EULER CLASS GROUPS OF POLYNOMIAL AND SUBINTEGRAL EXTENSIONS OF A NOETHERIAN RING

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To
My Parents

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Chapter 1

Introduction

Objective

The main objectives of this thesis are the following:

(i) To investigate the behaviour of the Euler class groups under integral and subintegral extensions. More precisely, given a subintegral (or integral) extension $R \hookrightarrow S$ of Noetherian rings, we are interested in finding out the relationship between the Euler class group of R and the Euler class group of S.

(ii) To develop a theory (namely, an extension of the theory of Euler class group to the Euler class group of $R[T]$ relative to a projective $R[T]$-module L of rank 1) in order to detect the precise obstruction for a projective $R[T]$-module P of rank n with determinant L to split off a free summand of rank one, where n is the Krull dimension of the (Noetherian) ring R.

The results on (i) will be discussed in Chapters 3, 4, 5 and the results on (ii) will be discussed in Chapters 6, 7 and 8. These results have been obtained in joint works with Mrinal Kanti Das. The results on (i) are based on the paper [D-Z 1] and the results on (ii) are based on the paper [D-Z 2].

We now give brief introductions to the problems tackled in this thesis and the statements of the main results that we obtained.

As both (i) and (ii) involve the theory of the Euler class groups, we start with a few words on its history and development.

Obstruction theory and the Euler class group

Let A be a Noetherian ring of (Krull) dimension n. A classical result of Serre [Se] asserts that if $\text{rank}(P) \geq n+1$, then $P \simeq Q \oplus A$ for some A-module Q (in other words, P splits off a free summand of rank one). There are well-known examples of rings A and indecomposable projective A-modules of rank $\leq \dim(A)$ to show that Serre's result is best possible. Most of the research in projective modules in last thirty years is centred around the following question.

Question 1. Let A be a Noetherian ring of dimension d and P be a projective A-module of rank $n \leq d$. What is the precise obstruction for P to split off a free summand of rank one?

To tackle the above question one would like to find a suitable "obstruction group" $G^n(A)$ so that given a projective A-module P of rank n, an element $x_n(P) \in G^n(A)$ can be associated such that $x_n(P)$ is trivial in $G^n(A)$ if and only if $P \simeq Q \oplus A$. This has been achieved in the case $d = n$ through the following path-breaking works.

(i) [MK-M, Mu] Let $X = \text{Spec}(A)$ be a smooth affine variety of dimension n over an algebraically closed field k. Then the Chow group $CH^n(X)$ is the obstruction group and $c_n(P)$ (the top Chern class of P) is the obstruction element. It is well-known that this result is no longer valid for arbitrary base field k.

(ii) [B-RS 1, B-RS 4] Let A be a Noetherian $\mathbb{Q}$-algebra of dimension n. The n-th Euler class group $E^n(A, L)$ of A with coefficients in a line bundle L (defined in [B-RS 4]) takes the role of $G^n(A)$. Given a projective A-module P of rank n, its Euler class $e(P, \chi)$ takes the role of $x_n(P)$, where $\chi : \wedge^n(P) \xrightarrow{\sim} L$ is an isomorphism.

(iii) [B-M, F, F-Sr, Mo] Let X be a smooth affine scheme of dimension n and L be a line bundle over X. Let $\mathcal{E}$ be a vector bundle of rank n with determinant L. Then, one can take the Chow-Witt group $\widetilde{CH^n}(X, L)$ (defined in [B-M]) as the obstruction group. The Euler class associated to $\mathcal{E}$ in this group works as the obstruction element.

It is not known if the groups in (ii) and (iii) are isomorphic for a smooth affine scheme.

In Chapters 3, 4 and 5 we shall be mostly working with the Euler class group as defined in [B-RS 1, B-RS 4]. Although the precise definitions and the main results from [B-RS 1, B-RS 4] will be recalled in Chapter 2, let us now give a quick illustration of the core idea of the Euler class theory in very simple terms. This sketch will also help us in understanding the motivation behind some of the questions that we are about to discuss.

