Studies on Interval Digraphs

M.Tech. Dissertation Report

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Abstract

The *intersection digraph* of a family of ordered pairs of sets $\{(S_v, T_v) : v \in V\}$ is the digraph $D(V, E)$ such that $uv \in E$ if and only if $S_u \cap T_v \neq \emptyset$. *Interval digraph* are those intersection digraphs for which the subsets are intervals on the real line. We study the characterization of interval digraphs in terms of zeros partition property Sen et.al. (1) , (2) of its adjacency matrix and in terms of ferrers digraphs in Sen et.al. (1). The important problem of characterizing interval digraphs by its forbidden subgraphs is still open. Algorithm for recognizing interval digraphs was given in Müller (3). We propose an efficient algorithm for recognizing interval digraphs based our approach to characterize the class of all interval digraph using forbidden subgraphs.

Acknowledgments

I would like to thank Prof. Sandip Das for suggesting this problem to me and also his support at all times made this dissertation possible. I got invaluable help and encouragement from Dr. Malay Sen at the later stage. The entire present work is an outcome of his previous works in the field of Interval Digraphs.

Contents

Chapter 1

Introduction

1.1 Basic definitions and Notation

Simple graphs are undirected graphs without loops and multiple edges, and denoted by: $G = (V, E)$ where $V = V(G)$ is the vertex-set of G and $E = E(G)$ is the edge-set of G. $v(G) =\mid V(G) \mid$ is the number of vertices in G (order), $e(G) =\mid E(G) \mid$ is the number of edges in G (size).

Digraphs are denoted by $D = (V, E)$. We use $A(D)$ for the adjacency matrix of a digraph D. The *complement* $\bar{D}$ of a digraph D has adjacency matrix obtained by converting 0's to 1's and 1's to 0's in $A(D)$. $N^+(v)$ and $N^-(v)$ denote the successor set (out-neighbors) and predecessor set (in-neighbors) of a vertex v in a digraph.

For a bipartite graph with source vertex set X and sink vertex set Y, the *biadjacency matrix* is the submatrix of the adjacency matrix consisting of the rows for X and columns for Y.

1.2 Intersection Graphs

An *intersection representation* of a graph G is a family of sets $\{S_v : v \in V(G)\}$, such that there is an edge between u, v if and only if $S_u \cap S_v \neq \emptyset$. If $\{S_v\}$ is an intersection representation of G, then G is the *intersection graph* of $\{S_v\}$. When $\{S_v\}$ is allowed to be an arbitrary family of sets, the class of graphs obtained as intersection graphs is simply all undirected graphs, Marczewski (4).

The problem of characterizing the intersection graphs of families of sets having some specific topological or other pattern is often very interesting and frequently has applications to the real world.

1.3 Interval Graphs

A graph is an *interval graph* if it is the intersection graph of a family of intervals on a linearly ordered set (like the real line).

Several characterizations are known for interval graphs. Property B in Theorem 1.1 is due to Gilmore and Hoffman (5), and property C is due to Fulkerson and Gross (6).

A $0,1$-matrix is said to have the *consecutive ones property (for rows)* if its columns can be permuted so that the ones in each row appear consecutively. The incidence matrix between the vertices and maximal complete subgraphs of a graph G is called *clique matrix* $\mathbf{M}$.

Theorem 1.1 (Gilmore and Hoffman (5),Fulkerson and Gross (6)) *The following equivalent conditions on a graph G characterize the interval graphs.*
A. *G has an interval representation.*
B. *G contains no chordless 4-cycle* [1] *and its complement $\bar{G}$ is a comparability graph*[2].
C. *The clique matrix* $\mathbf{M}$ *has consecutive 1′s property.* □

A recognition algorithm for interval graphs was obtained using the above consecutive 1′s property of the clique matix. The algorithm is a two-step process. First, verify that G is chordal and, if so, enumerate its maximal cliques. This can be excuted in time proportional to $\mid V \mid + \mid E \mid$ and will produce at most $n = \mid V \mid$ maximal cliques. Second, test whether or not the cliques can be ordered so that those which cantain vertex v occur consecutively for every $v \in V$. Booth and Leuker (7) have shown that this step can also be executed in linear time. Thus we have the following Theorem.

Theorem 1.2 (Booth and Leuker (7)) *Interval graphs can be recognized in linear time.* □

However, the earliest characterization of interval graphs was obtained by Lerkerkerker and Boland (8). Their result embodies the notion that an interval graph cannot branch into more than two directions, nor can it circle back onto itself.

[1] G is a *chordal graph*

[2] A *transitive orientation* of a graph G is an orientation F such the whenever xy and yz are edges in F, also there is an edge xz in G that is oriented from x to z in F.A simple graph G is a *comparability graph* if it has a transitive orientation.

