JOINT ASYMPTOTIC DISTRIBUTION OF U-STATISTICS AND ORDER STATISTICS

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SUMMARY. It is shown under some mild restrictions that the joint distribution of a U-statistic (Hoeffding) and the a_n-th order statistic tends to (i) the bivariate normal distribution if $\frac{a_n}{n} \to p$, $0 < p < 1$, (ii) the joint distribution of two independent variables, one of which is gamma and the other normal, in case $a_n \to$ constant or $n - a_n \to$ constant, (iii) the joint distribution of two independent normal variables if $a_n \to \infty$ such that $\frac{a_n}{n} \to 0$ or $\frac{n-a_n}{n} \to 0$. The above results are generalised to the case of several order statistics and several U-statistics. The generalisation to the case of several populations and generalised (Lehmann) U-statistics is also pointed out.

1. Introduction

Let $x_1, \ldots x_n$ be n independent observations on a random variable X. Writing down the observations in increasing order we get $x_{(1)} \leqslant x_{(2)} \leqslant \ldots < x_{(n)}$. If $\{a_n\}$ is a sequence of integers satisfying $1 \leqslant a_n \leqslant n$, then by the a_n-th order statistic we mean $x_{(a_n)}$. The random variable corresponding to it is $X_{(a_n)}$. We are interested in the joint asymptotic distribution of $X_{(a_n)}$ and any U-statistic. Sukhatme (1957) has shown that the joint asymptotic distribution of $X_{([\frac{n}{2}])}$ and any U-statistic with a bounded kernel, from a distribution whose density function is continuous at the median, is bivariate normal. We now proceed to prove the results stated in the summary.

2. The case $a_n \to \infty$; $\frac{a_n}{n} \to p$, $0 < p < 1$

Theorem 1 : *Let $x_1, \ldots x_n$ be n independent observations on a random variable X with distribution function $F(x)$ and density function $f(x)$. Let $f(x)$ be continuous at θ, the p-th quantile of the population. Let y be the a_n-th order statistic from the sample where $\frac{a_n}{n} \to p$, $0 < p < 1$. Let Y be the random variable corresponding to y. Let U_n be generated as a U-statistic from the bounded kernel*

$$\psi(w_1, \ldots w_t). \qquad \ldots \quad (2.1)$$

Let $$E(\psi(X_1, \ldots X_t)) = m.$$

Then the joint distribution of

$$\{\xi = \sqrt{n}(Y-\theta),\ \ \eta = \sqrt{n}\,(U_n - m)\}$$

tends to the bivariate normal.

Proof: Let $$\phi(x_1) = E(\psi(x_1, X_2, \ldots, X_t)) \qquad \ldots (2.2)$$

$$E(\phi(X)-m)^2 = \sigma^2 \qquad \ldots (2.3)$$

$$\left.\begin{aligned} &\int_{-\infty}^{\theta} (\phi(w)-m)\,f(w)dw = m' \\ &\int_{\theta}^{\infty} (\phi(w)-m)\,f(w)dw = m'' \\ &\int_{-\infty}^{\theta} (\phi(w)-m)^2 f(w)dw = \sigma_1^2 \\ &\int_{\theta}^{\infty} (\phi(w)-m)^2 f(w)dw = \sigma_2^2 \\ &q = 1-p \end{aligned}\right\} \qquad \ldots (2.4)$$

Then we have $$m'+m'' = 0, \qquad \sigma_1^2+\sigma_2^2 = \sigma^2. \qquad \ldots (2.5)$$

The characteristic function $\phi_n(t_1, t_2)$ of $\left\{ \xi, \quad \eta' = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\phi(X_i)-m) \right\}$

is given by $\phi_n(t_1, t_2) = E(\exp(it_1\xi + it_2\eta')) = E_\xi\, [E(\exp(it_1\xi + it_2\eta')\,|\,\xi)]. \quad \ldots (2.6)$

