass concentrated at the point x.

Lemma 3.2 : *X is homeomorphic to the set $D = \{p_x : x\epsilon X\}$.*

Proof : It is enough to prove that $x_n \to x$ if and only if $p_{x_n} \Longrightarrow p_x$. If $x_n \to x$, $\int\limits_X g dp_{x_n} = g(x_n)$ which tends to $g(x) = \int\limits_X g dp_x$ for each $g\epsilon C(X)$. This proves that $p_{x_n} \Longrightarrow p_x$. Conversely, suppose that $p_{x_n} \Longrightarrow p_{x_0}$. If x_n does not converge to x_0, there is an open set G and a subset x_m such that $x_0 \epsilon G$ and $x_m \epsilon X-G$ for all m. Take a continuous function g such that $0 \leqslant g(x) \leqslant 1$ for all x, $g(x_0) = 0$ and $g(x) = 1$ for $x\epsilon X-G$. Then $\int\limits_X g dp x_m = 1$ while $\int\limits_X g dp_{x_0} = 0$. This contradicts the assumption that $p_{x_m} \Longrightarrow p_{x_0}$ and hence we must have $x_n \to x_0$. This completes the proof of the lemma.

Lemma 3.3 : *D is a sequentially closed subset of $\mathfrak{M}^+$.*

Proof : Let $\{x_n\}$ be sequence of points in X such that $p_{x_n} \Longrightarrow q$. We first show that $\{x_n\}$ must have a convergent subsequence. If not, (we can assume in this case that all the x_n are distinct) then $S = \{x_1, x_2, \ldots\}$ is a closed subset of X, and so is every subset of S. Since $p_{x_n} \Longrightarrow q$, $q(C) \geqslant \limsup\limits_{n\to\infty} p_{x_n}(C)$ for each closed set C. Hence $q(S_1) = 1$ for every infinite set $S_1 \subset S$, which is a contradiction.

Thus for some x and some subsequence $\{x_{n_k}\}$, $x_{n_k} \to x$. In this case $q = p_x$ which completes the proof that D is sequentially closed. It may be worthwhile to note that in consequence of lemma 3.2, $\{x_n\}$ itself $\to x$.

Lemma 3.4 : *If X is a totally bounded metric space, then $U(X)$ is a separable Banach space.*

Proof : We recall that if a metric space X is totally bounded, then its completion is compact. Let X_1 be the completion. Any $g\epsilon U(X)$ can be extended to a $\bar{g}\epsilon C(X_1)$ and since X is dense in X_1, this extension is unique and we have also $\sup_{x \epsilon X} |g(x)| = \sup_{x \epsilon X_1} |\hat{g}(x)|$. In other words, the Banach spaces $U(X)$ and $C(X_1)$ are isomorphic. Since X_1 is a compact metric space, $C(X_1)$ is separable. This shows that $U(X)$ is separable and completes the proof of the lemma.

We now prove our metrization theorem.

Theorem 3.1 : (Metrization of $\mathfrak{M}^+$). *$\mathfrak{M}^+$ can be metrized as a separable metric space if and only if X is a separable metric space.*

Proof: We prove the 'if' part first. Since X is a separable metric space, it can, as a consequence of the celebrated theorem of Urysohn, be topologically imbedded in a countable product of unit intervals. Consequently there exists a totally bounded metrization of X which we will now impose on X. It follows from Lemma 3.4 that $U(X)$ is separable. Let $\{g_1, g_2, \ldots\}$ be a countable dense subset of $U(X)$ with $g_1(x) = 1$ for all x.

Let R be the countable product of the real lines and define the map T of $\mathfrak{M}^+$ into R as follows. For each $m \epsilon \mathfrak{M}^+$, $T(m) = (\int g_1 dm, \int_X g_2 dm, \ldots)$. We now show that T maps $\mathfrak{M}^+$ homeomorphically into R.

Firstly, T is one-one. For, if $T(m) = T(n)$, then $\int_X g_r dm = \int_X g_r dn$ for all r. Since $\{g_1, g_2, \ldots\}$ is dense in $U(X)$, this implies that $\int_X g dm = \int_X g dn$ for all $g \epsilon U(X)$. It now follows from Lemma 2.3 that $m = n$.