Let A be a commutative Noetherian ring of dimension $n \geq 2$. Let P be a projective A-module of rank n and for simplicity, assume that the determinant of P is trivial. Let us fix $\chi : A \simeq \wedge^n(P)$. By a theorem of Eisenbud-Evans [E-E], there exists a surjective map $\alpha : P \twoheadrightarrow J$, where $J \subset A$ is an ideal of height n. The map α will induce a surjection $\overline{\alpha} : P/JP \twoheadrightarrow J/J^2$. As $\dim(A/J) = 0$, the A/J-module P/JP is free. Let us choose an isomorphism $\sigma : (A/J)^n \simeq P/JP$ such that $\wedge^n(\sigma) = \chi \otimes A/J$ and take the composite surjection $\omega_J := \overline{\alpha}\sigma : (A/J)^n \simeq P/JP \twoheadrightarrow J/J^2$. Bhatwadekar and Raja Sridharan proves in [B-RS 4, Corollary 4.4] that if $\mathbb{Q} \subset R$, then $P \simeq Q \oplus A$ for some A-module Q if and only if ω_J can be lifted to a surjection $\theta : A^n \twoheadrightarrow J$. In [B-RS 1, B-RS 4], this phenomenon is formalized in the form of the theory of Euler class group.

Subintegral extensions and the Euler class groups

Let R be a commutative Noetherian ring. An extension $R \hookrightarrow S$ is called subintegral if: (1) it is integral, (2) the induced map $\text{Spec}(S) \to \text{Spec}(R)$ is bijective, and (3) the induced field extensions $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \hookrightarrow S_{\mathfrak{P}}/\mathfrak{P}S_{\mathfrak{P}}$ are all trivial, where $\mathfrak{P} \in \text{Spec}(S)$ and $\mathfrak{p} = \mathfrak{P} \cap R$.

Subintegral ring extensions, apart from their intrinsic appeal, played an important role in studying projective modules. As evidence, we mention a few results below.

Let $R \hookrightarrow S$ be a subintegral extension and P be a finitely generated projective R-module. In [I 2], Ischebeck studied the behaviour of some K-theoretic functors under the extension $R \hookrightarrow S$, there is a result [I 2, Proposition 8] which asserts that P is free if and only if $P \otimes_R S$ is a free S-module (P is assumed to have trivial determinant). Later, while answering a conjecture of Murthy, Swan proved in [Sw2, Theorem 14.1] that if the module $P \otimes_R S$ has a direct sum decomposition into projective S-modules, then there is a similar decomposition for P (see [Sw2] for the precise statement). We mention another result which inspired us to investigate the questions we are going to describe below. In [B 1], to address some question on existence of unimodular elements, Bhatwadekar implicitly proves that if $P \otimes_R S \simeq Q' \oplus S$ for some S-module Q', then there is an R-module Q such that $P \simeq Q \oplus R$ (in other words, P has a unimodular element if and only if so does $P \otimes_R S$). This result is not mentioned anywhere but it can be derived using the techniques of [B 1].

We now assume that $R \hookrightarrow S$ is a subintegral extension with $\dim(R) = n \geq 2$. Let $J \subset R$ be an ideal of height n such that $\mu(J/J^2) = n$, where $\mu(-)$ stands for the minimal number of generators. Assume that we are given: $J = (a_1, \cdots, a_n) + J^2$. Taking a cue from Bhatwadekar's (unstated) result in [B 1] and the illustration of Euler class theory discussed above, we ask the following question.

Question 2. Let $R \hookrightarrow S$ be a subintegral extension with $\dim(R) = n \geq 2$. Let $J \subset R$ be an ideal of height n such that $J = (a_1, \cdots, a_n) + J^2$. Assume further that $JS = (\beta_1, \cdots, \beta_n)$ such that $\beta_i - a_i \in J^2S$ for $i = 1, \cdots, n$. Then, can we find $b_1, \cdots, b_n \in J$ such that $J = (b_1, \cdots, b_n)$ with $b_i - a_i \in J^2$ for $i = 1, \cdots, n$?

Note that the generators of J/J^2 may not have been induced by a surjection from a projective R-module as there are examples of rings R of dimension n and ideals J of height n such that $\mu(J/J^2) = n$ but J is not even surjective image of a projective R-module. Therefore a combination of [B 1] and [B-RS 4, Corollary 4.4] would not work, whereas, an affirmative answer to Question 2 would imply Bhatwadekar's (unstated) result in [B 1] (provided $\mathbb{Q} \subset R$). Anyone familiar with the Euler class groups will readily understand that we are essentially asking if the natural map from the n-th Euler class group $E^n(R)$ to the n-th Euler class group $E^n(S)$ is injective or not. It requires some arguments to ascertain that there is a natural map $\Phi : E^n(R) \to E^n(S)$. In Chapter 3, we answer Question 2 in the affirmative. In fact, we prove the following result (Theorem 3.2.3).