Then proceeding as in Sukhatme (1957) it turns out that

$$\phi_n(t_1, t_2) = \frac{n!}{(a_n-1)!\ (n-a_n)!} \int_{-\infty}^{\infty} \exp\left(it_1\xi + it_2\frac{\phi(y)-m}{\sqrt{n}}\right) f(y)dy \times$$

$$\times \left[\int_{-\infty}^{y} \exp\left(it_2\frac{\phi(w)-m}{\sqrt{n}}\right) f(w)dw\right]^{a_n-1} \times \left[\int_{y}^{\infty} \exp\left(it_2\frac{\phi(w)-m}{\sqrt{n}}\right) f(w)dw\right]^{n-a_n} \qquad \ldots (2.7)$$

Putting $w = \theta + \frac{u}{\sqrt{n}}$ we find as in Sukhatme (1957) that

$$\left.\begin{aligned} &\int_{-\infty}^{y} \exp\left(it_2\frac{\phi(w)-m}{\sqrt{n}}\right) f(w)dw = p + it_2\frac{m'}{\sqrt{n}} - \frac{t_2^2}{2n}\sigma_1^2 + \frac{\lambda}{\sqrt{n}} + \frac{\mu i t_2}{n} + o\left(\frac{1}{n}\right) \\ &\int_{y}^{\infty} \exp\left(it_2\frac{\phi(w)-m}{\sqrt{n}}\right) f(w)dw = q + it_2\frac{m''}{\sqrt{n}} - \frac{t_2^2}{2n}\sigma_2^2 - \frac{\lambda}{\sqrt{n}} - \frac{\mu i t_2}{n} + o\left(\frac{1}{n}\right) \end{aligned}\right\} \ldots (2.8)$$

where
$$\left.\begin{aligned}\lambda &= \int_0^{\xi} f\left(\theta+\frac{u}{\sqrt{n}}\right)du \\ \mu &= \int_0^{\xi}\left\{\varphi\left(\theta+\frac{u}{\sqrt{n}}\right)-m\right\} f\left(\theta+\frac{u}{\sqrt{n}}\right)du\end{aligned}\right\} \quad \dots (2.9)$$

We also note that for a fixed ξ, $\lambda \to \xi f(\theta)$ as $n\to\infty$ since $f(w)$ is continuous at θ. Using (2.8) and simplifying (2.7) after expansion

$$\log\left\{\left[\int_{-\infty}^{v}\exp\left(it_2\frac{\varphi(w)-m}{\sqrt{n}}\right)f(w)dw\right]^{a_n-1}\times\left[\int_{v}^{\infty}\exp\left(it_2\frac{\varphi(w)-m}{\sqrt{n}}\right)f(w)dw\right]^{n-a_n}\right\}$$

$$= \text{const} - \frac{t_2^2}{2}\sigma^2 - \frac{\lambda^2}{2pq} + \frac{2it_2}{2}\lambda\left(\frac{m''}{q}-\frac{m'}{p}\right) + \frac{t_2^2}{2}\left(\frac{m'^2}{p}+\frac{m''^2}{q}\right) + o(1) \quad \dots (2.10)$$

Thus
$$\varphi_n(t_1, t_2) = \text{const} \times \int_{-\infty}^{\infty}\exp\left[it_1\xi + it_2\frac{\varphi\left(\theta+\frac{\xi}{\sqrt{n}}\right)-m}{\sqrt{n}} - \frac{t_2^2}{2}\sigma^2 + \right.$$

$$\left. + \frac{t_2^2}{2}\left(\frac{m'^2}{p}+\frac{m''^2}{q}\right) + \frac{2it_2}{2}\lambda\left(\frac{m''}{q}-\frac{m'}{p}\right) - \frac{\lambda^2}{2pq} + o(1)\right] f\left(\theta+\frac{\xi}{n}\right)d\xi. \quad \dots (2.11)$$

