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THE ADDITION OF RANDOM VECTORS

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SUMMARY. Expressions are obtained for the resultant of n vectors whose directions are completely random in s dimensions, but for each of which the distribution of the scalar square is given. This is most readily calculated from the vectorial cumulants, which are defined, and shown, besides their cumulative property, to measure departure from a gamma distribution.

Introduction

I consider the addition, or composition, of vectors whose direction is completely random in s dimensions. Randomness of direction is a very much simpler notion than randomness of a scalar number, as appears from the various formulations which Jeffreys (1933) has used to express his ignorance of the value of a scalar. By the statement that the direction of a displacement is completely random I mean that if a point is displaced through a distance x the probability that it will be displaced into any region of equal magnitude (length, area, volume, etc.) of the $(s-1)$ dimensional variety (circle, sphere, etc.) consisting of points at a distance x in Euclidean or other isotropic space from its original position, is equal. In particular the probability that a one dimensional vector will have a positive or negative value must be equal. When we have said that the direction is completely random we know nothing about the scalar magnitude. I shall suppose that the distribution of its square, that is to say of the squared magnitude of a displacement, or the scalar square of a vector, is specified by its cumulants κ_r. We might have used the moments μ_r of the square, or the even cumulants κ'_{2r} or moments μ'_{2r} of the displacement, but these lead to less compact expressions. I shall later introduce the vectorial cumulants θ_r which have several properties analogous to cumulants in one dimension. As however they are functions of the number s of dimensions it is best to show how they arise naturally in the course of the analysis.

It is worth while considering for a moment what happens if the directions of the vectors added are incompletely random. The simplest types of departure from complete randomness are of two kinds. The directions of all vectors may be unequally frequent in different directions, for example displacements approximately north or south may be commoner than those approximately east or west. Unless the probabilities of displacements in two opposite directions are equal there is no advantage in using scalar squares. But even if they are, it is often best to consider displacements in mutually orthogonal directions separately. Secondly the angle between two successively added vectors may have some other distribution than that appropriate to s dimensions. If so, addition is not in general associative, though we shall discover the conditions under which it is so. Here however there is no advantage in resolving the vectors orthogonally.

I have deliberately avoided such resolution because in the data which induced me (Haldane, 1960) to undertake this investigation, namely those of Alström (1958), there was a considerable positive correlation between the squares of displacements in orthogonal directions. The most general problem of the addition of random vectors would, I suppose, be something like this. We have added a series of vectors $[x_1, y_1, z_1 \ldots], [x_2, y_2, z_2 \ldots], \ldots [x_{n-1}, y_{n-1}, z_{n-1}, \ldots]$ in that order, x, y, z, etc. being resolutes in s mutually orthogonal directions. Given these vectors we can predict the frequency with which the resolutes $x_n, y_n, z_n, \ldots$ of the n-th vector lie between infinitesimal limits. We have to determine the distribution of the sum of these n vectors. If there are s dimensions then in the most general case the s scalars $x_n, y_n, z_n \ldots$ have a joint distribution whose parameters depend on the $s(n-1)$ components of the preceding vectors, and on an indefinite number of arbitrary constants. The problem is hopelessly complicated. In what follows I assume that successive vectors are independent as regards their magnitude, though the distribution of the angle between them, which plays a part comparable to correlation, depends on the number of dimensions.

THE ADDITION OF TWO RANDOM VECTORS

Let X, Y, be two random vectors in Euclidean space. Let the non-negative numbers x and y be their scalar squares X.X and Y.Y, and let ϕ be the angle between their directions. As the resolutes of the vectors may be negative we can suppose $0 \leqslant \phi \leqslant \frac{1}{2}\pi$. If we assumed $0 \leqslant \phi \leqslant \pi$ we should have to add that the distribution function of $\phi - \frac{1}{2}\pi$ was even, otherwise we could not operate with squares. Then if Z is the sum (resultant) of X and Y, and $z = \mathbf{Z} \cdot \mathbf{Z}$ is the scalar square of Z,

$$z = x + y \pm 2\sqrt{xy} \cos \phi. \qquad \ldots \quad (1)$$

Let κ_r, λ_r, ζ_r be the r-th cumulants of the distributions of x, y, and z. I assume that positive and negative values of the ambiguity are equiprobable, so that the expectations of odd powers of $\pm\sqrt{xy}\cos\phi$ are zero. Let

$$\mathcal{E}(\cos^{2r}\phi) = c_{2r}.$$

The distribution function of ϕ in s dimensions is

$$dF \propto \sin^{s-2}\phi \, d\phi \qquad (s > 1).$$

Thus
$$c_{2r} = \int_0^{\frac{\pi}{2}} \sin^{s-2}\phi \cos^{2r}\phi \, d\phi \div \int_0^{\frac{\pi}{2}} \sin^{s-2}\phi \, d\phi.$$