Theorem A. Let $R \hookrightarrow S$ be a subintegral extension. Then the natural map $\Phi : E^n(R) \longrightarrow E^n(S)$ is an isomorphism.

To prove Theorem A, we first argue that whenever required, we can always assume that the rings are reduced rings. It is well known that an extension $R \hookrightarrow S$ is subintegral if and only if S is the filtered union of subrings, each of which can be obtained from R by a finite number of 'elementarily subintegral' extensions (recall that an 'elementarily subintegral' extension is a ring extension of the form $R \hookrightarrow R[b]$, where $b^2, b^3 \in R$). In order to reduce our problem to finite subintegral extensions (and therefore to elementarily subintegral extensions), we first prove an interesting general result on the Euler class groups which roughly says that the Euler class group, under some suitable conditions, commutes with filtered direct limit of a directed system of rings (see Theorem 3.2.2 and the discussion preceding it for the details). If $R \hookrightarrow S$ is elementarily subintegral extension of reduced rings, then we observe that the conductor ideal C of R in S has height at least one and $(R/C)_{\text{red}} = (S/C)_{\text{red}}$. We then take advantage of a 'conductor diagram' to prove the result (see Theorem 3.2.1).

There is this notion of the Euler class group $E^n(R, L)$ of R with respect to a line bundle L, defined in [B-RS 4]. We generalize Theorem A in the following form (Theorem 3.2.4).

Theorem B. Let $R \hookrightarrow S$ be a subintegral extension and L be a projective R-module of rank one. Then $E^n(R, L)$ is isomorphic to $E^n(S, L \otimes_R S)$.

In Euler class theory there is a notion of the weak Euler class group $E_0^n(R)$ of a Noetherian ring R, which is a certain quotient of $E^n(R)$ (see Chapter 2 for the definition). We prove the following result on the weak Euler class group.

Theorem C. Let $R \hookrightarrow S$ be a subintegral extension and $\dim(R)$ is even. Then the natural map $\Phi_0 : E_0^n(R) \longrightarrow E_0^n(S)$ is an isomorphism.

An interesting offshoot of Theorem A is that if R is an affine algebra over a C_1-field of characteristic zero, and if $J \subset R$ is an ideal of height n with $\mu(J/J^2) = n$, then $\mu(J) = n$ if and only if $\mu(JS) = n$. We prove this result in (Theorem 3.2.11).

Recall that for a reduced ring A, there is a maximal subintegral extension contained in the total ring of fractions of A. This is called the *seminormalization* of A and is denoted by ^+A (see [Sw1] for details). From the category of commutative Noetherian rings we have the seminormalization functor, $R \mapsto {}^+(R_{red})$, to the category of seminormal rings. As a consequence of Theorem A, we conclude that the Euler class group behaves well with respect to this functor in the sense that $E^n(R) \simeq E^n({}^+(R_{red}))$.

Let us now assume that $R \hookrightarrow S$ is an integral extension. It requires more effort than in the subintegral case to show that there is a group homomorphism $\Psi : E(R) \longrightarrow E(S)$.

It is natural to ask whether Ψ is an isomorphism or not. In this context, we prove

Theorem D. Let $R \hookrightarrow S$ be an integral extension such that the extension $R_{\text{red}} \hookrightarrow S_{\text{red}}$ is birational. Then $\Psi : E(R) \longrightarrow E(S)$ is surjective. Further, if $R_{\text{red}} \hookrightarrow S_{\text{red}}$ is a subintegral extension, then Ψ is an isomorphism.

However, we give an example to show that Theorem A is not valid for arbitrary integral extension even if it is finite birational.

In Chapter 5, we answer a question of Ischebeck from [I 2] in the following form (Theorem 5.0.1).

Theorem E. Let R be a ring of dimension 2 and $R \hookrightarrow S$ be a subintegral extension. Let P and Q be two projective R-modules of rank 2 such that $det(P) \simeq det(Q)$ and $P \otimes S \simeq Q \otimes S$. Let $\chi : det(P) \xrightarrow{\sim} det(Q)$ and $\theta : P \otimes S \xrightarrow{\sim} Q \otimes S$ be isomorphisms. Assume that $\chi \otimes S = \wedge^2 \theta$. Then $P \simeq Q$.

