SIGNIFICANCE OF AN OBSERVED RANGE

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The Problem

HAVING taken a set of data we customarily are interested in at least two characteristics of the given group of observations, that is, some measure of the central tendency and some measure of the dispersion. From these two measures we try to form some estimate of the probability that a future average taken under supposedly the same conditions will fall within a given range and the probability that a single observation will fall within a given range. The most efficient way of determining such probabilities in the customary case is considered in a previous paper published in this JOURNAL.[1] That paper presented new material showing that the mean probability associated with a given range is less than the probability associated with the mean range and gave for the first time at least an approximate method of estimating the mean probability associated with a given range, which probability is almost always required in practical problems.

If we have all of the observed values and have the time to calculate the standard deviation in the most efficient way, naturally we make use of information such as that given in the previous paper. Sometimes, however, we find that the results of a set of measurements or observations are reported in the form of the arithmetic mean of n observations together with the maximum and minimum observed values. Several instances of this kind have come to my attention in recent studies of published data giving the physical properties of timbers. Similar cases have been observed in many fields. In such cases, it becomes necessary to make some estimate of the probability associated with the observed range between the maximum and minimum values in a sample of size n and some estimate of the probable error of either a single observation or of an average. In this case we are forced to make use of the average, maximum, and minimum values in doing this.

Another case where engineers as well as scientists are often called upon to make similar estimates of probabilities arises when they are faced with several series of raw data not previously analyzed, except possibly for the determination of the averages. Cases of this character arise around a conference table, out in the field or in the shop, or, in general, on the job. Tentative estimates of these probabilities are required and some quick and ready method must be used.

It is the object of the present paper to provide a chart which will give estimates of the probabilities required in such cases. An example of the use of the chart will be given first and then this will be followed by a discussion of the theory and experimental results underlying the proposed use of the chart. *It will be seen that the new experimental results presented here give for the first time a means of estimating the mean probability associated with an observed*

[1] Shewhart, W. A. Note on the Probability Associated with the Error of a Single Observation. JOURNAL OF FORESTRY, May, 1928.

range. Possibly it should again be emphasized that the present method is only to be used where the one referred to above cannot be used, because of some practical or economic reason.

We shall confine our attention in this note simply to the use of the proposed method for interpreting results published only in the form of the arithmetic mean of n observations together with the maximum and minimum observed values. Examples of this kind are Tables 4 and 12 in the first edition of the very interesting book, "Timber, Its Strength, Seasoning and Grading," by Harold S. Betts.[1]

A typical set of data of this character obtained from another source is presented in Table 1. Naturally the engineer interested in the strengths of poles of Species A, B, and C wants to gain from the tabulated data some indication of the range of variation to be expected in future samples of poles from these species. Should he assume, for example, that the probability of a pole of Species A having a modulus of rupture within the range 2980 to 5690 is the same as that for a pole of Species B within the range 4460 to 7090, and so forth? Specifically, the two general questions answered by this paper are:

TABLE 1

Species of pole	Number of poles in sample	Modulus of rupture in lbs. per sq. in. Average	Max.	Min.
A	4	3985	5690	2980
B	16	5978	7090	4460
C	100	5787	7790	3490

1. What is the probability associated with the range between maximum and minimum observed values in a sample of size n?

2. How can we obtain an estimate of the probable error of a single observation or of the average when only the mean, maximum, and minimum values of a set of n observations are given?

An Example

To illustrate the proposed method of answering these two questions we shall consider the data of Table 1. From curve I of Figure 1 we read directly the probabilities associated with the observed ranges for samples of 4, 16, and 100. These are given in the fourth column of Table 2.