Hence
$$c_2 = \frac{1}{s}, \; c_4 = \frac{1.3}{s(s+2)}, \; c_6 = \frac{1.3.5}{s(s+2)(s+4)}, \; c_{2r} = \frac{1.3.5\cdots(2r-1)}{s(s+2)(s+4)\cdots(s+2r)}. \quad \ldots \quad (2)$$

So
$$\mathcal{E}(z^r) = \mathcal{E}\left[(x+y)^r + 4\binom{r}{2} c_2 (x+y)^{r-2} xy + 16\binom{r}{4} c_4 (x+y)^{r-4}(xy)^2 + \cdots \right]. \quad \ldots \quad (3)$$

This is of course a terminating series with $\frac{1}{2}r+1$ terms if r is even, and $\frac{1}{2}(r+1)$ if r is odd. Before calculating the values of ζ_r it is instructive to consider the cases when $s = 1$ and $s = \infty$. The latter is very simple. As s tends to infinity almost all angles between X and Y are right angles, and their scalar squares become additive, so that $z = x+y$, and $\zeta_r = \kappa_r + \lambda_r$, the well-known additive property of cumulants. When $s = 1$, so that $\phi = 0$, the even cumulants of $x^{\frac{1}{2}}$ have this property. If κ'_{2r} be the $2r$-th cumulant of the distribution of $x^{\frac{1}{2}}$ we write $\kappa'_{2r} = \theta_r$, and find, inverting equations (3) of Haldane (1941), then :

$$\begin{aligned}
\theta_1 &= \kappa_1 \\
\theta_2 &= \kappa_2 - 2\kappa_1^2 \\
\theta_3 &= \kappa_3 - 12\kappa_1\kappa_2 + 16\kappa_1^3 \\
\theta_4 &= \kappa_4 - 24\kappa_1\kappa_3 - 32\kappa_2^2 + 272\kappa_1^2\kappa_2 - 272\kappa_1^4 \\
\theta_5 &= \kappa_5 - 40\kappa_1\kappa_4 - 200\kappa_2\kappa_3 + 880\kappa_1^2\kappa_3 + 2{,}400\kappa_1\kappa_2^2 - 9{,}920\kappa_1^3\kappa_2 + 7{,}936\kappa_1^5 \\
\theta_6 &= \kappa_6 - 60\kappa_1\kappa_5 - 480\kappa_2\kappa_4 - 452\kappa_3^2 + 4{,}200\kappa_1^2\kappa_4 + 22{,}368\kappa_1\kappa_2\kappa_3 + 15{,}360\kappa_2^3 - \\
&\quad - 29{,}344\kappa_1^3\kappa_3 - 496{,}128\kappa_1^2\kappa_2^2 + 1{,}364{,}528\kappa_1^4\kappa_2 - 1{,}430{,}552\kappa_1^6. \quad \ldots \quad (4)
\end{aligned}$$

The values of κ_r in terms of θ_r, θ_{r-1}, etc, have already been given by Haldane (1941). The expressions (4) are of no value in this investigation, except as special cases of the expressions (9) derived later which hold in any number of dimensions, on which they furnish a useful check.