The Euler class group of $R[T]$ relative to a line bundle

We now go back to our discussion on Question 1 at the beginning. As described there, Question 1 has a satisfactory solution in the case: $\text{rank}(P) = \dim(A)$. However, not much progress has been made for $\text{rank}(P) < \dim(A)$ (see [B-RS 5] for some results in this direction). The first case that one would like to investigate is obviously when $\text{rank}(P) = \dim(A) - 1$. In this context, it is most natural to inquire first what happens if A is a polynomial algebra, i.e., $A = R[T]$, where R is a Noetherian $\mathbb{Q}$-algebra of dimension n, and P is a projective $R[T]$-module of rank n. In this setup, we settle Question 1 here. A partial solution in the same setup has been obtained by Das in [D 1]. We give the details below.

Let R be a commutative Noetherian ring of dimension $n \geq 2$ containing $\mathbb{Q}$. Following the works of Bhatwadekar and Raja Sridharan [B-RS 1, B-RS 4] on the Euler class groups, the notion of the n-th Euler class group $E^n(R[T])$ has been defined and explored in detail in [D 1, D 2]. This group serves as an obstruction group to detect whether a given projective $R[T]$-module P of rank n, *with trivial determinant*, splits as $P \simeq Q \oplus R[T]$. To achieve this, given such a P and a trivialization $\chi : R[T] \xrightarrow{\sim} \wedge^n(P)$, an element of $E^n(R[T])$ was associated to the pair (P, χ), which is called the Euler class of (P, χ). It was then proved [D 1, Corollary 4.11] that $P \simeq Q \oplus R[T]$ if and only if this Euler class vanishes in $E^n(R[T])$.

Evidently the theory was limited, as it could only capture projective $R[T]$-modules with trivial determinant. Here we eliminate that restriction. We extend the theory to $E^n(R[T], L)$ (the n-th Euler class group of $R[T]$ with respect to a line bundle L over $R[T]$) and define the Euler class of a pair (P, χ), where P is a projective $R[T]$-module of rank n and $\chi : L \xrightarrow{\sim} \wedge^n(P)$ an isomorphism. We prove the following (Theorem 8.0.2).

Theorem F. The Euler class of the pair (P, χ) vanishes in $E^n(R[T], L)$ if and only if P splits off a free summand of rank one.

We carry this out in two steps. First, in Chapter 6 we tackle the case when the line bundle L is extended from R. We have been able to extend almost all the relevant results from [D 1] here. These results are then crucially used to develop the theory further, as discussed below.

But before we describe our next step, let us digress a bit. It is not hard to believe that for all practical purposes we may assume that R is reduced. Now let R_1 be the seminormalization of R. Then R_1 is seminormal and as a consequence, $\text{Pic}(R_1) \simeq \text{Pic}(R_1[T])$. In other words, line bundles over $R_1[T]$ are extended from R_1. We successfully exploit this phenomenon to define and study the n-th Euler class group $E^n(R[T], L)$ for $n \geq 4$, when L is not necessarily extended from R.

For an arbitrary L, the idea is to find a suitable finite subintegral extension S of R such that the projective $S[T]$-module $L \otimes S[T]$ is extended from S. Note that, $E^n(S[T], L \otimes S[T])$ is now well-understood. In Chapter 7 we introduce a machinery, which is modelled on a series of lemmas from [B 1], to descend from $S[T]$ to $R[T]$. Then we develop the theory of the Euler class group $E^n(R[T], L)$ by going forth and back by using these crucial 'descent lemmas'.

As an application of the descent lemmas, we also prove the following (Theorem 7.0.4) which generalizes Mandal's result ([M1, Theorem 1.2]).

Theorem G. Let $A = R[T]$ be a polynomial ring over a commutative Noetherian ring R with $\dim(R) = n \geq 4$. Let I be an ideal of A of height n that contains a monic polynomial. let $\mathbb{L}$ be a projective $R[T]$-module of rank 1. Write $\mathfrak{L} = \mathbb{L} \oplus R[T]^{n-1}$. Suppose that there exists $\alpha : \mathfrak{L} \twoheadrightarrow I/I^2$. Then there is a surjection $\beta : \mathfrak{L} \twoheadrightarrow I$ such that

β lifts α.