To obtain an estimate of the probable error we require an estimate σ of the standard deviation σ' of the universe of poles of this species. Now, an estimate σ can be obtained with the aid of curve II of Figure I in the following way:

$$\sigma = \frac{\text{Observed range between maximum and minimum values}}{\text{Ordinate of curve II of Figure 1 for a sample of size } n},$$

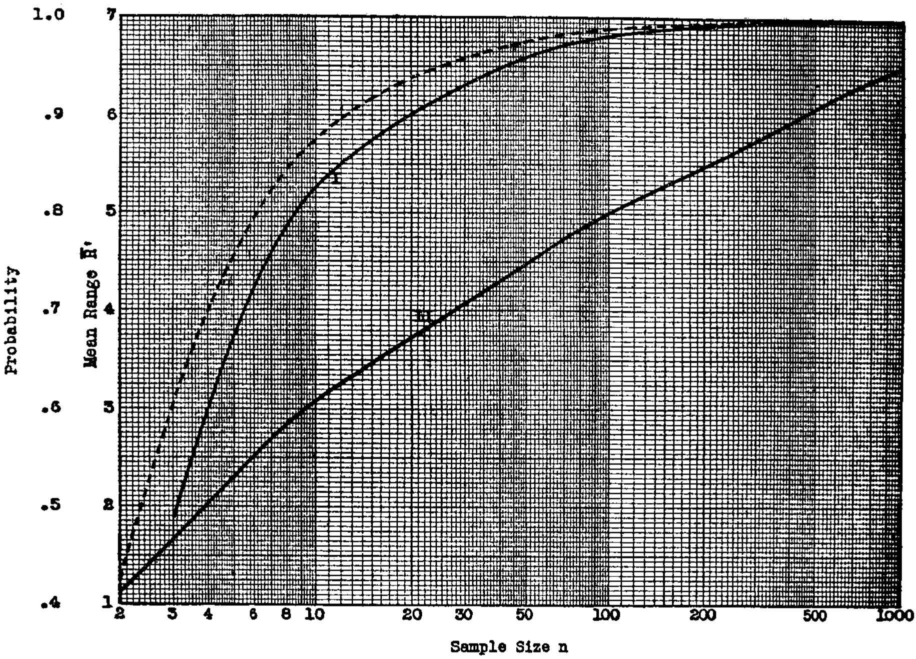
as is numerically illustrated in the fifth column of Table 2.

Theoretical Considerations

Stating the problem in general terms, let us assume that we have a set of n observed values of some chance variable X, which we may represent by X_1, X_2, $\ldots X_n$. To be perfectly definite we may think of these measurements as being observations of modulus of rupture on n telephone poles. We may assume quite properly that this sample of n poles is drawn from a universe which we may characterize by the equation

$$dy = f(X)dX, \qquad (1)$$

[1] Published by McGraw-Hill Book Company, 1919, pp. 34 and 91.



I. Mean Probability Associated with Observed Range.
II. Mean Range $\overline{R}'$.
Probability Associated with Mean Range $\overline{R}'$.

FIG. 1. Probability and range chart.

TABLE 2

Species of pole	Number of poles in sample (n)	Observed range max.-min.	Estimate of probability of another pole of this species falling within observed range	Estimate of standard deviation (σ)	Estimate of probable error of single observation ($.6745\sigma$)	Estimate of probable error of average $\left(.6745 \frac{\sigma}{\sqrt{n}}\right)$
A	4	2710	.60	$\frac{2710}{2.06} = 1316$	888	444
B	16	2630	.88	$\frac{2630}{3.53} = 745$	503	126
C	100	4300	.98	$\frac{4300}{5.02} = 857$	578	58

dy representing the probability of a pole having a value of modulus of rupture X within the range X to $X+dX$. Now, if σ' be the root mean square deviation of the universe, it can be shown that the expected range $\bar{R}'$ between maximum and minimum observed values in a sample of size n can be expressed as a function of σ' for the given universe. The functional relationship between $\bar{R}'$ and σ' of course depends upon the functional relationship $f(X)$ in equation 1. We shall limit the discussion of the present paper to a consideration of the significance of the observed range for the case where the function $f(X)$ in equation 1 is normal, or, in other words, where

$$f(X)=\frac{1}{\sigma'\sqrt{2\pi}}e^{-\frac{(X-\bar{X}')^2}{2\sigma'^2}}, \quad (2)$$

$\bar{X}'$ being the mean or expected value of the universe. We shall then discuss briefly the possible effect of the failure of the physical conditions to meet this assumption.