From (3) we can write down expressions for the expectations of powers of z, and hence for the cumulants of its distribution. If κ_r and λ_r are the r-th cumulants of the distributions of x and y, whilst ζ_r are those of z, we find :

$$\begin{aligned}
\zeta_1 &= \kappa_1 + \lambda_1 \\
\zeta_2 &= \kappa_2 + \lambda_2 + 4c_2\kappa_1\lambda_1 \\
\zeta_3 &= \kappa_3 + \lambda_3 + 12c_2(\kappa_1\lambda_2 + \lambda_1\kappa_2) \\
\zeta_4 &= \kappa_4 + \lambda_4 + 24c_2(\kappa_1\lambda_3 + \lambda_1\kappa_3 + 2\kappa_2\lambda_2) + 16c_4(\kappa_2 + \kappa_1^2)(\lambda_2 + \lambda_1^2) - 48c_2^2\kappa_1^2\lambda_1^2 \\
\zeta_5 &= \kappa_5 + \lambda_5 + 40c_2[\kappa_1\lambda_4 + \lambda_1\kappa_4 + 3(\kappa_2\lambda_3 + \lambda_2\kappa_3)] + 80c_4[\kappa_2\lambda_3 + \lambda_2\kappa_3 + \kappa_1^2\lambda_3 + \\
&\quad + \lambda_1^2\kappa_3 + 2(\kappa_1 + \lambda_1)\kappa_2\lambda_2 + 2\kappa_1\lambda_1(\kappa_1\lambda_2 + \lambda_1\kappa_2)] - 480c_2^2\kappa_1\lambda_1(\kappa_1\lambda_2 + \lambda_1\kappa_2) \quad \ldots \quad (5)
\end{aligned}$$

The expression for ζ_6 is considerably more complicated, containing terms in c_2^3, $c_2 c_4$, and c_6. I do not think it is likely to be required in practice. Inserting the values of c_2 and c_4 given by (2) we have :

$$\zeta_1 = \kappa_1+\lambda_1$$
$$\zeta_2 = \kappa_2+\lambda_2+4s^{-1}\kappa_1\lambda_1$$
$$\zeta_3 = \kappa_3+\lambda_3+12s^{-1}(\kappa_1\lambda_2+\lambda_1\kappa_2)$$
$$\zeta_4 = \kappa_4+\lambda_4+24s^{-1}(\kappa_1\lambda_3+\lambda_1\kappa_3)+48s^{-1}(s+2)^{-1}(s+3)\kappa_2\lambda_2+48s^{-1}(s+2)^{-1}(\kappa_1^2\lambda_2+\lambda_1^2\kappa_2)-96s^{-2}(s+2)^{-1}\kappa_1^2\lambda_1^2$$
$$\zeta_5 = \kappa_5+\lambda_5+40s^{-1}(\kappa_1\lambda_4+\lambda_1\kappa_4)+120s^{-1}(s+2)^{-1}(s+4)(\kappa_2\lambda_3+\lambda_2\kappa_3)+240s^{-1}(s+2)^{-1}(\kappa_1^2\lambda_3+\lambda_1^2\kappa_3)+480s^{-1}(s+2)^{-1}(\kappa_1+\lambda_1)\kappa_2\lambda_2-960s^{-2}(s+2)^{-1}\kappa_1\lambda_1(\kappa_1\lambda_2+\lambda_1\kappa_2). \quad \ldots \quad (6)$$

These are the fundamental equations for vectorial addition on which most of what follows is based. The values for $s = 2$ may be useful. They are :

$$\zeta_1 = \kappa_1+\lambda_1$$
$$\zeta_2 = \kappa_2+\lambda_2+2\kappa_1\lambda_1$$
$$\zeta_3 = \kappa_3+\lambda_3+6(\kappa_1\lambda_2+\lambda_1\kappa_2)$$
$$\zeta_4 = \kappa_4+\lambda_4+12(\kappa_1\lambda_3+\lambda_1\kappa_3)+30\kappa_2\lambda_2+6(\kappa_1^2\lambda_2+\lambda_1^2\kappa_2)-6\kappa_1^2\lambda_1^2$$
$$\zeta_5 = \kappa_5+\lambda_5+20(\kappa_1\lambda_4+\lambda_1\kappa_4)+90(\kappa_2\lambda_3+\lambda_2\kappa_3)+30(\kappa_1^2\lambda_3+\lambda_1^2\kappa_3)+60(\kappa_1+\lambda_1)\kappa_2\lambda_2-60\kappa_1\lambda_1(\kappa_1\lambda_2+\lambda_1\kappa_2). \quad \ldots \quad (7)$$

When $s = 1$ these expressions give the cumulants of the distribution of the variance of the sum of two random variables in terms of the cumulants of the distributions of the two variances.