Unfortunately the method we just described above used for $dim(R) \geq 4$, does not work so well in the case $\dim(R) = 3$ due to the lack of a suitable "subtraction principle". We treat this case separately, by defining a "restricted" Euler class group which serves most of our purposes. For instance, here also we prove that the Euler class of a projective $R[T]$-module P of rank 3 is the precise obstruction for P to split off a free summand of rank one. We also treat the case $\dim(R) = 2$ and extend most of the results from [D 1] for a two dimensional ring R.

Layout

The layout of this thesis is as follows.

In Chapter 2, we recall definitions and known results. Sometimes we have provided proofs of others' results due to two reasons: (1) there is no suitable proof available for the versions we are interested in, (2) to make the thesis as much self-contained as possible. We also prove some preliminary results in this chapter which will be required in subsequent chapters.

In Chapter 3 we consider subintegral extensions, prove Theorem A and other allied results and some applications of the main theorems.

Chapter 4 is about integral extensions of rings and the relations of the Euler class group under such extensions. In this chapter we prove a more general form of Theorem A in the form of Theorem D. Further, we give an example to show that Theorem A is not valid for arbitrary integral extension even if it is finite birational.

In Chapter 5 we carry out a delicate investigation of subintegral extension of two dimensional rings and consider a question of Ischebeck, posed in [I 2].

Chapters 6, 7, 8 are about developing the theory of n-th Euler class group for a polynomial algebra $R[T]$ with respect to a projective $R[T]$-module L of rank one. In Chapter 6 we consider the special case when L is extended from R. In Chapter 7 we prove some "descent lemmas", which we crucially use to build the theory in Chapter 8 in the general case when L is not necessarily extended from R.

Chapter 2

Preliminaries

All the rings considered in this thesis are commutative and Noetherian. By dimension of a ring we mean its Krull dimension. Modules are assumed to be finitely generated. Projective modules are assumed to have constant rank.

2.1 Definitions and general results

In this section we shall give some definitions and prove some preliminary results which will be used throughout this thesis. We start with the following definition.

Definition 2.1.1. Let R be a ring and P be an R-module. Then P is said to be *projective* if there exists an R-module Q such that $P \oplus Q \simeq R^n$ for some positive integer n, in other words $P \oplus Q$ is free.

Definition 2.1.2. Let R be a ring and P be a projective R-module. An element $p \in P$ is called *unimodular* if there is a surjective R-linear map $\phi : P \twoheadrightarrow R$ such that $\phi(p) = 1$. The set of all unimodular elements of P is denoted by $\mathrm{Um}(P)$. If $P = R^n$, then we write $\mathrm{Um}_n(R)$ for $\mathrm{Um}(R^n)$.

Remark 2.1.1. It is easy to see that if a projective R-module P has a unimodular element, then $P \simeq Q \oplus R$ for some R-module Q. We describe this phenomenon by saying that P splits off a free summand of rank one.

The following result is due to Serre [Se]. We shall refer to this result as "Serre's splitting theorem".

Theorem 2.1.1. *Let R be a ring and P be a projective R-module. If rank$(P) \geq$ dim$(R) + 1$, then P splits off a free summand of rank one, i.e., $P \simeq Q \oplus R$ for some R-module Q.*

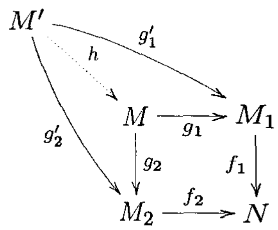
Let P be a projective R-module of rank n and φ be an R-linear endomorphism of P. Then ϕ induces an endomorphism $\wedge^n(\phi)$ of $\wedge^n(P)$ in a natural way. We call $\wedge^n(P)$ the *determinant* of P. We call the endomorphism $\wedge^n(\phi)$ the determinant of ϕ and denote it by $\det(\phi)$. As $\wedge^n(P)$ is a projective R-module of rank one, $\det(\phi) \in R$. It can be easily checked that ϕ is an automorphism if and only if $\det(\phi)$ is a unit of R.

Definition 2.1.3. Let P be a projective R-module. We define $SL(P)$ to be the group of automorphisms of P of determinant one. If $P = R^n$, then we write $SL_n(R)$ for $SL(R^n)$.