Theorem 1.3 (Lekerkerker and Boland (8)) *An undirected graph G is an interval graph if and only if the following two conditions are satisfied:*
A. *G is a chordal graph, and*
B. *any three vertices of G can be ordered in such a way that every path from the first vertex to the third vertex passes through a neighbor of the second vertex.* □

Three vertices which fail to satisfy B are called *astroidal triple*. They would have to be pairwise nonadjacent, but any two of them would have to be connected by a path which avoids neighborhood of the remaining vertex. Thus, G is an interval graph if and only if G is chordal and contains no astroidal triple. Lerkerkerker and Boland (8) also determined all the minimal forbidden induced subgraphs for the class of interval graphs.

Theorem 1.4 (Lekerkerker and Boland (8)) *The minimal forbidden induced subgraphs for the class of interval digraphs are: bipartite claw, n-net for every $n \geq 2$, umbrella, n-tent for every $n \geq 3$, and C_n for every $n \geq 4$ (cf. Fig. 1.1).* □

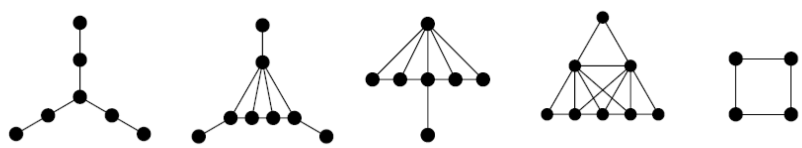

Figure 1.1: Minimal forbidden induced subgraphs for the class of interval graphs

1.4 Interval Digraphs/Bigraphs

Beineke and Zamfirescu (9) introduced the analogous concept of intersection digraph, under the name “connection digraph“. Let $\{S_v, T_v\}$ be a collection of ordered pairs of sets indexed by a set V; we call S_v the *source set* and T_v the *terminal set* for v. The *intersection digraph* of this collection is the digraph with vertex set V having edge from u to v if and only if $S_u \cap T_v \neq \emptyset$. The pairs of sets form an *intersection representation*.

Harary, Kabell, and McMorris (10) defined an equivalent intersection model for bipartite graphs. Treating the partite sets as source vertices and sink vertices, we represent each vertex by one set and take the intersection graph, but we ignore intersection between source sets or between sink sets to obtain a bipartite graph. Intersection digraphs correspond to intersection bigraphs by splitting each vertex v into a source copy x_v represented by S_v and a sink copy y_v represented by T_v, and optionally deleting source or sink vertices when the corresponding set in the representation is empty.

When source sets and sink sets are all intervals, we obtain an *interval digraph* or *interval bigraph*. Interval digraphs were characterized by Sen et.al. in (1) and (2). We discuss them in Chapter 3.

A recognition algorithm for interval bigraphs (interval digraphs) was given by Müller (3) based on dynamic programming approach. This algorithm recursively constructs a bipartite interval representation of a graph from bipartite interval representation of proper subgraphs. However, the overall running time of the algorithm is $O(nm^6(n+m)logn)$.

We propose a greedy algorithm for interval digraphs based on the characterization by Sen et.al. (1) and obtain a running time of $O(n^4)$. The algorithm is discussed in Chapter 4.

The problem of characterizing the whole class of interval digraphs by forbidden induced subgraphs is still open.

1.5 Ferrers Digraphs

Ferrers digraph was introduced independently by Guttman (11) and Riguet (12). A digraph is a *Ferrers Digraph* if its successor sets (or its predecessor sets) form a chain under inclusion.

The *Ferrers dimension* of D is defined to be the minimum number of Ferrers digraphs whose intersection is D. The digraphs of Ferrers dimension 2 have been characterized by Cogis (13) and Doignon, Ducamp, and Falmagne (14) in different contexts. This characterization yields a polynomial algorithm for testing whether a digraph has Ferrers dimension at most 2. These topics are discussed in Chapter 2.

In Sen et.al. (1) digraph D is characterized as interval digraph if and only if it is the intersection of two Ferrers digraphs whose union is complete digraph, thus the Ferrers dimension of interval digraphs is at most 2. However, it was shown that not every digraph of Ferrers dimension 2 is an interval digraph. Details are in Chapter 3.

Chapter 2

Ferrers Digraphs

Riguet (12) introduced Ferrers digraphs as “Ferrers relations” and proved the equivalence of A, B, C, D below. Doignon, Ducamp, and Falmagne (14) called them *biorders* and proved E.

In an arbitrary matrix, we define a *stair* to be a walk from the upper left corner to the lower right corner that moves rightward or downward between rows and between columns, crossing each row and column once. The *understair* consists of the positions below or to the left of the stair, and the *overstair* consists of the positions above or to the right of it.

Theorem 2.1 (Riguet (12), Doignon et.al. (14)) *For a digraph D, the following conditions are equivalent.*
A. *A(D) has no 2 by 2 submatrix that is a permutation matrix.*[1]
B. *The successor sets of D are linearly ordered by inclusion.*
C. *The predecessor sets of D are linearly ordered by inclusion.*
D. *The rows and columns of A(D) can be permuted independently so that some stair in the resulting matrix separates the 0's from the 1's.*
E. (Biorder representation) *There exists two real-valued functions f,g on V(D) such that* $uv \in E(D)$ *if and only if* $f(u) > g(u)$.