THE CONDITIONS THAT ADDITION SHOULD BE ASSOCIATIVE

It is clear from the symmetrical character of equations (5) that the addition of random vectors is commutative, that is to say $X+Y = Y+X$, no matter what are the mean values $c_{2\nu}$ of $\cos^{2\nu}\theta$. But unless $c_{2\nu}$ has the special values given in (2) this addition does not in general obey the associative law, that is to say the distributions of $(X+Y)+Z$ and $X+(Y+Z)$ are not the same. The first three cumulants are the same, but the fourth is not. To take one example, if μ_r are the cumulants of the distribution of $Z \,.\, Z$, then the coefficient of $\kappa_1\lambda_1\mu_2$ in $(X+Y)+Z$ is $32\ (6c_2^2+c_4+2c_2c_4)$, and that of $\kappa_2\lambda_1\mu_1$ or $\kappa_1\lambda_2\mu_1$ is $288c_2^2$, from equations (5). The condition for associative addition is that these should be equal, or

$$288c_2^2 = 32(6c_2^2+c_4+2c_2c_4)$$

whence
$$c_4 = \frac{3c_2^2}{1+2c_2}.$$

Thus if $c_2 = s^{-1}$, $c_4 = 3s^{-1}(s+2)^{-1}$.

The same condition can be derived by considering other coefficients. It is, I think, clear that the addition of random vectors whose directions are strictly random in s dimensions is associative. That is to say the addition of random vectors is associative provided equations (2) hold good for the moments of the cosine of the angle between successive vectors. Further (2) is necessary as well as sufficient, for as we consider successive cumulants, each new value of c_{2r} arises as its first power, so it is unambiguously given in terms of the preceding values of c_{2r}.

However there is no need that s should be a positive integer in equations (2). In fact these equations can be satisfied if s is not less than unity. In particular they can be satisfied in two dimensional space. We can postulate that the distribution of the angle ϕ between successive vectors should have the distribution of that between any pair of vectors in random directions in s dimensions, or the distribution whose moments are given by (2) for non-integral s.

The vectorial cumulants and the general equations for associative addition

We can now proceed to build up general equations from (6). Consider the addition of n random vectors, each with given cumulants; let ζ_r be the r-th cumulant of the distribution of the scalar square of their resultant. If $r > 2$, ζ_r includes some terms of a type not found in (6). For example if to the sum of the two random vectors giving the cumulants of (6) we add a third vector the r-th cumulant of whose scalar square is μ_r, we find :

$$\begin{aligned}\zeta_3 &= \kappa_3+\lambda_3+\mu_3+12s^{-1}[\kappa_1\lambda_2+\lambda_1\kappa_2+(\kappa_1+\lambda_1)\mu_2+\mu_1(\kappa_2+\lambda_2+4s^{-1}\kappa_1\lambda_1)]\\ &= \kappa_3+\lambda_3+\mu_3+12s^{-1}(\kappa_1\lambda_2+\kappa_2\lambda_1+\lambda_1\mu_2+\lambda_2\mu_1+\mu_1\kappa_2+\mu_2\kappa_1)+48s^{-2}\kappa_1\lambda_1\mu_1.\end{aligned}$$

These equations may be written in terms of symmetrical functions or power sums. In what follows $\Sigma\kappa_2\lambda_1$ means the sum of all products of the second cumulant of the distribution of the scalar square of one vector by the first cumulant of that of another, $\Sigma\kappa_1^2\kappa_2$ means the sum of all products of the second cumulant of one such distribution and the square of the first cumulant of the same distribution, and so on. We find:

$$\begin{aligned}\zeta_1 &= \Sigma\kappa_1\\ \zeta_2 &= \Sigma\kappa_2+4s^{-1}\Sigma\kappa_1\lambda_1\\ \zeta_3 &= \Sigma\kappa_3+12s^{-1}\Sigma\kappa_2\lambda_1+48s^{-2}\Sigma\kappa_1\lambda_1\mu_1, \text{ etc.}\end{aligned}$$

$$\begin{aligned}\zeta_1 &= \Sigma\kappa_1\\ \zeta_2 &= \Sigma\kappa_2+2s^{-1}[(\Sigma\kappa_1)^2-\Sigma\kappa_1^2]\\ \zeta_3 &= \Sigma\kappa_3+12s^{-1}[\Sigma\kappa_1\Sigma\kappa_2-\Sigma\kappa_1\kappa_2]+8s^{-2}[(\Sigma\kappa_1)^3-3\Sigma\kappa_1^2\Sigma\kappa_1+2\Sigma\kappa_1^3], \text{ etc.} \qquad \dots \quad (8)\end{aligned}$$