We now recall the definition of a subgroup of $SL(P)$. Given $\varphi \in P^*(= \text{Hom}_R(P, R))$ and $p \in P$, we define an endomorphism φ_p of P as the composite $P \xrightarrow{\varphi} R \xrightarrow{p} P$. If $\varphi(p) = 0$, then $\varphi_p^2 = 0$ and $1 + \varphi_p$ is an automorphism of P.

Definition 2.1.4. An automorphism of P is called a *transvection* if it is of the form $1 + \varphi_p$ where $\varphi(p) = 0$ and either φ is unimodular in P^* or p is unimodular in P. The subgroup of $SL(P)$ generated by all transvections will be denoted by $\mathcal{E}(P)$. If $n \geq 3$ and $P = R^n$, then, by a result of Suslin [Su 2, 1.4], $\mathcal{E}(R^n)$ can be identified with $\mathcal{E}_n(R)$, the group of $n \times n$ elementary matrices.

Definition 2.1.5. Let A be a ring and let $f_1 : M_1 \longrightarrow N$ and $f_2 : M_2 \longrightarrow N$ be A-linear maps. The *fiber product* of M_1 and M_2 over N is a triple (M, g_1, g_2), where M is an A-module, $g_1 : M \longrightarrow M_1$ and $g_2 : M \longrightarrow M_2$ are A-linear maps such that $f_1 \circ g_1 = f_2 \circ g_2$ and the triple is universal in the sense that given any other triple (M', g_1', g_2') where M' is an A-module, $g_1' : M' \longrightarrow M_1$ and $g_2' : M' \longrightarrow M_2$ are A-linear maps such that $f_1 \circ g_1' = f_2 \circ g_2'$, there is a unique homomorphism $h : M' \longrightarrow M$ such that $g_1 \circ h = g_1'$ and $g_2 \circ h = g_2'$.



Remark 2.1.2. We say that

$$\begin{array}{ccc} M & \xrightarrow{g_1} & M_1 \\ \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle f1} \\ M_2 & \xrightarrow{f_2} & N \end{array}$$

is a *fiber product diagram* or *Cartesian diagram* to mean M is the fiber product of M_1 and M_2 over N as in the above definition.

The following examples of Cartesian diagrams will be used later.

Example 2.1.1. Let A be a ring and let M be an A-module. Let $s, t \in A$ be such that $As + At = A$. Then

$$\begin{array}{ccc} A & \longrightarrow & A_s \\ \downarrow & & \downarrow \\ A_t & \longrightarrow & A_{st} \end{array} \qquad \begin{array}{ccc} M & \longrightarrow & M_s \\ \downarrow & & \downarrow \\ M_t & \longrightarrow & M_{st} \end{array}$$

are Cartesian diagrams of commutative rings and A-modules respectively.

Example 2.1.2. Let A be a ring and I, J be ideals of A. Let M be an A-module, then

$$\begin{array}{ccc} A/I \cap J & \longrightarrow & A/I \\ \downarrow & & \downarrow \\ A/J & \longrightarrow & A/I + J \end{array} \qquad \begin{array}{ccc} M/(IM \cap JM) & \longrightarrow & M/IM \\ \downarrow & & \downarrow \\ M/JM & \longrightarrow & M/(I + J)M \end{array}$$

are Cartesian diagrams of A-modules.

Definition 2.1.6. Let $R \hookrightarrow S$ be a ring extension. Then the ideal $C := \{a \in R | aS \subset R\}$ is called the *conductor ideal* of R in S. It is the largest ideal in R which is also an ideal in S.

Example 2.1.3. Let $R \hookrightarrow S$ be a ring extension and C be the conductor ideal of R in S. Let M be an R-module. Then the following are Cartesian diagrams

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R/C & \longrightarrow & S/C \end{array} \qquad \begin{array}{ccc} M & \longrightarrow & M \otimes_R S \\ \downarrow & & \downarrow \\ M \otimes_R R/C & \longrightarrow & M \otimes_R S/C \end{array}$$

Definition 2.1.7. Let $R \subset S$ be rings and $f \in R$ be a non-zerodivisor in both R and S. Then we say that the inclusion map $i : R \hookrightarrow S$ is an *analytic isomorphism along* f if $R/(f) \simeq S/(f)$ or equivalently $S = R + fR$ and $fS \cap R = fR$.