Proof: $B \Leftrightarrow A \Leftrightarrow C$. The successor sets fail to form an inclusion chain if and only if there exists u, v such that $x \in N^+(u) - N^+(v)$ and $y \in N^+(v) - N^+(u)$, which holds if and only if rows u, v and columns x, y form the forbidden submatrix. The analogous argument applies for predecessor sets.

$B, C \Rightarrow D$. It suffices to permute the rows and the columns so that every entry below or leftward of a 1 is a 1. Place the rows in increasing order of

[1]We call such a forbidden submatrix an *obstruction*.

out-degree and the columns in decreasing order of in-degree, breaking ties arbitrarily. If $A_{rs} = 1$, then the inclusion orders yield $v_s \in N^+(u_i)$ for all $i \geq r$ and $u_r \in N^-(v_j)$ for all $j \leq s$, as desired.

$D \Rightarrow E$. Consider such a permutation of $A(D)$. The stair takes $2n$ moves, crossing row u after its last 1 and column v above its first 1. Let $f(v) = r$ if row v is crossed on step r, and let $g(v) = r$ if column v is crossed on step r. Now $f(u) > g(v)$ corresponds to crossing row u after column v, meaning that row u is below the stair in column v, which holds if and only if $uv \in E(D)$.

$E \Rightarrow A$. If D has a biorder representation f, g and rows u, v and columns x, y of $A(D)$ form a permutation matrix with $A_{u,x} = A_{v,y} = 1$, then $f(u) > g(x)$ and $f(v) > g(y)$, but $f(u) \leq g(y)$ and $f(v) \leq g(x)$. Summing yields two contradictory inequalities. □

Cogis (13) defined a graph $\mathbf{H}(D)$ whose vertices correspond to the 0's of the adjacency matrix, with two such vertices joined by an edge if the correponding 0's belong to an obstruction. In the following Theorem Cogis charaterize the digraph of Ferrers dimension at most 2.

Theorem 2.2 (Cogis (13), Doignon et.al. (14)) *A digraph D has Ferrers dimension at most* 2 *if and only if* $\mathbf{H}(D)$ *is bipartite.* □

This equivalence yields a short proof of the permutation characterization of Ferrers dimension 2, because we can omit the more difficult step of showing that $\mathbf{H}(D)$ bipartite implies the other conditions.

Theorem 2.3 (Sen et.al. (1), Cogis (13), Doignon et.al. (14)) *The following conditions are equivalent:*
A. *D has Ferrers dimension at most* 2.
B. *The rows and columns of* $\mathbf{A}(D)$ *can be (independently) permuted so that no* 0 *has a* 1 *both below it and to its right.*
C. *The graph* $\mathbf{H}(D)$ *is bipartite.*

Proof: $A \Rightarrow B$. Let F_1, F_2 be two Ferrers digraphs whose intersection is D, with adjacency matrices A_1, A_2. Let $u_1, \ldots, u_n$ be the row ordering of A_1 that with some column ordering, puts the 0's of A_1 in the lower left and its 1's in the upper right. Let $w_1, \ldots, w_n$ be the column ordering of A_2 that, with some row ordering, puts the 0's of A_2 in the upper right and its 1's in the lower left. Put the rows of $\mathbf{A}(D)$ in the order $u_1, \ldots, u_n$ and its columns in the order $w_1, \ldots, w_n$. We denote the matrix position corresponding to vertex pair $u_i w_j$ as $M_{u_i w_j}$, where M is any of $A_1, A_2, \mathbf{A}(D)$. If $\mathbf{A}(D)_{u_i w_j} = 0$, then $D = F_1 \cap F_2$ implies $(A_1)_{u_i w_j} = 0$ or $(A_2)_{u_i w_j} = 0$. If $(A_1)_{u_i w_j} = 0$, then $(A_1)_{u_r w_j} = 0$ for all $r > i$, and hence $\mathbf{A}(D)_{u_r w_j} = 0$ for $r > i$, even though this column may be in a different position in A_1 and $\mathbf{A}(D)$. Similarly, if $(A_2)_{u_i w_j} = 0$, then the remainder of the row in $\mathbf{A}(D)$ is 0.

$B \Rightarrow C$. Permute the rows and columns of $\mathbf{A}(D)$ so that no 0 has a 1 both to its right and below. Let R be the set of 0's having a 1 somewhere below them, and let C be the set of 0's having a 1 somewhere to the right. For any 2 by 2 submatrix forming a couple, the 0's must be an R in the upper right and a C in the lower left; these are the only edges in $\mathbf{H}(D)$. Therefore H is bipartite, with the 0's having no 1 to the right or below generating isolated points.