Secondly, T is continuous. For, if for a net $\{m_n\}$, $m_n \Longrightarrow m$, then $\int_X g_r dm_n \to \int_X g_r dm$ as $n \to \infty$ for all r. This implies however that $T(m_n) \to T(m)$.

Lastly, T^{-1} is continuous. For, let $\{m_n\}$ be a net in $\mathfrak{M}^+$ and let $T(m_n) \to T(m)$, i.e., $\int_X g_r dm_n \to \int_X g_r dm$ for all r. We will show that $m_n \Longrightarrow m$. Since $g_1 \equiv 1$, we have $m_n(X) \to m(X)$, and hence $m_n(X) \leqslant c$ for all n after some n_0, where c is a constant. We then have, for any r, and n following n_0

$$\left| \int_X g dm_n - \int_X g dm \right| \leqslant 2c\|g - g_r\| + \left| \int_X g_r dm_n - \int_X g_r dm \right|$$

and consequently,
$$\limsup_{n \to \infty} \left| \int_X g dm_n - \int_X g dm \right| \leqslant 2c\|g - g_r\|$$

which can be made $<$ any ϵ for some r (since the set $\{g_1, \ldots\}$ is dense in $U(X)$). This proves that $\int g dm_n \to \int_X g dm$ for each $g \epsilon U(X)$, i.e, $m_n \Longrightarrow m$.

The proof that $\mathfrak{M}^+$ is a separable metric space is now complete, since R is a separable metric space and $\mathfrak{M}^+$ is shown to be homeomorphic to a subset of it.

For the 'only if' part, we observe that X is homeomorphic to $D = \{p_x : x \epsilon X\}$ and D is a separable metric space whenever $\mathfrak{M}^+$ is so. Consequently X itself is separable. This completes the proof.

Theorem 3.2 : *Let X be a separable metric space and $E \subset X$ dense in X. Then the set of all measures in $\mathfrak{M}^+$ which vanish outside finite subsets of E is dense in $\mathfrak{M}^+$*

Proof : It is obviously enough to prove that the set of all measures in $\mathfrak{M}^+$ which vanish outside finite subsets of X is dense in $\mathfrak{M}^+$. We can effect a further simplification by observing that any $\mu \epsilon \mathfrak{M}^+$ which vanishes outside some countable set of X is the weak limit of a sequence $\{\mu_n\}$ of measures each of which vanishes outside some finite subset of X. We will therefore prove that the set of all measures in $\mathfrak{M}^+$ which vanish outside countable subsets of X, is dense in $\mathfrak{M}^+$.

Choose and fix $\mu \epsilon \mathfrak{M}^+$. Since X is separable we can, for each integer n, write X as $\bigcup A_{nj}$ where $A_{nj} \cap A_{nk} = \phi$ for $j \neq k$, $A_{nj} \epsilon S$ for each j and diameter $(A_{nj}) \leqslant \frac{1}{n}$ for all j. Let $x_{nj} \epsilon A_{nj}$ be arbitrary. Define $\mu_n \epsilon \mathfrak{M}^+$ as the measure with masses $\mu(A_{nj})$ at the points x_{nj}. Let $g \epsilon U(X)$ be arbitrary and put

$$\alpha_{nj} = \inf_{x \epsilon A_{nj}} g(x), \quad \beta_{nj} = \sup_{x \epsilon A_{nj}} g(x).$$

Since g is uniformly continuous and since diameter $(A_{nj}) \to 0$ as $n \to \infty$ uniformly in j, $\beta_{nj} - \alpha_{nj} \to 0$ as $n \to \infty$ uniformly in j. Now

$$\left| \int_X g d\mu_n - \int_X g d\mu \right| = \left| \sum_j \int_{A_{nj}} (g - g(x_{nj})) d\mu \right| \leqslant \sup_j (\beta_{nj} - \alpha_{nj}) \to 0$$

as $n \to \infty$. Since $g \epsilon U(X)$ is arbitrary, this proves that $\mu_n \Longrightarrow \mu$ and completes the theorem.

Next we establish a theorem needed for the investigation of the topological completeness of $\mathfrak{M}^+$. The theorem can be discussed in situations more general than the present one (see for instance Kolmogorov and Prohorov, 1954.)