Now letting $n\to\infty$, and taking the lim sign inside the integral, which is valid in virtue of the bounded convergence theorem, we find that the right hand side of (2.11) without the constant tends to

$$f(\theta)\int_{-\infty}^{\infty}\exp\left[it_1\xi - \frac{f^2(\theta)\xi^2}{2pq} + \frac{2it_2}{2}f(\theta)\,\xi\left(\frac{m''}{q}-\frac{m'}{p}\right) - \frac{t_2^2}{2}\sigma^2 + \frac{t_2^2}{2}\left(\frac{m'^2}{p}+\frac{m''^2}{q}\right)\right]d\xi$$

$$= \sqrt{2\pi pq}\,\exp\left[-\frac{t_2^2}{2}\sigma^2 - \frac{2\,t_1t_2}{2}\left(\frac{m''}{q}-\frac{m'}{p}\right)\frac{pq}{f(\theta)} - \frac{t_1^2}{2}\frac{pq}{f^2(\theta)}\right]$$

and the constant on the right hand side of (2.11), namely

$$\frac{1}{\int_{-\infty}^{\infty}\exp\left(-\frac{\lambda^2}{2pq}+o(1)\right) f\left(\theta+\frac{\xi}{\sqrt{n}}\right)d\xi} \to \frac{1}{\sqrt{2\pi pq}}.$$

Thus $\varphi_n(t_1, t_2) \to \varphi(t_1, t_2) = \exp\left[-\frac{t_2^2}{2}\sigma^2 - \frac{2t_1t_2}{2}\left(\frac{m''}{q}-\frac{m'}{p}\right)\frac{pq}{f(\theta)} - \frac{t_1^2}{2}\frac{pq}{f^2(\theta)}\right]$. ... (2.12)

Thus, we see that the asymptotic distribution of (ξ, η') is bivariate normal. From Hoeffding (1948) we see that $E(\eta - t\eta')^2 \to 0$ as $n \to \infty$, so that η and $t\eta'$ are asymptotically equivalent. Thus the asymptotic distribution of (ξ, η) is bivariate normal with zero means and asymptotic variances $\frac{pq}{f^2(\theta)}$ and $t^2\sigma^2$ the correlation coefficient being

$$\left(\frac{m''}{q} - \frac{m'}{p}\right) \frac{\sqrt{pq}}{\sigma}. \quad \ldots \quad (2.13)$$

The above theorem was proved under the condition that $\psi(w_1, \ldots w_t)$ is bounded. It is easy to show that the theorem holds good provided $\psi(w_1, \ldots w_t)$ is bounded on any bounded interval of $(w_1, \ldots w_t)$ and

$$E\,(\psi^3(X_1, \ldots X_t)) < \infty. \quad \ldots \quad (2.14)$$

This condition is sometimes more useful in practice.

The above theorems can be easily extended to the case of several U-statistics each of which satisfies one of the conditions (2.1) or (2.14). Again, the result can also be extended to the case of several order statistics. The last extension is quite straightforward, but the proof involves heavy algebra and hence will not be given here.

Another kind of extension of the above results is as follows :

Let $(x_1, \ldots, x_{n_1})$ and $(y_1, \ldots, y_{n_2})$ be independent observations on two independent random variables X and Y respectively. Let $\{a_n\}$, $\{b_n\}$ be two sequences of integers and let Z_1, Z_2 be the random variables corresponding to $x_{(a_{n_1})}$ and $y_{(b_{n_2})}$. Let $\frac{a_n}{n} \to p_1$, $\frac{b_n}{n} \to p_2$ $0 < p_1, p_2 < 1$. Let θ_1 and θ_2 be the p_1-th, p_2-th, quantiles of X and Y respectively.

