Some Conjugacy Problems in Algebraic Groups

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To My Father

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Contents

CHAPTER 1

Introduction

In this thesis we address two problems related to the study of algebraic groups and Lie groups. The first one deals with computation of an invariant called the genus number of a connected reductive algebraic group over an algebraically closed field and that of a compact connected Lie group. The second problem is about characterisation of real elements in exceptional groups of type F_4 defined over an arbitrary field.

Let G be a connected reductive algebraic group over an algebraically closed field or a compact connected Lie group. Let $Z_G(x)$ denote the centralizer of $x \in G$. Define the **genus number** of G as the cardinality of the set $\{[Z_G(x)] : x \in G\}$, where $[Z_G(x)]$ denotes the conjugacy class of $Z_G(x)$ in G. It turns out that the number of conjugacy classes of centralizers of elements in a connected reductive algebraic group over an algebraically closed field is finite ([**St**]). It is therefore natural to pose the following problem: Given a connected reductive algebraic group G, compute the genus number. Although this problem may be implicit in Dynkin's papers [**D1**], [**D2**], the explicit knowledge of genus number is difficult to extract from these works.

Semisimple conjugacy classes for finite groups of Lie type have been studied by Fleischmann and Carter (see [**F**], [**C1**]). K. Gongopadhyay and R. Kulkarni have computed the number of conjugacy classes of centralizers in $I(\mathbb{H}^n)$ (the group of isometries of the hyperbolic $n-$space) [**GK**]. See [**K**] where the author discusses a related notion of $z-$classes. Conjugacy classes of centralizers in anisotropic groups of type G_2 over $\mathbb{R}$, have been explicitly calculated by A. Singh in [**Si**].

In this thesis we describe a method of computing this number by looking at the Weyl group of the group in question and its action on a fixed maximal torus. We explicitly compute the genus number for all the classical groups and for G_2 and F_4 among the exceptional ones, as far as semisimple elements are concerned.

Let G be a group. An element $x \in G$ is said to be **real** in G if there exists $g \in G$ such that $gxg^{-1} = x^{-1}$ and x is called **strongly real** in G if there exists $g \in G$ such that $g^2 = 1$ and $gxg^{-1} = x^{-1}$. Note that $x \in G$ is strongly real if and only if there exist elements $g_1, g_2 \in G$ such that $x = g_1g_2$ and $g_1^2 = g_2^2 = 1$. Let G be an algebraic group defined over a field k and $G(k)$ be the set of all k-rational points of G. We say that $x \in G(k)$ is **k-real** if there exists $g \in G(k)$ such that $gxg^{-1} = x^{-1}$ and x is called **strongly k-real** if there exists an element $g \in G(k)$ with $g^2 = 1$ and $gxg^{-1} = x^{-1}$.

The problem of characterising real elements in a group is directly related to studying the representation theory of the group. Let G be a finite group. observe that if $g \in G$ is real then every element in the conjugacy class of g is real. Such a conjugacy class is called a **real conjugacy class**. Consider representations of G over $\mathbb{C}$. A character χ of G is said to be **real** if $\chi(g) \in \mathbb{R}$ for all $g \in G$. A representation $\rho : G \longrightarrow GL(V)$ is said to be **realizable** if it is defined over $\mathbb{R}$. In fact, the number of real irreducible characters of G is equal to the number of real conjugacy classes of G ([**JL**], Theorem 23.1). In [**Pr1**] and [**Pr2**], Prasad has studied self- dual representations of finite groups of Lie type and p-adic groups.

It was proven by Wonenburger that for a field k, any element in $GL_n(k)$ is real if and only if it is strongly real in $GL_n(k)$ ([**W1**], Theorem 1). For $n \not\equiv 2(mod\ 4)$, an element of $SL_n(k)$ is real if and only if it is strongly real in $SL_n(k)$ ([**ST2**], Theorem 3.1.1). For a finite dimensional vector space V over a field k with a non degenerate quadratic form Q, every semisimple element in the special orthogonal group $SO(V, Q)$ is real if and only if it is strongly real in $SO(V, Q)$ ([**ST2**], Theorem 3.4.6). In [**W**], Wonenburger proved that in an anisotropic group of type G_2, which is obtained as the group of automorphisms of an octonion division algebra over a real closed field, every element is strongly real (Corollary 2, [**W**]). Reality for groups of type G_2 was further studied by Singh and Thakur in [**ST1**]. It is worthwhile to mention the reality properties known for the classical compact simple Lie groups: In the special unitary group $SU(n)$ with $n \not\equiv 2(mod\ 4)$, an element is real if and only if it is strongly real (Corollary 3.6.3, [**ST2**]). In the special orthogonal group $SO(n)$ of an n-dimensional real quadratic space, an element $t \in SO(n)$ is real if and only if it is strongly real (Theorem3.4.6, [**ST2**]). However, in compact symplectic groups $Sp(n)$, there exist real elements that are not strongly real ([**ST2**], refer to the remark following Theorem 3.5.3).

In an algebraic group G defined over a field k, an element $x \in G$ is called **strongly regular** if $Z_G(x)$ is a maximal torus in G. It is known that in a connected adjoint semisimple algebraic group over a perfect field, with -1 in the Weyl group, any strongly regular k-real element is strongly k-real ([**ST2**], Theorem 2.1.2). In this thesis we characterise real elements in groups of type F_4 which are not necessarily strongly regular.

Chapter 2 and Chapter 3 cover preliminary material for the chapters that follow. In Chapter 2 we give a brief exposition on the theory of Lie groups, algebraic groups and other related notions. Chapter 3 discusses the construction of exceptional groups of type G_2 and F_4 starting from octonion and Albert algebras respectively. We have briefly described the principle of triality for the norm on an octonion algebra in Section 3.2.1 as this principle is quite crucial in the study of these groups. For proofs of the main results one can directly look up Chapters 4 and 5.

Main Results

In this section, we state the main results proved in this thesis.

Results on computation of genus number

Let G be a compact connected Lie group or a connected algebraic group over an algebraically closed field. The cardinality of the set $\{[Z_G(x)] : x \in G,\ x \text{ semisimple}\}$, where $Z_G(x)$ is the centralizer of x in G, is defined as the **semisimple genus number** of G. We call this simply the genus number as we shall consider only semisimple elements here. If G is not simply connected, then the cardinality of the set $\{[Z_G(x)^\circ] : x \in G,\ x \text{ semisimple}\}$, is called the **connected genus number** of G. Here $Z_G(x)^\circ$ denotes the connected component of identity in $Z_G(x)$.

Theorem 4.2.4: For a simply connected compact Lie group G with maximal torus T and Weyl group W, there exists a bijection

$$\{[Z_G(x)] \colon x \in T\} \longrightarrow \{[W_x] \colon x \in T\}$$

given by

$$[Z_G(x)] \longmapsto [W_x]$$

Here $[Z_G(x)]$ and $[W_x]$ respectively denote the conjugacy class of the centralizer $Z_G(x)$ of x in G and the conjugacy class of the stabilizer W_x of x in W.

Theorem 4.2.7: For a simply connected algebraic group G over an algebraically closed field, with maximal torus T and Weyl group W, there exists a bijection

$$\{[Z_G(x)] \colon x \in T\} \longrightarrow \{[W_x] \colon x \in T\}$$

given by

$$[Z_G(x)] \longmapsto [W_x]$$

Here $[Z_G(x)]$ and $[W_x]$ respectively denote the conjugacy class of the centralizer $Z_G(x)$ of x in G and the conjugacy class of the stabilizer W_x of x in W.

Corollary 4.2.8: Let G be a compact simply connected Lie group (resp. a simply connected algebraic group over an algebraically closed field), $T \subset G$ a maximal torus. The genus number (resp. semisimple genus number) of G equals the number of orbit types of the action of the Weyl group $W(G,T)$ on T.

Theorem 4.2.12: Let G be a compact connected semisimple Lie group or a connected semisimple algebraic group over an algebraically closed field k. Let $\widetilde{G}$ be the simply connected cover of G. Then the connected genus number of G is equal to the genus number of $\widetilde{G}$.

Theorem 4.2.13: Let G be a connected reductive algebraic group over an algebraically closed field. Let G' be the commmutator subgroup of G. Then the connected genus number of G is equal to the connected genus number of G'.

Theorem 4.9.1: Let G be a compact connected Lie group (or a connected reductive algebraic group over an algebraically closed field) with the Lie algebra denoted by $\mathfrak{g}$. With respect to the action, $Ad : G \longrightarrow Aut(\mathfrak{g})$ defined by $g \mapsto Ad_g$, where $Ad_g(x) = gxg^{-1}$, (having embedded G in a suitable GL_n) there is a bijection between the conjugacy classes of centralizers of semisimple elements in $\mathfrak{g}$ in G and the conjugacy classes of centralizers of elements of a Cartan subalgebra in WG.

Apart from these general results, explicit computation of the genus number has been done for all the classical simple groups and groups of type G_2 and F_4 among the exceptional groups (refer to the table at the end of Section 4.9). The proofs of the above results make up Chapter 4 of this thesis.

Results on real elements in F_4

For real elements in groups of type F_4 we have the following results:

Theorem 5.2.4: Every element of the compact connected Lie group of type F_4 is strongly real.

Theorem 5.3.5: Let A be an Albert division algebra over a perfect field k and $G = Aut(A)$ be the corresponding algebraic group of type F_4. Then $G(k)$ does not have any k-real element.

Theorem 5.4.2: Let A be a reduced Albert algebra over a perfect field k ($char(k) \neq 2$) where -1 is a square and $G = Aut(A)$. If ϕ be a k-real automorphism of A, then either ϕ is strongly k-real in $G(k)$ or it is a product of two involutions in the group of norm similarities of A.

The proofs of these results are the contents of Chapter 5.

CHAPTER 2

Lie Groups and Algebraic groups

In this chapter, we give a brief introduction to the theory of Lie groups and linear algebraic groups. We start with definition and examples Lie groups. Section 2.2 deals with compact connected Lie groups. We introduce the notion of a maximal torus and the associated finite group called the Weyl group. We also define the simply connected cover of a connected Lie group and see some examples. Explicit descriptions of simply connected covers of the compact classical simple Lie groups are given in Section 2.3. From Section 2.4 onwards, we briefly discuss the structure theory of algebraic groups and we conclude this chapter with the classification of simple algebraic groups. For a detailed account of the theory, the reader may refer to [**BD**], [**FH**], [**H**], [**B1**] and [**Hu**].

2.1. Lie groups: Definition and examples

Let G be a group. Let $\mu : G \times G \longrightarrow G$ and $\iota : G \longrightarrow G$ denote the product and inverse operations respectively i.e., $\mu(a,b) = ab$ and $\iota(a) = a^{-1}$ for all $a, b \in G$. A group G is called a **topological group** if G is a topological space and the maps μ and ι are continuous. Here, one considers the space $G \times G$ equipped with the product topology.

A topological group G is called a **Lie group** if G is a C^∞-manifold and the operations μ and ι are C^∞-functions. By **dimension** of a Lie group G, we mean the dimension of the underlying manifold.

Let G_1 and G_2 be two Lie groups. A homomorphism of G_1 into G_2 is a map $f : G_1 \longrightarrow G_2$, such that f is a group homomorphism as well as a C^∞-map of manifolds. Given a Lie group G, the connected component at the identity is denoted by G°. We denote the center of G by $Z(G)$. For any element $g \in G$, let $Z_G(g) := \{x \in G : xg = gx\}$ denote the centralizer of g in G.

The most basic example of a Lie group is $GL_n(\mathbb{R})$, the group of $n \times n$ invertible real matrices. Clearly, $GL_n(\mathbb{R})$ is an open subset of the space of all $n \times n$ real matrices. This makes $GL_n(\mathbb{R})$ a C^∞-manifold and it can be checked that the operations of matrix multiplication and inversion are C^∞-maps. Other interesting examples occur as various closed subgroups of $GL_n(\mathbb{R})$ such as:

1. $SL_n(\mathbb{R}) := \{x \in GL_n(\mathbb{R}) : \ det(x) = 1\}$
2. $SO_n(\mathbb{R}) := \{x \in SL_n(\mathbb{R}) : \ xx^t = 1\}$, where x^t denotes the transpose of the matrix x.
3. The subgroup of all upper triangular matrices in $GL_n(\mathbb{R})$.
4. $D_n(\mathbb{R}) := \{diag(a_1, ..., a_n) : a_i \in \mathbb{R}^*, \ i = 1, ..., n\}$.

The latter example is of particular interest and we shall see some important properties of such Lie groups in the following section. These were some examples of real Lie groups. Similar constructions can be made with complex matrices i.e., $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$, etc.

2.2. Compact Lie groups

We now restrict ourselves to the study of Lie groups, which are compact and connected as topological spaces. Let G be a compact connected Lie group. A subgroup S of G is called a **torus** if there exists $n \in \mathbb{N}$, such that $S \cong (\mathbb{R}/\mathbb{Z})^n$ as Lie groups. A **maximal torus** of G is a torus $T \subset G$ such that, if H is another torus of G with $T \subset H$, then $T = H$. Note that, a torus is a compact, connected abelian Lie group. Also, maximal tori are maximal abelian subgroups of a given Lie group. However, it is worthwhile to note that not all maximal abelian subgroups are tori.

For example, consider the Lie group $SO_{2n}(\mathbb{R})$. A maximal torus of $SO_{2n}(\mathbb{R})$ can be described as follows: Consider the subgroup of all block diagonal matrices of the form $diag\ (B_1, ..., B_n)$, where

$$B_i = \begin{bmatrix} cos2\pi x_i & -sin2\pi x_i \\ sin2\pi x_i & cos2\pi x_i \end{bmatrix},$$

with $x_i \in (0,1)$, $i = 1, ..., n$. This subgroup forms a maximal torus in $SO_{2n}(\mathbb{R})$. However, the subgroup consisting of diagonal matrices $diag(\alpha_1, ..., \alpha_{2n})$, such that $\alpha_i = \pm 1$ for all i and $\alpha_1 ... \alpha_{2n} = 1$, is a maximal abelian subgroup of $SO_{2n}(\mathbb{R})$ but it is not a torus.

We are now in a position to state the following important theorem.

Theorem 2.2.1. *(*[**BD**]*, Chapter IV, Theorem 1.6) Let G be a compact, connected Lie group. Then, for every $g \in G$, there exists a maximal torus $T \subset G$, such that $g \in T$. If T_1 and T_2 are maximal tori in G, then there exists $h \in G$, such that $hT_1h^{-1} = T_2$.*

As an easy consequence of the above theorem, we have

Corollary 2.2.2. *Let G be a compact connected Lie group. Then $Z(G)$ is the intersection of all maximal tori in G.*

Corollary 2.2.3. *Let G be a compact connected Lie group. For $g \in G$, $Z_G(g)^\circ$ is the union of all maximal tori of G containing g.*

Also, since any two maximal tori in a compact connected Lie group G are conjugate, the dimension of a maximal torus is uniquely determined. This number is defined as the **rank** of G.

Now, let T be a maximal torus in a compact, connected Lie group G. Define the normalizer of T in G as $N_G(T) := \{g \in G : \; gTg^{-1} = T\}$. The group $W(G,T) = N_G(T)/T$ is called the **Weyl group** of G. By Theorem 2.2.1, since any two maximal tori are conjugate in G, different maximal tori give rise to isomorphic Weyl groups. Henceforth, whenever the choice of the maximal torus is clear from the context, we shall denote the Weyl group by W. Observe that, $N_G(T)$ acts on the maximal torus T by conjugation; $N_G(T) \times T \longrightarrow T, \; (n,t) \mapsto ntn^{-1}$. Since T acts on itself trivially by conjugation, one obtains an induced action of the Weyl group W on T as

$$W \times T \longrightarrow T, \; (nT, t) \mapsto ntn^{-1}.$$

Thus W acts on T by automorphisms. Let us denote the group of automorphisms of the maximal torus T by $Aut(T)$.

Theorem 2.2.4. *([**BD**], Chapter IV) Let G be a compact, connected Lie group and $T \subset G$, a maximal torus. Then the Weyl group W is finite and the homomorphism $W \longrightarrow Aut(T)$ defined by the action of W on T is injective.*

The Weyl group of $SO_{2n}(\mathbb{R})$ can be shown to be isomorphic to $(\mathbb{Z}/2)^{n-1} \ltimes S_n$, where S_n denotes the symmetric group corresponding to a set of n elements. However, the Weyl group of $SO(2n+1)$ is isomorphic to $(\mathbb{Z}/2)^n \ltimes S_n$. Detailed description of Weyl groups for the classical simple groups and their corresponding actions on maximal tori will be taken up in Chapter 4.

The following theorem will be needed in the sequel.

Theorem 2.2.5. *([**BD**], Chapter IV, Theorem 2.9) Let $f : G_1 \longrightarrow G_2$ be a surjective homomorphism of compact, connected Lie groups. Then, if $T \subset G_1$ be a maximal torus, so is $f(T) \subset G_2$. Furthermore, $ker(f) \subset T$ if and only if $ker(f) \subset Z(G_1)$. In this case, f induces an isomorphism of the Weyl groups of G_1 and G_2.*

Recall that a topological space X is said to be **simply connected** if the fundamental group $\pi_1(X)$ of X is trivial. Let G be a connected (not necessarily compact) Lie group. Then, a **universal cover** of G is a simply connected Lie group $\widetilde{G}$ together with a homomorphism of Lie groups $\rho : \widetilde{G} \longrightarrow G$, which is a covering map of topological spaces. Let us denote this universal cover by $(\widetilde{G}, \rho)$.

Theorem 2.2.6. *([**H**], Theorem 3.10) For a connected Lie group G, a universal cover exists. If $(\widetilde{G_1}, \rho_1)$ and $(\widetilde{G_2}, \rho_2)$ be two universal covers of G, then there exists a Lie group isomomorphism $\phi : \widetilde{G_1} \longrightarrow \widetilde{G_2}$ such that, $\phi \circ \rho_2 = \rho_1$.*

For example, let $G = S^1$, the unit circle in the plane. Here, the universal cover $\widetilde{G}$ is isomorphic to $\mathbb{R}$. The covering homomorphism is given by $\rho(\alpha) = e^{i\alpha}$ for all $\alpha \in \mathbb{R}$. For $SO(n)$, the universal cover is $Spin(n)$.

2.3. The Compact Classical Lie groups

There are four infinite families of compact connected Lie groups, which are called classical groups and are denoted by A_n, B_n, C_n and D_n. Apart from these, there are up to isomorphism, five exceptional groups G_2, F_4, E_6, E_7 and E_8. The subscripts appearing in the symbols, denote the rank of each group. In this section we shall describe all the classical simple Lie groups.

Type A_n: This family of compact simply connected classical groups are given by the special unitary groups $SU(n)$. Let $U(n) := \{x \in GL_n(\mathbb{C}) : \overline{x}^t x = 1\}$, where $\overline{x}^t$ denotes the conjugate transpose of the matrix x. Then $SU(n) := \{x \in U(n) : det(x) = 1\}$.

Types B_n and D_n: Consider the special orthogonal group$SO(n) := \{x \in GL_n(\mathbb{R}) : x^t x = 1 \textit{ and } det(x) = 1\}$. These groups are compact and connected but however, they are not simply connected. Simply connected cover of $SO(n)$ is the **Spin group** $Spin(n)$, which we shall describe in Section 2.8. Compact simply connected Lie groups of type B_n are given by $Spin(2n + 1)$ and those of type D_n are given by $Spin(2n)$.

Type C_n: The compact simply connected Lie group of type C_n, denoted by $Sp(n)$ and is defined as follows: Consider $U(n)$ the group of $n \times n$ unitary matrices. Define $Sp(n) := \{A \in U(2n) : A^t JA = J\}$, where $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$, I is the identity matrix in $GL_n(\mathbb{C})$.

2.4. Linear Algebraic Groups: Definitions and examples

We shall now give a brief exposition on the theory of linear algebraic groups and other related concepts. For details and proofs of the results discussed, the reader can refer to the books [**B1**], [**Hu**], [**S**], [**C2**].

Let k denote a field and $\bar{k}$ be its algebraic closure. Consider the polynomial ring $\bar{k}[x_1, ..., x_n]$ of n variables over $\bar{k}$. For a subset $S \subset \bar{k}[x_1, ..., x_n]$, define the zero locus of S as $\mathcal{V}(S) := \{x \in \bar{k}^n : f(x) = 0 \ \forall f \in S\}$. A subset of $\bar{k}^n$ of the form $\mathcal{V}(S)$ for some subset $S \subset \bar{k}[x_1, ..., x_n]$ is called an **affine variety**. The collection of subsets $\{\mathcal{V}(S) : S \subset \bar{k}[x_1, ..., x_n]\}$ satisfy the axioms of closed sets in a topology. The resulting topology on $\bar{k}^n$ is called the **Zariski topology**.

A group G is called an **affine algebraic group** if G is an affine variety and the maps $\mu : G \times G \longrightarrow G$ and $\iota : G \longrightarrow G$ defined by $\mu(a, b) = ab$ and $\iota(a) = a^{-1}$ for all $a, b \in G$ are variety morphisms. Let G_1, G_2 be affine algebraic groups. A map $f : G_1 \longrightarrow G_2$ is a homomorphism of affine algebraic groups if f is a group homomorphism as well as a morphism of varieties, f is an isomorphism if it is bijective and both f and f^{-1} are homorphism of algebraic groups. From now onwards, the mention of any topological property, associated to affine algebraic groups, will be with respect to the Zariski topology.

Interesting examples of affine algebraic groups can be obtained as groups of non-singular matrices over $\bar{k}$:

(1.) $GL_n(\bar{k}) := \{x \in \mathbb{M}_n(\bar{k}) : det(x) \neq 0\}$.

(2.) $SL_n(\bar{k}) := \{x \in GL_n(\bar{k}) : det(x) = 1\}$.

(3.) The subgroup of all diagonal matrices in $GL_n(\bar{k})$.

(4.) The subgroup of upper triangular matrices in $GL_n(\bar{k})$ with all eigen values equal to 1, i.e., upper triangular unipotent matrices.

(5.) $SO_{2n+1}(\bar{k}) := \{x \in SL_{2n+1}(\bar{k}) : x^t s x = s\}$, where $s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & J \\ 0 & J & 0 \end{bmatrix}$ and J is the $2n \times 2n$ matrix with all off diagonal entries equal to 1 and 0 otherwise.

(6.) $Sp_{2n}(\bar{k}) := \{x \in GL_{2n}(\bar{k}) : x^t a x = a\}$, where $a = \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}$, J as in example (5) above.

In fact we have,

Theorem 2.4.1. *(Theorem 8.6,* [**Hu**]*) Let G be an affine algebraic group. Then G is isomorphic to a closed subgroup of $GL_n(\bar{k})$ for some n.*

By virtue of Theorem 2.4.1, affine algebraic groups are also called linear algebraic groups. In this thesis, we shall deal with only linear algebraic groups and we will refer to such groups simply as algebraic groups.

Let $X \subset \bar{k}^n$ be an affine variety. We say that X is **defined over** k if there exists a subset S of $k[x_1, ..., x_n]$ such that $X = \mathcal{V}(S)$ (see [**Hu**], Chapter XII). We shall denote the set (possibly empty) of k-rational points of X by $X(k)$. Let $X_1 \subset \bar{k}^n,\ X_2 \subset \bar{k}^m$ be affine varieties defined over k. A morphism $\phi : X_1 \longrightarrow X_2$ is said to be **defined over** k if the cordinate functions of ϕ lie in $k[x_1, ..., x_n]$. Now let G be an affine algebraic group over $\bar{k}$. We say that G is a k**-group** or **defined over** k if G together with the morphisms $\mu : G \times G \longrightarrow G$ and $\iota : G \longrightarrow G$ are all defined over k. In this case, the subgroup of k-rational points in G is denoted by $G(k)$.

For an algebraic group G, there exists a unique irreducible component of G containing the identity element. We denote this irreducible component by G°. It can be shown that G° is a normal subgroup of finite index in G and every closed subgroup of finite index in G contains G° (see [**Hu**], §7.3). We say that an algebraic group G is connected if $G = G^\circ$. For example, consider the algebraic group $O_n(\bar{k}) := \{x \in GL_n(\bar{k}) : x^t s x = s\}$, where s is as in Example 5 above. This group is not connected and $O_n(\bar{k})^\circ = SO_n(\bar{k})$.

2.5. The Lie Algebra of an Algebraic Group

We now want to associate a Lie algebra to a given algebraic group. Let k be a field and $\bar{k}$ its algebraic closure. A k-vector space L, together with a binary operation $L \times L \longrightarrow L$ denoted by $(x, y) \mapsto [x, y]$, called the **bracket** of x and y, is called a **Lie algebra** over k if the following axioms hold:
(1) The bracket operation is k-bilinear.
(2) $[x, x] = 0$ for all $x \in L$.
(3) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$. This equation is called the **Jacobi identity**.

An immediate example is $\mathbb{M}_n(k)$, the algebra of all $n \times n$ matrices over k equipped with the bracket operation $[x, y] := xy - yx$ for all $x, y \in \mathbb{M}_n(k)$.

Now let G be an algebraic group. Consider the cordinate ring $\bar{k}[G]$ of G. Then G acts on $\bar{k}[G]$ in the following way: For any $f \in \bar{k}[G]$ and $x \in G$, define the map $\lambda : G \times \bar{k}[G] \longrightarrow \bar{k}[G]$ by $\lambda(x, f) = \lambda_x f$, where $\lambda_x f(y) := f(x^{-1}y)$ for all $y \in G$. Given a $\bar{k}$-algebra A, a $\bar{k}$**-derivation** of A is a $\bar{k}$-linear map $d : A \longrightarrow A$ such that $d(\alpha_1\alpha_2) = \alpha_1 d(\alpha_2) + \alpha_2 d(\alpha_1)$. Let $Der(A)$ denote the set of all $\bar{k}$-derivations of A.

Now define the set $\mathcal{L}(G) := \{\delta \in Der(\bar{k}[G]) : \delta\lambda_x = \lambda_x\delta \ \forall x \in G\}$. It can be easily checked that $[\delta_1, \delta_2] \in \mathcal{L}(G)$ whenever $\delta_1, \delta_2 \in \mathcal{L}(G)$. This space $\mathcal{L}(G)$ is called the **Lie algebra of** G.

Given an algebraic group G, consider the **tangent space** $\mathcal{T}(G)_e$ of G at the identity element e. This is defined as follows: Let $A = \bar{k}[G]$ and let $M = \mathcal{I}(e)$ be the maximal ideal of A at e. Consider the local ring $\mathcal{O}_e := A_M$ and its unique maximal ideal $\mathfrak{m}_e := MA_M$. Then the tangent space of G at e is defined as the dual vector space $(\mathfrak{m}_e/\mathfrak{m}_e^2)^*$ over the field $\mathcal{O}_e/\mathfrak{m}_e$. From now on, we shall denote the tangent space of G at e by $\mathfrak{g}$. Define a **point derivation** of $\mathcal{O}_e$ as a map $\delta : \mathcal{O}_e \longrightarrow \mathcal{O}_e/\mathfrak{m}_e$, such that δ is $\mathcal{O}_e/\mathfrak{m}_e$-linear and it satisfies $\delta(fg) = \delta(f)g(e) + f(e)\delta(g)$ for all $f, g \in \mathcal{O}_e$. Let $\mathcal{D}_e$ denote the space of all point derivations of $\mathcal{O}_e$. It can be shown that $\mathfrak{g} \cong \mathcal{D}_e$.

Let $\phi : G_1 \longrightarrow G_2$ be a homomorphism of algebraic groups. Let e_1 and e_2 be the identity elements of G_1 and G_2 respectively. Note that ϕ induces a homomorphism $\widetilde{\phi} : (\mathcal{O}_{e_2}, \mathfrak{m}_{e_2}) \longrightarrow (\mathcal{O}_{e_1}, \mathfrak{m}_{e_1})$ of the corresponding local rings. Recall that $\mathfrak{g}_1 = (\mathfrak{m}_{e_1}/\mathfrak{m}_{e_1}^2)^*$ and $\mathfrak{g}_2 = (\mathfrak{m}_{e_2}/\mathfrak{m}_{e_2}^2)^*$. Define the **differential** of the morphism ϕ as the map $d\phi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$, $d\phi(X)(a) := X(\widetilde{\phi}(a))$ for all $X \in \mathfrak{g}_1$ and $a \in \mathfrak{m}_{e_2}/\mathfrak{m}_{e_2}^2$.