$C \Rightarrow A$. By Theorem 2.2, see Cogis (13) or Doignon, Ducamp, and Falmagne (14). □

The graph $\mathbf{H}(D)$ may be disconnected and may have isolated vertices for 0's belonging to no obstruction. Deleting the isolated vertices yields a graph $\mathbf{H}^b(D)$ called the *bare graph* associated with D.

Let D be a digraph with Ferrers dimension 2, so $\mathbf{H}(D)$ is bipartite. Let $\mathbf{I}$ denote the set of isolated vertices in $\mathbf{H}(D)$. Let (R, C) denote a bicoloration of $\mathbf{H}(D)$, where a *bicoloration* of a graph is an ordered pair of (possibly empty) stable sets whose union is the vertex set. let $H_1, \ldots, H_p$ denote the components of H^b, with (R_i, C_i) denoting a bicloration of H_i.

In proving his result, Cogis obtained a bicoloration (R, C) of $\mathbf{H}^b(D)$ such that $\mathbf{R} \cup \mathbf{I}$ and $\mathbf{C} \cup \mathbf{I}$ are Ferrers digraphs; this is called a *satisfactory bicoloration*. It yields the complement $\bar{D}$ as the union of two Ferrers digraphs, not necessarily edge-disjoint.

Chapter 3

Interval Digraph/Bigraph

A $0,1$-matrix has a *zero-partition* if its 0's admit a partition into sets C and R such that every entry to the right of an R is an R and every entry below a C is a C. A $0,1$-matrix has the *partitionable zeros property* if its rows and columns can be permuted independently to obtain a matrix having a zero-partition. The interval digraphs are those whose adjacency matrices have the partitionable zeros property (see Sen et.al. (1)). The addition of rows or columns of 0's doesnot affect this property, so the same statement characterizes biadjacency matrices of interval bigraphs.

Another characterization of interval digraphs is given by Sen et.al. (2) which is a specialization of a characterization of circular-arc digraphs. Given a stair in a matrix, let V_i be the maximal set o consecutive positions in row i, begining immediately to the right of the stair, such that every position in V_i has a 1. Similarly, let W_j be the maximal set of consecutive positions in column j, begining immediately below the stair, such that evey position in W_j has a 1. We say that a matrix has the *stair-linear ones property* if and only if its rows and columns can be permuted independently to admit a stair such that every 1 in the matrix is covered by the union of the V_i's and W_j's. We have the following Theorem.

Theorem 3.1 (Sen et.al. (1), (2); West (15)) *For a digraph D, the following conditions are equivalent.*
A. *D is an interval digraph.*
B. *$\bar{D}$ is the edge-disjoint union of two Ferrers digraphs.*
C. *$A(D)$ has the partitionable zeros property.*
D. *$A(D)$ has the stair-linear ones property.*

Proof: $A \Rightarrow B$. Let $S_v = [a(v), b(v)]$ and $T_v = [c(v), d(v)]$ in an interval representation of D. When $uv \in E(\bar{D})$, we have $S_u \cap T_v = \emptyset$. We put $uv \in E(D_1)$ is $b(u) < c(v)$ and $uv \in E(D_2)$ if $d(v) < a(u)$; this expresses $\bar{D}$ as the edge-disjoint union of D_1 and D_2. Each satisfies the biorder characterization of ferrers digraphs.

$B \Rightarrow C$. Suppose $\bar{D}$ is the edge-disjoint union of Ferrers digraphs D_1, D_2. By the biorder characterization of Ferrers digraphs, there exist functions a,b,c,d such that $(b(u) < c(v) \Leftrightarrow uv \in E(D_1))$ and $d(v) < a(u) \Leftrightarrow uv \in E(D_2))$. Place the rows of $A(D)$ in increasing order of $a(u)$, and place the columns in increasing order of $c(v)$. Let R and C be the set of 0's in $A(D)$ corresponding to the edges of D_1 and D_2, respectively; this partitions the 0's. Since $b(u) < c(v)$ when $uv \in E(D_1)$, the column ordering guarantees that evry position to the right of an R is in R. Similarly, since $d(v) < a(u)$ when $uv \in E(D_2)$, the row ordering guarantees that evry position below a C is in C.

$C \Rightarrow D(2)$. Permute the rows and columns of $A(D)$ to exhibit a zero-partition. Let S be the set of positions that contain an R or lie somewhere above R. By the definition of zero-partition, S is an overstair that contains no C. Every) in the overstair is an R, and hence te positions to its right are all 0. Every 0 in the understair is in C, so the positions below it are 0. Hence the 1's are covered as required for the stair-linear ones property.