But these expressions soon become very cumbrous. The expression for ζ_5 in symmetrical functions consists of 12 terms, that in terms of power and product sums consists of 42 terms such as $+960s^{-2}(s+2)^{-1}(s+3)\Sigma\kappa_1(\Sigma\kappa_2)^2$. They are complicated for the same reason as the expression for the 10-th moment of the sum of n random variables with given moments about zero is complicated. These latter complications are of course greatly reduced by the use of cumulants. The question arises whether we can find functions of the cumulants κ_r appertaining to one vector which have suitable additive properties. We can, and they prove to have other properties analogous to those of the cumulants of the distribution of a one-dimensional random variable. I call them the vectorial cumulants θ_r. They can be calculated from (6) without much difficulty by the method of undetermined coefficients given that $\theta_1 = \kappa_1$. We can multiply κ_1 by any constant without affecting the properties of these cumulants, and it might be better to put $\theta_1 = s^{-1}\kappa_1$, but I think the terminology here adopted is preferable. It is easy to show that $\theta_2 = \kappa_2 - 2s^{-1}\kappa_1^2$. Given this, suppose that $\theta_3 = \kappa_3 - x\kappa_1\kappa_2 + y\kappa$, and assume additivity. Then from the third of equations (6)

$$\kappa_3+\lambda_3-x(\kappa_1\kappa_2+\lambda_1\lambda_2)+y(\kappa_1^3+\lambda_1^3) \equiv \zeta_3-x\zeta_1\zeta_2+y\zeta_1^3$$
$$\equiv \kappa_3+\lambda_3+12s^{-1}(\kappa_1\lambda_2+\lambda_1\kappa_2)-x(\kappa_1+\lambda_1)(\kappa_2+\lambda_2+4s^{-1}\kappa_1\lambda_1)+y(\kappa_1+\lambda_1)^3.$$

Hence $$(\kappa_1\lambda_2+\kappa_2\lambda_1)(12s^{-1}-x)+(\kappa_1^2\lambda_1+\lambda_1^2\kappa_1)(3y-4s^{-1}x) \equiv 0.$$

It follows that $x = 12s^{-1}$, $y = \frac{4}{3}s^{-1}x = 16s^{-2}$.

To determine the 4 coefficients of θ_4 by the same method we have 6 simultaneous linear equations which are consistent. Similarly for the 6 coefficients of θ_5 we have 10 equations. The values of θ_r are:

$$\theta_1 = \kappa_1$$
$$\theta_2 = \kappa_2-2s^{-1}\kappa_1^2$$
$$\theta_3 = \kappa_3-12s^{-1}\kappa_1\kappa_2+16s^{-2}\kappa_1^3$$
$$\theta_4 = \kappa_4-24s^{-1}\kappa_1\kappa_3-24s^{-1}(s+2)^{-1}(s+3)\kappa_2^2+48s^{-2}(s+2)^{-1}(5s+12)\kappa_1^2\kappa_2- -48s^{-3}(s+2)^{-1}(5s+12)\kappa_1^4$$
$$\theta_5 = \kappa_5-40s^{-1}\kappa_1\kappa_4-120s^{-1}(s+2)^{-1}(s+4)\kappa_2\kappa_3+240s^{-2}(s+2)^{-1}(3s+8)\kappa_1^2\kappa_3+ +1440s^{-2}(s+2)^{-1}(s+4)\kappa_1\kappa_2^2-960s^{-3}(s+2)^{-1}(7s+24)\kappa_1^3\kappa_2+768s^{-4}(s+2)^{-1}(7s+24)\kappa_1^5$$
$$= \kappa_5-40s^{-1}[\kappa_1\kappa_4+3(s+2)^{-1}(s+4)\kappa_2\kappa_3]+240s^{-2}(s+2)^{-1}\kappa_1[(3s+8)\kappa_1\kappa_3+6(s+4)\kappa_2]-192s^{-3}(s+2)^{-1}(7s+24)\kappa_1^3(5\kappa_2-4s^{-1}\kappa_1^2). \quad \ldots \quad (9)$$