We are now in a position to state the following useful theorem,

Theorem 2.5.1. *(*[**Hu**]*, Theorem 9.1) Let G be an algebraic group. Define $\theta : \mathcal{L}(G) \longrightarrow \mathfrak{g}$ by $\theta(\delta)(f) := (\delta f)(e)$ for all $\delta \in \mathcal{L}(G)$ and $f \in \bar{k}[G]$. Then θ is a vector space isomorphism. If $\phi : G_1 \longrightarrow G_2$ be a homomorphism of algebraic groups, then $d\phi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ is a homomorphism of Lie algebras ($\mathfrak{g}_1, \mathfrak{g}_2$ being given the bracket product of $\mathcal{L}(G_1)$ and $\mathcal{L}(G_2)$ respectively).*

Now, for an algebraic group G, consider the inner automorphism $Int_g : G \longrightarrow G$, defined by $Int_g(x) = gxg^{-1}$ for all $x \in G$. Define $Ad_g := d(Int_g) : \mathfrak{g} \longrightarrow \mathfrak{g}$. Therefore, by Theorem 2.5.1, $g \mapsto Ad_g$ gives a representation $Ad : G \longrightarrow Aut(\mathfrak{g}) \subset GL(\mathfrak{g})$. This is called the **adjoint representation** of G.

2.6. The Jordan-Chevalley decomposition

Let V be a finite dimensional vector space over $\bar{k}$ and $x \in End(V)$. We define x as **semisimple** if x is diagonalizable over $\bar{k}$. We say that x is **nilpotent** if $x^n = 0$ for some integer n and **unipotent** if $x - 1$ is nilpotent(or equivalently, all eigen values of x are eaqual to 1). Now let $x \in GL(V)$. Then there exists unique elements $x_s, x_u \in GL(V)$ such that x_s is semisimple, x_u is unipotent and $x = x_s x_u = x_u x_s$.

This is called the multiplicative Jordan decomposition for V. The elements x_s and x_u are called the semisimple and unipotent parts of x respectively.

Let $X \in End(V)$. Then there exists unique elements $X_s, X_n \in End(V)$ such that X_s is semisimple, X_n is nilpotent, $X = X_s + X_n$ and $X_s X_n = X_n X_s$. This is called the additive Jordan decomposition for V. The elements X_s and X_n are called the semisimple and nilpotent parts of X.

Remark: For any $x \in GL(V)$, the semisimple part x_s of x is same for both the additive and multiplicative Jordan decompositions. So, if $x = x_s x_u$ and $x = x_s + x_n$, then the unipotent and nilpotent parts of x are related as $x_u = 1 + x_s^{-1} x_n$.

Now let G be an algebraic group defined over a field k. Let V be finite dimensional vector space over $\bar{k}$. A **rational representation** of G is an algebraic group homomorphism $\rho : G \longrightarrow GL(V)$. We can now state the following theorem.

Theorem 2.6.1. *([**S**], Theorem 2.4.8) Let $x \in G$. There there exists unique elements $x_s, x_u \in G$ such that $x = x_s x_u = x_u x_s$. For any rational representation $\rho : G \longrightarrow GL(V)$, $\rho(x_s)$ is semisimple and $\rho(x_u)$ is unipotent.*

For each $x \in G$, call x_s and x_u as the semisimple and unipotent parts of x respectively.

2.7. Semisimple and Reductive groups

We now briefly describe the structure theory of simple, semisimple and reductive algebraic groups. First, we need the notion of an algebraic torus. An algebraic group T is called **diagonalizable** if it is isomorphic to a closed subgroup of the group of all $n \times n$ invertible diagonal matrices over $\overline{k}$ for some n, T will be called a **torus** if T is isomorphic to the group of all $n \times n$ invertible diagonal matrices over $\overline{k}$ for some n. It can be shown that any connected algebraic group T, consisting of only semisimple elements, is a torus. We have,

Theorem 2.7.1. *([**B1**], Proposition 8.7) Let G be diagonalizable group, defined over a field k. Then $G = G^\circ \times H$, where G° is a torus defined over k and H, a finite group of order prime to $char(k)$.*

Let G be a connected algebraic group. A subgroup T of G is called a **maximal torus** if T is a torus and for any torus $T' \subset G$, $T \subset T' \implies T = T'$. Any semisimple element $x \in G$ is contained in some maximal torus of G. Also, if T_1, T_2 be two maximal tori in G, then there exists $g \in G$, such that $gT_1g^{-1} = T_2$. Hence, the dimension of a maximal torus in a connected algebraic group is uniquely determined. We call this number the **rank** of G.

Now, given a connected algebraic group G and a maximal torus $T \subset G$, it can be shown that the quotient $N_G(T)/Z_G(T)$ is finite (see [**Hu**], Chapter IX). Here, $N_G(T)$ and $Z_G(T)$ are the normalizer and centralizer of T in G, respectively. Define this group as the **Weyl group** of G with respect to T. Since any two maximal tori are conjugate in G, different maximal tori gives rise to isomorphic Weyl groups. Hence, we shall refer to this finite group as the Weyl group of G.

Define the **radical** of a connected algebraic group G as the maximal closed, connected, solvable normal subgroup of G. Denote this subgroup by $R(G)$. We call G **semisimple** if $R(G)$ is trivial. The **unipotent radical** $R_u(G)$ of G is defined as the largest closed, connected, unipotent, normal subgroup of G. Note that, $R_u(G)$ is the subgroup of all unipotent elements in $R(G)$. If $R_u(G)$ is trivial, we say that G is **reductive**. Thus, any semisimple group is necessarily reductive but the converse is not true in general. For example, $GL_n(\bar{k})$ is reductive but not semisimple. In fact, we have,

Theorem 2.7.2. *([**Hu**], Theorem 27.5) Let G be a semsimple algebraic group. Then $G = [G, G]$, where $[G, G]$ denotes the commutator subgroup of G.*

It immediately follows that,

Corollary 2.7.3. *Let G be a connected reductive algebraic group. Then $G = [G, G].Z(G)^\circ$, where $Z(G)$ is the center of G, $Z(G)^\circ$ is a torus and $Z(G) \cap [G, G]$ is finite.*

2.8. Clifford algebras and Spin groups

In this section we shall introduce the notion of a spin group. A spin group is the universal cover of a special orthogonal group and is defined by certain structures called Clifford algebras. For a detailed exposition, the reader may refer to [**SV**].

Let Q be a non degenerate quadratic form on a finite dimensional vector space V over a field k. Consider the tensor algebra

$$T(V) := k \oplus V \oplus (V \otimes V) \oplus ...$$

Let $I := \langle v \otimes v - Q(v) \rangle$ be the ideal of V, generated by the elements $v \otimes v - Q(v)$, $v \in V$. We define the **Clifford algebra** of V with respect to Q as the quotient $C(V, Q) = T(V)/I$. Now V can be canonically identified as a subspace of $C(V, Q)$.

For a basis $\{e_1, ...e_n\}$ of V, it is easy to check that a basis of $C(V,Q)$ is given by $\{e_{i_1}...e_{i_l} : 1 \leq i_1 < ... < i_l \leq n\}$, $0 \leq l \leq n$. Therefore, if dimension of V is n, the dimension of $C(V,Q)$ is 2^n.

The **even Clifford algebra** $C(V,Q)^+$ is defined as the subalgebra of $C(V,Q)$ generated by the set $\{vw : v, w \in V\}$. Define the **Clifford group** of Q as the group $\Gamma(V,Q)$ of all invertible elements $x \in C(V,Q)$ such that $xVx^{-1} = V$. Then the **even Clifford group** is defined as $\Gamma^+(V,Q) = \Gamma(V,Q) \cap C(V,Q)^+$. For every $x \in \Gamma(V,Q)$, define $t_x : V \longrightarrow V$ by $v \mapsto xvx^{-1}$, for all $v \in V$. Then we have an exact sequence

$$1 \to k^* \to \Gamma^+(V,Q) \xrightarrow{\chi} SO(V,Q) \to 1,$$

where χ denotes the homomorphism $x \mapsto t_x$ and $SO(V,Q)$ denotes the orthogonal group of V with respect to Q. Thus, every element of $\Gamma^+(V,Q)$ is of the form $x = v_1...v_{2l}$ for $v_1, ..., v_{2l} \in V$ with $Q(v_i) \neq 0$ and each such $x \in \Gamma^+(V,Q)$ is determined up to a scalar factor in k^* by the map t_x.

Let $\iota : C(V,Q) \longrightarrow C(V,Q)$ defined by $\iota(v_1...v_r) = v_r...v_1$, for $v_1, ..., v_r \in V$, denote the **main involution** (anti automorphism of order 2) of $C(V,Q)$. Now, for $x = v_1...v_{2l} \in \Gamma^+(V,Q)$, define $N(x) := x\iota(x) = Q(v_1)...Q(v_{2l}) \in k^*$. It can be easily checked that $N : \Gamma^+(V,Q) \longrightarrow k^*$, $x \mapsto N(x)$ is a homomorphism. Define $ker(N)$ as the the **spin group** $Spin(V,Q)$. Alternatively, we denote the spin group of an n-dimensional vector space V over k by $Spin_n(k)$.

Now, consider $V_{\overline{k}} = \overline{k} \otimes_k V$ together with the quadratic form $Q_{\overline{k}}$, which is just the extension of Q to $V_{\overline{k}}$. It follows that $C(V_{\overline{k}}, Q_{\overline{k}}) = \overline{k} \otimes_k C(V,Q)$. We thus have an algebraic group $\Gamma(V_{\overline{k}}, Q_{\overline{k}})$ which is defined over k, $\Gamma^+(V_{\overline{k}}, Q_{\overline{k}})$ and $Spin(V_{\overline{k}}, Q_{\overline{k}})$ are closed subgroups of $\Gamma(V_{\overline{k}}, Q_{\overline{k}})$.

The covering map from $Spin(V,Q)$ to $SO(V,Q)$ is given by the restriction of the homomorphism χ to $Spin(V,Q)$, i.e., $\rho : Spin(V,Q) \longrightarrow SO(V,Q)$, defined by $\rho(v_1...v_{2l}) = s_{v_1}...s_{v_{2l}}$, where $v_1, ..., v_{2l} \in V$ and $Q(v_1)...Q(v_{2l}) = 1$ and s_v denotes the reflection in the hyperplane orthogonal to $v \in V$.

In particular, if we consider the vector space $V = \mathbb{R}^n$ over $\mathbb{R}$ with the Euclidean inner product, the resulting spin group is the compact simply connected simple Lie group of type B_n or D_n, according as the dimension of V is $2n+1$ or $2n$ respectively.

2.9. Classification of simple algebraic groups

For the material described in this section, the reader may consult [**B1**], [**Hu**] and [**KMRT**].

A connected algebraic group G defined over a field k, is said to be **simple** if there does not exist any proper closed connected normal subgroup of G. A homomorphism of algebraic groups $f : G_1 \longrightarrow G_2$ is called an **isogeny** if $ker(f)$ is finite. If this finite kernel is contained in the center of G_1, we call f a **central isogeny**. We need the notion of a root datum associated to a connected algebraic group, which we now describe.

Let G be a connected reductive algebraic group. Fix a maximal torus $T \subset G$ and let W be the Weyl group. By a **character** of T we mean an algebraic group homomorphism $\chi : T \longrightarrow \mathbb{G}_m$, where $\mathbb{G}_m := \overline{k}^*$ and a **cocharacter** of T is defined as a homomorphism $\gamma : \mathbb{G}_m \longrightarrow T$. Let $X(T) := Hom(T, \mathbb{G}_m)$ be the group of all characters of T under the multiplication $(\chi_1 + \chi_2)(t) = \chi_1(t)\chi_2(t)$, for all $\chi_1, \chi_2 \in X(T)$ and $t \in T$. Simialarly, we have the group $Y(T) := Hom(\mathbb{G}_m, T)$, of all cocharacters of T and the group operation is given by $(\gamma_1 + \gamma_2)(\alpha) = \gamma_1(\alpha)\gamma_2(\alpha)$, for all $\alpha \in \mathbb{G}_m$ and $\gamma_1, \gamma_2 \in Y(T)$. If the rank of T is n, then it easily follows that $X(T)$ and $Y(T)$ are free abelian groups of rank n.

Now, for $\chi \in X(T)$ and $\gamma \in Y(T)$, we can define an integer $\langle \chi, \gamma \rangle$ in the following way: note that $\chi \circ \gamma$ is a homomorphism of $\mathbb{G}_m$ to itself. Since any homomorphism $f : \mathbb{G}_m \longrightarrow \mathbb{G}_m$ is given by $f(\alpha) = \alpha^n$ for all $\alpha \in \mathbb{G}_m$, we define $\langle \chi, \gamma \rangle$ to be the integer such that $\chi(\gamma(\alpha)) = \alpha^{\langle \chi, \gamma \rangle}$ for all $\alpha \in \mathbb{G}_m$. Thus, we have a bilinear map $\langle , \rangle : X(T) \times Y(T) \longrightarrow \mathbb{Z}$. One can check that this map is non degenerate and hence, we have $X(T) \cong Hom(Y(T), \mathbb{Z})$ and $Y(T) \cong Hom(X(T), \mathbb{Z})$.

The maximal torus T acts on the Lie algebra $\mathfrak{g}$ of G via the adjoint action. So $\mathfrak{g}$ decomposes as a direct sum of T-invariant subspaces,

$$\mathfrak{g} = \bigoplus_{\chi \in X(T)} \mathfrak{g}_\chi,$$

where $\mathfrak{g}_\chi := \{x \in \mathfrak{g} : Ad_t(x) = \chi(t)x, \ \forall t \in T\}$. Now those $\chi \in X(T)$ for which $\mathfrak{g}_\chi \neq 0$, are called the **weights** of T in G and the non zero ones are called **roots** of G with respect to T. Let Φ denote the set of all roots. It turns out that Φ is independent of the choice of maximal torus T. We define Φ to be the **root system** of G. Given a group G, the root system Φ is unique up to isomorphism. Also $X(T)$ and $Y(T)$ are independent of the maximal torus T . So we shall denote them respectively by X and Y.

Every root $r \in \Phi$ of a connected reductive algebraic group G gives rise to a unique (up to scalars) homomorphism $\epsilon_r : \mathbb{G}_a \longrightarrow G$ such that $t\epsilon_r(x)t^{-1} = \epsilon_r(r(t)x)$ for all $x \in \mathbb{G}_a$ and $t \in T$. Also, G is generated by the groups U_r and T ([**Hu**], Theorem 26.3).

Here, $\mathbb{G}_a := \overline{k}$. The image U_r of ϵ_r is called the **root subgroup** of G corresponding to $r \in \Phi$.

To each root $r \in \Phi$, in a canonical way, one can associate a cocharacter $r^* \in Y(T)$, such that $\langle r, r^* \rangle = 2$. Call $\Phi^* := \{r^* : r \in \Phi\}$ the set of all **coroots** of G. Consider the vector space $V := \mathbb{R} \otimes X$. It can be shown that Φ generates V over $\mathbb{R}$. Now there exists a subset Δ of Φ such that Δ is a basis of V and also every element $r \in \Phi$ can be uniquely expressed as $r = \sum c_i \delta_i$, where $\delta_i \in \Delta$ and c_i are integers having the same sign. Call Δ the set of **simple roots** of G. Define the set Φ^+ (resp. Φ^-) of **positive roots** (resp **negative roots**) as those roots in Φ, which are obtained as non-negative (resp. non-positive) linear combinations of Δ. For a root $r \in \Phi$, let s_r denote the reflection in the hyperplane orthogonal to r in the vector space V. Reflections with respect to simple roots are called **simple reflections**. It can be shown that the Weyl group W of G is isomorphic to the group generated by all simple reflections. Thus, W acts on the root system Φ. There exists a unique element $w_0 \in W$ of order 2, such that $w_0(\Phi^+) = \Phi^-$. Define w_0 to be the **longest element** of W. The root system Φ is called **reducible** if there exist proper subsets Φ_1, Φ_2 of Φ such that $\Phi = \Phi_1 \cup \Phi_2$ and each root in Φ_1 is orthogonal to each root in Φ_2. Otherwise, we call Φ **irreducible**.

We now are in a position to define the **Dynkin diagram** of the group G. It is a graph $\Gamma(G)$ with the set of vertices being $\Delta = \{\delta_1, ..., \delta_n\}$, n being the rank of G. For any two vertices $\delta_i, \delta_j \in \Delta$, the number of edges is $\langle \delta_i, \delta_j^* \rangle \langle \delta_j, \delta_i^* \rangle$. An arrow is put from δ_i to δ_j if δ_i has a bigger length than δ_j. Here, the length is given by the norm in the $\mathbb{R}$-vector space V.

Let G be a semisimple algebraic group defined over a field k. By the **type** of G we mean the Cartan-Killing type of the root system of the group $G_{\overline{k}}$ obtained by extension of scalars to an algebraic closure $\overline{k}$ of k. For a reductive group G, its type is defined as the type of its commutator subgroup $[G, G]$.

A connected semisimple algebraic group G is called **simply connected** if the character group X is isomorphic to $Hom(\mathbb{Z}\Phi^*, \mathbb{Z})$ and it is called **adjoint** if $X \cong \mathbb{Z}\Phi$. It turns out that a connected semisimple algebraic group G over $\overline{k}$ is simple if and only if its Dynkin diagram is connected ([**KMRT**], Proposition 25.8). Up to central isogeny, there are only finitely many classes of connected simple algebraic groups, which we now enumerate.

The classical groups: There exists four infinite families of simple algebraic groups which are denoted by the symbols A_n, B_n, C_n and D_n, where the subscript n denotes the rank of the group. They are also called Classical groups.

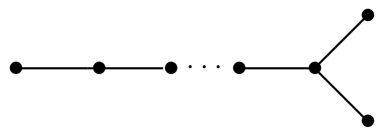
Groups of **type** A_n $(n \geq 1)$ are given by the $SL_{n+1}(\overline{k})$, the group of all $n + 1 \times n + 1$ matrices over $\overline{k}$ with determinant 1. This group is simply connected. The corresponding adjoint group is $PSL_{n+1}(\overline{k})$ which is $SL_{n+1}(\overline{k})$ modulo its center. The Weyl group of $SL_{n+1}(\overline{k})$ is isomorphic to S_{n+1}, the symmetric group corresponding to a set with $n + 1$ elements. The Dynkin diagram is given by:

Groups of **type** B_n $(n \geq 2)$ correspond to the special orthogonal groups $SO_{2n+1}(\bar{k}) := \{x \in SL_{2n+1}(\bar{k}) : x^t s x = s\}$, where $s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & J \\ 0 & J & 0 \end{bmatrix}$ and J is the $2n \times 2n$ matrix with all off diagonal entries equal to 1 and 0 otherwise. This group is adjoint. The simply connected cover of $SO_{2n+1}(\overline{k})$ is $Spin_{2n+1}(\overline{k})$. The Weyl group of $Spin_{2n+1}(\overline{k})$ is isomorphic to $(\mathbb{Z}/2)^n \rtimes S_n$ and the Dynkin diagram is given by:

The classical groups of **type** C_n $(n \geq 3)$ are the symplectic groups $Sp_{2n}(\overline{k}) = \{x \in GL_{2n}(\overline{k}) : x^t a x = a\}$, where $a = \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}$, where J is the matrix used in the definition of $SO_{2n+1}(\overline{k})$ above. These are simply connected groups. The corresponding adjoint group in this class is the projective conformal symplectic group $PCSp_{2n}(\overline{k})$, which is the conformal symplectic group $CSp_{2n}(\overline{k})$ modulo its center. $CSp_{2n}(\overline{k})$ is defined as the group of all symplectic similitudes of a $2n$-dimensional vector space V over $\overline{k}$, equipped with a non singular skew symmetric form $<,>$. A nonsingular endomorphism $T : V \longrightarrow V$ is called a symplectic similitude if there exists $\alpha \in \overline{k}$, such that $< T(x), T(y) >= \alpha < x, y >$ for all $x, y \in V$. The Weyl group in this case, is isomorphic to $(\mathbb{Z}/2)^n \rtimes S_n$ and the Dynkin diagram is given by:

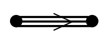
Finally, the simple groups of **type** D_n $(n \geq 4)$ are given by $SO_{2n}(\overline{k})$. This group is neither simply connected nor adjoint. The simply connected cover of $SO_{2n}(\overline{k})$ is $Spin_{2n}(\overline{k})$. The adjoint group in this isogeny class is the projective group of the connected component of $CO_{2n}(\overline{k})$, the group of all orthogonal similitudes on a $2n$-dimensional orthogonal space V over $\overline{k}$ with maximal Witt index n. If $<,>$ be the non degenerate symmetric bilinear form on V, a non singular endomorphism $T : V \longrightarrow V$ is called an orthogonal similitude if $< T(x), T(y) >= \beta < x, y >$ for all $x, y \in V$

and a fixed β independent of x, y. The Weyl group of $Spin_{2n}(\overline{k})$ is isomorphic to $(\mathbb{Z}/2)^{n-1} \rtimes S_n$. The Dynkin diagram is given by:

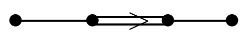


The exceptional groups: In addition to the four families of classical groups, there are five exceptional groups denoted by G_2, F_4, E_6, E_7 and E_8. In Chapter 3 we shall describe the groups of type G_2 and F_4. However, the remaining exceptional groups E_6, E_7 and E_8 are beyond the scope of this thesis. Presently, we shall give the Dynkin diagrams of these groups:

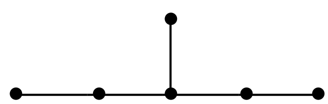
G_2 :



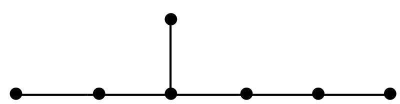
F_4 :



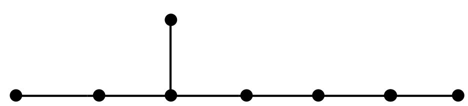
E_6 :



E_7 :



E_8 :



CHAPTER 3

Groups of type G_2 and F_4

In this chapter we give a brief description of the exceptional groups of type G_2 and F_4. These groups are obtained as automorphism groups of Octonion algebras and Albert algebras respectively. For a detailed exposition on these constructions, one may refer to [**SV**] and [**KMRT**]. Throughout this chapter unless otherwise stated, we consider a field k with $char(k) \neq 2$.

3.1. Octonion algebras and groups of type G_2

A **composition algebra** $\mathfrak{C}$ over a field k is a non associative algebra over k with an identity element 1, equipped with a non degenerate quadratic form N such that N is multiplicative, i.e., $N(xy) = N(x)N(y)$ for all $x, y \in \mathfrak{C}$.

The bilinear form associated to N is given by $\langle x, y\rangle := N(x+y) - N(x) - N(y)$. We call the quadratic form N the **norm** on $\mathfrak{C}$ and the bilinear form $\langle , \rangle$ as the **inner product**. A k-subspace $\mathfrak{D}$ of $\mathfrak{C}$ is called a **subalgebra** of the composition algebra $\mathfrak{C}$, if it is non singular with respect to the inner product, closed under multiplication and contains the identity element 1 of $\mathfrak{C}$.

Any element $x \in \mathfrak{C}$ satisfies the equation $x^2 - \langle x, 1\rangle x + N(x)1 = 0$, also called the **minimum equation** of x, if x is not a scalar multiple of the identity $1 \in \mathfrak{C}$. **Conjugation** on a composition algebra $\mathfrak{C}$ is a map $^- : \mathfrak{C} \longrightarrow \mathfrak{C}$, defined by $\overline{x} = \langle x, 1\rangle 1 - x$, for all $x \in \mathfrak{C}$. This map is an **involution** (anti automorphism of order 2) on $\mathfrak{C}$. It is easy to see that an element $x \in \mathfrak{C}$ is invertible if and only if $N(x) \neq 0$ and in this case, $x^{-1} = N(x)^{-1}\overline{x}$.

We are now in a position to state the following two results about the structure and dimension of a composition algebra.

Theorem 3.1.1. *([**SV**], Proposition 1.5.3) Let $\mathfrak{D}$ be a composition algebra over a field k, with norm N and $\lambda \in k^*$. Define on $\mathfrak{C} = \mathfrak{D} \oplus \mathfrak{D}$ a product by*

$$(x, y)(u, v) := (xu + \lambda\overline{v}y, vx + y\overline{u}),\ \forall\ x, y, u, v \in \mathfrak{D}$$

and a quadratic form N_1 *by*

$$N_1((x,y)) := N(x) - \lambda N(y),\ \forall\ x, y \in \mathfrak{D}.$$

If $\mathfrak{D}$ *is associative, then* $\mathfrak{C}$ *is a composition algebra.* $\mathfrak{C}$ *is associative if and only if* $\mathfrak{D}$ *is commutative and associative.*

The process of constructing a composition algebra $\mathfrak{C}$, starting from a given composition algebra $\mathfrak{D}$ as seen in the above theorem, is known as doubling. In fact, we have,

Theorem 3.1.2. *(*[**SV**]*, Theorem 1.6.2) Every composition algebra* $\mathfrak{C}$ *is obtained by repeated doubling starting from* $k1$ *in characteristic* $\neq 2$ *and from a* 2*-dimensional composition algebra in characteristic* 2*. The possible dimensions of a composition algebra are* 1 *(in characteristic* $\neq 2$ *only),* 2*,* 4 *and* 8*. Composition algebras of dimension* 1 *or* 2 *are commutative and associative, those of dimension* 4 *are associative but not commutative and those of dimension* 8 *are neither commutative nor associative.*

Composition algebras of dimension 4 are called **quaternion algebras** and those of dimension 8 are called **octonion algebras**.

Let $\mathfrak{C}$ be an octonion algebra over a field k. Consider the group of its automorphisms $Aut(\mathfrak{C})$. Any automorphism of $\mathfrak{C}$ is necessarily an isometry of the norm N on $\mathfrak{C}$ ([**SV**], Corollary 1.2.4). Therefore, $Aut(\mathfrak{C}) \subset O(\mathfrak{C}, N)$, the orthogonal group of $\mathfrak{C}$ with respect to N. In fact,

Theorem 3.1.3. *(*[**SV**] *Theorem 2.3.5 and Proposition 2.4.6) Let* $\mathfrak{C}$ *be an octonion algebra over* k *and* $\mathfrak{C}_{\overline{k}} := \mathfrak{C} \otimes \overline{k}$*, where* $\overline{k}$ *is the algebraic closure of* k*. Then the group* $\mathcal{G} := Aut(\mathfrak{C}_{\overline{k}})$ *is the connected, simple algebraic group of type* G_2*, defined over* k*. Also, any algebraic group of type* G_2 *defined over a field* k *is isomorphic to* $Aut(\mathfrak{C}_{\overline{k}})$ *for some octonion algebra* $\mathfrak{C}$ *over* k*.*

Compact real form of G_2: Let us now consider an octonion algebra defined over $\mathbb{R}$. This is constructed by the doubling method as seen in Theorem 3.1.1 starting from $\mathbb{R}$. Let $\mathbb{H} := \{a + bi + cj + dk :\ a, b, c, d \in \mathbb{R}\}$ denote the space of real quaternions, where $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. For a typical element $x = x_1 + x_2 i + x_3 j + x_4 k \in \mathbb{H}$, define the conjugate of x as $\overline{x} = x_1 - x_2 i - x_2 j - x_3 k$ and a norm on $\mathbb{H}$ by $N(x) := x\overline{x} = x_1^2 + x_2^2 + x_3^2 + x_3^2$.

Now, consider the $\mathbb{R}$-vector space $\mathfrak{C} = \mathbb{H} \oplus \mathbb{H}$ and we define a multiplication on $\mathfrak{C}$ by $(x, y)(u, v) := (xu - \overline{v}y, vx + y\overline{u})$ for all $x, y, u, v \in \mathbb{H}$. Define a norm on $\mathfrak{C}$ by

$N_1(x, y) := N(x) + N(y)$, for all $x, y \in \mathbb{H}$. Thus, if $x = (x_1, ..., x_8)$, with $x_i \in \mathbb{R}$, be a typical element in the 8-dimensional space $\mathfrak{C}$, then $N_1(x) = x_1^2 + ... + x_8^2$. It is easy to see that, the above multiplication together with the norm N_1 gives the structure of an octonion division algebra on $\mathfrak{C}$. The group $\mathcal{G} = Aut(\mathfrak{C})$ is the compact connected simple Lie group of type G_2 (see [**P**], Lecture 14).