$D \Rightarrow A$. Consider a permutation and stair exhibiting the stair-linear ones property. Let $u_1, \ldots, u_n$ be the vertex ordering by rows, and let $v_1, \ldots, v_n$ be the ordering by columns. We produce an interval representation. Let $a(u_i) = r$ if the stair crosses row i on move r, and let $c(v_j) = r$ if the stair crosses cloumn j on move r. let $b(u_i) = a(u_i)$ when V_i is empty, and otherwise let $b(u_i) = c(v_j)$, where j is the column of the rightmost position in V_i. Similarly, let $d(v_j) = c(v_j)$ when W_j is empty, and otherwise let $d(v_j) = a(u_i)$, where i is the row of the lowest position in W_j. Now let $S_u = [a(u), b(u)]$ and $T_v = [c(v), d(v)]$. If position (i, j) is in the overstair, then $S_u \cap T_v \neq \emptyset$ if and only if j is small enough that $(i, j) \in V_i$. Similarly, if (i, j) is in the understair, then $S_u \cap T_v \neq \emptyset$ if and only if i is samll enough that $(i, j) \in W_j$. Thus $S_u \cap T_v \neq \emptyset$ if and only if $uv \in E(D)$. □

The above Theorem implies that Ferrers dimension at most 2 is a necessary condition for an interval digraph. But it is not a suffient condition.

Theorem 3.2 (Sen et.al. (1)) *The interval digraphs are properly contained in the set of digraphs with Ferrers dimension at most* 2.

Proof: Any permutation of $\mathbf{A}(D)$ that satisfies condition C of Theorem 3.1 also satisfies condition B of Theorem 2.3, so inclusion holds. For proper containment, we show that the digraph D below, of Ferrers dimension 2, is not an interval digraph.

$$D = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

We claim that there is no way to permute the rows and columns of $\mathbf{A}(D)$ so as to satisfy condition C of Theorem 3.1. First, note that 0's of any obstruction must receive different labels; i.e., they cannot be both R or both C. Therefore, when we consider the bipartite $\mathbf{H}(D)$, the partite sets of each component must be all R's or all C's. For this D, $\mathbf{H}(D)$ consists of one nontrivial component and one isolated vertex corresponding to $D_{6,6}$. Leaving the assignment of this label unspecified, the two possibilities we must consider for the nontrivial component yield the assignments below.

$$\begin{pmatrix} 1 & 1 & 1 & R & R & R & R \\ 1 & 1 & 1 & 1 & 1 & R & R \\ 1 & 1 & 1 & 1 & 1 & 1 & R \\ C & 1 & 1 & 1 & 1 & 1 & 1 \\ C & 1 & 1 & 1 & 1 & C & 1 \\ C & C & 1 & 1 & R & 0 & R \\ C & C & C & 1 & 1 & C & 1 \end{pmatrix} \quad or \quad \begin{pmatrix} 1 & 1 & 1 & C & C & C & C \\ 1 & 1 & 1 & 1 & 1 & C & C \\ 1 & 1 & 1 & 1 & 1 & 1 & C \\ R & 1 & 1 & 1 & 1 & 1 & 1 \\ R & 1 & 1 & 1 & 1 & R & 1 \\ R & R & 1 & 1 & C & 0 & C \\ R & R & R & 1 & 1 & R & 1 \end{pmatrix}$$

Next we obtain a forbidden configuration that appears in each of these assignments. Let a, b, c, d be rows and A, B, C, D be the columns satisfying the following properties:

(1) R appears in positions $(a, D), (b, D), (b, C)$, and the rest of rows a, b is 1.

(2) C appears in positions $(d, A), (d, B), (c, B)$, and the rest of columns A, B is 1.

(3) Row c has at least two R's, and column C has at least two C's.

We claim that no ordering of rows and columns of a labeled matrix containing rows a, b, c, d and columns A, B, C, D as specified can have only R's to the right of each R and only C's below each C. Suppose there is such an ordering. Row a forces column D to be right-most, and then row b forces column C to be next to it. Similarly, column A forces row d at the bottom, and then column B forces row c immediately above it. But no the next to last diagonal position must be both R and C, since c has at least two R's and C has at least two C's.

Consider the potential assignments of R and C in $\mathbf{A}(D)$. For the assignment on the left, choose a, b, c, d to be rows $3, 1, 6, 7$, respectively, and A, B, C, D to be columns $3, 1, 6, 7$, respectively. For the assignment on the right, choose a, b, c, d to be rows $4, 5, 6, 1$, respectively, and A, B, C, D to be columns $4, 5, 6, 1$, respectively. In each case, these choices satisfy the requirement for the forbidden configuration. □

From previous chapter, we know that satisfactory bicoloration of $\mathbf{H}(D)$ is equivalent to Ferrers dimension 2. But for interval digraphs we need more. Hence Theorem 3.1 impies that D is an interval digraph if and only if $\mathbf{H}^b(D)$ has a satisfactory bicoloration such that $\mathbf{I}$ can be distributed to $\mathbf{R}$ and $\mathbf{C}$ to from two disjoint Ferrers digraphs.

Chapter 4

Interval Digraph Recognition Algorithm

As mentioned in the introduction, a recognition algorithm for interval digraphs (interval bigraphs) was given by Müller (3) based on dynamic programming approach. The overall running time of the algorithm is $O(nm^6(n+m)logn)$.