Estimate σ of Standard Deviation σ' of Universe

Tippett[1] has recently tabulated the mean or expected range $\bar{R}'$ as a function of the size of the sample, $\bar{R}'$ being measured in terms of σ'. A form of these results is presented graphically (curve II) in Figure 1. The way in which this curve may be used is illustrated by application to the data for Species A of Table 1. For this case the observed range is $R=5690-2980=2710$. From this range we may determine an estimate σ of the true standard deviation σ' of the universe. For example, curve II of Figure 1 shows that for $n=4$ the expected range $\bar{R}'$ is $2.059\sigma'$. Hence we may obtain an estimate σ of the true standard deviation σ' from the relation

$$\sigma=\frac{R}{2.059}=\frac{2710}{2.059}=1316.$$

We have at once, therefore, the estimate of the probable error of a single observation $.6745\sigma$ and an estimate of the probable error of the arithmetic mean $\frac{.6745\sigma}{\sqrt{n}}$, these errors being interpretable in the customary way. In fact, the observed average $\bar{X}$ and the observed standard deviation σ may be substituted in equation 2 and the tabulated integral of this normal law function may be used to find approximately the probability associated with any range.

Estimate of Probability Associated With Observed Range

As a first approximation of the probability associated with an observed range we may take the probability associated with the expected range for a given sample size as determined for the normal law as shown in Figure 1. A specific example for the case $n=4$ will illustrate the method. Curve II of Figure 1 shows that the expected range is $2.059\sigma'$. We may, therefore, take twice the integral of the normal law over the range of 0 to 1.029 as an approximation for the probability sought. For the case $n=4$ this probability is .697. In this way probabilities associated with the different expected ranges have been calculated and are presented graphically in the dotted curve of Figure 1. Thus this dotted

[1] Tippett, L. H. C. Range Between Extreme Individuals. Biometrika, Vol. XVII, Parts III and IV, December, 1925, pp. 364-387.

curve of Figure 1 may be used to read off directly first approximations for the probabilities associated with observed ranges for samples of size n. We see at once that for $n=4$, 16, and 100 used in Table 1, the associated probabilities are .697, .923, and .988. Enough has been said to show definitely that the probability associated with a given observed range is a function of the sample size.

However, the statistically trained analyst will be quite seriously disturbed by the assumptions which have been made in attaining the first approximation indicated in Figure 1, because it is obvious that the expected probability associated with the observed range for a sample of given size drawn from a normal population is less than the probability associated with the expected range as given in Figure 1, although it can be shown that this difference is a decreasing function of the sample size. Hence to form some estimate of the correction to be applied in practical cases to the dotted probability curve in Figure 1 the following experiment was made.

One thousand samples[1] of size 4 were drawn from a normal universe and the range between maximum and minimum observations for each sample determined. By means of the normal law integral table the percentages in the universe confined between the limits established by each of the thousand ranges were then obtained. The frequency distribution of these one thousand observed values of percentage (or we may say probability) are presented in Figure 2. The average of the thousand observed values of probability gives a quite accurate estimate of the expected probability associated with the range for samples of size 4 drawn from a normal universe. The average probability[2] was, in this case, .599 or roughly a difference of .10 from the probability .697 associated with the expected range $2.059\sigma'$ for a sample of size $n=4$. It will be observed that the probabilities associated with five of the thousand observed ranges were .06 and the probabilities of 27 of the thousand observed ranges were .98, whereas the average was .599. Thus about .5 per cent of the time, when we would have expected a probability of .599 associated with the observed range for $n=4$, we actually observed a probability as low as .06 and similarly about 2.7 per cent of the time when we would have expected a probability of .599 we actually observed a probability of approximately .98. Of course this dispersion of the distribution of probability associated with a given range decreases with sample size n.