3.2. The principle of triality

We now describe the principle of triality in the group of similarities and the orthogonal group of the norm N on an octonion algebra $\mathfrak{C}$ over k. For a detailed exposition on this principle, refer to [**SV**], Chapter 3.

Let $\mathfrak{C}$ be an octonion algebra over a field k and N be the norm on $\mathfrak{C}$. A **similarity** of $\mathfrak{C}$ with respect to N is a linear map $t : \mathfrak{C} \longrightarrow \mathfrak{C}$, such that $N(t(x)) = n(t)N(x)$, for all $x \in \mathfrak{C}$, where $n(t) \in k^*$ is called the **multiplier** of t. An immediate consequence of the definition is that any similarity t is necessarily a bijective linear transformation. Denote the group of all similarities of $\mathfrak{C}$ with respect to N by $GO(N)$. Note that, the map $n : GO(N) \longrightarrow k^*$, $t \mapsto n(t)$ is a homomorphism. Therefore, the kernel of n is the orthogonal group $O(N)$ of $\mathfrak{C}$ with respect to N. The **principle of triality** states the following:

Theorem 3.2.1. *([**SV**], Theorem 3.2.1) Let $\mathfrak{C}$ be an octonion algebra over k with norm N.*

(i) The elements $t_1 \in GO(N)$ such that there exist $t_2, t_3 \in GO(N)$ with

$$t_1(xy) = t_2(x)t_3(y) \ \forall \ x, y \in \mathfrak{C} \ .... \ (*)$$

*form a normal subgroup $SGO(N)$ of index 2 in $GO(N)$, called the **special similarity group**. If (t_1, t_2, t_3) and (s_1, s_2, s_3) satisfy $(*)$, then so do (t_1s_1, t_2s_2, t_3s_3) and $(t_1^{-1}, t_2^{-1}, t_3^{-1})$.*

(ii) If $t_1 \in GO(N)$, there exist $t_2, t_3 \in GO(N)$ such that

$$t_1(xy) = t_2(y)t_3(x) \ \forall \ x, y \in \mathfrak{C} \ .... \ (**)$$

if and only if $t_1 \notin SGO(N)$.

(iii) The elements t_2 and t_3 in $(*)$ and $(**)$ are uniquely determined by t_1 up to scalar factors λ and λ^{-1} in k^*.

(iv) If a triple (t_1, t_2, t_3) satisfy either $(*)$ or $(**)$, then $n(t_1) = n(t_2)n(t_3)$.

(v) Every element $t \in GO(N)$ is a product of a left multiplication by an invertible element of $\mathfrak{C}$ and an orthogonal transformation t'. Then $t \in SGO(N)$ if and only if t' is a rotation.

(vi) If t_1, t_2, t_3 are bijective linear transformations of $\mathfrak{C}$ such that they satisfy $(*)$ or $(**)$, then they are necessarily similarities of $\mathfrak{C}$ with respect to N.

From now on, we shall refer to any triple (t_1, t_2, t_3) of similarities of $\mathfrak{C}$, satisfying $(*)$, as a **related triple**. We shall see in Chapter 4, that the principle of triality helps us define two automorphisms of the spin group of an octonion algebra. These automorphisms are outer and they generate a group isomorphic to S_3.

3.3. Albert algebras and groups of type F_4

To define groups of type F_4, we need the notion of an Albert algebra. Let $\mathfrak{C}$ be an octonion algebra over a field k with norm N and $x \mapsto \overline{x}$ be the canonical involution on $\mathfrak{C}$ as in Section 3.1. Let $\gamma_1, \gamma_2, \gamma_3 \in k^*$ be fixed scalars. Denote the k-algebra (non associative) of all 3×3 matrices over $\mathfrak{C}$ by $\mathbb{M}_3(\mathfrak{C})$. Define an involution $\sigma : \mathbb{M}_3(\mathfrak{C}) \longrightarrow \mathbb{M}_3(\mathfrak{C})$, by $X \mapsto \Gamma^{-1}\overline{X}^t\Gamma$ for all $X \in \mathbb{M}_3(\mathfrak{C})$, where $\Gamma = diag(\gamma_1, \gamma_2, \gamma_3)$ and $\overline{X} = (\overline{X_{ij}})$ and X^t denotes the transpose of X. Let us denote the subset of all σ-**hermitian** matrices in $\mathbb{M}_3(\mathfrak{C})$ by $H_3(\mathfrak{C}, \Gamma)$, i.e., $H_3(\mathfrak{C}, \Gamma) = \{X \in \mathbb{M}_3(\mathfrak{C}) : X = \sigma(X)\}$. Then it can be shown that any $X \in H_3(\mathfrak{C}, \Gamma)$ is of the form

$$X = \begin{bmatrix} \alpha_1 & c_3 & \gamma_1^{-1}\gamma_3\overline{c_2} \\ \gamma_2^{-1}\gamma_1\overline{c_3} & \alpha_2 & c_1 \\ c_2 & \gamma_3^{-1}\gamma_2\overline{c_1} & \alpha_3 \end{bmatrix}$$

where $\alpha_i \in k$, $c_i \in \mathfrak{C}$ for $1 \leq i \leq 3$. Clearly, $H_3(\mathfrak{C}, \Gamma)$ is a 27- dimensional k-vector space and we define a multiplication on it by

$$XY := \frac{1}{2}(X.Y + Y.X)$$

where $X.Y$ denotes the usual product of matrices. With this multiplication, $H_3(\mathfrak{C}, \Gamma)$ is a commutative, non associative algebra over k. Define a **trace** T on $H_3(\mathfrak{C}, \Gamma)$ by $T(X) = \alpha_1 + \alpha_2 + \alpha_3$. This trace map defines a quadratic form Q on $H_3(\mathfrak{C}, \Gamma)$ as

$$Q(X) := \frac{1}{2}T(X^2) = \frac{1}{2}(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + \gamma_3^{-1}\gamma_2 n(c_1) + \gamma_1^{-1}\gamma_3 n(c_2) + \gamma_2^{-1}\gamma_1 n(c_3)$$

for every X as above. There exists a cubic form N on $H_3(\mathfrak{C}, \Gamma)$ defined as

$$N(X) = \alpha_1\alpha_2\alpha_3 - \gamma_3^{-1}\gamma_2\alpha_1 n(c_1) - \gamma_1^{-1}\gamma_3\alpha_2 n(c_2) - \gamma_2^{-1}\gamma_1\alpha_3 n(c_3) + n(c_1c_2, \bar{c_3}),$$

where $n(,)$ is the bilinear form associated to the norm n of $\mathfrak{C}$ (Chapter 5, [**SV**]). We call $H_3(\mathfrak{C},\Gamma)$ a **reduced Albert algebra** over k. More generally, an **Albert algebra** over k is defined to be a k-algebra A, such that $A\otimes_k L\cong H_3(\mathfrak{C},\Gamma)$ over some field extension L of k, where $\mathfrak{C}$ is an octonion algebra over L and Γ, an invertible diagonal matrix in $\mathbb{M}_3(L)$. Albert algebras are given up to isomorphism by Tits' first and second constructions, which we now describe briefly.

Tits' first construction: Let k be a field with $char(k)\neq 2,3$. Let D be a central simple associative algebra over k of degree 3 and let $\mu\in k^*$ be arbitrary. Consider the vector space

$$J(D,\mu):=D_0\oplus D_1\oplus D_2,$$

$D_i=D$ for $i=0,1,2$ and define a multiplication on $J(D,\mu)$ by

$$(a_0,a_1,a_2)(b_0,b_1,b_2)$$

$$=(a_0\cdot b_0+\widetilde{a_1b_2}+\widetilde{b_2a_2},\tilde{a}_0b_1+\tilde{b}_0a_1+(2\mu)^{-1}a_2\times b_2,a_2\tilde{b}_0+b_2\tilde{a}_0+\frac{1}{2}\mu a_1\times b_1),$$

where, for $a,b\in D$,

$$a\cdot b=\frac{1}{2}(ab+ba),\ a\times b=2a\cdot b-T_D(a)b-T_D(b)a+T_D(a)T_D(b)-T_D(a\cdot b),$$

$\tilde{a}=\frac{1}{2}(T_D(a)-a)$ and T_D denotes the reduced trace on D. The algebra $J(D,\mu)$ is an Albert algebra over k. It is a division algebra if and only if μ is not a norm from D (Theorem 20, Chapter IX, [**J1**])

Tits' second construction: Let K/k be a quadratic extension and let (B,σ) be a central simple K-algebra of degree 3 with a unitary involution σ over K. Let $u\in B^*$ be such that $\sigma(u)=u$ and $N_B(u)=\mu\bar{\mu}$ for some $\mu\in K^*$. Here, N_B is the reduced norm on B and bar denotes the nontrivial k-automorphism of K. Let $H(B,\sigma)$ denote the subspace of σ-symmetric elements of B, with multiplication defined by $xy:=(x\cdot y+y\cdot x)/2$, for $x,y\in H(B,\sigma)$. With the notation as in Tits' first construction, define a multiplication on the vector space $J(B,\sigma,u,\mu):=H(B,\sigma)\oplus B$ by

$$(a_0,a)(b_0,b):=(a_0\cdot b_0+\widetilde{au\sigma(b)}+\widetilde{bu\sigma(a)},\tilde{a}_0b_0+\tilde{b}_0a+\bar{\mu}(\sigma(a)\times\sigma(b))u^{-1}).$$

With this multiplication, $J(B,\sigma,u,\mu)$ is an Albert algebra and and it is a division algebra if and only if μ is not a norm from B.

Algebraic groups of type F_4 are determined by Albert algebras. We have

Theorem 3.3.1. *([**SV**], Theorem 7.2.1) Let A be an Albert algebra over a field k and K be the algebraic closure of k. Then $\mathcal{G} = Aut(A_K)$ is the connected simple algebraic group of type F_4 defined over k. Conversely, any algebraic group of type F_4 defined over k is isomorphic to $Aut(A_K)$ for some Albert algebra A over k.*

Compact real form of F_4: For compact connected Lie groups of type F_4, let us first consider the octonion division algebra $\mathfrak{C}$ over $\mathbb{R}$ as in Section 3.1. Let $A = H_3(\mathfrak{C}) := H_3(\mathfrak{C}, I)$, where I is the 3×3 identity matrix. Hence, A is the set of all 3×3 matrices in $\mathbb{M}_3(\mathfrak{C})$ of the form $\begin{bmatrix} \alpha_1 & c_3 & \overline{c_2} \\ \overline{c_3} & \alpha_2 & c_1 \\ c_2 & \overline{c_1} & \alpha_3 \end{bmatrix}$, where $\alpha_i \in \mathbb{R}$, $c_i \in \mathfrak{C}$ and $x \mapsto \overline{x}$ is the canonical involution in $\mathfrak{C}$. As before, we define a multiplication on $H_3(\mathfrak{C})$ by $xy := \frac{1}{2}(x.y + y.x)$, where $x.y$ denotes the usual matrix multiplication. Then $Aut(H_3(\mathfrak{C}))$ is the compact connected simple Lie group Lie group of type F_4 (see [**P**], Lecture 16).

Groups of type E_6 : Let A be an Albert algebra over a field k. Then $A_{\bar{k}} \cong H_3(\mathfrak{C}, \Gamma)$ is a reduced Albert algebra over $\bar{k}$. Let N be the cubic norm on $A_{\bar{k}}$. Consider $H := \{f \in GL(A_{\bar{k}}) : N(f(X)) = N(X)\}$, the full group of isometries of N. Then we have,

Theorem 3.3.2. *([**SV**], Theorem 7.3.2) H is a connected, quasisimple, simply connected algebraic group of type E_6 defined over k.*

The following result gives an embedding of a group of type F_4 in that of type E_6 (see [**SV**], Chapter 7).

Theorem 3.3.3. *$Aut(A_{\bar{k}}) = \{f \in H : f(1_{A_{\bar{k}}}) = 1_{A_{\bar{k}}}\}$, where $1_{A_{\bar{k}}}$ denotes the identity element in $A_{\bar{k}}$.*

CHAPTER 4

Genus number of Lie groups and algebraic groups

4.1. Introduction

Let G be a group acting on a set M. Let for $x \in M$, G_x denote the stabilizer of x in G. Two elements $x, y \in M$ are said to have the same orbit type if the orbits of x and y are isomorphic as G-sets. This can be seen to be equivalent to the conjugacy of G_x and G_y in G. When one considers the action of a group G on itself by conjugacy, this amounts to computing the conjugacy classes of centralizers of elements in G. In the 1950s Mostow proved that for a compact Lie group acting on a compact manifold, the number of orbit types is finite [**M**], which was initially conjectured by Montgomery ([**E**], problem 45). Though this problem for Lie groups has been studied by Dynkin in his exhaustive works [**D1**], [**D2**], there is still considerable interest in the subject.

It is known that the number of conjugacy classes of centralizers of elements in a reductive algebraic group G over an algebraically closed field (with char G good), is finite ([**St**], Corollary 1 of Theorem 2, Chapter 3). From the expanse of Dynkin's aforementioned works, the number of conjugacy classes of centralizers of elements in Lie groups, though implict, is difficult to extract. In this chapter we compute the number of conjugacy classes of centralizers in a compact simply connected simple Lie group as well as for a simply connected simple algebraic group (for semisimple elements) over an algebraically closed field. We also compute the number of orbit types of the adjoint action of G on its Lie algebra $\mathfrak{g}$. We mainly do this for all classical groups and for G_2 and F_4 among the exceptional groups. We hope that, apart from the explicit computations, the techniques and some of the results proved are new and would be of interest to the community. The results proved in this chapter can be found in [**Bo**].

4.2. Preliminaries

Let G denote a compact simply connected Lie group or a simply connected algebraic group over an algebraically closed field and $T \subset G$ be a maximal torus of G. Let

W be the Weyl group of G with respect to T, i.e. $W = N_G(T)/T$, where $N_G(T)$ denotes the normalizer of T in G. Conjugation induces an action of W on T. For $x \in T$ let W_x denote the stabilizer of x in W for this action i.e. $W_x = \{g \in W\colon gxg^{-1} = x\}$. In what follows, we shall denote the conjugacy class of a subgroup $H \subset G$, in G by $[H]$. The cardinality of the set $\{[Z_G(x)] : x \in G,\ x \text{ semisimple}\}$, where $Z_G(x)$ is the centralizer of x in G, is defined as the **semisimple genus number** of G. Since we shall deal with only semisimple elements, we call this number simply as the **genus number** of G. If G is not simply connected, then the cardinality of the set $\{[Z_G(x)^\circ] : x \in G,\ x \text{ semisimple}\}$, is called the **connected genus number** of G. Here $Z_G(x)^\circ$ denotes the connected component of identity in $Z_G(x)$.
The following results are known:

Proposition 4.2.1. *([**B**], Theorem 3.4) Let G be a simply connected compact Lie group and $\sigma \in Aut(G)$. Then the set F of all fixed points of σ in G is connected. In particular, if σ is the inner conjugation by an element $x \in G$, then the centralizer $Z_G(x)$ is connected.*

Proposition 4.2.2. *([**Hu1**], Theorem 2.11) If G be a simply connected algebraic group over an algebraically closed field, the centralizer of any semisimple element of G is connected.*

For a compact connected Lie group G with maximal torus T and Weyl group W, define the following subsets with respect to a reflection $s \in W$: T^s is the the subset of T fixed by the action of $s \in W$ and $(T^s)^\circ$ is the connected component at the identity of T^s. Let $K(s) = \{x^2 \in T | x \in N_G(T),\ xT = s \in W\}$ and $\sigma(s) = (T^s)^\circ \cup K(s)$. Then we have,

Proposition 4.2.3. *([**DW**], Theorem 8.2) Suppose that G is a compact connected Lie group with maximal torus T and Weyl group W. Then the centre of G is equal to the intersection $\bigcap_s \sigma(s)$, where s runs through the reflections in W.*

We have the following basic result:

Theorem 4.2.4. *For a simply connected compact Lie group G with maximal torus T and Weyl group W, there exists a bijection*

$$\{[Z_G(x)]\colon x \in T\} \longrightarrow \{[W_x]\colon x \in T\}$$

given by

$$[Z_G(x)] \longmapsto [W_x]$$

Here $[Z_G(x)]$ and $[W_x]$ respectively denote the conjugacy class of the centralizer of x in G and the conjugacy class of the stabilizer of x in W.

Proof. First we show that the map is well-defined.
Let $x, y \in T$ such that $[Z_G(x)] = [Z_G(y)]$ i.e. there exists some $g \in G$ such that $gZ_G(x)g^{-1} = Z_G(y)$. Since T is a maximal torus in $Z_G(x)$ containing x, $gTg^{-1} \subset Z_G(y)$ and also $T \subset Z_G(y)$. Hence there exists $g_1 \in Z_G(y)$ such that $g_1 gTg^{-1}g_1^{-1} = T$. Let $g_1 g = h \in G$. Then $[h] = hT \in W$ and $[h]W_x[h^{-1}] = W_y$ since, for $[h_1] \in W_x$ we have

$$\begin{aligned}(hh_1h^{-1})y(hh_1^{-1}h^{-1}) =& (g_1gh_1g^{-1}g_1^{-1})y(g_1gh_1^{-1}g^{-1}g_1^{-1})\\ =& (g_1(gh_1g^{-1})g_1^{-1})y(g_1(gh_1^{-1}g^{-1})g_1^{-1})\\ =& y,\end{aligned}$$

since $h_1 \in Z_G(x)$ and $gZ_G(x)g^{-1} = Z_G(y)$. Hence $gh_1g^{-1} \in Z_G(y)$. Also, $g_1 \in Z_G(y)$. Therefore $[hh_1h^{-1}] \in W_y$. Similarly we have the other inclusion. Thus the given map is well defined.

Surjectivity of the map is clear from the definition. Hence we only need to check injectivity.

Let $x, y \in T$ such that W_x is conjugate to W_y, i.e. for some $[h] \in W$, $[h]W_x[h^{-1}] = W_y$, i.e. $W_{hxh^{-1}} = W_y$, where $h \in N_G(T)$ is a representative of $[h] \in W$. We denote $hxh^{-1} \in T$ by a. We intend to show that $Z_G(a) = Z_G(y)$. Clearly for any element $x \in T$, $W_x = N_{Z_G(x)}(T)/T$. Therefore by Proposition 4.2.3, $Z(Z_G(a)) = \bigcap_{s\in W_a} \sigma(s)$ and $Z(Z_G(y)) = \bigcap_{s\in W_y} \sigma(s)$. Since $W_a = W_y$, we have

$$Z(Z_G(a)) = Z(Z_G(y))..........(*).$$

Observe that for any $x \in T$, $Z_G(x)$ is the union of all maximal tori of G containing x. So let T_1 be any maximal torus in $Z_G(a)$. Since $y \in Z(Z_G(a))$ by $(*)$, $y \in T_1$, which implies $T_1 \subset Z_G(y)$. Similarly any maximal torus of $Z_G(y)$ is contained in $Z_G(a)$. Therefore $Z_G(y) = Z_G(a) = Z_G(hxh^{-1}) = hZ_G(x)h^{-1}$. This shows that the map is injective.

□

Next we prove an analogue of Theorem 4.2.4 for simply connected algebraic groups over an algebraically closed field. But before that, we note the following results:

Proposition 4.2.5. *(*[**C2**]*, Theorems 3.5.3 and 3.5.4) Let G be a connected reductive algebraic group, with maximal torus T, Weyl group W and root system Φ,*

then, for a semisimple element $x \in G$, $Z_G(x)^\circ$ *is a reductive group and* $Z_G(x)^\circ = < T, U_\alpha, \alpha(x) = 1 >$, *where* $\alpha \in \Phi$ *and* U_α *is the root subgroup corresponding to* α.
The root system of $Z_G(x)^\circ$ *is* $\Phi_1 = \{\alpha \in \Phi | \alpha(x) = 1\}$.
The Weyl group of $Z_G(x)^\circ$ *is* $W_1 = < w_\alpha | \alpha \in \Phi_1 >$, *where* w_α *is the reflection at* α.

Lemma 4.2.6. *Let* G *be a simply connected algebraic group with maximal torus* T *and Weyl group* W. *If* w_α *be a reflection in* W, *such that* $w_\alpha \in W_x$, *where* $x \in T$ *and* $\alpha \in \Phi$, *the root system of* G, *then* $\alpha(x) = 1$.

Proof. Let $(X(T), \Phi, Y(T), \Phi^*)$ be the root datum for G. Since G is simply connected,
$X(T) = Hom(\mathbb{Z}\Phi^*, \mathbb{Z})$ and $Y(T) = \mathbb{Z}\Phi^*$. Therefore for a system of simple roots $\{\alpha_i\}$ of G, there exists a basis $\{\lambda_j\}$ of $X(T)$ such that $< \lambda_i, \alpha_j^* >= \delta_{ij}$, α_j^* being the coroot corresponding to α_j (see [**SSt**], Chapter 2, Section 2.)

Now let $w_\alpha \in W$ be a reflection such that, $w_\alpha \in W_x$, i.e. $w_\alpha(x) = x$. There exists $s \in W$ such that $s(\alpha)$ is a simple root. Consider $\lambda \in X(T)$ such that $< \lambda, s(\alpha)^* >= 1$. Note that,

$$w_{s(\alpha)}(s(x)) = s w_\alpha s^{-1}(s(x)) = s w_\alpha(x) = s(x) \text{..........(1)}.$$

Applying λ to equation (1) we get,

$$\begin{aligned}
&\lambda(w_{s(\alpha)}(s(x))) = \lambda(s(x))\\
\Rightarrow &(w_{s(\alpha)}\lambda)(s(x)) = \lambda(s(x))\\
\Rightarrow &(\lambda - < \lambda, s(\alpha)^* > s(\alpha))(s(x)) = \lambda(s(x))\\
\Rightarrow &\lambda(s(x)) s(\alpha)(s(x))^{-1} = \lambda(s(x))\\
\Rightarrow &s(\alpha)(s(x)) = 1\\
\Rightarrow &\alpha(s^{-1}(s(x)) = 1\\
\Rightarrow &\alpha(x) = 1.
\end{aligned}$$

□

Theorem 4.2.7. *For simply connected algebraic group* G *over an algebraically closed field, with maximal torus* T *and Weyl group* W, *there exists a bijection*

$$\{[Z_G(x)] \colon x \in T\} \longrightarrow \{[W_x] \colon x \in T\}$$

given by

$$[Z_G(x)] \longmapsto [W_x]$$

Here $[Z_G(x)]$ and $[W_x]$ respectively denote the conjugacy class of the centralizer of x in G and the conjugacy class of the stabilizer of x in W.

Proof. The proof of well-definedness and surjectivity of the map is same as that in Theorem 4.2.4. We prove that this map is injective.

Let $x, y \in T$ such that W_x is conjugate to W_y, i.e. for some $[h] \in W$, $[h]W_x[h^{-1}] = W_y$, i.e. $W_{hxh^{-1}} = W_y$, where $h \in N_G(T)$ is a representative of $[h] \in W$. We denote $hxh^{-1} \in T$ by a. We intend to show that $Z_G(a) = Z_G(y)$. To achieve this, we first show that $Z_G(a)$ and $Z_G(y)$ have the same roots. Let Φ_a and Φ_y respectively denote the root systems of $Z_G(a)$ and $Z_G(y)$ with respect to the common maximal torus T. Since G is simply connected, by Proposition 4.2.2, both $Z_G(a)$ and $Z_G(y)$ are connected. Hence by Proposition 4.2.5, we have, $\Phi_a = \{\alpha \in \Phi | \alpha(a) = 1\}$ and $\Phi_y = \{\beta \in \Phi | \beta(y) = 1\}$.

Let $\alpha \in \Phi_a$. Hence $w_\alpha \in W_a = W_y$. Therefore by Lemma 4.2.6, $\alpha(y) = 1$ which implies $\alpha \in \Phi_y$. This shows that $\Phi_a \subset \Phi_y$. Similarly the other inclusion. Hence $\Phi_a = \Phi_y$ which implies $Z_G(a) = Z_G(y)$ by Proposition 4.2.5. □

Corollary 4.2.8. *Let G be a compact simply connected Lie group (resp. a simply connected algebraic group over an algebraically closed field), $T \subset G$ a maximal torus. The genus number (resp. semisimple genus number) of G equals the number of orbit types of the action of $W(G,T)$ on T.*

Proof. By Theorem 4.2.4 and Theorem 4.2.7, the number of orbit types of elements belonging to a fixed maximal torus T is equal to the number of orbit types of elements from T in the Weyl group. Any (semisimple) element $x \in G$ is contained in some maximal torus of G. Let $y \in G$ be any other (semisimple) element and let T' be a maximal torus of G such that $y \in T'$. Now T is conjugate to T', i.e. $\exists\ g \in G$ such that $gTg^{-1} = T'$. Therefore $Z_G(y)$ is conjugate to $Z_G(x)$, where $x = g^{-1}yg \in T$. Hence each (semisimple) element of G is orbit equivalent to an element of T. The result now follows. □

Next we want to investigate connected groups which are not necessarily simply connected. It turns out that the connected genus number of a connected semisimple group is equal to the genus number of its simply connected cover, which we shall see (Theorem 4.2.12). We note the following two results, which are known:

Proposition 4.2.9. *([**BD**], Chapter 4, Theorem 2.9) Let $f; G \to H$ be a surjective homomorphism of compact Lie groups. If $T \subset G$ is a maximal torus, then $f(T) \subset H$*

is a maximal torus. Furthermore, $ker(f) \subset T$ *iff* $ker(f) \subset Z(G)$*. In this case* f *induces an isomorphism of Weyl groups.*

A similar result holds for algebraic groups also, which we now quote ([**Hu**], Chapter 9, Proposition B),

Proposition 4.2.10. *Let* $\phi : G \to G'$ *be an epimorphism of connected algebraic groups, with* T *and* $T' = \phi(T)$ *respective maximal tori. Then* ϕ *induces a surjective map* $WG \to WG'$*, which is also injective in case* $Ker\ \phi$ *lies in all Borel subgroups of* G*. Here,* WG *and* WG' *denote the Weyl groups of* G *and* G' *respectively.*

Let G be a compact connected semisimple Lie group or a connected semisimple algebraic group over an algebraically closed field. Let $\widetilde{G}$ be the simply connected cover of G with the covering map,

$$\rho : \widetilde{G} \longrightarrow G.$$

Then, for a maximal torus $\widetilde{T} \subset \widetilde{G}$, $\rho(\widetilde{T}) = T$ is a maximal torus in G. Since $ker\rho$ is contained in all the maximal tori of $\widetilde{G}$, ρ induces an isomorphism of $W\widetilde{G}$ and WG by the above cited propositions.