4.1 A Greedy Recognition Algorithm

Here we propose a greedy algorithm for interval digraphs based on the characterization given by Sen et.al. (1). If D is an interval digraph, we obtain a (R,C) coloring of the adjacency matrix $\mathbf{A}(D)$ such that some permutation of $\mathbf{A}(D)$ satisfies the partitionable zeros property, i.e. every entry to the right of an R is an R and every entry below a C is a C (this is same as obtaining the (R,C) bicoloration of the $\mathbf{H}(D)$). Otherwise, we decide that such an R,C coloring is not possible. In our algorithm we incrementally color the 0's of $\mathbf{A}(D)$ whenever it is possible to do so; otherwise if there exists no such 0, we make a random color choice. Once a color is assigned to a 0 in $\mathbf{A}(D)$, we call it to be *Fixed*.

To determine the color of a 0 in the position $A_{i,j}$, we consider all the 2×2 sub-matrices of $\mathbf{A}(D)$ with $A_{i,j}$ as one of its elements. We use rules R_1, R_2, R_3, described below, to fix the color of 0 at $A_{i,j}$. Here, the colors mentioned in the 2×2 sub-matrices are already *Fixed*. We get rule R_1, due to the fact that $\mathbf{H}(D)$ is bipartite. Rule R_2, R_3 is derived from the result that D is interval digraph if and only if $\mathbf{A}(D)$ has the partitionable zeros property.

Rule (R_1).

$$\begin{pmatrix} R & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} R & 1 \\ 1 & C \end{pmatrix} \tag{4.1a}$$

$$\begin{pmatrix} C & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} C & 1 \\ 1 & R \end{pmatrix} \tag{4.1b}$$

Rule (R_2).

$$\begin{pmatrix} R & 1 \\ 0 & R \end{pmatrix} \rightarrow \begin{pmatrix} R & 1 \\ R & R \end{pmatrix} \tag{4.2a}$$

$$\begin{pmatrix} C & 1 \\ 0 & C \end{pmatrix} \rightarrow \begin{pmatrix} C & 1 \\ C & C \end{pmatrix} \tag{4.2b}$$

Rule (R_3).

$$\begin{pmatrix} R & 1 \\ C & 0 \end{pmatrix} \rightarrow \begin{pmatrix} R & 1 \\ C & C \end{pmatrix} \tag{4.3a}$$

$$\begin{pmatrix} C & 1 \\ R & 0 \end{pmatrix} \rightarrow \begin{pmatrix} C & 1 \\ R & R \end{pmatrix} \tag{4.3b}$$

The above rules are applicable for any row or column permuation of the above 2×2 sub-matrices.

Algorithm: **INTERVAL-DIGRAPH-RECOG($\mathbf{A}(D)$)**

Step 1: While traversing row-wise from $A_{0,0}$ assign a *random color* to the first zero (i.e. not assigned any color, either R or C) of the $\mathbf{A}(D)$ matrix. Let it be $A_{i,j}$. Then we apply the rules R_1, R_2, R_3 to all the 2×2 sub-matrices containing $A_{i,j}$, to find the 0's in $\mathbf{A}(D)$ which can be colored. We call them *Tentative* elements and put them in *Queue*.

Step 2: If the *Queue* is not empty, *Dequeue* one element and use the rules R_1, R_2, R_3 to check for conflicts. If no conflict occurs, fix its color; upon fixing its color, we again apply the rules R_1, R_2, R_3 and similary *Enqueue* only the new elements (not already existing in the *Queue*). Loop again.
If conflict occurs, then D is not interval digraph.Stop.
If *Queue* is empty, go to **Step 1**.

Step 3: D is an interval graph. □

Claim 4.1 (Correctness of Recognition algorithm) *D is an interval digraph if and only if **INTERVAL-DIGRAPH-RECOG** returns a R, C coloring of $\mathbf{A}(D)$.*

Proof:
only if. If **INTERVAL-DIGRAPH-RECOG** returns a R, C coloring of $\mathbf{A}(D)$, then from equivalence $C \Leftrightarrow A$ in Theorem 3.1 it obvious that D is an interval digraph.
if. This is the difficult part of the proof. We need to show that if D is an interval digraph then **INTERVAL-DIGRAPH-RECOG** will return a proper R, C coloring of $\mathbf{A}(D)$. We can try to prove the contrapositive, i.e.

Subclaim: *If **INTERVAL-DIGRAPH-RECOG** fail to return a R, C coloring of $\mathbf{A}(D)$, then D is not an interval digraph.* □

If we prove this statement, then we are done. However, this means we have to give an algorithmic approach to the still open forbidden submatrices problem for the interval digraphs class. □

It is easy to see that our rules R_1, R_2, R_3 are exhaustive set of rules, as any other configuration of 2×2 sub-matrices donot force a 0 to take any particular color. So, we won't miss any conflicting configuration which might arise during the coloring process.

However, if the algorithm stops due to a conflict, it might be possible that some different *random choice* of color would have avoided this present conflict. Hence, our subclaim seems hard to prove.