In a similar way points were determined for 160 samples of $n=25$, 80 samples of $n=50$, 53 samples of $n=75$, 40 samples of $n=100$, and 4 samples of $n=1000$. Curve I of Figure 1 was then drawn through these points and hence the ordinate of this curve gives us an empirically determined estimate of the probability associated with the range between maximum $X_{\max.}$ and minimum

[1] I am indebted to Miss M. B. Cater and Miss M. S. Harold for making this experiment and carrying out the calculations.

[2] It is approximately 99 per cent certain that the correction factor lies within the range $.10 \pm .01$.

$X_{min.}$ for samples for size n up to $n = 1000$.

Thus, in the general case, given a sample of any size n, we can read from curve I of Figure 1 an approximate estimate of the probability associated with the observed range $X_{max.} - X_{min.}$. As already indicated, there are more efficient ways of determining this probability, namely by making the most efficient estimates of the expected value or average of the universe and of the standard deviation σ' of the universe and then using the normal law integral table to determine the probability associated with the range, certain corrections being made, as already noted, to take account of the reasoning *a posteriori.* Further discussion in this paper of the details of the methods of doing this are not entered into because they are of interest to the analyst of the data preparing the results for publication and cannot be applied by the reader to data presented in the form of Table 1 because the separate observations are not given.

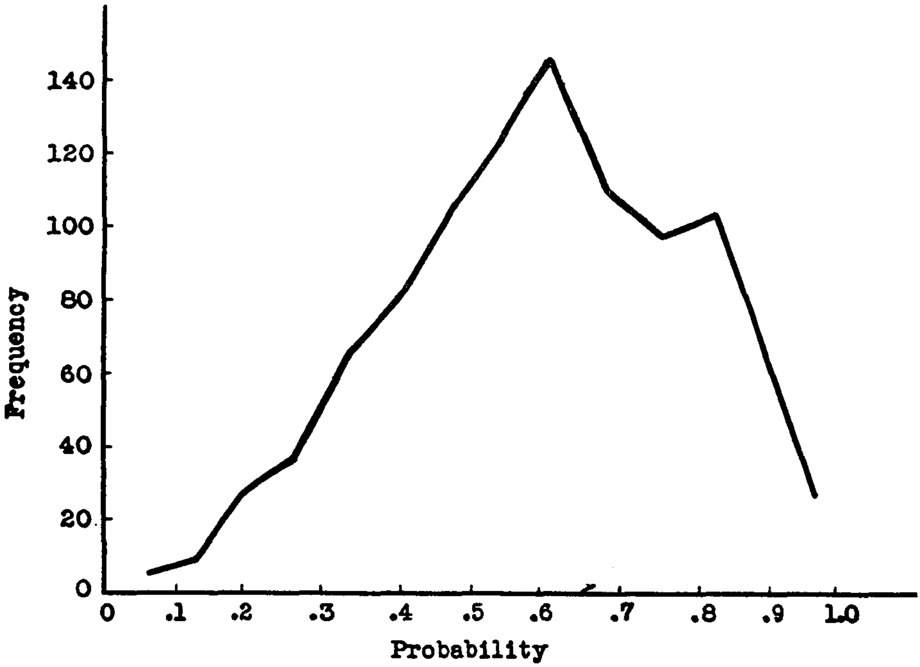


FIG. 2. Observed probabilities associated with ranges between maximum and minimum in samples of four drawn from a normal universe.

SUMMARY

It is, of course, highly desirable that the analyst of the data publish the most efficient estimates of the probabilities associated with the given ranges. *Where this has not been done, as in the cited cases, the method of the present paper makes possible a more efficient use of the published data than could otherwise be* *made.* Results of the character already referred to certainly show the necessity of making due allowance for the size of the sample in the interpretation of data. Enough has been said to show, for example, that it would indeed be a serious mistake for the engineer using the data of Table 1 to assume, as is sometimes done, that the probabilities associated with these three ranges are approximately equal. Work has also been done to indicate that the probability curve of Figure 1 may be used without serious trouble even when there is reason to believe that the universe from which the sample is drawn differs but little from normal.