Let $(X(T), \Phi, Y(T), \Phi^*)$ be the root datum of G. Let $V = (Y(T) \otimes \mathbb{R})$ and $\overline{Y(T)} = \{v \in V : \alpha(v) \in \mathbb{Z},\ \forall \alpha \in \Phi\}$. We associate a finite group $C := \overline{Y(T)}/\mathbb{Z}\Phi^*$ with the isogeny class of G. Then C is a finite abelian group. Let $C'(G) := Y(T)/\mathbb{Z}\Phi^* \subset C$. It can be shown that any subgroup of C is of the form $C'(H)$, for some group H belonging to the isogeny class of G. (see [**T**], Section 1.5)

We first make the following observation:

Lemma 4.2.11. *Let* G *be a compact connected semisimple Lie group or a connected semisimple algebraic group over an algebraically closed field* K *and* $\widetilde{G}$ *be its simply connected cover. Let* $\rho : \widetilde{G} \to G$ *be the covering map. Assume that,* $char(K)$ *does not divide the order of* $C(G)$*. Then* $\rho(Z_{\widetilde{G}}(\tilde{x})) = Z_G(x)^\circ$*, where* $\tilde{x} \in \widetilde{T}$*, a fixed maximal torus in* $\widetilde{G}$ *and* $x = \rho(\tilde{x})$*.*

Proof. For an algebraic group or a Lie group G, let us denote the corresponding Lie algebra by $\mathbf{L}(G)$. Since $char(K)$ does not divide the order of $C'(G)$, ρ is a separable morphism. Hence, the differential $d\rho : \mathbf{L}(\widetilde{G}) \to \mathbf{L}(G)$, is an isomorphism of Lie algebras. Since $Z_{\widetilde{G}}(\tilde{x})$ is connected, $\rho(Z_{\widetilde{G}}(\tilde{x})) \subset Z_G(x)^\circ$. If we show that the dimensions are equal, we would be through. For this, we look at the corresponding Lie algebras. Now since $Ad_x v = v$ for all $v \in \mathbf{L}(Z_G(x)^\circ)$, $d\rho Ad_{\tilde{x}} d\rho^{-1} v = Ad_x v = v$. Therefore, for every $v \in \mathbf{L}(Z_G(x)^\circ)$, $Ad_{\tilde{x}} d\rho^{-1} v = d\rho^{-1} v$. Hence $d\rho^{-1}(\mathbf{L}(Z_G(x)^\circ)) \subset \mathbf{L}(\rho(Z_{\widetilde{G}}(\tilde{x})))$. Since $d\rho$ is an isomorphism, we have $dim(\mathbf{L}(Z_G(x)^\circ)) \leq dim(\mathbf{L}(\rho(Z_{\widetilde{G}}(\tilde{x}))))$. Therefore $dim(Z_G(x)^\circ) \leq dim(\rho(Z_{\widetilde{G}}(\tilde{x})))$. Hence the equality.

□

Remark: Note that, the covering map $\rho : SL_2(K) \longrightarrow PSL_2(K)$, is not separable if $char(K) = 2$, since $C'(PSL_2(K)) = \mathbb{Z}_2$. Hence in this case, $d\rho$ is not an isomorphism.

Theorem 4.2.12. *Let G be a compact connected semisimple Lie group or a connected semisimple algebraic group over an algebraically closed field k. Let $\widetilde{G}$ be the simply connected cover of G. Then the connected genus number of G is equal to the genus number of $\widetilde{G}$.*

Proof. Let $\rho : \widetilde{G} \longrightarrow G$ be the covering map. Fix a maximal torus $\widetilde{T}$ in $\widetilde{G}$. Then $T = \rho(\widetilde{T}) \subset G$ is a maximal torus. If $\tilde{g} \in \widetilde{G}$, then we shall denote $\rho(\tilde{g})$ by g. To prove the result, it suffices to show that the map

$$\{[Z_{\widetilde{G}}(\tilde{t})] : \tilde{t} \in \widetilde{T}\} \to \{[Z_G(x)^\circ] : x \in T\}$$

defined by,

$$[Z_{\widetilde{G}}(\tilde{t})] \mapsto [Z_G(\rho(\tilde{t}))^\circ],$$

is a bijection.

We first show that the map is well-defined. So let, $[Z_{\widetilde{G}}(\tilde{t})] = [Z_{\widetilde{G}}(\tilde{t_1})]$ with $\tilde{t}, \tilde{t_1} \in \widetilde{T}$. Therefore there exists $\tilde{g} \in \widetilde{G}$ such that, $Z_{\widetilde{G}}(\tilde{t}) = \tilde{g} Z_{\widetilde{G}}(\tilde{t_1}) \tilde{g}^{-1} = Z_{\widetilde{G}}(\tilde{g}\tilde{t_1}\tilde{g}^{-1})$. Take $a \in Z_G(t)^\circ$, where $\rho(\tilde{t}) = t$. Consider any lift $\tilde{a} \in Z_{\widetilde{G}}(\tilde{t})$ of a (such a lift exists by Lemma 2.2). Therefore, $\tilde{a}\tilde{g}\tilde{t_1}\tilde{g}^{-1}\tilde{a}^{-1} = \tilde{g}\tilde{t_1}\tilde{g}^{-1}$. Applying ρ on both sides we get, $agt_1g^{-1}a^{-1} = gt_1g^{-1}$. Thus, $Z_G(t)^\circ \subset Z_G(gt_1g^{-1})^\circ$. Similarly $Z_G(gt_1g^{-1})^\circ \subset Z_G(t)^\circ$.

That the map is onto is clear from the definition.

To prove that the map is injective, let $Z_G(t_1)^\circ = gZ_G(t_2)^\circ g^{-1} = Z_G(gt_2g^{-1})^\circ$ for some $g \in G$. If $\tilde{a} \in Z_{\widetilde{G}}(\tilde{t_1})$, the $a = \rho(\tilde{a}) \in Z_G(t_1)^\circ = Z_G(gt_2g^{-1})^\circ$. Therefore, $agt_2g^{-1}a^{-1} = gt_2g^{-1}$. If we show that $\tilde{a} \in Z_{\widetilde{G}}(\tilde{g}\tilde{t_2}\tilde{g}^{-1})$ then we are through. So let $\tilde{a_1}$ be any lift of a in $Z_{\widetilde{G}}(\tilde{g}\tilde{t_2}\tilde{g}^{-1})$. Then, $\rho(\tilde{a}\tilde{a_1}^{-1}) = 1 \Rightarrow \tilde{a}\tilde{a_1}^{-1} \in Ker\rho \subset Z(\widetilde{G})$. Therefore, $\tilde{a}\tilde{a_1}^{-1}\tilde{g}\tilde{t_2}\tilde{g}^{-1}\tilde{a_1}\tilde{a}^{-1} = \tilde{g}\tilde{t_2}\tilde{g}^{-1} \Rightarrow \tilde{a}\tilde{g}\tilde{t_2}\tilde{g}^{-1}\tilde{a}^{-1} = \tilde{g}\tilde{t_2}\tilde{g}^{-1}$. Hence, $\tilde{a} \in Z_{\widetilde{G}}(\tilde{g}\tilde{t_2}\tilde{g}^{-1})$, which shows that $Z_{\widetilde{G}}(\tilde{t_1}) \subset Z_{\widetilde{G}}(\tilde{g}\tilde{t_2}\tilde{g}^{-1})$. Similarly the other inclusion follows. This completes the proof. □

Remark: It is important to note that if the group is not simply connected, then the number of classes of centralizers might be larger than the number of isotropy classes of the Weyl group. For example if we consider the group $PSL_2(K)$($char(K) \neq$

2), the number of isotropy subgroups in the Weyl group S_2 is 2 but the number of conjugacy classes of centralizers is 3. However, by Theorem 4.2.12, the connected genus number of $PSL_2(K)$ is 2 which is equal to the genus number of its simply connected cover $SL_2(K)$.

We have the following result on reductive algebraic groups:

Theorem 4.2.13. *Let G be a connected reductive algebraic group over an algebraically closed field. Let G' be the commmutator subgroup of G. Then the connected genus number of G is equal to the connected genus number of G'.*

Proof. Since G is reductive, we have $G = G'.Z(G)^\circ$, where $Z(G)^\circ$ is the connected component of the centre of G. For any $g \in G$, we shall write $g = g's_g$, with $g' \in G'$ and $s_g \in Z(G)^\circ$. Observe that for any $g' \in G'$ and $s \in Z(G)^\circ$, $Z_G(g's) = Z_G(g')$.......$(**)$.

Define a map:

$$\{[Z_G(x)^\circ] : x \textit{ is semisimple}\} \to \{[Z_{G'}(x')^\circ] : x' \textit{ is semisimple}\}$$

by, $[Z_G(x)^\circ] \mapsto [Z_{G'}(x')^\circ]$, where $x = x's_x$, $x' \in G'$ and $s_x \in Z(G)^\circ$. We prove that this map is a bijection.

To show that the above map is well defined, assume that $Z_G(x)^\circ = Z_G(gyg^{-1})^\circ$, for some $g \in G$. Then by $(**)$, $Z_{G'}(x')^\circ \subset Z_G(x')^\circ = Z_G(x)^\circ = Z_G(gyg^{-1})^\circ = Z_G(gy'g^{-1})^\circ$. Hence $Z_{G'}(x')^\circ \subset Z_{G'}(gy'g^{-1})^\circ$. Similarly $Z_{G'}(gy'g^{-1})^\circ \subset Z_{G'}(x')^\circ$, which shows that the above map is well defined.

It is clear from the definition that the map is onto.

We now prove the injectivity. So assume that, $Z_{G'}(x')^\circ = Z_{G'}(g'y'g'^{-1})^\circ$, for some $g' \in G'$. Let $a \in Z_G(x')^\circ$, where $a = a's_a$. Then $a' \in Z_G(x')$ as s_a is central. Also note that $s_a \in Z(G)^\circ \subset Z_G(x')^\circ$. Therefore, $a' = as'^{-1} \in Z_G(x')^\circ$. In particular, $a' \in Z_{G'}(x')$.

We claim that $a' \in Z_{G'}(x')^\circ$. If a' is unipotent, then $a' \in Z_{G'}(x')^\circ$, since G' is a connected semisimple group(see [**Hu1**], Chapter 1, Section 12). So let a' be semisimple. Choose a maximal torus $T \in Z_G(x')^\circ$ such that $a' \in T$. Let $T = T'.Z(G)^\circ$, where T' is a maximal torus in G. Therefore, $T' \subset Z_{G'}(x')^\circ$. Write $a' = a_1 b$ with $a_1 \in T'$ and $b \in Z(G)^\circ$. Since both a_1 and b are in $Z_{G'}(x')^\circ$, so is a'. Hence the claim. Therefore, by assumption, $a' \in Z_{G'}(x')^\circ = Z_{G'}(g'y'g'^{-1})^\circ \subset Z_G(g'y'g'^{-1})^\circ$. Since $a_s \in Z(G)^\circ$, $a = a'a_s \in Z_G(g'y'g'^{-1})^\circ$. Thus we have shown that, $Z_G(x')^\circ \subset Z_G(g'y'g'^{-1})^\circ$. Similarly the other inclusion follows. Hence the map is injective. $\square$

Remark: By Theorem 4.2.13, the genus number of $GL_n(k)$ is equal to the genus number of $SL_n(k)$.

Disconnected centralizers. In general, for a connected semisimple group we can derive a necessary and sufficient condition for connectedness of centralizers of semisimple elements. Let G be a connected semisimple algebraic group, with the simply connected cover $\widetilde{G}$ and $\rho : \widetilde{G} \longrightarrow G$ be the covering map. Let $T \subset G$ be a fixed maximal torus. Consider $t \in T$ and let $\rho^{-1}(t) = \{\tilde{t_1}, ..., \tilde{t_l}\} \subset \widetilde{G}$. Then we have the following:

Theorem 4.2.14. *Fix a lift $\tilde{t_1} \in \widetilde{G}$ of $t \in T$. Then $Z_G(t)$ is disconnected if and only if there exists $\tilde{g} \in \widetilde{G}$ such that, $\tilde{g}\tilde{t_1}\tilde{g}^{-1} = \tilde{t_i}$, for some $i \neq 1$.*

Proof. Let $Z_G(t)$ be disconnected. Therefore, there exists $g \in Z_G(t) \setminus Z_G(t)^\circ$. Let $\tilde{g} \in \widetilde{G}$ be a lift of g. Observe that $\rho(\tilde{g}\tilde{t_1}\tilde{g}^{-1}) = gtg^{-1} = t$. So, $\tilde{g}\tilde{t_1}\tilde{g}^{-1} \in \rho^{-1}(t)$. Also note that $\tilde{g}\tilde{t_1}\tilde{g}^{-1} \neq \tilde{t_1}$. For else, $\tilde{g} \in Z_{\widetilde{G}}(\tilde{t_1})$, which implies $\rho(\tilde{g}) \in \rho(Z_{\widetilde{G}}(\tilde{t_1})) \Rightarrow g \in Z_G(t)^\circ$ (since $Z_{\widetilde{G}}(\tilde{t_1})$ is connected). Hence $\tilde{g}\tilde{t_1}\tilde{g^{-1}} = \tilde{t_i}$ for some $i \neq 1$.

Conversely, let there exist $\tilde{g} \in \widetilde{G}$, such that, $\tilde{g}\tilde{t_1}\tilde{g}^{-1} = \tilde{t_i}$, for some $i \neq 1$. Therefore $g = \rho(\tilde{g}) \in Z_G(t)$. Define $S_j = \{x \in Z_G(t) | \tilde{x}\tilde{t_1}\tilde{x}^{-1} = \tilde{t_j}\}$, where $\rho(\tilde{x}) = x$. Then clearly, $Z_G(t) = \bigcup_{j=1}^{n} S_j$. Note that, $S_1 = \rho(Z_{\widetilde{G}}(\tilde{t_1})) = Z_G(t)^\circ$ and by hypothesis, S_i is non empty. Hence $Z_G(t)$ is not connected. □

In what follows, we shall compute the genus number of all the compact simply connected simple Lie groups and simply connected simple algebraic groups of Classical type and of types G_2 and F_4.

Notation: We define a **partition** of a positive integer n as a set $\{n_1, ..., n_k : \ n_i \in \mathbb{N}\}$ such that $n_1 + ... + n_k = n$. Define $p(n)$ to be number of all partitions of n and we set $p(0) = 1$. The function p is sometimes referred to as the unrestricted partition function (see [**N**]).

4.3. A_n

In this section, we compute the genus number for the compact Lie group $SU(n+1)$ and the semisimple genus number of the algebraic group $SL(n+1)$ over an algebraically closed field. We fix a maximal torus T of $SU(n+1)$ consisting of all matrices of the form

$$\begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_{n+1} \end{bmatrix},$$

where $z_i \in S^1$ and $z_1 ... z_{n+1} = 1$. If we write $z_l = exp(2\pi i \gamma_l)$, then the above matrix can be represented by the $(n+1)$-tuple $(\gamma_1, \gamma_2 ..., \gamma_{n+1})$, where $\gamma_i \in \mathbb{R}/\mathbb{Z}$. The Weyl group of $SU(n+1)$ is S_{n+1} and it acts on the diagonal maximal torus in the following way: let $\alpha \in S_{n+1}$ and $\gamma = (\gamma_1, \gamma_2, ..., \gamma_{n+1}) \in T$, then $\alpha^{-1}(\gamma_1, \gamma_2, ..., \gamma_{n+1})$ $=(\gamma_{\alpha(1)}, \gamma_{\alpha(2)}, .., \gamma_{\alpha(n+1)})$.

We wish to compute the number of conjugacy classes of isotropy subgroups of S_{n+1} with respect to its action on T.

Let $\gamma \in T$. By the action of a suitable element of S_{n+1} we can assume γ to be such that, $\gamma_1 = \gamma_2 = ... = \gamma_{k_1}$; $\gamma_{k_1+1} = ... = \gamma_{k_1+k_2}$;...; $\gamma_{k_1+...+k_{l-1}+1} = ... = \gamma_{k_1+...k_l}$ and $k_1 + k_2 + ... k_l = n$, with $\gamma_1 \neq \gamma_{k_1+1} \neq ... \neq \gamma_{k_1+...+k_{l-1}+1}$. Hence, for this γ, the isotropy subgroup in S_{n+1} is $S_{k_1} \times S_{k_2} \times ... \times S_{k_l} \subset S_{n+1}$, where $S_{k_i} = \{\rho \in S_{n+1} | \, \rho(j) = j \; for \; j = 1, ..., (k_1 + ... + k_{i-1}), (k_1 + ... + k_i + 1), ..., n+1\}$. Note that $S_{k_i} \cap S_{k_j} = \{1\}$ for $i \neq j$ and $S_{k_i} S_{k_j} = S_{k_j} S_{k_i}$. So, $S_{k_i} S_{k_j}$ is a subgroup of S_n and hence by induction $S_{k_1} ... S_{k_n}$ is a subgroup of S_n.

More precisely, any element $\rho \in W_\gamma$, necessarily has a cycle decomposition of the type $(k_1, ..., k_l)$, i.e. $\rho \in S_{k_1} S_{k_2} ... S_{k_l}$ and conversely any element of $S_{k_1} \times S_{k_2} \times ... \times S_{k_l}$ is clearly a stabilizer of γ. In other words, we have the following isomorphism :

$$W_\gamma \longrightarrow S_{k_1} ... S_{k_l}$$

$$\rho \longmapsto (\rho \,|_{k_1} . \rho \,|_{k_2} ... \rho \,|_{k_l}),$$

where $\rho \,|_{k_i}$ denotes the restriction of ρ on to the k_i many entries of γ, which are equal modulo $\mathbb{Z}$.

Let $(n_1, ..., n_l)$ and $(m_1, ..., m_k)$ be two ordered partitions of $n+1$ and suppose they correspond to elements $\gamma_1, \gamma_2 \in T$ respectively. If $l = k$ and $n_i = m_i$ for all $1 \leq i \leq l$, clearly $W_{\gamma_1} = W_{\gamma_2}$. Now suppose that the two partitions are different. Then $n_i \neq m_i$ for some i. We observe that any element in W_{γ_1} has a cycle type $(n_1, ..., n_l)$ and any element in W_{γ_2} has cycle type $(m_1, ..., m_k)$ and since conjugation in S_n must preserves cycle types, W_{γ_1} is not conjugate to W_{γ_2}.

Thus the number of conjugacy classes of isotropy subgroup is precisely $p(n+1)$, i.e. the number of partitions of $n+1$.

For $SL(n+1)$ over an algebraically closed field k, the semisimple genus number is similarly obtained by computing the number of isotropy subgroups of the Weyl group (up to conjugacy) with respect to its action on a maximal torus. In this situation again we consider the diagonal maximal torus $T \subset SL(n+1)$, i.e the subgroup of matrices of the form $diag(a_1, ..., a_{n+1})$ such that $a_1...a_{n+1} = 1,\ a_i \in k$. Following a similar argument as in the case of $SU(n+1)$, we see that the number of conjugacy classes of isotropy subgroups of Weyl group is $p(n+1)$.

We record this as :

Theorem 4.3.1. *The genus number of a compact simply connected Lie group or a simply connected algebraic group over an algebraically closed field, of type A_n is $p(n+1)$.*

4.4. B_n

We consider the simply connected group $Spin(2n+1)$ and a maximal torus

$$T = \{\prod_{i=1}^{n}(cost_i - e_{2i-1}e_{2i}sint_i) : 0 \leq t_i \leq 2\pi\}.$$

To simplify notations let us denote a typical element of T by $t = (t_1, ..., t_n)$, with $0 \leq t_i \leq 2\pi$.

For a description of the Weyl group of $Spin(2n+1)$, we fix the following notation:

$$t_{-i} = -t_i, \quad \text{for} \quad \text{i} = 1, ..., \text{n}.$$

The Weyl group of $Spin(2n+1)$ is $W = (\mathbb{Z}/2)^n \rtimes S_n$, where S_n acts on $(\mathbb{Z}/2)^n$ by permuting the coordinates. The group W can be identified with the group of permutations ϕ of the set $\{-n, ..., -1, 1, ..., n\}$, which satisfy $\phi(-i) = -\phi(i)$. W acts on the fixed maximal torus T of $Spin(2n+1)$ in the following way:

$$\phi(t_1, ..., t_n) = (t_{\phi^{-1}(1)}, ..., t_{\phi^{-1}(n)}),$$

where $\phi \in W$ and $(t_1, ..., t_n) \in T$.

A useful interpretation: The action of W on the maximal torus of $Spin(2n+1)$ can be described in the following way:

An element $\phi \in G(n)$ acts on a toral element $t \in T$ by permuting the parameters and changing the sign of some of them. If $\phi = (\alpha, \beta)$, with $\alpha \in (\mathbb{Z}/2)^n$ and $\beta \in S_n$, then β permutes the parameters of t and α changes the signs of the parameters.

In order to compute the number of conjugacy classes of isotropy subgroups of W, we start with an element $t = (t_1, ..., t_n) \in T$ and find the isotropy subgroup W_t.

Let $n = n_1 + ... + n_k$, where, $t_i = 0$ or π, for $i = 1, ..., n_1'$, $t_i = \pi/2$ or $3\pi/2$, for $i = n_1' + 1, ..., n_1$, and $t_i \neq 0, \pi, \pi/2, 3\pi/2$ for $i \geq n_1 + 1$. The remaining integers $n_2, ..., n_k$ denote the number of parameters which are equal.

Note that, for $i = 1, ..., n_1'$, a non- trivial $(\mathbb{Z}/2)^n$ action on t_i fixes the factor $(cost_i - e_{2i-1}e_{2i}sint_i)$, which is 1 or -1 according as $t_i = 0$ or π. However, for $i = n_1' + 1, ..., n_1$, a non-trivial $(\mathbb{Z}/2^n)$ action on t_i inverts the factor $(cost_i - e_{2i-1}e_{2i}sint_i)$, which is $e_{2i-1}e_{2i}$ or $-e_{2i-1}e_{2i}$, according as $t_i = \pi/2$ or $3\pi/2$. For the rest of the parameters, only the S_n part of the Weyl group contributes to the isotropy. Therefore the isotropy subgroup for such an element of T is

$$((\mathbb{Z}/2)^{n_1'} \rtimes S_{n_1'}) \times ((\mathbb{Z}/2)^{n_1 - n_1' - 1} \rtimes S_{n_1 - n_1'}) \times S_{n_2} \times ... \times S_{n_k}, \text{ .... } (*)$$

Therefore for each choice of n_1 we have $(n_1 + 1)p(n - n_1)$ many non-conjugate isotropy subgroups. (Here we assume $n \geq 3$ since for $n = 1, 2$, the above enumeration does not give distinct subgroups. This happens because the first two factors in $(*)$ are not symmetric. For example, if we take $n = 2$ and $n_1 = 2$, then we get the isotropy subgroups as $\mathbb{Z}/2 \rtimes S_2$, $(\mathbb{Z}/2)^2 \rtimes S_2$ and $\mathbb{Z}/2$ and for $n_1 = 1$, we get the isotropy subgroup $\mathbb{Z}/2$, which is a repetition. So we explicitly enumerate the isotropy subgroups for B_1 and B_2 below.) Hence the total number of conjugacy classes of isotropy subgroups of W for $SO(2n + 1)$ $(n \geq 3)$, is

$$\sum_{i=0}^{n} (i + 1)p(n - i).$$

When we consider $Spin(2n + 1)$ over an algebraically closed field k, we take a maximal torus $T = \{\prod_{i=1}^{n}(t_i^{-1} + (t_i - t_i^{-1}e_{2i-1}e_{2i}), t_i \in k^*\}$. We can calculate the number of conjugacy classes of isotropy subgroups of the Weyl group using similar arguments.

Now, for $n = 1, 2$ by $(*)$, we list all the isotropy subgroups (up to conjugacy). For B_1, the isotropy subgroups are $\{1\}$ and $\mathbb{Z}/2$. For B_2, the isotropy subgroups are $\{1\}$, S_2, $\mathbb{Z}/2$, $(\mathbb{Z}/2)^2 \rtimes S_2$ and $\mathbb{Z}/2 \rtimes S_2$.

We record this discussion as:

Theorem 4.4.1. *The genus number of a compact simply connected Lie group or a simply connected algebraic group over an algebraically closed field, of type* B_n *is* $\sum_{i=0}^{n}(i + 1)p(n - i)$, *for* $n \geq 3$. *The genus number for* B_1 *is* 2 *and that for* B_2 *is* 5.

Corollary 4.4.2. *The connected genus number of $SO(2n+1)$ is equal to the genus number of $Spin(2n+1)$.*

Proof. Follows from Theorem 4.2.12. □

4.5. C_n

Let k be an algebraically closed field. The symplectic group over k of rank n, is defined as $Sp(n,k) := \{A \in GL_{2n}(k) : A^t JA = J\}$, where $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$, I being the identity matrix in $GL_n(k)$. $Sp(n,k)$ is the simply connected algebraic group of type C_n.

When $k = \mathbb{C}$, the field of complex numbers, $Sp(n,\mathbb{C})$ is the complex symplectic group of rank n. The compact simply connected Lie group of type C_n, denoted by $Sp(n)$ is defined as follows: let $U(n)$ denote the group of $n \times n$ unitary matrices. Define $Sp(n) := \{A \in U(2n) : A^t JA = J\}$, where $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$, I is the identity matrix in $GL_n(\mathbb{C})$. Therefore, $Sp(n) = Sp(2n,\mathbb{C}) \cap U(2n)$. We have the inclusion $U(n) \longrightarrow Sp(n)$, given by $A \mapsto \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix}$.

Consider the maximal torus

$$T(n) = \left\{ \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix} \in U(n) : z_i \in S^1 \right\} \subset U(n).$$

Then the image of this maximal torus in $Sp(n)$ under the above inclusion gives a maximal torus $T \subset Sp(n)$, a typical element of which is of the form,

$$t = \begin{bmatrix} z_1 & & & & & & & \\ & z_2 & & & & & & \\ & & \ddots & & & & & \\ & & & z_n & & & & \\ & & & & \overline{z_1} & & & \\ & & & & & \overline{z_2} & & \\ & & & & & & \ddots & \\ & & & & & & & \overline{z_n} \end{bmatrix}.$$

Let $z_k = exp(2\pi i t_k)$. Then we can represent each $t \in T$ by an n-tuple $(t_1, ..., t_n)$, where $t_k \in \mathbb{R}/\mathbb{Z}$.

The Weyl group of $Sp(n)$ is $W = (\mathbb{Z}/2)^n \rtimes S_n$, where S_n acts on $(\mathbb{Z}/2)^n$ by permuting the coordinates, as noted in Section 4. The action of W on T is given by, $\phi(t_1, ..., t_n) = (t_{\phi^{-1}(1)}, ..., t_{\phi^{-1}(n)})$, where $\phi \in W$ and $(t_1, ..., t_n) \in T$. We follow the same convention: $t_{-i} = -t_i$, for $i = 1, ..., n$ (see Section 4.4).

To compute the isotropy subgroup of $t \in T$ in W, first note that, if $t_i = 0$ or $1/2$, a non-trivial $(\mathbb{Z}/2)^n$ action fixes t_i. Therefore, we can assume without loss of generality that, $t_i \neq -t_j$ unless $t_i = t_j = 0, 1/2$. For, if there exist $t_i = -t_j$ for some i, j with $t_i, t_j \neq 0, 1/2$ then we can change the sign of t_j by suitable element from $(\mathbb{Z}/2)^n$.

Let $n = n_1 + ... + n_k$ be a partition of n with n_1 being the total number of 0's and 1/2's and $n_2, ..., n_k$ are the sizes of the blocks of parameters t_i which are equal. The isotropy subgroup for this particular t is

$$((\mathbb{Z}/2)^i \rtimes S_i) \times ((\mathbb{Z}/2)^{n_1 - i} \rtimes S_{n_1 - i}) \times S_{n_2} \times ... \times S_{n_k},$$

where i and $n_1 - i$ respectively denote the number of 0's and 1/2's in t. Therefore for this partition of n, we have $([n_1/2]+1)p(n-n_1)$ many distinct isotropy subgroups (by varying the number of 0's). Hence the total number of conjugacy classes of isotropy subgroups is

$$\sum_{i=0}^{n}([i/2] + 1)p(n - i).$$

Over an algebraically closed field k, the diagonal maximal torus of $Sp(n)$ can again be parametrized by n coordinates $(a_1, ..., a_n)$ $a_i \in k^*$. The calculation for genus number follows exactly as above.Thus we have the following:

Theorem 4.5.1. *The genus number of a compact simply connected Lie group or a simply connected algebraic group over an algebraically closed field, of type C_n is $\sum_{i=0}^{n}([i/2] + 1)p(n - i)$.*

4.6. D_n

Here, as in the case of $Spin(2n + 1)$, we work with the maximal torus $T = \{\prod_{i=1}^{n}(cost_i - e_{2i-1}e_{2i}sint_i) : 0 \leq t_i \leq 2\pi)\}$. The Weyl group is $W = (\mathbb{Z}/2)^{n-1} \rtimes S_n$, the subgroup of even permutations in the Weyl group of $Spin(2n+1)$ and it acts on a typical element $(t_1, ..., t_n) \in T$, by permuting the entries and changing the signs of even number of them. We discuss two separate cases:

Case 1: n is odd.