4.1.1 Analysis

$\mathbf{A}(D)$ consists of $O(n^2)$ number of 0's. For each 0 in the matrix $\mathbf{A}(D)$ we consider 2×2 sub-matrices to fix its color and after that again apply the rules R_1, R_2, R_3 on $O(n^2)$ 0's. As there are fixed number of rules, overall rule checking takes constant amount of time. Thus it takes $O(n^2) \times (2 \times O(n^2))$ overall time, i.e. $O(n^4)$.

Appendix A

Interval Digraph Recognition Algorithm: C Implementation

A.1 Source Code

```c
#include<stdio.h>
#include<stdlib.h>
#include<errno.h>

typedef struct pos {
        int x;
        int y;
        struct pos *nxt;
     } pos;

pos *Q=NULL;
int rows=0,cols=0;

void putQ(int i,int j){
   if(Q!=NULL) {
      pos *tmp=Q;
      while(tmp->nxt!=NULL)
         tmp=tmp->nxt;
      tmp->nxt=(pos *)malloc(sizeof(pos));
      tmp->nxt->x=i;
      tmp->nxt->y=j;
      tmp->nxt->nxt=NULL;
   }
   else {
      Q=(pos *)malloc(sizeof(pos));
      Q->x=i;
      Q->y=j;
```

```c
      Q->nxt=NULL;
   }
}

pos *remQ() {
   pos *tmp=Q;
   Q=Q->nxt;
   tmp->nxt=NULL;
   return tmp;
}

int checkQ(pos *tmp) {
   pos *ptr=Q;
   while(ptr!=NULL) {
      if(ptr->x==tmp->x && ptr->y==tmp->y)
         return 1;
      else
         ptr=ptr->nxt;
   }
   return 0;
}

pos *check_matrix(char **);
void apply_rules(char **,pos *);
int check_rules(char **,int,int,int,int);
int fix_color(char **,int,int);

int main(int argc,char *argv[]) {

   FILE *fp;
   if(argc!=2) {
      printf("specify a single m?.txt file\n");
      return 0;
   }

   if((fp=fopen(argv[1],"r"))==NULL) {
      perror("fopen");
      return -1;
   }
   int in=0;
   int i=0,j=0;
   while((in=getc(fp))!=EOF) {
      if((char)in=='\n')
         ++i;
      else
         if((char)in=='1' || (char)in=='0')
            ++j;
   }