Theorem 3.3 : *Let X be a compact metric space. Then $K \subset \mathfrak{M}$ is conditionally sequentially compact if and only if*

$$\sup_{\phi \epsilon K} |\phi| (X) < \infty.$$

Proof : Since X is a compact metric space, $\mathfrak{M}$ can be regarded (in view of the Riesz theorem) as the adjoint space of $C(X)$ and weak convergence in $\mathfrak{M}$ as weak convergence in the adjoint space. Since $C(X)$ is separable, Theorem 3.3 is now seen to be a corollary of the theorem of Banach (Banach, 1932) which states that a subset of the adjoint space of a separable Banach space is conditionally sequentially compact if and only if it is norm-bounded.

Theorem 3.4 : *When X is compact, $\mathfrak{M}^+$ can be metrized as a separable and complete metric space. $\mathfrak{M}_p$ is compact if and only if X is compact.*

Proof : Since X is compact $C(X)$ is separable. Let $\{g_1, g_2, \ldots\}$ with $g_1 \equiv 1$ be a dense subset of $C(X)$ and consider the map T defined in the proof of theorem 3.1 which imbeds $\mathfrak{M}^+$ into R. We first show that $T(\mathfrak{M}^+)$ is a closed subset of R. In fact let $T(m_n) \to \xi$ where $\xi = (\alpha_1, \alpha_2, \ldots)$ is a point of R and $\{m_n\}$ a net in $\mathfrak{M}^+$.

Thus $\alpha_r = \lim_{n \to \infty} \int_X g_r dm$. Since $g_1 \equiv 1$ and $\{g_1, g_2 \ldots\}$ is dense in $C(X)$, it follows, on using the argument given in the proof of Theorem 3.1, that $\lim_{n \to \infty} \int_X g dm_n$ exists for each $g \epsilon C(X)$. If the limit is written as $\Lambda(g)$, then Λ is a non-negative linear functional on $C(X)$ and hence for some $m \epsilon \mathfrak{M}^+$, $\Lambda(g) = \int_X g dm$ for all $g \epsilon C(X)$. This shows that $T(m) = \xi$ and completes the proof that $T(\mathfrak{M}^+)$ is closed in R. Since R itself is a separable and complete metric space, any closed subset of it is likewise a separable and complete metric space. Hence the first part of the theorem.

For the second part of the theorem, we note that if X is compact, then by virtue of Theorems 3.1 and 3.3 $\mathfrak{M}_p$ is a compact metric space. Conversely, if $\mathfrak{M}_p$ is a compact metric space, X also is a compact metric space since it is (via Lemmas 3.2 and 3.3) homeomorphic to a closed, and hence compact, subset of $\mathfrak{M}_p$. This completes the proof of the theorem.

The implication from Theorem 3.4 that $\mathfrak{M}_p$ is sequentially compact whenever X is compact is proved by a different method in Wald's book on decision functions (Wald, 1950).

We now prove a theorem on the topological completeness of $\mathfrak{M}^+$. We recall that a metric space is called topologically complete if it is homeomorphic to a complete metric space. We shall require the following theorem (Vaidyanathaswamy, 1947). A metric space is topologically complete if and only if it is a G_δ in some complete metric space, in which case it is a G_δ in every complete metric space into which it can be topologically embedded.

Theorem 3.5 : *Suppose that X is separable. Then $\mathfrak{M}^+$ is topologically complete if and only if X is so.*