Let $t = (t_1, ..., t_n) \in T$ be an arbitrary element of the torus. As in the case of B_n, we consider a partition of n as $n = n_1 + ... + n_k$, where the n_i's are as in §4. Thus looking at the torus element t, we can read off the isotropy subgroup, which is

$$((\mathbb{Z}/2)^{n_1'-1} \rtimes S_{n_1'})) \times ((\mathbb{Z}/2)^{n_1-n_1'-1} \rtimes S_{n_1-n_1'}) \times S_{n_2} \times ... \times S_{n_k},$$

Thus for each n_1 the number of non-conjugate isotropy subgroups is $([n_1/2] + 1)p(n-n_1)$. This is because the number of partitions of n_1 which give non -conjugate isotropy subgroups for a fixed choice of $n_2, .., n_3$ is $[n_1/2]$. Hence the total number is

$$\sum_{i=0}^{n}([i/2]+1)p(n-i).$$

Case 2: n is even.

First let us investigate the following situation: $t = (t_1, ..., t_n) \in T$, where $t_1 = ... = t_{n-1} = -t_n$ and $t_i \neq 0, \pi, \pi/2, 3\pi/2$, for $1 \leq i \leq n$. We have the Weyl group $W = (\mathbb{Z}/2)^{n-1} \rtimes S_n$. The action of an element $(\tau, \rho) \in W$ on any $t \in T$ is given by,

$$(\tau, \rho)(t_1, ..., t_n) = (t_{(\rho)^{-1}(\tau)^{-1}(1)}, ..., t_{(\rho)^{-1}(\tau)^{-1}(n)}),$$

If $(\tau, \rho) \in W_t$, then $(\tau, \rho)(t_1, ..., t_n) = (t_{(\rho)^{-1}(\tau)^{-1}(1)}, ..., t_{(\rho)^{-1}(\tau)^{-1}(n)}) = (t_1, ..., t_n)$. Therefore,

(a) if $\rho(n) = n$ then $\tau = (0, ..., 0) \in (\mathbb{Z}/2)^{n-1}$

(b) if $\rho(n) = i \neq n$ then necessarily τ is an n-tuple with 1 at the n-th and $\rho(n)$-th positions and 0 everywhere else.

The isotropy subgroup of t therefore has exactly $n!$ many elements and as we will see, is not conjugate to S_n (since S_n is the only other isotropy subgroup of order $n!$).

Let if possible $(\tau, \rho) \in W$ be such that

$$(\tau, \rho)S_n(\tau, \rho)^{-1} = W_t.$$

Then, for an arbitrary $(1, \sigma) \in S_n \subset W$ we have,

$$\begin{aligned} &(\tau, \rho)(1, \sigma)(\rho^{-1}(\tau), \rho^{-1}) \\ =&(\tau, \rho\sigma)(\rho^{-1}(\tau), \rho^{-1}) \\ =&(\tau\rho\sigma\rho^{-1}(\tau), \rho\sigma\rho^{-1}) \quad \in W_t. \end{aligned}$$

Note that τ cannot be $(0,...,0)$ or $(1,...,1)$ because in that case $\tau\rho\sigma\rho^{-1}(\tau)$ is necessarily equal to $(0,...,0)$ for any chosen σ; and we can suitably choose a $\sigma \in S_n$ such that $\rho\sigma\rho^{-1}(n) \neq n$, in which case the above element cannot belong to W_t. Thus τ must contain both 0 and 1 as its parameters. Moreover, since $(\tau,\rho) \in W$, τ must be a permutation changing an even number of signs. Since there is at least one 1 in the n-tuple representing τ, there must be at least two of them. Similar argument holds for the number of 0's occurring in τ. Now let the n-th and the i-th positions in τ be 1. Then we simply choose a suitable σ such that $\rho\sigma\rho^{-1} = (1\ n)$ (the transposition flipping 1 and n). This shows that the element $(\tau\rho\sigma\rho^{-1}(\tau), \rho\sigma\rho^{-1}) \notin W_t$ because $\tau\rho\sigma\rho^{-1}(\tau) = (1,...,1)$ in this case again.

With this in hand, we carry out the computation for the number of conjugacy classes in a way similar to that of $Spin(2n+1)$. If $n = n_1 + ... + n_k$ is a partition consisting of at least one odd integer, then by the action of a suitable Weyl group element the computation can be carried out as in Case 1.

If the partition $n = n_1 + ... + n_k$ consists of only even integers, and also let us assume that none of the parameters are 0 or π, then we can have the following possibility:
$t_1 = ... = t_{n_1-1} = -t_{n_1}$ and the remaining blocks containing equal parameters with $t_i \neq -t_j$ for $n_1 < i, j \leq n$. By the argument at the beginning of Case 2, the isotropy subgroup for such an element is obtained as: Let $n = 2l$. If $l = l_1 + ... + l_k$, then $W_t = H_{2k_1}.S_{2k_2}...S_{2k_l}$, where H_{2k_1} is a subgroup of order $(2k_1)!$ as described in the beginning of Case 2.

So if $n = 2l$ then the total number of conjugacy classes of isotropy subgroups is :

$$\begin{aligned}&(\sum_{i=1}^{n}([i/2]+1)p(n-i)) + p(n) - p(l) + 2p(l)\\ =&(\sum_{i=0}^{n}([i/2]+1)p(n-i)) + p(l).\end{aligned}$$

As noted in the previous section, over an algebraically closed field, the number of conjugacy classes of isotropy subgroups of the Weyl group can be obtained exactly as above. Thus we have the following theorem:

Theorem 4.6.1. *The genus number of a compact simply connected Lie group or a simply connected algebraic group over an algebraically closed field, of type D_n is*
$\sum_{i=0}^{n}([i/2]+1)p(n-i)$ *for n odd and*
$\sum_{i=0}^{n}([i/2]+1)p(n-i)) + p(l)$ *for $n = 2l$.*

Corollary 4.6.2. *The connected genus number of $SO(2n)$ is equal to the genus number of $Spin(2n)$.*

Proof. Follows from Theorem 4.2.12. □

4.7. F_4

Let $\mathfrak{C}$ be the octonion division algebra over $\mathbb{R}$ with norm N. We fix an orthogonal basis $\mathfrak{B} = \{v_1, v_2, ..., v_8\}$, where $v_1 = 1$, $v_6 = v_2 v_5$, $v_7 = v_3 v_5$ and $v_8 = v_4 v_5$ ([**P**], Lecture 14). Let $Spin(N)$ and $SO(N)$ respectively denote the spin group and the special orthogonal group of $(\mathfrak{C}, N)$. With respect to the basis $\mathfrak{B}$, the matrix of the bilinear form associated with N is diagonal.

Consider the $\mathbb{R}$-algebra $A := H_3(\mathfrak{C})$, consisting of all 3×3 matrices of the form

$$\begin{bmatrix} \alpha_1 & c_3 & \bar{c_2} \\ \bar{c_3} & \alpha_2 & c_1 \\ c_2 & \bar{c_1} & \alpha_3 \end{bmatrix},$$

where $\alpha_i \in \mathbb{R}$, $c_i \in \mathfrak{C}$ and $x \mapsto \bar{x}$ is the canonical involution on $\mathfrak{C}$. The multiplication in A is given by

$$xy = (x \cdot y + y \cdot x)/2,$$

where dot denotes the standard matrix multiplication.

Then $Aut(A)$ is the compact connected Lie group of type F_4 (see Chapter 3). For this discussion we need an explicit embedding of $Spin(N)$ in F_4. Consider the subalgebra $S = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \subset A$. Then $Spin(N)$ sits inside $Aut(A)$ as the subgroup of all automorphisms ϕ, such that $\phi(s) = s$ for all $s \in S$ ([**J**], Theorem 6).

We first discuss an explicit description of $Spin(N)$. Let as before $\mathfrak{C}$ denote an octonion algebra over $\mathbb{R}$ and consider a subgroup $RT(\mathfrak{C}) \subset SO(N)^3$, defined as,

$$RT(\mathfrak{C}) := \{(t_1, t_2, t_3) \in SO(N)^3 |\ t_1(xy) = t_2(x) t_3(y) \quad \forall x, y \in \mathfrak{C}\}$$

Any element of $RT(\mathfrak{C})$ is called a related triple (see Chapter 3) We need the following result from [**SV**] (Proposition 3.6.3).

Proposition 4.7.1. *There is an isomorphism,*

$$\Phi : Spin(N) \longrightarrow RT(\mathfrak{C})$$

defined by ,

$$\Phi(a_1 \circ b_1 \circ ... \circ a_r \circ b_r) = (s_{a_1} s_{b_1} ... s_{a_r} s_{b_r}, l_{a_1} l_{\overline{b_1}} ... l_{a_r} l_{\overline{b_r}}, r_{a_1} r_{\overline{b_1}} ... r_{a_r} r_{\overline{b_r}}),$$

where $a_i, b_i \in \mathfrak{C}, \prod_i N(a_i)N(b_i) = 1$,(N *being the norm on the octonion algebra),* s_v *is the reflection in the hyperplane orthogonal to* $v \in \mathfrak{C}$, l_v *and* r_v *are the left and right homotheties on* $\mathfrak{C}$ *respectively.*

Remark: Henceforth in the subsequent discussion we shall identify the groups $Spin(N)$ and $RT(\mathfrak{C})$ via the above isomorphism. We note that a related triple $t = (t_1, t_2, t_3) \in RT(\mathfrak{C})$ acts on an element of A as

$$t \begin{bmatrix} \alpha_1 & c_3 & \bar{c_2} \\ \bar{c_3} & \alpha_2 & c_1 \\ c_2 & \bar{c_1} & \alpha_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 & t_1(c_3) & t_2(\bar{c_2}) \\ \overline{t_1(c_3)} & \alpha_2 & t_3(c_1) \\ \overline{t_2(\bar{c_2})} & \overline{t_3(c_1)} & \alpha_3 \end{bmatrix}$$

(refer to [**J**], §6).

Consider the following automorphisms of $RT(\mathfrak{C})$:

$$\begin{aligned} &\tau_1 : (t_1, t_2, t_3) \mapsto (\hat{t_1}, \hat{t_3}, \hat{t_2}), \\ &\tau_2 : (t_1, t_2, t_3) \mapsto (t_3, \hat{t_2}, t_1), \\ &\tau_3 : (t_1, t_2, t_3) \mapsto (t_2, t_1, \hat{t_3}), \end{aligned} \tag{4.7.1}$$

where $\hat{t}(x) = \overline{t(\overline{x})}$, for $t \in SO(N)$ and $x \in \mathfrak{C}$. We note the following result from [**SV**] (Proposition 3.6.4).

Proposition 4.7.2. τ_2 *and* τ_3 *generate a group of automorphisms of* $RT(\mathfrak{C})$ *isomorphic to* S_3 *and the non trivial elements of this group are outer automorphisms.*

.

Lemma 4.7.3. *Let* T *be a maximal torus in* $SO(N)$*. Then*

$$\widetilde{T} := \{(t_1, t_2, t_3) \in T^3 | \ (t_1, t_2, t_3) \textit{ is a related triple}\}$$

is a maximal torus in $Spin(N)$*.*

Proof. If we take $t_1 \in T$, then the fiber of t_1 in a maximal torus $\widetilde{T}$ of $Spin(N)$ consists of (t_1, t_2, t_3) and $(t_1, -t_2, -t_3)$, such that (t_1, t_2, t_3) is a related triple. Since the Weyl group acts on the maximal torus, $\tau_3(t_1, t_2, t_3) = (t_2, t_1, \hat{t_3}) \in \widetilde{T}$, which when projected onto $SO(N)$ via the two sheeted covering map, we gives $t_2 \in T$. Similarly by considering the automorphism τ_2 we can conclude $t_3 \in T$. Hence the proof. □

Lemma 4.7.4. *For a maximal torus* $\widetilde{T} \subset F_4$, $A^{\widetilde{T}} \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R}$*. Here* $A^{\widetilde{T}}$ *denotes the subalgebra of* A*, fixed point wise by* $\widetilde{T}$*.*

PROOF. Let T be the diagonal maximal torus of $SO(N)$. If $\widetilde{T_1}$ and $\widetilde{T_2}$ be two maximal tori in F_4, then $A^{\widetilde{T_1}} \cong A^{\widetilde{T_2}}$ since $\widetilde{T_1}$ and $\widetilde{T_2}$ are conjugate. So we can assume without loss of generality that, $\widetilde{T} \subset Spin(N)$ and hence by Lemma 4.7.3, $\widetilde{T} = \{(t_1, t_2, t_3) \in Spin(N)|\ t_i \in T \subset SO(N)\}$. Now suppose that

$$t \begin{bmatrix} \alpha_1 & c_3 & \bar{c_2} \\ \bar{c_3} & \alpha_2 & c_1 \\ c_2 & \bar{c_1} & \alpha_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 & t_1(c_3) & t_2(\bar{c_2}) \\ \overline{t_1(c_3)} & \alpha_2 & t_3(c_1) \\ \overline{t_2(\bar{c_2})} & \overline{t_3(c_1)} & \alpha_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 & c_3 & \bar{c_2} \\ \bar{c_3} & \alpha_2 & c_1 \\ c_2 & \bar{c_1} & \alpha_3 \end{bmatrix}$$

holds for all $t \in \widetilde{T}$. This means that $t_1(c_3) = c_3$ for all $t_1 \in T$. Note that t_1 is a block diagonal matrix consisting of 2×2 rotation matrices along the diagonal. Let if possible $c_3 \neq 0$. We can assume without loss of generality that at least one of the first two coordinates of c_3 (say x_1, x_2) with respect to the basis $\mathfrak{B}$ of $\mathfrak{C}$, is non zero.

Now if we take the first 2×2 diagonal block of t_1 as

$$\begin{bmatrix} cos2\theta_1 & -sin2\theta_1 \\ sin2\theta_1 & cos2\theta_1 \end{bmatrix},$$

then $t_1(c_3) = c_3$ implies that

$$\begin{bmatrix} cos2\theta_1 & -sin2\theta_1 \\ sin2\theta_1 & cos2\theta_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which forces $cos2\theta_1 = 1$. But we can choose a t_1 with $\theta_1 \neq 0$, for which $cos2\theta_1 \neq 1$. Hence $c_3 = 0$. By similar arguments we can say the same for c_1 and c_2. Hence the proof. □

Lemma 4.7.5. *The Weyl group of F_4 is $WSpin(N) \rtimes S_3$, $WSpin(N)$ being the Weyl group of $Spin(N)$.*

PROOF. Let us denote the group F_4 by G. Consider the $\mathbb{R}-$subalgebra $S = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \subset A$ and define,

$$Aut(A/S) := \{\phi \in Aut(A) : \phi(s) = s,\ \forall s \in S\},$$

$$Aut(A, S) := \{\phi \in Aut(A) : \phi(S) = S\}.$$

Then $Aut(A, S) \cong Aut(A/S) \rtimes Aut(S)$ ([**J**], Theorem 8). We have $Aut(A/S) = Spin(N)$ and $Aut(S) = S_3$ and therefore, $Aut(A, S) = Spin(N) \rtimes S_3$.

First let us fix a maximal torus $T \subset G$. Then $A^T \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R}$(by Lemma 4.7.4). Let $\phi \in N_G(T)$. Then $\phi \in Aut(A, A^T)$, since, for $s \in A^T$ and for any $t \in T$ we have $t(\phi(s)) = (t\phi)(s) = \phi(\phi^{-1}t\phi)(s) = \phi(s)$ (as $\phi^{-1}t\phi \in T$ and $s \in A^T$). Hence

$\phi(s) \in A^T$. Therefore we have shown that $N_G(T) \subset Aut(A, A^T) = Spin(N) \rtimes S_3$. Thus $N_G(T) \subset N_{Spin(N)}(T) \rtimes S_3$, which implies that $WG = N_G(T)/T \subset WSpin(N) \rtimes S_3$. Both the groups being finite and of the same order, are therefore equal.

□

Remark: Note that, the S_3 factor arising in the Weyl group of F_4 is the group of outer automorphisms of $Spin(N)$ and its action on the maximal torus is given by $\tau_1, \tau_2, \tau_3 \in Aut(RT(\mathfrak{C}))$ (refer to the remark preceding Proposition 4.7.2).

Computation of the genus number for F_4:

Let us denote the maximal torus in F_4 by $\widetilde{T}$ and the Weyl group by W. We work with the chosen orthogonal basis $\mathfrak{B} = \{v_1, ..., v_8\}$ of $\mathfrak{C}$, such that, $v_1 = 1$, $v_6 = v_2v_5$, $v_7 = v_3v_5$ and $v_i^2 = -1$, $1 \leq i \leq 8$. Let $T \subset SO(N)$ be the diagonal maximal torus and without loss of generality we can assume $\widetilde{T} \subset Spin(N)$. If $t = (t_1, t_2, t_3) \in \widetilde{T}$, with $t_1 = (\theta_1/\pi, \theta_2/\pi, \theta_3/\pi, \theta_4/\pi)$, $\theta_i/2\pi \in \mathbb{R}/\mathbb{Z}$, we wish to compute t_2 and t_3 in terms of the θ_is.

First note that for $t = (\gamma_1/\pi, \gamma_2/\pi, \gamma_3/\pi, \gamma_4/\pi) \in T$, $\hat{t} = (-\gamma_1/\pi, \gamma_2/\pi, \gamma_3/\pi, \gamma_4/\pi)$. This is evident from the following calculation: Let $x = (x_1, ..., x_8) \in \mathfrak{C}$, $x_i \in \mathbb{R}$. Then $\bar{x} = (x_1, -x_2, ..., -x_8)$ (considered as a column vector). By definition, $\hat{t}(x) = \overline{t(\bar{x})}$. Now, $t = (\gamma_1/\pi, \gamma_2/\pi, \gamma_3/\pi, \gamma_4/\pi)$ is an 8×8 block diagonal matrix with the i-th diagonal block being:

$$\begin{bmatrix} cos2\gamma_i & -sin2\gamma_i \\ sin2\gamma_i & cos2\gamma_i \end{bmatrix}.$$

Let $s = (-\gamma_1/\pi, \gamma_2/\pi, \gamma_3/\pi, \gamma_4/\pi) \in \widetilde{T}$. Then, a direct computation shows that,

$$\hat{t}(x) = \overline{t(\bar{x})} = \begin{bmatrix} cos2\gamma_1 x_1 + sin2\gamma_1 x_2 \\ -sin2\gamma_1 x_1 + cos2\gamma_1 x_2 \\ cos2\gamma_2 x_3 - sin2\gamma_2 x_4 \\ sin2\gamma_2 x_3 + cos2\gamma_2 x_4 \\ cos2\gamma_3 x_5 - sin2\gamma_3 x_6 \\ sin2\gamma_3 x_5 + cos2\gamma_3 x_6 \\ cos2\gamma_4 x_7 - sin2\gamma_4 x_8 \\ sin2\gamma_4 x_7 + cos2\gamma_4 x_8 \end{bmatrix} = s(x).$$

Therefore,

$$\hat{t} = (-\gamma_1/\pi, \gamma_2/\pi, \gamma_3/\pi, \gamma_4/\pi). \tag{4.7.2}$$

If $t_1 = (\theta_1/\pi, 0, 0, 0)$ then a direct computation gives $t_1 = s_a s_b$, with $a = sin\theta_1 v_1 - cos\theta_1 v_2$ and $b = v_2$. We now calculate t_2 and t_3. Recall that t_1 in matrix notation is an 8×8 matrix consisting of four 2×2 identity diagonal blocks, the first block being

$$\begin{bmatrix} cos2\theta_1 & -sin2\theta_1 \\ sin2\theta_1 & cos2\theta_1 \end{bmatrix}$$

and 2×2 identity blocks in the next three diagonal positions. So in order to calculate t_2 and t_3 we just evaluate these on the basis vectors, look at the matrices and get the parameters. We have,

$$\begin{aligned}
l_a l_{\bar{b}}(v_1) &= a\bar{b} = (sin\theta_1 v_1 - cos\theta_1 v_2)(-v_2) = -cos\theta_1 v_1 - sin\theta_1 v_2 \\
l_a l_{\bar{b}}(v_2) &= a(\bar{b}v_2) = -a(v_2^2) = sin\theta_1 v_1 - cos\theta_1 v_2 \\
l_a l_{\bar{b}}(v_3) &= a(\overline{v_2}v_3) = -av_4 = -cos\theta_1 v_3 - sin\theta_1 v_4 \\
l_a l_{\bar{b}}(v_4) &= -a(v_2 v_4) = av_3 = sin\theta_1 v_3 - cos\theta_1 v_4 \\
l_a l_{\bar{b}}(v_5) &= -a(v_2 v_5) = av_6 = -cos\theta_1 v_5 - sin\theta_1 v_6 \\
l_a l_{\bar{b}}(v_6) &= -a(v_2 v_6) = av_5 = sin\theta_1 v_5 - cos\theta_1 v_6 \\
l_a l_{\bar{b}}(v_7) &= -a(v_2 v_7) = av_8 = -cos\theta_1 v_7 + sin\theta_1 v_8 \\
l_a l_{\bar{b}}(v_8) &= -a(v_2 v_8) = -a(v_7) = -sin\theta_1 v_7 - cos\theta_1 v_8.
\end{aligned}$$

This gives us t_2. Next we compute t_3 as:

$$\begin{aligned}
r_a r_{\bar{b}}(v_1) &= -v_2 a = -cos\theta_1 v_1 - sin\theta_1 v_2 \\
r_a r_{\bar{b}}(v_2) &= -v_2^2 a = sin\theta_1 v_1 - cos\theta_1 v_2 \\
r_a r_{\bar{b}}(v_3) &= -(v_3 v_2)a = v_4 a = -cos\theta_1 v_3 + sin\theta_1 v_4 \\
r_a r_{\bar{b}}(v_4) &= -(v_4 v_2)a = -v_3 a = -sin\theta_1 v_3 - cos\theta_1 v_4 \\
r_a r_{\bar{b}}(v_5) &= -(v_5 v_2)a = v_6 a = -cos\theta_1 v_5 + sin\theta_1 v_6 \\
r_a r_{\bar{b}}(v_6) &= -(v_6 v_2)a = -v_5 a = -sin\theta_1 v_5 - cos\theta_1 v_6 \\
r_a r_{\bar{b}}(v_7) &= -(v_7 v_2)a - v_8 a = -cos\theta_1 v_7 - sin\theta_1 v_8 \\
r_a r_{\bar{b}}(v_8) &= -(v_8 v_2)a = v_7 a = sin\theta_1 v_7 - cos\theta_1 v_8
\end{aligned}$$

So t_1, t_2, t_3 in their possible parametric forms are given as follows:

$$t_1 = (\theta_1/\pi, 0, 0, 0)$$
$$t_2 = ((\pi+\theta_1)/2\pi, (\pi+\theta_1)/2\pi, (\pi+\theta_1)/2\pi, -(\pi+\theta_1)/2\pi)$$
$$t_3 = ((\pi+\theta_1)/2\pi, -(\pi+\theta_1)/2\pi, -(\pi+\theta_1)/2\pi, (\pi+\theta_1)/2\pi)$$

$$t_1 = (0, \theta_2/\pi, 0, 0)$$
$$t_2 = ((\pi+\theta_2)/2\pi, (\pi+\theta_2)/2\pi, -(\pi+\theta_2)/2\pi, (\pi+\theta_2)/2\pi)$$
$$t_3 = (-(\pi+\theta_2)/2\pi, (\pi+\theta_2)/2\pi, -(\pi+\theta_2)/2\pi, (\pi+\theta_2)/2\pi)$$

$$t_1 = (0, 0, \theta_3/\pi, 0)$$
$$t_2 = ((\pi+\theta_3)/2\pi, -(\pi+\theta_3)/2\pi, (\pi+\theta_3)/2\pi, (\pi+\theta_3)/2\pi)$$
$$t_3 = (-(\pi+\theta_3)/2\pi, -(\pi+\theta_3)/2\pi, (\pi+\theta_3)/2\pi, (\pi+\theta_3)/2\pi)$$

$$t_1 = (0, 0, 0, \theta_4/\pi)$$
$$t_2 = (-(\pi+\theta_4)/2\pi, (\pi+\theta_4)/2\pi, (\pi+\theta_4)/2\pi, (\pi+\theta_4)/2\pi)$$
$$t_3 = ((\pi+\theta_4)/2\pi, (\pi+\theta_4)/2\pi, (\pi+\theta_4)/2\pi, (\pi+\theta_4)/2\pi)$$

Therefore in general we have,

$$\begin{aligned} t_1 &= (\theta_1/\pi, \theta_2/\pi, \theta_3/\pi, \theta_4/\pi) \\ t_2 &= ((\theta_1+\theta_2+\theta_3-\theta_4)/2\pi, (\theta_1+\theta_2-\theta_3+\theta_4)/2\pi, \\ &\qquad (\theta_1-\theta_2+\theta_3+\theta_4)/2\pi, (-\theta_1+\theta_2+\theta_3+\theta_4)/2\pi) \\ t_3 &= ((\theta_1-\theta_2-\theta_3+\theta_4)/2\pi, (-\theta_1+\theta_2-\theta_3+\theta_4)/2\pi, \\ &\qquad (-\theta_1-\theta_2+\theta_3+\theta_4)/2\pi, (\theta_1+\theta_2+\theta_3+\theta_4)/2\pi) \end{aligned}$$

We record the above set of equations as $(*)$. These parameters are written modulo $\mathbb{Z}$. Now we analyse all the possibilities for θ_i's to compute the non conjugate isotropy classes.

Case1:(At least one θ_i is 0 or 1/2)

(a) If $\theta_i = 0 \quad \forall i$, then by $(*)$, $t_1 = t_2 = t_3 = (0,0,0,0)$ and hence $W_t = W$.

(b) If $\theta_i/\pi = 1/2 \quad \forall i$, then by $(*)$, we have,

$t_1 = t_2 = (1/2, 1/2, 1/2, 1/2)$ and $t_3 = (0,0,0,0)$. Note that only τ_3 from $S_3 = Out(Spin(N))$ occurs in the stabilizer since it leaves t stable and any other element from S_3 brings t_3 in the first place from which we cannot get back t_1 by the action of any element from $WSpin(N)$ (see 4.7.1, 4.7.2). Thus $W_t = ((\mathbb{Z}/2)^3 \rtimes S_4) \rtimes \{1, \tau_3\}$.

(c) If $t_1 = (0, 0, 0, 1/2)$, then by $(*)$,

$$t_2 = (-1/4, 1/4, 1/4, 1/4)$$
$$t_3 = (1/4, 1/4, 1/4, 1/4)$$

Note here that $\tau_1(t) = t$ and hence $\tau_1 \in W_t$ and no other element from S_3 can occur because t_1 has 0's as parameters but t_2, t_3 do not (see 4.7.1, 4.7.2). Hence $W_t = ((\mathbb{Z}/2)^2 \rtimes S_3) \rtimes \{1, \tau_1\}$.
(d) If $t_1 = (1/2, 1/2, 1/2, 0)$, then by $(*)$,

$$t_2 = (3/4, 1/4, 1/4, 1/4)$$
$$t_3 = (3/4, 3/4, 3/4, 1/4)$$

Here $W_t = ((\mathbb{Z}/2)^2 \rtimes S_3)$, because any element from $Out(Spin(N))$ will alter t_2, t_3 and as a result we cannot get back t by a subsequent action of $WSpin(N)$ (see 4.7.1, 4.7.2).
(e) If $t_1 = (0, 0, 1/2, 1/2)$ then by $(*)$, $t_1 = t_2 = t_3$ and the isotropy is $((\mathbb{Z}/2 \rtimes S_2) \times (\mathbb{Z}/2)) \rtimes S_2) \rtimes S_3$.
(f) If $t_1 = (0, 0, 0, \theta_4/\pi)$ with $\theta_4/\pi \neq 0, 1/2$ then by $(*)$,

$$t_2 = (-\theta_4/2\pi, \theta_4/2\pi, \theta_4/2\pi, \theta_4/2\pi)$$
$$t_3 = (\theta_4/2\pi, \theta_4/2\pi, \theta_4/2\pi, \theta_4/2\pi).$$

In this case apart from τ_1 no other element from S_3 can contribute to the isotropy since t_1 contains 0 and t_2, t_3 do not (see 4.7.1. 4.7.2). So $W_t = ((\mathbb{Z}/2)^2 \rtimes S_3) \rtimes \{1, \tau_1\}$, being same as case (**c**).
(g) If $t_1 = (1/2, 1/2, 1/2, \theta_4/\pi)$, then by $(*)$,

$$t_2 = (3/4 - \theta_4/2\pi, 1/4 + \theta_4/2\pi, 1/4 + \theta_4/2\pi, 1/4 + \theta_4/2\pi)$$
$$t_3 = (-1/4 + \theta_4/2\pi, -1/4 + \theta_4/2\pi, -1/4 + \theta_4/2\pi, 3/4 + \theta_4/2\pi)$$

Here, just as in **(d)**, we have $W_t = ((\mathbb{Z}/2)^2 \rtimes S_3) \subset Spin(N)$.
(h) If $t_1 = (0, 0, \theta/\pi, \theta/\pi)$, then by $(*)$, $t_1 = t_2 = t_3$.