Let U_{n_1, n_2} be a generalised (Lehmann) U-statistic with kernel $\psi(x_1, \ldots x_{t_1}; y_1, \ldots y_{t_2})$ which is either bounded or bounded in any bounded interval of its arguments and possesses a third moment. If $E(\psi) = m$ then the joint distribution of

$$\{\sqrt{n_1}(Z_1 - \theta_1), \sqrt{n_2}(Z_2 - \theta_2), \sqrt{n_1}(U_{n_1, n_2} - m)\}$$

tends to the trivariate normal distribution as $n_1, n_2 \to \infty$ such that $\frac{n_1}{n_2} \to c$, $0 < c < \infty$. The proof depends on the fact that

$$\sqrt{n_1}\,(U_{n_1, n_2} - m) \text{ and } \frac{1}{\sqrt{n_1}}\, t_1 \sum_{i=1}^{n_1} \psi_1(X_i) + \frac{\sqrt{n_1}}{\sqrt{n_2}} \frac{t_2}{\sqrt{n_2}} \sum_{j=1}^{n_2} \psi_2(Y_j)$$

are asymptotically equivalent (see Fraser, 1957) where

$$\psi_1(x_1) = E(\psi(x_1, X_2, \ldots, X_{t_1};\; Y_1, \ldots, Y_{t_2}) - m$$

and

$$\psi_2(y_1) = E(\psi(X_1, \ldots, X_{t_1};\; y_1, Y_2, \ldots, Y_{t_2}) - m.$$

3. The case $a_n \to s$ or $n-a_n \to (r-1)$ where s and r are constants

Since $\{a_n\}$ is a sequence of integers, it is obvious that and $a_n \to s$ and $n-a_n \to (r-1)$ implies that after a certain stage, and $a_n = s$ and $n-a_n = (r-1)$ respectively, so that we may take them to be s and $(r-1)$ respectively for all n. Thus we will have to find the joint asymptotic distribution of the s-th and $(n-r+1)$-th order statistics and U-statistic.

Theorem 2 : *Let $x_1, \dots x_n$ be n independent observations made on a random variable X with a distribution function $F(x)$ which is continuous. Let y and z be the s-th and $(n-r+1)$-th order statistics of the sample. Let U_n be a U-statistic generated out of a bounded symmetric kernel $\psi(w_1, \dots w_t)$.*

Let
$$E(\psi(X_1, \dots X_t)) = m$$

$$\left.\begin{aligned} \xi &= n\ F(Y) \\ \eta &= n(1-F(Z)) \\ \zeta &= \sqrt{n}(U_n-m) \end{aligned}\right\} \qquad \dots \quad (3.1)$$

Then, in the asymptotic distribution of (ξ, η, ζ) the variables are independent and the marginal distributions are gamma, gamma and normal, respectively.

Proof: It is obvious that we need prove the theorem for the case of the rectangular distribution only.

Let

$$\varphi(x_1) = E(\psi(x_1, X_2, \dots, X_t)); \qquad E(\varphi(X)-m)^2 = \sigma^2 \qquad \dots \quad (3.2)$$

$$\zeta' = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\varphi(X_i)-m).$$

Then the characteristic function of (ξ, η, ζ') is given by

$$\varphi_n(t_1, t_2, t_3) = E\{\exp(it_1\xi+it_2\eta+it_3\zeta')\} = \underset{\xi,\eta}{E}[E\{\exp(it_1\xi+it_2\eta+it_3\zeta')|\xi, \eta\}]. \quad \dots \quad (3.3)$$

Arguing as before

$\varphi_n(t_1, t_2, t_3) =$

$$\frac{n!}{(s-1)!(n-r-s)!(r-1)!} \iint\limits_{0<\xi+\eta<n} \exp\left[it_1\xi+it_2\eta+it_3\left(\frac{\varphi\left(\frac{\xi}{n}\right)-m}{\sqrt{n}}+\frac{\varphi\left(1-\frac{\eta}{n}\right)-m}{\sqrt{n}}\right)\right]$$

$$\times\left[\int_0^{\xi/n} \exp\left(it_3\frac{\varphi(w)-m}{\sqrt{n}}\right)dw\right]^{s-1} \times \left[\int_{\xi/n}^{1-\eta/n} \exp\left(it_3\frac{\varphi(w)-m}{\sqrt{n}}\right)dw\right]^{n-r-s} \times$$

$$\times\left[\int_{1-\eta/n}^{1} \exp\left(it_3\frac{\varphi(w)-m}{\sqrt{n}}\right)dw\right]^{r-1} \frac{d\xi d\eta}{n^2}. \qquad \dots \quad (3.4)$$