The equations for κ_r in terms of $\theta_1, \theta_2 \ldots, \theta_r$ are :

$$\kappa_1 = \theta_1$$
$$\kappa_2 = \theta_2+2s^{-1}\theta_1^2$$
$$\kappa_3 = \theta_3+12s^{-1}\theta_1\theta_2+8s^{-2}\theta_1^3$$
$$\kappa_4 = \theta_4+24s^{-1}\theta_1\theta_3+24s^{-1}(s+2)^{-1}(s+3)\theta_2^2+144s^{-2}\theta_1^2\theta_2+48s^{-3}\theta_1^4$$
$$\kappa_5 = \theta_5+40s^{-1}\theta_1\theta_4+120s^{-1}(s+2)^{-1}(s+4)\theta_2\theta_3+480s^{-2}\theta_1^2\theta_3+ +960s^{-2}(s+2)^{-1}(s+3)\theta_1\theta_2^2+1920s^{-3}\theta_1^3\theta_2+384s^{-4}\theta_1^5. \quad \ldots \quad (10)$$

It will be seen that when $s = 1$, equations (9) agree with equations (4) which were in fact worked out as a check on them, and to enable others to extend them further if desired. Similarly (10) agree with Haldane's (1941) equations (3). They are a good deal simpler than (9).

The distribution of the resultant of a sample of n random vectors, and that of n unit vectors

Consider a sample of n random vectors, each drawn from a population in which the r-th cumulant of the distribution of the scalar square is κ_r. To find the cumulants of the distribution of the scalar square of their resultant, we have only to multiply the values of θ_r in (9) by n, and then to apply equations (10). We find :

$$\begin{aligned}
\zeta_1 &= n\kappa_1 \\
\zeta_2 &= 2n(n-1)s^{-1}\kappa_1^2+n\kappa_2 \\
\zeta_3 &= 4n(n-1)s^{-1}\kappa_1[2(n-2)s^{-1}\kappa_1^2+3\kappa_2]+n\kappa_3 \\
\zeta_4 &= 24n(n-1)s^{-1}[2\{(n-2)(n-3)-s(s+2)^{-1}\}s^{-2}\kappa_1^4+2\{3(n-2)+s(s+2)^{-1}\}s^{-1}\kappa_1^2\kappa_2+ \\
&\qquad +\{1+(s+2)^{-1}\}\kappa_2^2+\kappa_1\kappa_3]+n\kappa_4 \\
\zeta_5 &= 8n(n-1)s^{-1}[48(n-2)\{(n-3)(n-4)-5s(s+2)^{-1}\}s^{-3}\kappa_1^5+120\{2(n-2)(n-3)+ \\
&\qquad +(2n-5)s(s+2)^{-1}\}s^{-2}\kappa_1^3\kappa_2+60\{3(n-2)-(n-3)s(s+2)^{-1}\}s^{-1}\kappa_1\kappa_2^2+ \\
&\qquad +30\{2(n-2)+s(s+2)^{-1}\}\kappa_1^2\kappa_3+15\{1+2(s+2)^{-1}\}\kappa_2\kappa_3+5\kappa_1\kappa_4]+n\kappa_5. \qquad \dots \quad (11)
\end{aligned}$$

Since these equations may be of use in the theory of random migration on a plane surface, I give the values for $s = 2$. They are :

$$\begin{aligned}
\zeta_1 &= n\kappa_1 \\
\zeta_2 &= n(n-1)\kappa_1^2+n\kappa_2 \\
\zeta_3 &= 2n(n-1)\kappa_1[(n-2)\kappa_1^2+3\kappa_2]n\kappa_3 \\
\zeta_4 &= 3n(n-1)[(2n^2-10n+11)\kappa_1^4+2(6n-11)\kappa_1^2\kappa_2+5\kappa_2^2+4\kappa_1\kappa_3]+n\kappa_4 \\
\zeta_5 &= 2n(n-1)[3(n-2)(2n^2-14n+19)\kappa_1^5+30(4n^2-18n+19)\kappa_1^3\kappa_2+15(7n-12)\kappa_1\kappa_2^2+ \\
&\qquad +15(4n-7)\kappa_1^2\kappa_3+45\kappa_2\kappa_3+10\kappa_1\kappa_4]+n\kappa_5. \qquad \dots \quad (12)
\end{aligned}$$