Clearly here, the whole of S_3 leaves t stable (by 4.7.1, 4.7.2) and hence $W_t = ((\mathbb{Z}/2 \rtimes S_2) \times S_2) \rtimes S_3$.
(i) If $t_1 = (1/2, 1/2, \theta/\pi, \theta/\pi)$, then by $(*)$,

$$t_2 = (1/2, 1/2, \theta/\pi, \theta/\pi)$$
$$t_3 = (0, 0, 1/2 + \theta/\pi, 1/2 + \theta/\pi)$$

Now $(t_1, t_2, t_3) = \tau_2(s_1, s_2, s_3) = (s_3, \hat{s_2}, s_1)$, (by 4.7.2) where,

$$s_1 = (0, 0, 1/2 + \theta/\pi, 1/2 + \theta/\pi)$$
$$s_2 = (1/2, 1/2, \theta/\pi, \theta/\pi)$$
$$s_3 = (1/2, 1/2, \theta/\pi, \theta/\pi).$$

If $s = (s_1, s_2, s_3)$, W_t is conjugate to W_s in W. Since any element of S_3 other than τ_1 removes s_1 from the first position, τ_1 is the only element from S_3 which contributes to the isotropy of s (see 4.7.1) Hence $W_s = ((\mathbb{Z}/2 \rtimes S_2) \times S_2) \rtimes \{1, \tau_1\}$.

(j) If $t_1 = (0, \theta/\pi, \theta/\pi, \theta/\pi)$, then by $(*)$,

$$t_2 = (\theta/2\pi, \theta/2\pi, \theta/2\pi, 3\theta/2\pi)$$
$$t_3 = (-\theta/2\pi, \theta/2\pi, \theta/2\pi, 3\theta/2\pi)$$

Here $\tau_1(t) = t$ and no other element from $S_3 = Out(Spin(N))$ can contribute to the isotropy, since t_1 has a 0 and $\hat{t_2} = t_3$ (4.7.1. 4.7.2). Thus $W_t = S_3 \rtimes \{1, \tau_1\}$.

(k) If $t_1 = (1/2, \theta/\pi, \theta/\pi, \theta/\pi)$, then by $(*)$,

$$t_2 = (1/4 + \theta/2\pi, 1/4 + \theta/2\pi, 1/4 + \theta/2\pi, -1/4 + 3\theta/2\pi)$$
$$t_3 = (1/4 - \theta/2\pi, -1/4 + \theta/2\pi, -1/4 + \theta/2\pi, 1/4 + 3\theta/2\pi).$$

Here, $\theta/\pi \neq 0, 1/2$. Therefore t_2, t_3 does not contain 0 or 1/2 as parameters. Hence, $\tau_2, \tau_3 \in S_3$ does not contribute to the isotropy. As $t_2 \neq \hat{t_3}$, $\tau_1 \in S_3$ cannot belong to the isotropy (see 4.7.1, 4.7.2). Therefore, $W_t = S_3 \subset WSpin(N)$.

(l) If $t_1 = (0, 0, \theta_3/\pi, \theta_4/\pi)$, then by $(*)$,

$$t_2 = ((\theta_3 - \theta_4)/2\pi, (-\theta_3 + \theta_4)/2\pi, (\theta_3 + \theta_4)/2\pi, (\theta_3 + \theta_4)/2\pi)$$
$$t_3 = ((-\theta_3 + \theta_4)/2\pi, (-\theta_3 + \theta_4)/2\pi, (\theta_3 + \theta_4)/2\pi, (\theta_3 + \theta_4)/2\pi)$$

We assume here $\theta_3/\pi \neq \theta_4/\pi$ modulo $\mathbb{Z}$. Therefore 0 does not occur in t_2 and t_3, so the only non trivial element from S_3 which lies in the isotropy is τ_1 (see 4.7.1, 4.7.2). Thus, $W_t = (\mathbb{Z}/2 \rtimes S_2) \rtimes \{1, \tau_1\}$

(m) If $t_1 = (1/2, 1/2, \theta_3/\pi, \theta_4/\pi)$, then by $(*)$,

$$t_2 = (1/2 + (\theta_3 - \theta_4)/2\pi, 1/2 + (\theta_4 - \theta_3)/2\pi, (\theta_3 + \theta_4)/2\pi, (\theta_3 + \theta_4)/2\pi)$$
$$t_3 = ((\theta_4 - \theta_3)/2\pi, (\theta_4 - \theta_3)/2\pi, 1/2 + (\theta_3 + \theta_4)/2\pi, 1/2 + (\theta_3 + \theta_4)/2\pi)$$

Here $\hat{t_3} \neq t_2$ and $\hat{t_2} \neq t_3$ and t_1, contains 1/2 as a parameter. So $S_3 = Out(Spin(N))$ does not contribute to the isotropy (see 4.7.1, 4.7.2). Hence $W_t = \mathbb{Z}/2 \rtimes S(2)$.

(n) If $t_1 = (0, \theta/\pi, \theta/\pi, \theta_4/\pi)$, then by $(*)$,

$$t_2 = ((2\theta - \theta_4)/2\pi, \theta_4/2\pi, \theta_4/2\pi, (2\theta + \theta_4)/2\pi)$$
$$t_3 = ((-2\theta + \theta_4)/2\pi, \theta_4/2\pi, \theta_4/2\pi, (2\theta + \theta_4)/2\pi).$$

We have $W_t = S_2 \rtimes \{1, \tau_1\}$ in this case, because again $\hat{t_2} = t_3$ and $\hat{t_3} = t_2$. And if $\theta/\pi = \theta_4/2\pi$, we have by $(*)$, $t_1 = t_2 = t_3$ and $W_t = S_2 \rtimes S_3$ (see 4.7.1, 4.7.2).
(o) If $t_1 = (1/2, \theta/\pi, \theta/\pi, \theta_4/\pi)$, then by $(*)$,

$$t_2 = (1/4 + (2\theta - \theta_4)/2\pi, 1/4 + \theta_4/2\pi, 1/4 + \theta_4/2\pi, -1/4 + (2\theta + \theta_4)/2\pi)$$
$$t_3 = (1/4 + (-2\theta + \theta_4)/2\pi, -1/4 + \theta_4/2\pi, -1/4 + \theta_4/2\pi, 1/4 + (2\theta + \theta_4)/2\pi).$$

Here $W_t = S_2 \subset WSpin(N)$ because no element from S_3 can contribute to the isotropy of this element, as we have taken $\theta/\pi \neq \theta_4/\pi$ and hence $1/2$ does not occur in t_2 and t_3 (see 4.7.1, 4.7.2).
(p) If $t_1 = (0, \theta_2/\pi, \theta_3/\pi, \theta_4/\pi)$, then by $(*)$,

$$t_2 = ((\theta_2 + \theta_3 - \theta_4)/2\pi, (\theta_2 - \theta_3 + \theta_4)/2\pi, (-\theta_2 + \theta_3 + \theta_4)/2\pi, (\theta_2 + \theta_3 + \theta_4)/2\pi)$$
$$t_3 = ((-\theta_2 - \theta_3 + \theta_4)/2\pi, (\theta_2 - \theta_3 + \theta_4)/2\pi, (-\theta_2 + \theta_3 + \theta_4)/2\pi, (\theta_2 + \theta_3 + \theta_4)/2\pi).$$

If none of the coordinates in t_2, t_3 are $0, 1/2$ then $W_t = \{1, \tau_1\}$, otherwise the only non trivial possibility is $W_t = S_3 \subset WSpin(N)$, which occurs if $(\theta_2 + \theta_3)/\pi = \theta_4/\pi$, in which case $t_1 = t_2 = t_3$ holds by $(*)$ (refer to 4.7.1, 4.7.2).
Case 2:(no θ_i in t_1 are $0, 1/2$) Here, however the isotropy subgroups for various possibilities for θ_i are conjugate to certain subgroups already occurring in Case 1, except the situation when all θ_i's are distinct, which yields the trivial isotropy subgroup.
(a) If $t_1 = (\theta/\pi, \theta/\pi, \theta/\pi, \theta/\pi)$, then by $(*)$,

$$t_2 = (\theta/\pi, \theta/\pi, \theta/\pi, \theta/\pi)$$
$$t_3 = (0, 0, 0, 2\theta/\pi)$$

Then clearly $W_t = S_4 \rtimes \{1, \tau_3\}$ since τ_3 contributes to the isotropy from S_3 (see 4.7.1, 4.7.2) and this isotropy is conjugate to that in case 1(c).
(b) If $t_1 = (\theta_1/\pi . \theta_1/\pi, \theta_2/\pi, \theta_2/\pi)$, then by $(*)$,

$$t_2 = (\theta_1/\pi . \theta_1/\pi, \theta_2/\pi, \theta_2/\pi)$$
$$t_3 = (0, 0, (\theta_2 - \theta_1)/\pi, (\theta_1 + \theta_2)/\pi).$$

Note that, $(t_1, t_2, t_3) = \tau_2(s_1, s_2, s_3)$, where,

$$\begin{aligned} s_1 &= (0, 0, (\theta_2 - \theta_1)/\pi, (\theta_1 + \theta_2)/\pi) \\ s_2 &= (-\theta_1/\pi.\theta_1/\pi, \theta_2/\pi, \theta_2/\pi) \\ s_3 &= (\theta_1/\pi, \theta_1/\pi, \theta_2/\pi, \theta_2/\pi) \end{aligned}$$

which case has already been considered before (case 1(l)).

(c) If

$t_1 = (\theta_1/\pi, \theta_1/\pi, \theta_3/\pi, \theta_4/\pi)$, then by $(*)$,

$$\begin{aligned} t_2 &= ((2\theta_1 + \theta_3 - \theta_4)/2\pi, (2\theta_1 - \theta_3 + \theta_4)/2\pi, (\theta_3 + \theta_4)/2\pi, (\theta_3 + \theta_4)/2\pi) \\ t_3 &= ((\theta_4 - \theta_3)/2\pi, (\theta_4 - \theta_3)/2\pi, (-2\theta_1 + \theta_3 + \theta_4)/2\pi, (2\theta_1 + \theta_3 + \theta_4)/2\pi) \end{aligned}$$

If $\theta_1/\pi \neq (\theta_3 + \theta_4)/2\pi$ or $\theta_1/\pi \neq (\theta_4 - \theta_3)/2\pi$ modulo $\mathbb{Z}$, then $W_t = S_2$(which has already occurred in case (o) of case 1). If θ_1/π is equal to any one of the above two elements (modulo $\mathbb{Z}$) then t_2 or t_3 has 0 as one of it's co-ordinates. Accordingly t_2 or t_3 can be brought to the first position of the related triple (see 4.7.1). Note that for all related triples (t_1, t_2, t_3) such that t_1 has at least one 0 as a parameter, the isotropy subgroups have been computed in Case 1. Hence, this does not give us any new isotropy subgroup.

Now we consider (t_1, t_2, t_3) such that t_i has all the parameters distinct and not equal to zero. For this situation we record the following lemmas.

Lemma 4.7.6. *If $t_i \in SO(N)$ does not have any of the parameters equal to zero, then $\mathfrak{C}^{t_i} = \{0\}$.*

Proof. Let $x \in \mathfrak{C}^{t_i}$ with $x \neq 0$ for some i. Without loss of generality we can assume that $x_1 \neq 0$, where x_1 denotes the first coordinate of x with respect to the chosen basis $\mathfrak{B} = \{v_1, ..., v_8\}$. Hence the first 2×2 block

$$\begin{bmatrix} cos2\theta_1 & -sin2\theta_1 \\ sin2\theta_1 & cos2\theta_1 \end{bmatrix}$$

of t_1 has a non zero eigenvector (x_1, x_2) which implies that $\theta_1/\pi = 0$, which is a contradiction to the assumption that no parameter of t_i is 0. □

An element x in a connected group G is called strongly regular if $Z_G(t) = T$.

Lemma 4.7.7. *If $t_1 \in SO(N)$ be strongly regular then (t_1, t_2, t_3) is strongly regular in $Spin(N)$.*

Proof. Let $t_1 \in SO(N)$ be strongly regular and $T \subset SO(N)$ be the maximal torus containing t_1. Then $Z_{SO(N)}(t_1) = T$. Let $s = (s_1, s_2, s_3) \in Spin(N)$ and $st = ts$. Therefore,
$s_1t_1 = t_1s_1 \Rightarrow s_1 \in T \Rightarrow s_2, s_3 \in T \Rightarrow (s_1, s_2, s_3) \in \widetilde{T}$ (by Lemma 4.7.3) $\Rightarrow Z_{Spin(N)}(t) = \widetilde{T}$. Hence (t_1, t_2, t_3) is strongly regular in $Spin(N)$.

□

Theorem 4.7.8. *If t_i does not have any parameter equal to* 0, *and all parameters in t_i are distinct, $1 \le i \le 3$, then (t_1, t_2, t_3) is strongly regular in F_4 and hence $W_t = \{1\}$.*

Proof. Since t_i does not have 0 for all i, by Lemma 4.7.6, $\mathfrak{C}^{t_i} = \{0\}$ $\forall i$. Hence by this and the remark preceding Proposition 8.2, $A^t = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. So if $\phi \in Z_{F_4}(t)$, then $\phi(\mathbb{R} \times \mathbb{R} \times \mathbb{R}) = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$
$\Rightarrow \phi \in Aut(A, \mathbb{R} \times \mathbb{R} \times \mathbb{R}) \cong Spin(N) \rtimes S_3$(by [**J**], Theorem 8.)
$\Rightarrow Z_{F_4}(t) \subset Spin(N) \rtimes S_3$
$\Rightarrow Z_{F_4}(t) \subset Spin(N)$ (since F_4 is simply connected, $Z_{F_4}(t)$ is connected by Proposition 4.2.1).
$\Rightarrow Z_{F_4}(t) \subset Z_{Spin(N)}(t)$.

Since all parameters of t_1 are distinct and none of them is 0, the isotropy subgroup of t_1 in $WSO(N)$ is trivial. Note that $WSO(N)_{t_1} = Z_{SO(N)}(t_1)/T$, where T is the diagonal maximal torus in $SO(N)$. Therefore, $WSO(N)_{t_1} = \{1\} \Rightarrow Z_{SO(N)}(t_1) = T$, which means t_1 is strongly regular in $SO(N)$. Hence by Lemma 4.7.7, $t = (t_1, t_2, t_3)$ is strongly regular in $Spin(N)$. Therefore, $Z_{F_4}(t) \subset Z_{Spin(N)}(t) = \widetilde{T}$. This is in fact an equality since, $\widetilde{T} \subset Z_{F_4}(t)$ for all $t \in \widetilde{T}$. Thus t is strongly regular in F_4. □

We now proceed to calculate the semisimple genus number of a connected algebraic group of type F_4 over an algebraically closed field k of characteristic different from 2. Let $\mathfrak{C}$ and $\mathbb{H}$ be respectively the (split) octonion and quaternion algebras over k, i.e. $\mathfrak{C} := \mathbb{H} \oplus \mathbb{H}$, where

$$\mathbb{H} := \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in k\},$$

under the usual matrix addition and multiplication with the norm $N : H \to k$, defined as $N(x) = det(x)$, for $x \in \mathbb{H}$. The norm for $\mathfrak{C}$ is given by $N((x, y)) = det(x) - det(y)$, for $x, y \in \mathbb{H}$. The conjugation in $\mathbb{H}$ is given by

$$\overline{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The multiplication and conjugation in $\mathfrak{C}$ are as follows:

$$(x, y)(u, v) := (xu + \bar{v}y, vx + y\bar{u}),$$
$$\overline{(x, y)} := (\bar{x}, -y),$$

where $x, y, u, v \in \mathbb{H}$.
We consider the following basis $\{v_1, ..., v_8\}$ of $\mathfrak{C}$:-

$$v_1 = (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 0),\ v_2 = (\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, 0),\ v_3 = (0, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}),\ v_4 = (0, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}),$$
$$v_5 = (0, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}),\ v_6 = (0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}),\ v_7 = (\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, 0),\ v_8 = (\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, 0).$$

The multiplication table for $\mathfrak{C}$ with respect to this basis is:

$\cdot$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_1	v_1	v_2	v_3	0	v_5	0	0	0
v_2	0	0	v_4	0	$-v_6$	0	$-v_1$	v_2
v_3	0	$-v_4$	0	0	v_7	$-v_1$	0	v_3
v_4	v_4	0	0	0	$-v_8$	$-v_2$	v_3	0
v_5	0	v_6	$-v_7$	$-v_1$	0	0	0	v_5
v_6	v_6	0	$-v_8$	v_2	0	0	$-v_5$	0
v_7	v_7	$-v_8$	0	$-v_3$	0	v_5	0	0
v_8	0	0	0	v_4	0	v_6	v_7	v_8

With respect to the above basis of $\mathfrak{C}$ the matrix of the bilinear form for the norm N is

$$\begin{bmatrix} & & 1 \\ & \ddots & \\ 1 & & \end{bmatrix}$$

and

$$T := \{diag(a, b, c, d, 1/d, 1/c, 1/b, 1/a) \in SO(N) | a, b, c, d \in k^*\} \subset SO(N)$$

is a maximal torus. With the notation used for compact F_4, any element of $Spin(N)$ corresponds uniquely to $(t_1, t_2, t_3) \in SO(N)^3$ such that $t_1(xy) = t_2(x)t_3(y)$ for all $x, y \in \mathfrak{C}$.

Let $t_1 = diag(a, b, c, d, 1/d, 1/c, 1/b, 1/a) \in T$. We can write $t_1 = s_{x_1}s_{y_1}...s_{x_4}s_{y_4}$, where s_{x_i} denotes the reflection in the hyperplane perpendicular to x_i and

$$x_1 = \sqrt{a}v_1 + \sqrt{a}^{-1}v_8,\ y_1 = v_1 + v_8,\ x_2 = \sqrt{b}v_2 + \sqrt{b}^{-1}v_7,\ y_2 = v_2 + v_7$$
$$x_3 = \sqrt{c}v_3 + \sqrt{c}^{-1}v_6,\ y_3 = v_3 + v_6,\ x_4 = \sqrt{d}v_4 + \sqrt{d}^{-1}v_5,\ y_4 = v_4 + v_5.$$

Therefore, by Proposition 7.1, the corresponding t_2, t_3 are given by $t_2 = l_{x_1}l_{\bar{y_1}}...l_{x_4}l_{\bar{y_4}}$ and
$t_3 = r_{x_1}r_{\bar{y_1}}...r_{x_4}r_{\bar{y_4}}$. So if we calculate t_2 and t_3 using these formulas and the above multiplication table we get (henceforth we shall denote an 8×8 diagonal matrix of the form $diag(a, b, c, d, 1/d, 1/c, 1/b, 1/a)$ by (a, b, c, d)),

$$t_1 = (a, b, c, d),$$
$$t_2 = (\sqrt{a}\sqrt{b}\sqrt{c}/\sqrt{d}, \sqrt{a}\sqrt{b}\sqrt{d}/\sqrt{c}, \sqrt{a}\sqrt{c}\sqrt{d}/\sqrt{b}, \sqrt{b}\sqrt{c}\sqrt{d}/\sqrt{a}),$$
$$t_3 = (\sqrt{a}\sqrt{d}/\sqrt{b}\sqrt{c}, \sqrt{b}\sqrt{d}/\sqrt{a}\sqrt{c}, \sqrt{c}\sqrt{d}/\sqrt{a}\sqrt{b}, \sqrt{a}\sqrt{b}\sqrt{c}\sqrt{d}).$$

Let us denote the above equations by $(**)$.

Now we can compute the isotropy classes in the Weyl group with respect to a maximal torus in F_4. Let T denote the diagonal maximal torus in $SO(N)$. Since any a maximal torus of F_4 sits inside a copy of $Spin(N) \subset F_4$, we may work with $\widetilde{T} := \{(t_1, t_2, t_3) \in T^3 | t_1(xy) = t_2(x)t_3(y), \forall x, y \in T\} \subset RT(\mathfrak{C}) \cong Spin(N)$.

With this we can compute the isotropy subgroups of the Weyl group (the action of the Weyl group on the torus had already been discussed before and we shall follow the same notations here). Recall that $W = ((\mathbb{Z}/2)^3 \rtimes S_4) \rtimes S_3$ is the Weyl group of F_4. In all the following cases the arguments for W_t are exactly similar to the ones we had in the case for compact F_4, only the roles played by 0 and $1/2$ are replaced by 1 and -1 respectively. With each of the following possibilities we refer to the corresponding calculation done in the discussion on compact F_4. In what follows, we denote a fixed square root of -1 by i.

1. $t_1 = (1, 1, 1, 1) = t_2 = t_3$. In this situation clearly $W_t = W$ (case 1(a)).

2.

$$t_1 = t_2 = (-1, -1, -1, -1)$$
$$t_3 = (1, 1, 1, 1)$$

$W_t = ((\mathbb{Z}/2)^3 \rtimes S_4) \rtimes \{1, \tau_3\}$ (case 1(b)).

3.

$$t_1 = (1, 1, 1, -1)$$
$$t_2 = (-i, i, i, i)$$
$$t_3 = (i, i, i, i)$$

$W_t = ((\mathbb{Z}/2)^2 \rtimes S_3) \rtimes \{1, \tau_1\}$ (case 1(c)).

4.

$$t_1 = t_2 = t_3 = (1, 1, -1, -1)$$

Note that all elements of S_3 fix this element t and hence we have $W_t = (((\mathbb{Z}/2) \rtimes S_2) \times ((\mathbb{Z}/2) \rtimes S_2)) \rtimes S_3$ (case 1(e)).

5.

$$t_1 = (-1, -1, -1, 1)$$
$$t_2 = (-i, i, i, i)$$
$$t_3 = (-i, -i, -i, -i)$$

Clearly no element from S_3 can belong to the isotropy, therefore $W_t = (\mathbb{Z}/2)^2 \rtimes S_3$. (case 1(d)).

6.$t_1 = t_2 = t_3 = (1, 1, c, c)$, where $c \neq 1, -1$. Since any S_3 element leaves this fixed, we have $W_t = ((\mathbb{Z}/2 \rtimes S_2) \times S_2) \rtimes S_3$ (case 1(h)).

7.

$$t_1 = t_2 = (-1, -1, c, c)$$
$$t_3 = (1, 1, c, c)$$

Here we observe that only $\tau_3 \in S_3$ can contribute to the isotropy. Hence $W_t = ((\mathbb{Z}/2 \rtimes S_2) \times S_2) \rtimes \{1, \tau_3\}$ (case 1(i)).

8.

$$t_1 = (1, b, b, b)$$
$$t_2 = (\sqrt{b}, \sqrt{b}, \sqrt{b}, b\sqrt{b})$$
$$t_3 = (1/\sqrt{b}, \sqrt{b}, \sqrt{b}, b\sqrt{b})$$

For this $W_t = S_3 \rtimes \{1, \tau_1\}$ (case 1(j)).

9.

$$t_1 = (-1, b, b, b)$$
$$t_2 = (i\sqrt{b}, i\sqrt{b}, i\sqrt{b}, -ib\sqrt{b})$$
$$t_3 = (i/\sqrt{b}, -i\sqrt{b}, -i\sqrt{b}, ib\sqrt{b})$$

where $b \neq 1, -1$. $W_t = S_3$ (case 1(k)).

10.

$$t_1 = (1, 1, c, d)$$
$$t_2 = (\sqrt{c}/\sqrt{d}, \sqrt{d}/\sqrt{c}, \sqrt{c}\sqrt{d}, \sqrt{c}\sqrt{d})$$
$$t_3 = (\sqrt{d}/\sqrt{c}, \sqrt{d}/\sqrt{c}, \sqrt{c}\sqrt{d}, \sqrt{c}\sqrt{d})$$

$W_t = (\mathbb{Z}/2 \rtimes S_2) \rtimes \{1, \tau_1\}$ (case 1(l)).

11.

$$t_1 = (-1, -1, c, d)$$
$$t_2 = (-\sqrt{c}/\sqrt{d}, -\sqrt{d}/\sqrt{c}, \sqrt{c}\sqrt{d}, \sqrt{c}\sqrt{d})$$
$$t_3 = (\sqrt{d}/\sqrt{c}, \sqrt{d}/\sqrt{c}, -\sqrt{c}\sqrt{d}, -\sqrt{c}\sqrt{d})$$

$W_t = \mathbb{Z}/2 \rtimes S_2$ (case 1(m)).

12.

$$t_1 = (1, b, b, d)$$
$$t_2 = (b/\sqrt{d}, \sqrt{d}, \sqrt{d}, b\sqrt{d})$$
$$t_3 = (\sqrt{d}/b, \sqrt{d}, \sqrt{d}, b\sqrt{d})$$

$W_t = S_2 \rtimes \{1, \tau_1\}$ and if $b = \sqrt{d}$, we have $t_1 = t_2 = t_3$ and hence $W_t = S_2 \rtimes S_3$ (case 1(n)).

13.

$$t_1 = (1, b, c, d)$$
$$t_2 = (\sqrt{b}\sqrt{c}/\sqrt{d}, \sqrt{b}\sqrt{d}/\sqrt{c}, \sqrt{c}\sqrt{d}/\sqrt{b}, \sqrt{b}\sqrt{c}\sqrt{d})$$
$$t_3 = (\sqrt{d}/\sqrt{b}\sqrt{c}, \sqrt{b}\sqrt{d}/\sqrt{c}, \sqrt{c}\sqrt{d}/\sqrt{b}, \sqrt{b}\sqrt{c}\sqrt{d})$$

$W_t = \{1, \tau_1\}$ and if $\sqrt{b}\sqrt{c} = \sqrt{d}$ then $t_1 = t_2 = t_3$ and $W_t = S_3$ (case 1(p)).

14.

$$t_1 = (-1, b, b, d)$$
$$t_2 = (ib/\sqrt{d}, i\sqrt{d}, i\sqrt{d}, -ib\sqrt{d})$$
$$t_3 = (i\sqrt{d}/b, -i\sqrt{d}, -i\sqrt{d}, ib\sqrt{d})$$

$W_t = S_2$ (case 1(o)).

Next we consider (t_1, t_2, t_3) such that none of the coordinates have 1 as a parameter and all parameters of t_i are distinct. Since we are over an algebraically closed field k, Theorem 4.7.8 holds in this case with the following modification:

Theorem 4.7.9. *If t_i does not have* 1 *as a parameter and all parameters in t_i are distinct, $1 \leq i \leq 3$, then (t_1, t_2, t_3) is strongly regular in F_4.*

PROOF. Note that with the hypothesis on t_i, $\mathfrak{C}^{t_i} = \{0\}$ for all i. For if not, let $x(\neq 0) \in \mathfrak{C}^{t_i}$ for some i. Then $t_i(x) = x \Rightarrow$ some parameter of t_i is 1 since x is assumed to be non zero, a contradiction. Also note that Lemma 4.7.7 holds in this case too. The rest of the proof is the same as that of Theorem 4.7.8, with $\mathbb{R}$ replaced by k. □

We record the above discussion as

Theorem 4.7.10. *The genus number of a compact simply connected Lie group or a simply connected algebraic group over an algebraically closed field, of type F_4 is* 17.

4.8. G_2

Definition 4.8.1. Let $\mathfrak{C}$ denote the octonion division algebra over $\mathbb{R}$. Then $Aut(\mathfrak{C})$ is the compact connected Lie group of type G_2.