It is easily seen that

$$\left.\begin{aligned}
&\int_0^{\xi/n} \exp\left(it_3 \frac{\varphi(w)-m}{\sqrt{n}} \right) dw = \frac{\xi}{n} + o\left(\frac{1}{n}\right) \\
&\int_{\xi/n}^{1-\eta/n} \exp\left(it_3 \frac{\varphi(w)-m}{\sqrt{n}} \right) dw = 1-\frac{t_3^2}{2n}\sigma^2-\frac{\xi}{n}-\frac{\eta}{n}+o\left(\frac{1}{n}\right) \\
&\int_{1-\eta/n}^{1} \exp\left(it_3 \frac{\varphi(w)-m}{\sqrt{n}} \right) dw = \frac{\eta}{n} + o\left(\frac{1}{n}\right)
\end{aligned}\right\} \quad \dots \quad (3.5)$$

so that

$$\varphi_n(t_1, t_2, t_3) = \frac{n!}{(s-1)!(n-r-s)!(r-1)!n^{r+s}} \iint_{0<\xi+\eta<n} \exp\left[it_1\xi+it_2\eta+it_3\left(\frac{\varphi\left(\frac{\xi}{n}\right)-m}{\sqrt{n}} + \right.\right.$$

$$\left.\left. +\frac{\varphi\left(1-\frac{\eta}{n}\right)-m}{\sqrt{n}} \right)\right] \times[\xi+o(1)]^{s-1} \times\left[1-\frac{t_3^2}{2n}\sigma^2-\frac{\xi}{n}-\frac{\eta}{n}+o\left(\frac{1}{n}\right)\right]^{n-r-s} \times[\eta+o(1)]^{r-1}d\xi d\eta. \quad \dots \quad (3.6)$$

Now letting $n\to\infty$, and taking the lim sign inside the integral which is valid in virtue of the bounded convergence theorem, we have

$$\lim_{n\to\infty} \varphi_n(t_1, t_2, t_3) = \text{const} \times \int_0^\infty \int_0^\infty \exp\left[it_1\xi+it_2\eta-\frac{t_3^2}{2}\sigma^2-\xi-\eta \right] \xi^{s-1}\eta^{r-1}d\xi d\eta$$

$$= \text{const} \times \frac{1}{(1-it_1)^s} \times \frac{1}{(1-it_2)^r} \times e^{-\frac{t_3^2}{2}\sigma^2} \quad \dots \quad (3.7)$$

where again, we can easily see that the constant is unity and hence the theorem is proved. Extensions to the case of several U-statistics and generalised U-statistics are obvious. Incidentally we note that the s-th order statistic and the $(n-r+1)$-th order statistic are independent in the limit, if r and s are constants.

4. THE CASE $a_n\to\infty$; $\frac{a_n}{n}\to 0$ or $\frac{n-a_n}{n}\to 0$

Theorem 3 : *Let $x_1, \dots, x_n$ be n independent observations on a random variable X with distribution function $F(x)$ which is continuous. Let $\{a_n\}$, $\{b_n\}$ be two sequences of integers such that*

$$1 \leqslant a_n < b_n \leqslant n \text{ and } a_n\to\infty,\ b_n\to\infty,\ \frac{a_n}{n}\to 0,\ \frac{b_n}{n}\to 1,\ \frac{c_n}{a_n} < K \textit{ a constant}, \quad \dots \quad (4.1)$$

where $c_n = n-b_n$.