Finally I give the cumulants of the resultant of n unit vectors with random direction. Here $\kappa_1 = 1$, $\kappa_r = 0 (r > 1)$. They are :

$$\begin{aligned}
\zeta_1 &= n \\
\zeta_2 &= 2n(n-1)s^{-1} \\
\zeta_3 &= 8n(n-1)(n-2)s^{-2} \\
\zeta_4 &= 48n(n-1)[(n-2)(n-3)(s+2)-s]s^{-3}(s+2)^{-1} \\
\zeta_5 &= 384n(n-1)(n-2)[(n-3)(n-4)(s+2)-5s]s^{-4}(s+2)^{-1} \qquad \dots \quad (13)
\end{aligned}$$

It will be seen that in (11) ζ_r tends to $2^r(r-1)!\, n^r s^{1-r}\kappa_1^r$ as n tends to infinity, that is to say the distribution approximates to a Gamma, or Pearsonian Type III, distribution.

The vectorial cumulants as measures of departure from a gamma distribution

It can be shown that if $\kappa_r = 2^{r-1}(r-1)!\, s^{1-r}\kappa_1^r$ then all the vectorial cumulants after the first vanish. If the vector X has s orthogonal components each normally distributed with mean zero and standard deviation σ, then $\kappa_1 = s\sigma^2$, and $\kappa_r = s2^r(r-1)!\, \sigma^{2r}$, $\sigma^{-2}x$ has a χ^2 distribution with s degrees of freedom. Haldane (1960) gives some values of κ_r for other "parent" distributions and a test for the significance of the departure of κ_2 (which is all that can be estimated with any accuracy in the sample there considered) from zero. If $s = 1$ it is readily seen that $\theta_2 = \mu_4 - 3\mu_2^2$. ($\mu_2$ and μ_4 being moments of x), $\theta_3 = \mu_6 - 15\mu_2\mu_4 + 30\mu_2^3$ and so on, so that θ_r is the $2r$-th cumulant of the distribution of x.

$\theta_r\theta_1^{-r}$ can thus be used as a shape parameter which becomes equal to Fisher's γ_2, when $s = 1$. From (9) we see that the sum of n vectors has

$$\theta_1 = \Sigma\kappa_1$$
$$\theta_2 = \Sigma\kappa_2 - 2s^{-1}\Sigma\kappa_1^2$$

etc. Hence $\theta_2\theta_1^{-2}$ tends to zero subject to the conditions of the central limit theorem.

In the case of n unit vectors

$$\theta_1 = n,\ \theta_2 = -2s^{-1}n,\ \theta_3 = +16s^{-2}n,\ \theta_4 = -48s^{-2}(s+2)^{-1}(5s+12)n, \text{ etc.}$$

So $\theta_2\theta_1^{-2} = -2/ns$. The greater the number of dimensions the more rapid the approach to zero. $\theta_r\theta_1^{-r}$ tends to zero with n^{1-r}, but can be very large when r is large.

DISCUSSION

Beckmann (1959) has considered the case of the resultant of a number of unit vectors whose directions in two dimensions have a given distribution. This is complementary to my general case. The present investigation arose from a consideration of Alström's (1958) work on distances between birthplaces of parents and offspring. It is possible however that it may find applications in physics as well as in demography. The velocity of a molecule is a vector, and its scalar square is the proportional to the kinetic energy, so in this case it is quite natural to consider the distribution of the squared velocity, which was of course done by Maxwell. In a perfect gas $s = 3$, and $\theta_r = 0(r > 1)$. Unfortunately for many purposes we are interested in the small fraction of molecules whose energy exceeds a critical value, and the first few moments of a distribution give little information about them.

I have deliberately kept this paper short, since it is possible that some different symbolism may prove more efficient than my own. However it seems possible that the study of random vectors may throw light on several problems of practical and theoretical statistics. Perhaps the most obvious extension is to consider the resultant of random vectors on the surface of a sphere, and more generally in various kinds of non-Euclidean space.

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