Conjugacy classes of centralizers in anisotropic forms of G_2 have been explicitly calculated in [**Si**]. Here we count the number of such classes using a different technique. Consider a maximal torus $T \subset G_2$. Then T sits inside a copy of $SU(3) \subset G_2$. If $K \subset \mathfrak{C}$ be a quadratic extension of $\mathbb{R}$, then $Aut(\mathfrak{C}/K) \cong SU(3)$, where $Aut(\mathfrak{C}/K)$ is the group of automorphisms of $\mathfrak{C}$ fixing K point wise. The Weyl group of G_2 is $WG_2 \cong WSU(3) \rtimes S_2$, note that $S_2 = Out(SU(3))$. Let us consider the diagonal maximal torus T in $SU(3)$ i.e. the one consisting of all diagonal matrices $t = (z_1, z_2, z_3)$, $z_i \in S^1$ and $z_1 z_2 z_3 = 1$. The action of WG_2 on T is given by

$$(\alpha, \beta)(z_1, z_2, z_3) = (\beta z_{\alpha^{-1}(1)}, \beta z_{\alpha^{-1}(2)}, \beta z_{\alpha^{-1}(3)}),$$

where $\alpha \in S_3$, $\beta \in S_2$ and $\beta(z_i) = \overline{z_i}$ for $\beta \neq 1 \in S_2$. With this action, we now consider the various possibilities for an element $diag(z_1, z_2, z_3) \in SU(3)$ and calculate their stabilizers in WG_2.
(a) If $z_1 \neq z_2 \neq z_3, z_i$ then clearly $(WG_2)_t = \{1\}$.
(b) If $z_1 = z_2 = z_3 \in \mathbb{R}$ then $(WG_2)_t = S_3 \rtimes S_2$.
(c) If $z_1 = z_2 = z_3 \in \mathbb{C} - \mathbb{R}$ then $(WG_2)_t = S_3$, since $Out(SU(3))$ acts non trivially.
(d) If $z_1 = z_2 \neq z_3, z_i \in \mathbb{C} - \mathbb{R}$ then $(WG_2)_t = S_2 \subset WSU(3))$ as $Out(SU(3))$ acts non trivially.
(e) If $z_1 = z_2 \neq z_3, z_i \in \mathbb{R}$ then $(WG_2)_t = S_2 \rtimes S_2$ as S_2 leaves this element fixed and $S_2 \subset WSU(3)$ further acts trivially on it.
(f) If $t = (1, exp(i\theta), exp(-i\theta))$ with $\theta \neq k\pi$ for any integer k, then $(WG_2)_t = \{(1,1), (\alpha, \beta)\} \equiv \mathbb{Z}/2$, where $\alpha \in S_3$ is the transposition $(2\ 3)$ and $\beta \in S_2$ is the transposition $(1\ 2)$.

If we consider a connected algebraic group of type G_2 over an algebraically closed field k, the semisimple genus number is the same. In this case, we work with the Zorn matrix model of split octonions and consider $k \times k \subset \mathfrak{C}$ as the diagonal subalgebra. Then $Aut(\mathfrak{C})/(k \times k) \cong SL(3.)$ Consider the diagonal maximal torus $T := \{diag(a_1, a_2, a_3) \in SL(3) | a_1 a_2 a_3 = 1\} \subset SL(3)$, then T is a maximal torus in G_2. The Weyl group G_2 is $WG_2 \cong WSL(3) \rtimes S_2 \cong S_3 \rtimes S_2$. The action of WG_2 on T is given by

$$(\alpha, \beta)(a_1, a_2, a_3) = (\beta a_{\alpha^{-1}(1)}, \beta a_{\alpha^{-1}(2)}, \beta a_{\alpha^{-1}(3)}),$$

where $\alpha \in S_3$, $\beta \in S_2$ and $\beta(a_i) = 1/a_i$ for $\beta \neq 1 \in S_2$. The conjugacy classes of isotropy subgroups of WG_2 are as listed below: (the arguments being same as the previous ones.)
(a) If $a_1 \neq a_2 \neq a_3$,$a_i \neq 1, -1$ and $a_i \neq 1/a_j$ for $i \neq j$, then $(WG_2)_t = \{1\}$
(b) If $a_i = 1$ for all i, with , $W_t = (WG_2)$.
(c) If $a_i = \omega$ for all i,where ω is a cube root of unity other than 1, $(WG_2)_t = S_3$.
(d) If $a_1 = a_2 \neq a_3$ with $a_1 \neq 1, -1$, $(WG_2)_t = S_2$.
(e) If $a_1 = a_2 = 1 = -a_3$ then $(WG_2)_t = S_2 \rtimes S_2$.
(f) If $a_1 = 1, a_2 = 1/a_3$ with $a_2 \neq 1, -1$ then $(WG_2)_t = \{(1,1), (\alpha, \beta)\} \equiv \mathbb{Z}/2$, where $\alpha \in S_3$ is the transposition $(2\ 3)$ and $\beta \in S_2$ is the transposition $(1\ 2)$.

The preceding discussion is recorded as,

Theorem 4.8.2. *The genus number of a compact simply connected Lie group or a simply connected algebraic group over an algebraically closed field, of type G_2 is* 6.

4.9. Computations for the Lie algebras

If G be a compact connected Lie group (or a connected reductive algebraic group over an algebraically closed field) with the Lie algebra denoted by $\mathfrak{g}$, the orbit structure of the action of Ad_G on $\mathfrak{g}$ can be neatly described in terms of the action of WG on the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. In this section we calculate the conjugacy classes of isotropy subgroups of WG with respect to its action on $\mathfrak{t}$. We begin with the following result ;

Theorem 4.9.1. *With respect to the action, $Ad : G \longrightarrow Aut(\mathfrak{g})$ defined by $g \mapsto Ad_g$, where $Ad_g(x) = gxg^{-1}$, (having embedded G in a suitable GL_n) there is a bijection between the conjugacy classes of centralizers of semisimple elements in $\mathfrak{g}$ in G and the conjugacy classes of centralizers of elements of a Cartan subalgebra in WG.*

Proof. Consider the map $[G_x] \mapsto [WG_x]$, where $x \in \mathfrak{t}$. To show this map a bijection we follow exactly the same line of argument as in Theorems 4.2.4 and 4.2.7. □

For determining the stabilizers in the Weyl group we follow the same line of argument as in the case of groups in the previous sections.

4.9.1. A_n**.** When G is the Lie group $SU(n+1)$, the corresponding Lie algebra $\mathfrak{su}(n+1)$ is the set of all $(n+1)\times(n+1)$ trace zero skew-hermitian matrices,while for $G = SL(n+1)$, $\mathfrak{g}$ consists of all trace zero $(n+1)\times(n+1)$ matrices. The Cartan subalgebra in the above cases are given by:

$$\mathfrak{t} = \{(a_1 i, ..., a_{n+1} i) \in \mathbb{M}_n(\mathbb{C}) | a_1 + ... + a_{n+1} = 0\} \subset \mathfrak{su}(n+1)$$

and,

$$\mathfrak{t} = \{(a_1, ..., a_{n+1}) \in \mathbb{M}_n(k) | a_1 + ... + a_{n+1} = 0\} \subset \mathfrak{sl}(n+1).$$

We have $WG = S_{n+1}$ and it acts on $\mathfrak{t}$ by permuting the entries in both cases. Hence by the argument followed in Section 4.3, we see that the number of conjugacy classes of isotropy subgroups is $p(n+1)$. The subgroups are of the form $S_{n_1}...S_{n_k}$ for a partition $(n_1, ..., n_k)$ of $(n+1)$.

4.9.2. B_n**.** For the Lie algebra of type B_n, the Cartan subalgebra $\mathfrak{t}$ consists of all block diagonal matrices of the form $(A_1, ..., A_n, 0)$, where

$$A_i = \begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix}$$

is the $i-$th block with $a_i \in \mathbb{R}$. And for B_n over an algebraically closed field k the Cartan subalgebra consists of all diagonal matrices of the form $(a_1, ..., a_n, -a_1, ..., -a_n, 0)$, where $a_i \in k$. So in either situation we note that the elements of the Cartan subalgebra can be parametrized by the n-tuples $(a_1, ..., a_n)$ with $a_i \in k$. The Weyl group $W = (\mathbb{Z}/2)^n \rtimes S_n$ acts on $\mathfrak{t}$ by permuting the elements , followed by a change of sign.

Let $(n_1, ..., n_k)$ be a partition of n such that n_1 denotes the number of 0's and n_i for $i \neq 1$ denotes the number of equal parameters. For such an element the isotropy subgroup is $((\mathbb{Z}/2)^{n_1-1} \rtimes S_{n_1}) \times S_{n_2} \times ... \times S_{n_k}$ by an argument similar to one seen in Section 4.4. Hence the number of isotropy classes is

$$\sum_{i=0}^{n} p(n-i).$$

4.9.3. C_n**.** The Cartan subalgebra $\mathfrak{t}$ consists of all diagonal matrices of the form $(a_1, ..., a_n, -a_1, ..., -a_n)$ with $a_i \in k$. The Weyl group being the same as that of B_n, we have the same number of isotropy classes in this case also,i.e

$$\sum_{i=0}^{n} p(n-i)$$

.

4.9.4. D_n**.** Here the Cartan subalgebra is same as that of B_n and the Weyl group $W = (\mathbb{Z}/2)^{n-1} \rtimes S_n$ acts on $\mathfrak{t}$ by permuting the parameters and changing the signs of an even number of them.

If n is odd, then for a partition $(n_1, ..., n_k)$ of n, where n_i's are as in Section 9.2, the isotropy subgroup of the Weyl group is $((\mathbb{Z}/2)^{n_1-1} \rtimes S_{n_1}) \times S_{n_2} \times ... \times S_{n_k}$ and hence the total number of isotropy classes is

$$\sum_{i=0}^{n} p(n-i).$$

However if $n = 2k$,then if at least one zero occurs as one of the parameters of $t \in \mathfrak{t}$, then the isotropy subgroup is obtained as above. But if no zero occurs i.e $n_1 = 0$, then for each partition of n containing only even integers we have a isotropy subgroup not conjugate to any one of the above, as we have seen in the group case (see Section 4.6). Thus the total number of isotropy classes for $n = 2k$ is

$$\sum_{i=0}^{n} p(n-i) + p(k).$$

4.9.5. G_2**.** In this case, we consider a subalgebra $\mathfrak{su}(3)$ (over reals) or $\mathfrak{sl}(3)$ (over an algebraically close field k) inside $\mathfrak{g}_2$ and a Cartan subalgebra of $\mathfrak{g}_2$ embeds in one such subalgebra. Hence, each element of the Cartan subalgebra can be considered as all tuples (a_1, a_2, a_3), $a_i \in k$, such that $a_1+a_2+a_3 = 0$. The Weyl group $WG_2 \cong S_3 \rtimes S_2$ (see Section 4.8) acts on these tuples as,

$$(\alpha, \beta)(a_1, a_2, a_3) = (\beta a_{\alpha^{-1}(1)}, \beta a_{\alpha^{-1}(2)}, \beta a_{\alpha^{-1}(3)}),$$

where $\alpha \in S_3$, $\beta \in S_2$ and $\beta(a_i) = -a_i$ for $\beta \neq 1 \in S_2$. Thus we have the following possibilities:
(a) If $t = (0, 0, 0)$ then clearly, $(WG_2)_t = WG_2$.
(b) If $t = (a, a, -2a)$ then $(WG_2)_t = S_2 \subset WSL(3)$ since the other S_2 factor acts non trivially.
(c) If $t = (a, b, -a-b)$ with $a \neq b \neq -(a+b)$, then clearly, $(WG_2)_t = \{1\}$.
(d) If $t = (0, a, -a)$ with $a \neq 0$ then $(WG_2)_t = \{(1, 1), (\alpha, \beta)\} \cong \mathbb{Z}/2$, where $\alpha = (2\ 3) \in S_3$ and $\beta = (1\ 2) \in S_2$.

4.9.6. F_4**.** Here we will use the notations used in Section 4.7. We work with the basis $\{v_1, ..., v_8\}$ of $\mathfrak{C}$. We reorder this basis as $e_1 = v_1, e_2 = v_2, e_3 = v_3, e_4 = v_4, e_5 = v_8, e_6 = v_7, e_7 = v_6, e_8 = v_5$ so that with respect to the new ordered basis $\{e_1, ..., e_8\}$, the matrix of the bilinear form associated with the norm N of $\mathfrak{C}$ becomes

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Also, the Cartan subalgebra of $\mathfrak{so}(N)$ is in the diagonal form with respect to the above bilinear form, i.e. $\mathfrak{t} \subset \mathfrak{so}(N)$ will consist of all diagonal matrices of the form $(a_1, ..., a_4, -a_1, ..., -a_4)$, $a_i \in k$. Henceforth we shall parametrize this diagonal matrix as (a_1, a_2, a_3, a_4), $a_i \in k$. The Cartan subalgebra of $\mathfrak{f}_4$ is contained in a copy of the Lie algebra of $Spin(N)$, i.e. $\mathfrak{spin}(N) \cong \mathbf{L}(RT(\mathfrak{C}))$, where $\mathbf{L}(RT(\mathfrak{C})) = \{(t_1, t_2, t_3) \in \mathfrak{so}(8)^3 | t_1(xy) = t_2(x)y + xt_3(y), x, y \in \mathfrak{C}\}$. It is known that $\mathfrak{so}(N)$ is generated as a vector space by $t_{a,b}$, $a, b \in \mathfrak{C}$; $t_{a,b}$ is defined as $t_{a,b}(x) = \langle x, a\rangle b - \langle x, b\rangle a$ for $x \in \mathfrak{C}$ where $\langle , \rangle$ is the bilinear form of the norm N ([**SV**], Chapter 3).

If $t_1 = t_{a,b}$, then $t_2 = 1/2(l_b l_{\overline{a}} - l_a l_{\overline{b}})$ and $t_3 = 1/2(r_b r_{\overline{a}} - r_a r_{\overline{b}})$ satisfy the property,

$$t_1(xy) = t_2(x)y + xt_3(y). \tag{4.9.1}$$

Also note that if (t_1, t_2, t_3) and (s_1, s_2, s_3) are related triples (in the Lie algebra sense) then so is $(t_1 + s_1, t_2 + s_2, t_3 + s_3)$. With this, we can now carry out the computation.

Let $t_1 = (a_1, a_2, a_3, a_4)$. Then by a direct computation using the multiplication table for the basis $\{v_i\}$ in Section 4.7 and (4.9.1), one can show that $t_1 = \sum_{i=1}^{4} t_{x_i,y_i}$, where x_i, y_i are given by $x_i = a_i(e_i + e_{4_i})$ and $y_i = (e_i - e_{4+i})/2$. Using this, the above formulas for t_2 and t_3 and the multiplication table for the v_i's, we get,

$$
\begin{aligned}
&t_1 = (a_1, a_2, a_3, a_4) \\
&t_2 = ((a_1 + a_2 + a_3 - a_4)/2, (a_1 + a_2 - a_3 + a_4)/2, \\
&(a_1 - a_2 + a_3 + a_4)/2, (-a_1 + a_2 + a_3 + a_4)/2) \\
&t_3 = ((a_1 - a_2 - a_3 + a_4)/2, (-a_1 + a_2 - a_3 + a_4)/2, \\
&(-a_1 - a_2 + a_3 + a_4)/2, (a_1 + a_2 + a_3 + a_4)/2)
\end{aligned}
$$

Also note that if $t = (a_1, a_2, a_3, a_4)$ then $\hat{t} = (-a_1, a_2, a_3, a_4)$. This is evident from the fact that $\bar{e_1} = e_5$ and $\bar{e_i} = -e_i$ whenever $i \neq 1, 5$ and the definition of $\hat{t}$ i.e. $\hat{t}(x) = \overline{t(\bar{x})}$, $x \in \mathfrak{C}$. We refer to the above set of equations by (A). Recall that the Weyl group of F_4 is $W \cong WSpin(N) \rtimes S_3 \cong ((\mathbb{Z}/2)^3 \rtimes S_4) \rtimes S_3$ and the action of W on $\mathbf{L}RT(\mathfrak{C})$ is given by (4.7.1).

We now calculate the stabilizers of elements of $\mathbf{L}(RT(\mathfrak{C}))$ in W, the arguments being similar to those for the group F_4.

(1) By (A),

$$t_1 = t_2 = t_3 = 0$$

Then clearly $W_t = WF_4$.

(2) If

$t_1 = (0, 0, 0, a_4)$, then by (A),

$$
\begin{aligned}
t_2 &= (-a_4/2, a_4/2, a_4/2, a_4/2) \\
t_3 &= (a_4/2, a_4/2, a_4/2, a_4/2)
\end{aligned}
$$

Here we observe that only τ_1 fixes t since t_2, t_3 do not have 0 as a parameter, no other element from $S_3 = Out(Spin(N))$ can contribute to the isotropy (see 4.7.1). Thus $W_t = ((\mathbb{Z}/2)^2 \rtimes S_3) \rtimes \{1, \tau_1\}$.

(3) If $t_1 = (0, 0, a_3, a_3)$, then by (A),

$$t_1 = t_2 = t_3 = (0, 0, a_3, a_3)$$

Therefore, $\hat{t_1} = \hat{t_2} = \hat{t_3}$. Hence all of $S_3 = Out(Spin(N))$ fixes t (see 4.7.1). Therefore, $W_t = ((\mathbb{Z}/2 \rtimes S_2) \times S_2) \rtimes S_3$

(4) If $t_1 = (0, 0, a_3, a_4)$, then by (A)

$$t_2 = \hat{t_3} = ((a_3 - a_4)/2, (a_4 - a_3)/2, (a_3 + a_4)/2, (a_3 + a_4)/2).$$

We have, $W_t = (\mathbb{Z}/2 \rtimes S_2) \rtimes \{1, \tau_1\}$, because apart from τ_1 any other element of S_3 sends t_2 or t_3 to the first position (see 4.7.1) and hence they cannot fix t.

(5) If $t_1 = (0, a_2, a_2, a_2)$, then by (A),

$$t_2 = \hat{t_3} = (a_2/2, a_2/2, a_2/2, a_2/2).$$

Since $t_2 = \hat{t_3}$, only $\tau_1 \in S_3$ appears in the isotropy subgroup (see 4.7.1). Therefore, $W_t = S_3 \rtimes \{1, \tau_1\}$.

(6) If $t_1 = (0, a_2, a_2, a_4)$, then by (A),

$$t_2 = \hat{t_3} = ((2a_2 - a_4)/2, a_4/2, a_4/2, (2a_2 + a_4)/2).$$

We have, $W_t = S_2 \rtimes \{1, \tau_1\}$ if $2a_2 \neq a_4$ and if $a_4 = 2a_2$ then $t_1 = t_2 = t_3$ and S_3 will clearly fixes t (see 4.7.1). Hence $W_t = S_2 \rtimes S_3$.

(7) If $t_1 = (0, a_2, a_3, a_4)$, then by (A),

$$t_2 = ((a_2 + a_3 - a_4)/2, (a_2 - a_3 + a_4)/2, (-a_2 + a_3 + a_4)/2, (a_2 + a_3 + a_4)/2)$$
$$t_3 = \hat{t_2}$$

If t_2, t_3 does not contain 0 as a parameter, then $W_t = \{1, \tau_1\} \subset S_3$ since any other element of S_3 removes t_1 from the first position of the related triple by 4.7.1. Otherwise, let $a_2 + a_3 - a_4 = 0$, then by (A), $t_1 = t_2 = t_3$ and therefore, S_3 stabilizes t. In this case, $W_t = \{1\} \rtimes S_3$. For the other three possibilities the related triple can be made Weyl group equivalent to the latter by a suitable permutation of a_2, a_3, a_4.

(8) If $t_1 = (a_1, a_1, a_3, a_4)$, then by (A),

$$t_2 = ((2a_1 + a_3 - a_4)/2, (2a_1 - a_3 + a_4)/2, (a_3 + a_4)/2, (a_3 + a_4)/2)$$
$$t_3 = ((-a_3 + a_4)/2, (-a_3 + a_4)/2, (-2a_1 + a_3 + a_4)/2, (2a_1 + a_3 + a_4)/2)$$

We have $W_t = S_2 \subset WSpin(N)$, since every element of S_3 other than 1, acts non trivially on t (see 4.7.1).

(9) If $t_1 = (a_1, a_1, a_1, a_4)$, then by (A)

$$t_2 = ((3a_1 - a_4)/2, (a_1 + a_4)/2, (a_1 + a_4)/2, (a_1 + a_4)/2)$$
$$t_3 = ((-a_1 + a_4)/2, (-a_1 + a_4)/2, (-a_1 + a_4)/2, (3a_1 + a_4)/2).$$

We have, $W_t = S_3 \subset Spin(N)$ because $a_1 \neq a_4$ and hence only elements from $WSpin(N)$ fixes t (see 4.7.1).

(10) If $t_1 = (a_1, a_2, a_3, a_4)$, then by (A),

$$t_2 = ((a_1 + a_2 + a_3 - a_4)/2, (a_1 + a_2 - a_3 + a_4)/2,$$
$$(a_1 - a_2 + a_3 + a_4)/2, (-a_1 + a_2 + a_3 + a_4)/2)$$
$$t_3 = ((a_1 - a_2 - a_3 + a_4)/2, (-a_1 + a_2 - a_3 + a_4)/2,$$
$$(-a_1 - a_2 + a_3 + a_4)/2, (a_1 + a_2 + a_3 + a_4)/2)$$

Here, the isotropy subgroup is trivial if none of the t_i's contain 0 as parameter, because in that case all non trivial elements of S_3 act non trivially on (t_1, t_2, t_3) (see 4.7.1).

Hence there are 12 conjugacy classes of isotropy subgroups in the Weyl group.

We conclude this chapter by collecting the results obtained thus far in the following tables.

Computation of the number of orbit types for Lie algebras:

Lie algebra	**Weyl group**	**Stabilizers**	**number of orbit types**
A_n	S_{n+1}	$S_{n_1}...S_{n_k}$ for a partition $n_1, ..., n_k$ of $n+1$	$p(n+1)$
B_n	$(\mathbb{Z}/2)^n \rtimes S_n$	$((\mathbb{Z}/2)^{n_1-1} \rtimes S_{n_1}) \times S_{n_2} \times ... \times S_{n_k}$	$\sum_{i=0}^{n} \quad p(n-i)$
C_n	$(\mathbb{Z}/2)^n \rtimes S_n$	$((\mathbb{Z}/2)^{n_1} \rtimes S_{n_1}) \times S_{n_2} \times ... \times S_{n_k}$	$\sum_{i=0}^{n} \quad p(n-i)$
$D_n$for $n$odd	$(\mathbb{Z}/2)^{n-1} \rtimes S_n$	$((\mathbb{Z}/2)^{n_1-1} \rtimes S_{n_1}) \times S_{n_2} \times ... \times S_{n_k}$	$\sum_{i=0}^{n} \quad p(n-i)$
D_n for $n = 2k$	$(\mathbb{Z}/2)^{n-1} \rtimes S_n$	$((\mathbb{Z}/2)^{n_1-1} \rtimes S_{n_1}) \times S_{n_2} \times ... \times S_{n_k}$, and for each partition $k_1, .., k_s$ of k, $H_{2k_1}.S_{2k_2}...S_{2k_s}$, where H_{2k_1} is a subgroup of order $(2k_1)!$ not conjugate to S_{2k_1}.	$\sum_{i=0}^{n} \quad p(n-i) + p(k)$
G_2	$S_3 \rtimes S_2$	As noted in § 4.9.5	4
F_4	$((\mathbb{Z}/2)^3 \rtimes S_4) \rtimes S_3$	As noted in § 4.9.6	12

Computation of genus number:

Group	**Weyl group**	**Stabilizers**	**Genus Number**
A_n	S_{n+1}	$S_{n_1}...S_{n_k}$, where $n_1+...n_k=n+1$	$p(n+1)$
B_1	$\mathbb{Z}/2$	$\{1\},\ \mathbb{Z}/2$	2
B_2	$(\mathbb{Z}/2)^2 \rtimes S_2$	$\{1\}$, S_2, $\mathbb{Z}/2$, $(\mathbb{Z}/2)^2 \rtimes S_2$ and $\mathbb{Z}/2 \rtimes S_2$	5
B_n, $n \geq 3$	$(\mathbb{Z}/2)^n \rtimes S_n$	$(((\mathbb{Z}/2)^{i-1} \rtimes S_i) \times ((\mathbb{Z}/2)^{n_1-i} \rtimes S_{n_1-i})) \times S_{n_2} \times ... \times S_{n_k}$, where, $n_1+...+n_k=n$	$\sum_{i=0}^{n}(i+1)p(n-i)$
C_n	$(\mathbb{Z}/2)^n \rtimes S_n$	$(((\mathbb{Z}/2)^{i} \rtimes S_i) \times ((\mathbb{Z}/2)^{n_1-i} \rtimes S_{n_1-i})) \times S_{n_2} \times ... \times S_{n_k}$, where $n_1+...n_k=n$	$\sum_{i=0}^{n}([i/2]+1)p(n-i)$
D_n, n odd	$(\mathbb{Z}/2)^{n-1} \rtimes S_n$	$(((\mathbb{Z}/2)^{i-1} \rtimes S_i) \times (\mathbb{Z}/2)^{n_1-i-1} \rtimes S_{n_1-i})) \times S_{n_2} \times ... \times S_{n_k}$, where $n_1+...+n_k=n$	$\sum_{i=0}^{n}([i/2]+1)p(n-i)$
D_n, $n=2k$	$(\mathbb{Z}/2)^{n-1} \rtimes S_n$	$(((\mathbb{Z}/2)^{i-1} \rtimes S_i) \times ((\mathbb{Z}/2)^{n_1-i-1} \rtimes S_{n_1-i}) \times S_{n_2} \times ... \times S_{n_l}$, where $n_1+...+n_l=n$ with at least one n_i odd and $H(2k_1) \times S_{2k_2} \times ... \times S_{2k_s}$, where $k_1+...+k_s=k$ and $H(2k_1)$ is a subgroup of order $(2k)!$ not conjugate to S_{2k_1}	$\sum_{i=0}^{n}([i/2]+1)p(n-i)+p(k)$
G_2	$S_3 \rtimes S_2$	As noted in Section 4.8	6
F_4	$((\mathbb{Z}/2)^3 \rtimes S_4) \rtimes S_3$	As noted in Section 4.7	17

CHAPTER 5

Real elements in F_4

5.1. Introduction

Let G be a group. An element $x \in G$ is called **real** if x is conjugate to x^{-1} in G. For an algebraic group G defined over a field k, let $G(k)$ denote the group of k-rational points in G. An element $g \in G(k)$ is said to be k**-real** if g is conjugate to g^{-1} in $G(k)$. An **involution** in $G(k)$ is an element $g \in G(k)$ such that $g^2 = 1$. An element $g \in G(k)$ is called **strongly** k**-real** if there exists involutions $h_1, h_2 \in G(k)$ such that, $g = h_1h_2$. It follows that an element $g \in G(k)$ is strongly k-real if and only if there exists an involution $h \in G(k)$ such that $hgh^{-1} = g^{-1}$.

The study of conjugacy classes of real elements in algebraic groups is important from representation theoretic point of view. In this chapter we study real elements in certain groups of type F_4. In Section 5.2, we prove that in the compact connected Lie group of type F_4, every element is strongly real. In Section 5.3, we consider algebraic groups of type F_4, that occur as groups of automorphisms of Albert division algebras over a field k. We show that there are no non trivial k-real elements in such groups. This was conjectured by A. Singh in his doctoral thesis. Finally, in Section 5.4, we give a characterisation of k-real elements in anisotropic algebraic groups of type F_4, that are obtained from reduced Albert algebras.

The results proved in this chapter can be found in [**Bo1**].

5.2. Reality in compact F_4

Let $\mathfrak{C}$ be the octonion division algebra over $\mathbb{R}$ with norm N. We fix an orthogonal basis $\mathfrak{B} = \{v_1, v_2, ..., v_8\}$, where $v_1 = 1$, $v_6 = v_2v_5$, $v_7 = v_3v_5$ and $v_8 = v_4v_5$ ([**P**], Lecture 14). Let $Spin(N)$ and $SO(N)$ respectively denote the spin group and the special orthogonal group of $(\mathfrak{C}, N)$. With respect to the basis $\mathfrak{B}$, the matrix of the bilinear form associated with N is diagonal.

Consider the reduced Albert algebra $A := H_3(\mathfrak{C})$ over $\mathbb{R}$. We have seen in Chapter 3, that $Aut(A)$ is the compact connected Lie group of type F_4.