Let U_n be any U-statistic generated by a bounded kernel $\psi(w_1, \ldots w_t)$ and let

$$\left.\begin{aligned} \varphi(x_1) &= E(\psi(x_1, X_2, \ldots X_t)) \\ E(\varphi(X)) &= m \\ E(\varphi(X)-m)^2 &= \sigma^2 \end{aligned}\right\} \qquad \ldots \quad (4.2)$$

Then the asymptotic distribution of (ξ, η, ζ) where

$$\left.\begin{aligned} \xi &= \frac{n}{\sqrt{a_n}}\left(F(X_{(a_n)})-\frac{a_n}{n}\right) \\ \eta &= \frac{n}{\sqrt{c_n}}\left(\frac{b_n}{n}-F(X_{(b_n)})\right) \\ \zeta &= \sqrt{n}(U_n-m) \end{aligned}\right\} \qquad \ldots \quad (4.3)$$

is given by the density function, constant $\times e^{-\frac{1}{2}\left(\xi^2+\eta^2+\frac{\zeta^2}{\sigma^2}\right)}$.

Proof: It is obvious that we need prove the theorem for the rectangular case only. If $\zeta' = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\varphi(X_i)-m)$ then $\varphi_n(t_1, t_2, t_3)$ the characteristic function of (ξ, η, ζ') is given by

$$\begin{aligned} \varphi_n(t_1, t_2, t_3) &= E\{\exp(it_1\xi+it_2\eta+it_3\zeta')\} \\ &= E_{\xi,\eta}[E\{\exp(it_1\xi+it_2\eta+it_3\zeta') \mid \xi, \eta\}]. \qquad \ldots \quad (4.4) \end{aligned}$$

Proceeding on the same lines as in the previous cases

$$\varphi_n(t_1, t_2, t_3) = \frac{n!}{(a_n-1)!(n-a_n-c_n+1)!(c_n)!}\,\frac{\sqrt{a_n c_n}}{n^2}\times$$

$$\times \iint\limits_{0 \leqslant \sqrt{a_n}\xi+\sqrt{c_n}\eta \leqslant n-a_n-c_n} \exp\left[\, it_1\xi+it_2\eta+it_3\left(\frac{\varphi\left(\frac{a_n}{n}+\frac{\sqrt{a_n}}{n}\,\xi\right)-m}{\sqrt{n}}+\frac{\varphi\left(1-\frac{c_n}{n}-\frac{\sqrt{c_n}}{n}\,\eta\right)-m}{\sqrt{n}}\right)\right]\times$$

$$\times\left[\int_0^{\frac{a_n}{n}+\frac{\sqrt{a_n}}{n}\xi} \exp\left(it_3\,\frac{\varphi(w)-m}{\sqrt{n}}\right)dw\right]^{a_n-1}\times$$

$$\times\left[\int_{\frac{a_n}{n}+\frac{\sqrt{a_n}}{n}\xi}^{1-\frac{c_n}{n}-\frac{\sqrt{c_n}}{n}\eta} \exp\left(it_3\,\frac{\varphi(w)-m}{\sqrt{n}}\right)dw\right]^{n-a_n-c_n+1}\times$$

$$\times\left[\int_{1-\frac{c_n}{n}-\frac{\sqrt{c_n}}{n}\eta}^{1} \exp\left(it_3\,\frac{\varphi(w)-m}{\sqrt{n}}\right)dw\right]^{c_n} d\xi d\eta \qquad \ldots \quad (4.5)$$

which on simplifying, as done in previous paras, reduces to

$$\phi_n(t_1, t_2, t_3) = \text{const} \times \underset{\sqrt{a_n}\xi+\sqrt{c_n}\eta<n-a_n-c_n}{\int\int} \exp\Bigg[it_1\xi+it_2\eta+it_3\Bigg(\frac{\psi\left(\frac{a_n}{n}+\frac{\sqrt{a_n}}{n}\xi\right)-m}{\sqrt{n}}$$

$$+\frac{\psi\left(1-\frac{c_n}{n}-\frac{\sqrt{c_n}}{n}\eta\right)-m}{\sqrt{n}}\Bigg)-\frac{t_3^2}{2}\sigma^2-\frac{\xi^2}{2}-\frac{\eta^2}{2}+o(1)\Bigg]\, d\xi d\eta. \quad \dots \quad (4.6)$$