```

```
  rows=i;
  cols=j/i;
  printf("rows:%d,cols:%d\n",rows,cols);


  //allocate a contiguous block of memory for the matrix
  char *m=(char *)malloc(rows*cols*sizeof(char));
  char **M=(char **)malloc(rows*sizeof(char *));

  for(i=0;i<rows;++i)
    *(M+i)=m+i*cols;

//   char M[rows][cols];

  //initialize the matrix with 0s
  for(i=0;i<rows;++i)
    for(j=0;j<cols;++j)
      M[i][j]=0;

  //rewind(fp);
  fseek(fp,0L,SEEK_SET);

  in=0;
  i=0;j=0;
  while((in=getc(fp))!=EOF) {
    if((char)in=='\n') {
      ++i;
      //printf("\n");
    }
    else {
      if((char)in=='1' ||(char)in=='0') {
        M[i][j]=(char)in;
        j=(j+1)%cols;
      }
    }
  }//end-of-while

  //print the matrix
  for(i=0;i<rows;++i) {
    for(j=0;j<cols;++j)
      printf("%c ",M[i][j]);
    printf("\n");
  }

  fclose(fp);

  pos *tmp=NULL;
  while((tmp=check_matrix(M))!=NULL) {
    printf("enter color M[%d][%d]:",tmp->x,tmp->y);
```

```c
        scanf(" %c",&(M[tmp->x][tmp->y]));
        //printf("%c\n",M[tmp->x][tmp->y]);

        apply_rules(M,tmp);
        free(tmp);
        while(Q!=NULL) {
            tmp=remQ();
            if(fix_color(M,tmp->x,tmp->y)) {
                //no error, carry on
                apply_rules(M,tmp);
            }
            else {
                //conflicting color...not an interval digraph..exit
                printf("conflict...(%d,%d)!\n",tmp->x,tmp->y);
                return -1;
            }
            free(tmp);

            for(i=0;i<rows;++i) {
            for(j=0;j<cols;++j)
                printf("%c ",M[i][j]);
            printf("\n");
            }
        }
    }

    printf("final coloring:\n\n");
    for(i=0;i<rows;++i) {
        for(j=0;j<cols;++j)
            printf("%c ",M[i][j]);
        printf("\n");
    }

    return 0;
}

pos *check_matrix(char **M) {
    int i=0,j=0;
    for(;i<rows;++i) {
        for(j=0;j<cols;++j)
            if(M[i][j]=='0') {
                pos *tmp=(pos *)malloc(sizeof(pos));
                tmp->x=i;
                tmp->y=j;
                tmp->nxt=NULL;
                return tmp;
            }
    }
    return NULL;
```

```
}

void apply_rules(char **M,pos *tmp) {
  int i=0,j=0;
  for(;i<rows;++i)
    for(j=0;j<cols;++j)
      if(M[i][j]=='0') {
        if(check_rules(M,tmp->x,tmp->y,i,j)) {
          if(checkQ(tmp))
            continue;
          else {
            putQ(i,j);
            printf("(%d,%d)\n",i,j);
          }
        }
      }
}

int check_rules(char **M,int x1,int y1,int x2,int y2) {

  if(M[x1][y1]=='R') {
    //rule 1a
    if(M[x1][y2]=='1' && M[x2][y1]=='1')
      return 1;

    //rule 2a
    if(x1==x2) {
      int i=0;
      for(;i<rows;++i)
        if(M[i][y1]=='1' && M[i][y2]=='R')
          return 1;
    }
    if(y1==y2) {
      int j=0;
      for(;j<cols;++j)
        if(M[x1][j]=='1' && M[x2][j]=='R')
          return 1;
    }

    //rule 3a
    if(M[x1][y2]=='C' && M[x2][y1]=='1')
      return 1;
    if(M[x1][y2]=='1' && M[x2][y1]=='C')
      return 1;

    //rule 3b
    if(x1==x2) {
      int i=0;
      for(;i<rows;++i)
```

```c
      if(M[i][y1]=='C' && M[i][y2]=='1')
        return 1;
  }
  if(y1==y2) {
    int j=0;
    for(;j<cols;++j)
      if(M[x1][j]=='C' && M[x2][j]=='1')
        return 1;
  }
}
else {
  //rule 1a
  if(M[x1][y2]=='1' && M[x2][y1]=='1')
    return 1;

  //rule 2a
  if(x1==x2) {
    int i=0;
    for(;i<rows;++i)
      if(M[i][y1]=='1' && M[i][y2]=='C')
        return 1;
  }
  if(y1==y2) {
    int j=0;
    for(;j<cols;++j)
      if(M[x1][j]=='1' && M[x2][j]=='C')
        return 1;
  }

  //rule 3a
  if(M[x1][y2]=='R' && M[x2][y1]=='1')
    return 1;
  if(M[x1][y2]=='1' && M[x2][y1]=='R')
    return 1;

  //rule 3b
  if(x1==x2) {
    int i=0;
    for(;i<rows;++i)
      if(M[i][y1]=='R' && M[i][y2]=='1')
        return 1;
  }
  if(y1==y2) {
    int j=0;
    for(;j<cols;++j)
      if(M[x1][j]=='R' && M[x2][j]=='1')
        return 1;
  }
}
```

```
  return 0;
}

int fix_color(char **M, int x, int y) {
  int i=0, j=0;
  int clrflg=0; //R:1,C:2
  for(;i<rows;++i)
    for(j=0;j<cols;++j) {
      if(x!=i) {
        if(y!=j) {
          //rule 1
          if(M[x][j]=='1' && M[i][y]=='1') {
            if(M[i][j]=='R') {
              if(clrflg==0 || clrflg==2) {
                clrflg=2;
                continue;
              }
              if(clrflg==1)
                return 0;
            }
            if(M[i][j]=='C') {
              if(clrflg==0 || clrflg==1) {
                clrflg=1;
                continue;
              }
              if(clrflg==2)
                return 0;
            }
          }
          //rule 2
          if(M[x][j]=='R' && M[i][y]=='R' && M[i][j]=='1') {
            if(clrflg==0 || clrflg==1) {
              clrflg=1;
              continue;
            }
            if(clrflg==2)
              return 0;
          }
          if(M[x][j]=='C' && M[i][y]=='C' && M[i][j]=='1') {
            if(clrflg==0 || clrflg==2) {
              clrflg=2;
              continue;
            }
            if(clrflg==1)
              return 0;
          }
          //rule 3
          if(M[x][j]=='1') {
            if(M[i][y]=='C' && M[i][j]=='R') {
```

```c
              if(clrflg==0||clrflg==2) {
                clrflg=2;
                continue;
              }
              if(clrflg==1)
                return 0;
            }
            if(M[i][y]=='R' && M[i][j]=='C') {
              if(clrflg==0||clrflg==1) {
                clrflg=1;
                continue;
              }
              if(clrflg==2)
                return 0;
            }
          }
          if(M[i][y]=='1') {
            if(M[x][j]=='C' && M[i][j]=='R') {
              if(clrflg==0||clrflg==2) {
                clrflg=2;
                continue;
              }
              if(clrflg==1)
                return 0;
            }
            if(M[x][j]=='R' && M[i][j]=='C') {
              if(clrflg==0||clrflg==1) {
                clrflg=1;
                continue;
              }
              if(clrflg==2)
                return 0;
            }
          }
        }
      }
    }
  if(clrflg==1)
    M[x][y]='R';
  if(clrflg==2)
    M[x][y]='C';
  return 1;
}
```

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