Proof : The 'if' part is proved first. As in the proof of Theorem 3.1, we assume that X is totally bounded and hence that X_1, its completion, is compact. Since X is topologically complete, X is a G_δ in X_1. Let $\mathfrak{M}_1^+$ denote the space of measures on X_1 and $\mathfrak{M}_0 = \{m : m \epsilon \mathfrak{M}_1^+, m(X_1 - X) = 0\}$. Then $\mathfrak{M}_0$ and $\mathfrak{M}^+$ are homeomorphic. Since $\mathfrak{M}_1^+$ is a separable and topologically complete metric space (Theorem 3.4), it is enough to show that $\mathfrak{M}_0$ is a G_δ in $\mathfrak{M}_1^+$. Since X is a G_δ in X_1, $X = \bigcap_k G_k$, each G_k being open in X_1 and $G_1 \supset G_2 \supset \ldots$. Hence $\mathfrak{M}_0 = \bigcap_k \{m : m(X_1 - G_k) = 0\}$ $= \bigcap_k \bigcap_r \left\{ m : m(X_1 - G_k) < \frac{1}{r} \right\}$. It remains to show that, for each k and $r(>0)$, $\left\{m : m(X_1 - G_k) < \frac{1}{r}\right\}$ is open. The complement of this set is $\left\{m : m(X_1 - G_k) \geq \frac{1}{r}\right\}$ which is closed because the relations $m_n(X_1 - G_k) \geq \frac{1}{r}$, $m_n \Longrightarrow m$ imply that $m(X_1 - G_k)$ $\geq \limsup_{n \to \infty} m_n(X_1 - G_k) \geq \frac{1}{r}$. This completes the proof of the 'if' part.

For the 'only if' part, we note that the topological completeness of $\mathfrak{M}^+$ implies that of D which is a closed subset of $\mathfrak{M}^+$ (Lemma 3.3). Since X, in view of Lemma

3.2 is homeomorphic to D, it follows that X is topologically complete. This proves the theorem.

Remark 1 : Since $\mathfrak{M}_p$ is a closed subset of $\mathfrak{M}^*$ it follows that $\mathfrak{M}_p$ is topologically complete if and only if X is so.

Remark 2 : The above result and Theorem 3.1 applied to $\mathfrak{M}_p$, form the generalizations of the results of P. Levy for probability measures on the real line.

4. CONVERGENCE IN $\mathfrak{M}$

In this section, we study the conditions under which $\mathfrak{M}$ is metrizable. Defining for each $\phi \epsilon \mathfrak{M}$, $\|\phi\| = |\phi|(X)$, we see that $\|\ \|$ is a norm for $\mathfrak{M}$. $\mathfrak{M}$ is a Banach space under that norm and $\|\phi_n - \phi\| \to 0$ as $n \to \infty$ if and only if $|\phi_n(A) - \phi(A)| \to 0$ as $n \to \infty$ uniformly for all $A \epsilon \mathcal{S}$. The topology of this Banach space will be referred to as the norm topology for $\mathfrak{M}$.

Theorem 4.1 : *Suppose that X is separable. Then, the weak topology on $\mathfrak{M}$ is metrizable if and only if the norm and weak topologies on $\mathfrak{M}$ are identical.*

Proof : Evidently if the weak topology coincides with the norm topology, it is metrizable. We have therefore to show the converse, i.e., we must show that if the weak topology on $\mathfrak{M}$ is metrizable, then every set $V_{\phi_0, \lambda} = \{\phi : \|\phi - \phi_0\| < \lambda\}$ (for $\phi_0 \epsilon \mathfrak{M}$ and $\lambda > 0$) is weakly open. Since weakly open subsets remain so under translation in $\mathfrak{M}$ and under scalar multiplication, it is enough to show that for some λ, 0 is an interior point (in the weak topology) of $V_{0, \lambda}$.

Let d be the distance function metrizing the weak topology on $\mathfrak{M}$ and let $\mathcal{S}_n = \left\{ \phi : d(0, \phi) < \frac{1}{n} \right\}$ We assert that for some n, $\sup_{\phi \epsilon S_n} \|\phi\| < \infty$. If not, for each n we can find $\phi_n \epsilon \mathcal{S}_n$, such that $\|\phi_n\| > n$. This however is a contradiction since $\phi_n \Longrightarrow 0$ and hence $\limsup_{n \to \infty} \|\phi_n\| < \infty$ by the Banach–Steinhaus theorem (Banach, 1932, p. 80). Thus for some $n = n_0$, $\sup_{\phi \epsilon S_n} \|\phi\| < \infty$. Defining $\lambda = \sup_{\phi \epsilon S n_0} \|\phi\|$ we see that $0 \epsilon S_{n_0} \subset V_{0,\lambda}$. This shows that 0 is an interior point of $V_{0,\lambda}$ in the weak topology and completes the proof of the theorem.

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