Letting $n\to\infty$, (the limits for ξ, η become $(0, \infty)$, $(0, \infty)$ because $\frac{c_n}{a_n} < K$), the above integral without the constant tends to

$$\int_0^\infty\int_0^\infty \exp\left[-\frac{t_3^2}{2}\sigma^2+it_1\xi+it_2\eta-\frac{\xi^2}{2}-\frac{\eta^2}{2}\right] d\xi d\eta = 2\pi \exp\left[-\frac{t_1^2}{2}-\frac{t_2^2}{2}-\frac{t_3^2}{2}\sigma^2\right] \quad \dots \quad (4.7)$$

and the constant which is

$$\frac{1}{\underset{\sqrt{a_n}\xi+\sqrt{c_n}\eta<n-a_n-c_n}{\int\int} \exp(-\xi^2/2-\eta^2/2+o(1))d\xi d\eta} \to \frac{1}{2\pi}.$$

Thus $$\phi_n(t_1, t_2, t_3) \to \exp\left[-\frac{t_1^2}{2}-\frac{t_2^2}{2}-\frac{t_3^2}{2}\sigma^2\right] \quad \dots \quad (4.8)$$

and hence the theorem is proved.

The extensions to several U-statistics and generalised U-statistics are immediate. We note that the a_n-th and b_n-th order statistics when suitably standardised are asymptotically normally and independently distributed if condition (4.1) holds.

In section 4 and section 5 we have shown that U_n and $F(Y)$, where Y is the a_n-th order statistic, have a certain asymptotic distribution. To make a similar statement about U_n and Y we need the following, (in case $\frac{a_n}{n}\to 0$).

$F(x)$ has a density function $f(x)$ and $\lim_{F(x)\to 0} f(x)$ exists and is not equal to zero. For an example where this occurs, we may cite the exponential distribution.

5. Remarks[1]

In the above proofs the use of characteristic functions has partly blurred out the picture of the exact process by which the limit distributions were attained. To gain some insight into this aspect we reason as follows: Suppose we fix the a_n-th order statistic, i.e., the normalised variable corresponding to it. If $\eta' = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\varphi(X_i)-m)$ we see that it splits up into 3 parts. One part being the sum of (a_n-1) independent random variables where x ranges on $(-\infty, x_{(a_n)})$, another part being the sum of $(n-a_n)$ independent random variable where x ranges on $(x_{(a_n)}, \infty)$ and a third part fixed at $\frac{\varphi(x_{(a_n)})-m}{\sqrt{n}}$. From an application of the Central Limit Theorem we see that the limit of the conditional distribution of η' for a fixed ξ is normal. It can also be shown, after some algebra, that the mean and variance of this distribution are

$$\left(-f(0)\,\xi\left(\frac{m''}{q}-\frac{m'}{p}\right),\ \sigma^2-\left(\frac{m'^2}{p}+\frac{m''^2}{q}\right)\right) \qquad \text{in the case 2}$$

$$(0, \sigma^2) \qquad \text{in the case 3}$$

$$(0, \sigma^2) \qquad \text{in the case 4}$$

and in each of these cases we know the limiting marginal distribution of ξ so that the nature of the limiting joint distribution can be concluded to be bivariate normal in case 2, the distribution of independent normal and gamma variables in case 3, and the distribution of two independent normal variables in case 4. The conclusion can be justified using the following

Lemma : *Let (X_n, Y_n) be a sequence of random variables. Let $F_n(y/x)$, the conditional distribution of Y_n given $X_n = x$, tend weakly to a distribution $F(y/x)$. Also let $G_n(x)$ the marginal distribution of X_n, tend weakly to a distribution function $G(x)$. Then under some conditions $F_n(x, y)$ the joint distribution of (X_n, Y_n) tends to the distribution $\int_{-\infty}^{x} F(y/x)dG(x)$.*

A set of sufficient conditions are (1) $F(y/x)$ is continuous in y for each x, (2) $G_n(x)$, $G(x)$ admit of probability densities $g_n(x)$, $g(x)$ respectively, and $g_n(x) \to g(x)$ uniformly in any bounded interval of x. Both, that these conditions are sufficient and that these conditions are satisfied in our case, can be easily verified.

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