Consider the diagonal subalgebra $S = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \subset A$. Then $Spin(N)$ sits in $Aut(A)$ as the subgroup of all automorphisms ϕ of A, such that $\phi(s) = s$ for all $s \in S$ ([**J**], Theorem 6). We consider an explicit description of $Spin(N)$ in the following way: Let, as before, $\mathfrak{C}$ denote an octonion algebra over $\mathbb{R}$ and consider the subgroup $RT(\mathfrak{C}) \subset SO(N)^3$, defined as,

$$RT(\mathfrak{C}) := \{(t_1, t_2, t_3) \in SO(N)^3 | \; t_1(xy) = t_2(x)t_3(y) \quad \forall x, y \in \mathfrak{C}\}$$

Any element of $RT(\mathfrak{C})$ is called a related triple (Chapter 3, Section 3.2). We need the following result from [**SV**] (Proposition 3.6.3).

Proposition 5.2.1. *There is an isomorphism,*

$$\Phi : Spin(N) \longrightarrow RT(\mathfrak{C})$$

defined by ,

$$\Phi(a_1 \circ b_1 \circ ... \circ a_r \circ b_r) = (s_{a_1}s_{b_1}...s_{a_r}s_{b_r}, l_{a_1}l_{\overline{b_1}}...l_{a_r}l_{\overline{b_r}}, r_{a_1}r_{\overline{b_1}}...r_{a_r}r_{\overline{b_r}}),$$

where $a_i, b_i \in \mathfrak{C}$, $\prod_i N(a_i)N(b_i) = 1$, (N being the norm on the octonion algebra), s_v is the reflection in the hyperplane orthogonal to $v \in \mathfrak{C}$, l_v and r_v are respectively the left and right homotheties with respect to v on $\mathfrak{C}$.

Remark: Henceforth, in the subsequent discussion, we shall identify the groups $Spin(N)$ and $RT(\mathfrak{C})$ via the above isomorphism.

Lemma 5.2.2. *(Lemma 4.7.3) Let T be a maximal torus in $SO(N)$. Then*

$$\widetilde{T} := \{(t_1, t_2, t_3) \in T^3 | \; (t_1, t_2, t_3) \textit{ is a related triple}\}$$

is a maximal torus in $Spin(N)$.

Lemma 5.2.3. *Every element in $Spin(N)$ is strongly real.*

Proof. We fix the maximal torus $T = SO(2) \times SO(2) \times SO(2) \times SO(2) \subset SO(N)$, consisting of block diagonal matrices, with blocks belonging to $SO(2)$. Denote a typical element of T by $(\gamma_1/\pi, \gamma_2/\pi, \gamma_3/\pi, \gamma_4/\pi)$, which represents a block diagonal matrix, where the i-th block is

$$\begin{bmatrix} cos2\gamma_i & -sin2\gamma_i \\ sin2\gamma_i & cos2\gamma_i \end{bmatrix}, \; i = 1, 2, 3, 4.$$

Therefore, by Lemma 5.2.2,

$$\widetilde{T} := T^3 \cap RT(\mathfrak{C}) = \{(t_1, t_2, t_3) \in T^3 | \; (t_1, t_2, t_3) \textit{ is a related triple}\}$$

is a maximal torus in $Spin(N)$.

So, let $(t_1, t_2, t_3) \in \widetilde{T}$. Consider the block diagonal matrix $m_1 \in SO(N)$, made up of four 2×2 blocks, each of which is equal to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Clearly, $m_1^2 = 1$ and there exists $s_1 \in T$, such that m_1 is conjugate to s_1 in $SO(N)$. This is because $SO(N)$ is a compact connected Lie group and hence any two maximal tori in $SO(N)$ are conjugate. Observe that, the characteristic polynomial of m_1 is $(x-1)^4(x+1)^4$. Therefore, m_1 is conjugate to the element $s_1 = (0, 0, \frac{1}{2}, \frac{1}{2}) \in T$ in $SO(N)$. Following ([**Bo**], Section 7), there exists an element $s = (s_1, s_2, s_3) \in \widetilde{T}$ with $s_1 = (0, 0, \frac{1}{2}, \frac{1}{2})$, $s_2 = (\frac{1}{2}, \frac{1}{2}, 0, 0)$ and $s_3 = (\frac{1}{2}, \frac{1}{2}, 0, 0)$. Since s_1 is conjugate to m_1 in $SO(N)$, it is clear that m_1 lifts to an involution $(m_1, m_2, m_3) \in \widetilde{T}$.

Now $m_1 t_1 = diag(B_1, B_2, B_3, B_4)$, a block diagonal matrix in $SO(N)$, where

$$B_i = \begin{bmatrix} sin2\gamma_i & cos2\gamma_i \\ cos2\gamma_i & -sin2\gamma_i \end{bmatrix}, \ i = 1, 2, 3, 4.$$

Observe that the characteristic polynomial of $m_1 t_1$ is again $(x-1)^4(x+1)^4$. Therefore $m_1 t_1$ is also conjugate to the involution $(0, 0, \frac{1}{2}, \frac{1}{2}) \in T$ in $SO(N)$. Hence, $m_1 t_1 \in SO(N)$ also lifts to the involution $(m_1 t_1, m_2 t_2, m_3 t_3)$ in $Spin(N)$. Thus any element $(t_1, t_2, t_3) \in \widetilde{T} \subset Spin(N)$ is a product of two involutions i.e., $(t_1, t_2, t_3) = (m_1, m_2, m_3)(m_1 t_1, m_2 t_2, m_3 t_3)$ and hence strongly real. Since any element of $Spin(N)$ is contained in a maximal torus and any two maximal tori of $Spin(N)$ are conjugate (since $Spin(N)$ is a compact connected Lie group), the result follows.

□

Theorem 5.2.4. *Every element of the compact connected Lie group of type F_4 is strongly real.*

Proof. Let $G = Aut(H_3(\mathfrak{C}))$ be the compact connected Lie group of type F_4. Let $g \in G$ be any element. Since G is a compact Lie group, every element is semisimple and is contained in a maximal torus. Thus there exists a maximal torus $T_1 \subset G$, such that $g \in T_1$. Now $Spin(N) \subset G$ is a maximal rank subgroup of G. Let $T \subset Spin(N)$ be a maximal torus. Then there exists $h \in G$ such that $hT_1h^{-1} = T$ and hence $hgh^{-1} \in T$. By Lemma 5.2.3, every element of $Spin(N)$ is strongly real. Hence the result holds for any element of G.

□

5.3. F_4 from Albert division algebras

Let k be any field and A an Albert algebra over k. Then either A is a division algebra or it is reduced. In the latter case, A is isomorphic to $H_3(\mathfrak{C}, \gamma)$, the space of all γ-hermitian matrices in $\mathbb{M}_3(\mathfrak{C})$ for a 3×3 invertible diagonal matrix γ over $\mathfrak{C}$. We have seen in Chapter 3 that, up to isomorphism, Albert algebras are given by Tits' first and second constructions. We first record the following result due to Jacobson ([**J**], Theorem 13).

Proposition 5.3.1. *Let A be an Albert algebra over a field k and suppose A possesses an automorphism η of order 2. Then A is reduced i.e., $A \cong H_3(\mathfrak{C}, \gamma)$ for an octonion algebra $\mathfrak{C}$ over k and either η is a reflection in a subalgebra $B = H_3(\mathfrak{D}, \Gamma)$, where $\mathfrak{D}$, a quaternion subalgebra of $\mathfrak{C}$ or η is in the center of a subgroup $Aut(A/ke) = \{f \in Aut(A) : f(e) = e\}$, where e is a primitive idempotent in A.*

Corollary 5.3.2. *If A is an Albert division algebra, then $Aut(A)$ does not contain an involution other than the identity.*

PROOF. The result follows directly from Proposition 5.3.1. □

Lemma 5.3.3. *Let D be a division algebra of degree 3 over k. Let K be a quadratic extension of k and B, a central simple K-algebra of degree 3 with a unitary involution σ over K. Let $G = SL_1(D)$ or $SU(B, \sigma)$. Then any automorphism θ of G, defined over k, is given by conjugation by an element of D^* or B^* respectively.*

PROOF. Let $\theta : G \longrightarrow G$ be an automorphism and $d\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$, the induced Lie algebra automorphism. Then by Theorems 10 and 11, of [**J2**], Chapter X, $d\theta$ extends to automorphism of D or (B, σ) according as $G = SL_1(D)$ or $SU(B, \sigma)$. Thus, there exists $a \in D^*$ or $b \in B^*$ such that $d\theta(x) = axa^{-1}$ or $d\theta(x) = bxb^{-1}$ respectively, for all $x \in \mathfrak{g}$. Therefore, if $G = SL_1(D)$ or $SU(B, \sigma)$, then θ is given by $\theta(g) = aga^{-1}$ or $\theta(g) = bgb^{-1}$ respectively, for all $g \in G$.

□

Before we prove the next theorem, we need the following result from [**Ho**], which rules out A_1 type subgroups from the possible types of a non-toral reductive k-subgroup of $Aut(A)$ for an Albert division algebra A. For the sake of completeness, we include from [**Ho**] the proof that A_1 type subgroups do not occur in $Aut(A)$.

Theorem 5.3.4. *([**PST**], Proposition 6.1, [**Ho**], Theorem 3.10) Let A be an Albert division algebra over a field k. Let $H \subset Aut(A)$ be a proper connected reductive non toral subgroup defined over k. Then $[H,H]$ is of type A_2, $A_2 \times A_2$ or D_4.*

Proof. In ([**PST**], Proposition 6.1), it was shown that other than the types listed above, one may have H as $R_{L/k}(S)$, where L/k is a cubic field extension, S is a simple group of type A_1 defined over L and $R_{L/k}$ denotes the Weil's restriction of scalars. We rule out this possibility now. Observe that such a subgroup H has a maximal k-torus of dimension 3. Hence, by (Lemma 2.3, [**GG**]), it follows that $Aut(A)$ then contains a rank 1 k-torus. Any rank 1 k-torus is of the form $K^{(1)}$, the norm torus of a quadratic extension K of k ([**V**], Chapter II, § IV, Example 6) and hence splits over K. Therefore, $Aut(A)$ is isotropic over K. Hence, A becomes reduced over a quadratic extension K of k ([**PR**], page 205), which is impossible since A, being an Albert division algebra, cannot be reduced over an extension of degree 2^l ([**PR**], Corollary, page 205). □

Theorem 5.3.5. *Let A be an Albert division algebra over a perfect field k and $G = Aut(A)$ be the corresponding algebraic group of type F_4. Then $G(k)$ does not have any non trivial k-real element.*

Proof. Let $g \in G(k)$ be a non trivial k-real element. Then there exists $h \in G(k)$ such that $hgh^{-1} = g^{-1}$. This implies that $h^2gh^{-2} = g$. Note that $h^2 \neq 1$ by Corollary 5.3.2. Thus, $h, g \in Z_G(h^2)$, the centralizer of h^2 in G. Now since A is an Albert division algebra, $G = Aut(A)$ is an anisotropic group of type F_4, defined over k (see [**KMRT**], Chapter IX). Therefore, since k is perfect, every element of $G(k)$ is semisimple by (Propostion 6.3, [**R**]). Since G is a group of type F_4, it is simply connected and hence the centralizer of any semisimple element in G is a connected subgroup of G (see [**SSt**], Chapter II, Section 3). Thus, $Z_G(h^2)$ is a connected reductive k-subgroup of G of maximal rank.

Observe that $Z_G(h^2)$ is not a torus. For if not, then $hgh^{-1} = g$ since $h, g \in Z_G(h^2)$. Therefore, we have $g^{-1} = hgh^{-1} = g$, which implies that g is an involution in $G(k)$, a contradiction by Corollary 5.3.2. Hence by Theorem 5.3.4, $Z_G(h^2)$ has the following possible types: $A_2 \times A_2$, D_4 or A_2.

Case1. Let $Z_G(h^2)$ be of type $A_2 \times A_2$ or D_4. Now since $Z_G(h^2)$ is reductive, $Z_G(h^2) = [Z_G(h^2), Z_G(h^2)].Z(Z_G(h^2))^\circ$, where $[Z_G(h^2), Z_G(h^2)]$ is semisimple and $Z(Z_G(h^2))^\circ$ is a torus. In this case, $[Z_G(h^2), Z_G(h^2)]$ is of type $A_2 \times A_2$ or D_4, and hence, a maximal rank subgroup of $Aut(A)$. Therefore, the rank of the torus $Z(Z_G(h^2))^\circ$ is 0, i.e., $Z(Z_G(h^2))^\circ$ is trivial. Thus, $Z_G(h^2)$ is semisimple and has a finite center. Now, observe that $h^2 \in Z(Z_G(h^2))$. Let the order of (h^2) be n. If $n = 2k$ for some integer k, then h^k is an involution, which is a contradiction by Corollary 5.3.2. If $n = 2k + 1$ for some integer k, then $h = h^{2k+2}$. Hence, $g^{-1} = hgh^{-1} = h^{2k+2}gh^{-(2k+2)} = g$. Therefore, $g^2 = 1$, again a contradiction by Corollary 5.3.2.

Case 2. Let $Z_G(h^2)$ be of type A_2. Therefore, $[Z_G(h^2), Z_G(h^2)]$ is semisimple of type A_2 and $Z(Z_G(h^2))^\circ$ is torus of rank 2. Thus, $[Z_G(h^2), Z_G(h^2)]$ is isomorphic to either $SL_1(D)$ or $SU(B,\sigma)$ or the corresponding adjoint groups, where D, B and σ are as in Lemma 5.3.3.

Let us first assume that $[Z_G(h^2), Z_G(h^2)] \cong SL_1(D)$. Observe that, $g, h \in Z_G(h^2)$, hence, $hgh^{-1}g^{-1} = g^{-2} \in SL_1(D)$. Now if possible, let $g^{-2} \in Z(SL_1(D))$. Let $h = ab$, where $a \in SL_1(D)$ and $b \in Z(Z_G(h^2))^\circ$. Then clearly, $hg^{-2}h^{-1} = a(bg^{-2}b^{-1})a^{-1} = ag^{-2}a^{-1} = g^{-2}$. But on the other hand, $hgh^{-1} = g^{-1} \Rightarrow hg^{-2}h^{-1} = g^2$. Therefore, $g^2 = g^{-2} \Rightarrow g^2$ is an involution and hence a contradiction. Hence g^{-2} does not belong to the center of $SL_1(D)$, in particular $g^{-2} \notin k$. Hence $k(g^2)/k$ is an extension of k of degree 3 since D is a division algebra of degree 3 over k.

Now, note that, for any $g \in Z_G(h^2)$, $(hgh^{-1})h^2(hg^{-1}h^{-1}) = h^2$. Therefore, $hZ_G(h^2)h^{-1} = Z_G(h^2)$. Hence, $h[Z_G(h^2), Z_G(h^2)]h^{-1} = [Z_G(h^2), Z_G(h^2)]$. Thus, we have an automorphism of $SL_1(D)$, which is given by conjugation by h. By Lemma 5.3.3, this automorphism is given by conjugation by some element $a \in D^*$. Therefore, $hg^2h^{-1} = ag^2a^{-1} = g^{-2}$. Thus we have a k-automorphism $f : k(g^2) \longrightarrow k(g^2)$, given by $g^2 \mapsto ag^2a^{-1} = g^{-2}$. Clearly, f is of order 2. But we have seen that the extension $k(g^2)/k$ is of degree 3 over k, a contradiction. A similar line of argument holds for the case when $[Z_G(h^2), Z_G(h^2)] \cong SU(B,\sigma)$. Here one argues with respect to K, the quadratic extension of k, such that B is a central simple K-algebra of degree 3.

Now let $[Z_G(h^2), Z_G(h^2)] \cong PSL_1(D) = SL_1(D)/Z(SL_1(D))$, the adjoint group corresponding to $SL_1(D)$. Let $h = ab$, where $a \in PSL_1(D)$ and $b \in Z(Z_G(h^2))^\circ$. Then $g^{-2} = hg^2h^{-1} = abg^2b^{-1}a^{-1} = ag^2a^{-1}$ since $b \in Z(Z_G(h^2))$ and hence it commutes with all elements of $Z_G(h^2)$. As before, g^2 cannot belong to the center of $PSL_1(D)$. Now, there exist elements $x, y \in D^*$ such that, $a = \bar{x}$ and $g^2 = \bar{y}$, where $\bar{x}$ and $\bar{y}$ denote the cosets of x and y respectively in $PSL_1(D)$. Therefore, the equation $ag^2a^{-1} = g^{-2}$ implies $xyx^{-1} = \alpha y^{-1}$, where $\alpha \in Z(SL_1(D)) \subset k^*$. Hence, $x^2yx^{-2} = \alpha(\alpha^{-1}y) = y$. Therefore, we get an automorphism $f : k(y) \longrightarrow k(y)$, given by $f(y) = xyx^{-1}$ of order 2. But D being a division algebra of degree 3 over k, we have a contradiction. The proof for the case $[Z_G(h^2), Z_G(h^2)] \cong PGU(B, \sigma)$ is along similar lines. $\square$

5.4. F_4 from reduced Albert algebras

Let us consider exceptional groups of type F_4, which are given by automorphisms of reduced Albert algebras. So let k be a field with $char(k) \neq 2$ and $\mathfrak{C}$ be an octonion algebra over k, $\gamma = diag\ (\gamma_1, \gamma_2, \gamma_3)$, where $\gamma_i \in k^*$. Following the notation of Chapter 3, Section 3.3, let $A = H_3(\mathfrak{C}, \gamma)$ be a reduced Albert algebra over a perfect field k. We further assume that A has no non trivial nilpotents and let $G = Aut(A)$ be the group of automorphisms of A. Then G is an anisotropic group of type F_4, defined over k. Therefore, by Proposition 6.3, [**R**], every element in $G(k)$ is semisimple. Hence, if $x \in G(k)$, $\overline{\langle x \rangle}$ is a diagonalisable group. Therefore, $\overline{\langle x \rangle} = H \times \overline{\langle x \rangle}^{\circ}$, where H is a finite group and $\overline{\langle x \rangle}^{\circ}$ is a torus (see [**Hu**], 16.2).

A reduced Albert algebra $A = H_3(\mathfrak{C}, \gamma)$, is equipped with a cubic form N, called the norm of A. Let

$$X = \begin{bmatrix} \alpha_1 & c_3 & \gamma_1^{-1}\gamma_3\bar{c_2} \\ \gamma_2^{-1}\gamma_1\bar{c_3} & \alpha_2 & c_1 \\ c_2 & \gamma_3^{-1}\gamma_2\bar{c_1} & \alpha_3 \end{bmatrix}$$

be a typical element in $H_3(\mathfrak{C}, \gamma)$. Then the norm of X is given by,

$$N(X) = \alpha_1\alpha_2\alpha_3 - \gamma_3^{-1}\gamma_2\alpha_1 n(c_1) - \gamma_1^{-1}\gamma_3\alpha_2 n(c_2) - \gamma_2^{-1}\gamma_1\alpha_3 n(c_3) + n(c_1c_2, \bar{c_3}),$$

where n is the norm on $\mathfrak{C}$ and $n(,)$ is the bilinear form associated to n (Chapter 5, [**SV**]). We define a quadratic form Q on A by

$$Q(X) = \frac{1}{2}(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + \gamma_3^{-1}\gamma_2 n(c_1) + \gamma_1^{-1}\gamma_3 n(c_2) + \gamma_2^{-1}\gamma_1 n(c_3).$$

A bijective k-linear mapping $f : A \longrightarrow A$ is called a **norm similarity** of A if there exists $\alpha \in k^*$, such that $f(N(X)) = \alpha N(X)$ for all $X \in A$. We call α the **multiplier** of f. Let $M(A)$ denote the group of all norm similarities of the Albert algebra A. Observe that, a norm similarity f of A with multiplier 1, is an isometry. Let

$$H := \{f \in M(A) : f(N(X)) = N(X)\ \forall\ X \in A\}.$$

Then H is a closed connected subgroup of $M(A)$ and it is a simply connected algebraic group of type E_6 defined over k (Theorem 7.3.2, [**SV**]). We are now in a position to prove the following crucial lemma.

Lemma 5.4.1. *Let $\phi \in G(k)$ be a k-real automorphism. Then there exists an element $\psi \in G(k)$ of finite order such that $\psi\phi\psi^{-1} = \phi^{-1}$.*

Proof. Let $x \in G(k)$ be such that $x\phi x^{-1} = \phi^{-1}$. Observe that $\overline{\langle\phi\rangle}$ and $\overline{\langle x\rangle}$ are diagonalisable groups since ϕ and x are semisimple. If x is of finite order, we have nothing to prove. So let x be of infinite order. Therefore, $\overline{\langle x\rangle} = D \times S$, where D is a finite abelian group and $S = \overline{\langle x\rangle}^{\circ}$ is a non trivial torus.

Now $x\phi x^{-1} = \phi^{-1} \implies x\overline{\langle\phi\rangle}x^{-1} = \overline{\langle\phi\rangle}$. Therefore, $x \in N_{G(k)}(\overline{\langle\phi\rangle})$ which implies that $\overline{\langle x\rangle} \subset N_{G(k)}(\overline{\langle\phi\rangle})$. By rigidity of the diagonalisable group $\overline{\langle\phi\rangle}$, we have $N_{G(k)}(\overline{\langle\phi\rangle})^{\circ} = Z_{G(k)}(\overline{\langle\phi\rangle})^{\circ}$. Hence $S = \overline{\langle x\rangle}^{\circ} \subset Z_{G(k)}(\overline{\langle\phi\rangle})^{\circ}$. Let $x = (d, s)$, where $d \in D$, $s \in S$. Therefore, $\phi^{-1} = x\phi x^{-1} = (d, 1)\phi(d^{-1}, 1)$ since S acts trivially on ϕ by conjugation. We have thus produced a finite order element $\psi = (d, 1) \in G(k)$ such that $\psi\phi\psi^{-1} = \phi^{-1}$.

□

Thus, if the order of ψ is $2k+1$, $\phi^{-1} = \psi\phi\psi^{-1} = \psi^{2k+1}\phi\psi^{-(2k+1)} = \phi$. Therefore ϕ is an involution in G. If ψ is of order $2k$ for k odd, then ψ^k is an involution in G, which conjugates ϕ to ϕ^{-1}. Hence, without loss of generality, we can assume that the order of ψ is 2^l with $l \geq 2$. Let us further assume that -1 is a square in the field k.

Let $\theta = \psi^{2^{l-1}}$. Then $\theta^2 = 1$ and $\phi, \psi \in Z_G(\theta)$. Therefore, by Proposition 5.3.1, either $Z_G(\theta) \cong Spin(9)$ or $Z_G(\theta)$ is of type $A_1 \times C_3$. Here, $Spin(9) \cong Aut(A/ke) := \{f \in Aut(A) : f(e) = e\}$, where $e \in A$ is a primitive idempotent. In the other case when $Z_G(\theta)$ is of type $A_1 \times C_3$, it is given by $Aut(H_3(\mathfrak{D}, \gamma))$ up to isomorphism, where $\mathfrak{D}$ is a quaternion subalgebra of the octonion algebra $\mathfrak{C}$.

If $Z_G(\theta)$ is of type $A_1 \times C_3$, then ϕ is strongly real in $Z_G(\theta)$ and hence in $G(k)$. This follows from the proof of (Theorem 1.2, [**AE**]).

Suppose that $Z_G(\theta) \cong Spin(9)$. Here, $Spin(9)$ is the Spin group of the 9-dimensional subspace $V \subset A$, consisting of matrices of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha & c \\ 0 & \gamma_3^{-1}\gamma_2\bar{c} & -\alpha \end{bmatrix}, \ \alpha \in k, c \in \mathfrak{C},$$

with the quadratic form being the restriction of the quadratic form Q to V, where Q is as defined before Lemma 5.4.1. We shall now denote $Spin(9)$ by $Spin(V, Q)$. Let $\Gamma^+(V, Q)$ be the even Clifford group associated to the quadratic space (V, Q). Also, $\Gamma^+(V, Q) \subset M(A)$, the group of norm similarities of the Albert algebra A (refer to Chapter IX, [**J1**]). We have $Spin(V, Q) \subset \Gamma^+(V, Q)$ and an exact sequence

$$1 \to k^* \to \Gamma^+(V, Q) \xrightarrow{\rho} SO(V, Q) \to 1$$

Now, ϕ is a real element in $Spin(V,Q)(k)$. Hence $\rho(\phi)$ is strongly real in $SO(V,Q)(k)$. Thus there exist $a_1, a_2 \in SO(V,Q)(k)$ with $a_1^2 = a_2^2 = 1$, such that $\rho(\phi) = a_1 a_2$. Since ρ is onto, there exist elements $b_1, b_2 \in \Gamma^+(V,Q)(k)$ such that $\rho(b_i) = a_i$, $b_i^2 = \pm 1$. Therefore, $b_1 b_2 = \phi$. If $b_i^2 = -1$, for $i = 1, 2$, consider $c \in k$ such that $c^2 = -1$. Then $\phi = (cb_1)(c^{-1}b_2)$, a product of involutions in $\Gamma^+(V,Q)$.

We claim that $b_1^2 = b_2^2 = \pm 1$. So if possible, let $b_1^2 = -1$ and $b_2^2 = 1$. Then we have

$$b_1 \phi b_1^{-1} = b_1(b_1 b_2)(-b_1) = b_2 b_1 = -(b_2^{-1} b_1^{-1}) = -\phi^{-1} \text{ .... } (*)$$

Let 1_A denote the identity element in the Albert algebra A. Then by $(*)$, we have $b_1 \phi b_1^{-1}(1_A) = -\phi^{-1}(1_A) = -1_A$. Therefore, $\phi(b_1^{-1}(1_A)) = -b_1^{-1}(1_A)$. Taking norm of both sides, we have $N(\phi(b_1^{-1}(1_A))) = N(-b_1^{-1}(1_A)) \implies N(b_1^{-1}(1_A)) = -N(b_1^{-1}(1_A)) \implies N(b_1^{-1}(1_A)) = 0$. But $N(b_1^{-1}(1_A)) = \beta N(1_A) \neq 0$, where β is the multiplier of the norm similarity b_1^{-1}. Hence we have a contradiction.

Thus we have shown,

Theorem 5.4.2. *Let A be a reduced Albert algebra over a perfect field k with $char(k) \neq 2$, such that -1 is a square in k and $G = Aut(A)$. If ϕ be a k-real automorphism of A, then either ϕ is strongly k-real in $G(k)$ or it is a product of two involutions in $M(A)$.*

CHAPTER 6

Further Questions

Genus number: Theorems 4.2.4 and 4.2.7 give a recipe for computing the genus number of a compact simply connected Lie group as well as for simply connected algebraic group over an algebraically closed field. However, for the exceptional groups of type $E_6,\ E_7$ and E_8, this method is quite hard. One has to find a possibly different way of computing the genus number of these groups. In [**BDS**], Borel and De Siebenthal describe an algorithm to calculate the maximal rank maximal subgroups (which turn out to be centralizers of certain elements) of a compact Lie group by looking at the root datum of the group. As a result, one can list up to isomorphism, all maximal rank subgroups in a compact Lie group. The problem of computing semisimple genus number is to list all maximal rank subgroups up to conjugacy. So it will be interesting to know if one can extract the genus number directly from the root datum. For a finite group of Lie type, Fleischmann described in [**F**] a method of computing genus number by considering stable subsystems of the root system of the group. It will be interesting to investigate whether similar interpretations work in the general case.

Reality: Chapter 5 deals with reality in certain forms of the exceptional group of type F_4. This description is far from being complete. Let A be a reduced Albert algebra over a field k with $char(k) \neq 2$. Then if A has no non zero nilpotents, Theroem 5.4.2 asserts that every k-real element $\phi \in Aut(A)$ is either strongly k-real in $Aut(A)$ or strongly k-real in a bigger group $M(A)$ containing $Aut(A)$. It perhaps can be shown that ϕ is necessarily strongly k-real in $Aut(A)$. Characterization of real elements in $Aut(A)$ for a reduced Albert algebra having non zero nilpotents and for a split group of type F_4 over $\mathbb{R}$ is yet to be done. We also wish to characterize real elements in the exceptional groups $E_6,\ E_7$ and E_8 in the future.

We hope that the work done in this thesis will be of interest to the community.

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