The following result is due to Nashier [N, Proposition 1.3].

Proposition 2.1.1. *Let $R \subset S$ be rings. Let I be an ideal of S and $J = I \cap R$. Suppose there is an element f in J such that the inclusion map $i : R \hookrightarrow S$ is an analytic isomorphism along f. Then*

(i) $R/J \simeq S/I$,

(ii) $I = JS$,

(iii) $J/J^2 \simeq I/I^2$ *as* R/J^2 *or* S/I^2*-modules.*

The following classical result is due to Bass [Ba].

Theorem 2.1.2. *Let A be a ring and let P be a projective A-module such that rank$(P) >$ dim(A). Then the group $\mathcal{E}(P \oplus A)$ of transvections of $P \oplus A$ acts transitively on Um$(P \oplus A)$.*

The following result is due to Lindel [L 2, Theorem 2.6].

Theorem 2.1.3. *Let A be a ring with* $\dim(A) = d$ *and* $R = A[T_1, \cdots, T_n]$. *Let P be a projective R-module of rank* $\geq \max(2, d+1)$. *Then $\mathcal{E}(P \oplus R)$ acts transitively on the set of unimodular elements of $P \oplus R$.*

Proposition 2.1.2. *Let R be a ring and P be a projective R-module such that P has a unimodular element. Let $\alpha, \beta \in$ Um(P^*) be such that $\alpha \equiv \beta$ modulo the nil radical $\mathfrak{n}$ of R. Then there is a transvection θ of P such that $\beta = \alpha\theta$.*

Proof. Applying [MK-M-R, Remark 2.3] it follows that there is $\Theta \in \mathcal{E}(P^*)$ such that $\Theta(\alpha) = \beta$. Therefore, Θ is a finite product of transvections of the projective module P^*. For simplicity, we prove this proposition by assuming that Θ itself is a transvection. The general case can be worked out in a similar manner.

Let $\Theta = 1 + \psi_\phi$, where $\psi \in P^{**}$ and $\phi \in P^*$ such that $\psi(\phi) = 0$. Since P is a projective module, P^{**} can be identified with P. Therefore, we may assume $\psi = p$ for some $p \in P$. With this identification we have $\psi(\phi) = \phi(p) = 0$. Now from the definition of a transvection, we have, either $\psi \in Um(P^{**})$ or $\phi \in Um(P^*)$. If $\psi \in Um(P^{**})$, then note that $p \in \mathrm{Um}(P)$. Therefore $1 + \phi_p$ is a transvection of P (as $\phi(p) = 0$).

Now we have $\Theta(\alpha) = (1 + \psi_\phi)(\alpha) = \beta$. Therefore, for any $q \in P$, we have $(1 + \psi_\phi)(\alpha)(q) = \beta(q)$. But $(1 + \psi_\phi)(\alpha)(q) = \alpha(q) + (\phi\psi)(\alpha)(q) = \alpha(q) + (\phi\alpha(p))(q) = \alpha(q) + \alpha(p)\phi(q)$.

On the other hand, $\alpha(1 + \phi_p)(q) = \alpha(q + p\phi(q)) = \alpha(q) + \alpha(p)\phi(q)$ and hence $\alpha(1 + \phi_p)(q) = \beta(q)$ for all $q \in P$. Therefore, if we write $\theta = 1 + \phi_p$, then θ is a transvection of P and $\alpha\theta = \beta$. □

The following result is due to Bhatwadekar and Roy and is about lifting of a transvection of a projective module.

Proposition 2.1.3. *[B-R 1, Proposition 4.1] Let A be a ring, $J \subset A$ be an ideal and P be a projective A-module of rank n. Then any transvection $\widetilde{\theta} \in \mathcal{E}(P/JP)$ can be lifted to a (unipotent) automorphism θ of P. If in addition, the map $Um(P) \to Um(P/JP)$ is surjective, then the map $\mathcal{E}(P) \to \mathcal{E}(P/JP)$ is surjective.*

Let S be a ring and C be an ideal of S. Let P be a projective S-module. The above result of Bhatwadekar and Roy asserts that any transvection σ of P/CP can be lifted to a unipotent automorphism of P. We need a variant of their result in the following form.

Proposition 2.1.4. *Let S be a ring and J, C be ideals of S such that $J + C = S$. Let P be a projective S-module and σ be a transvection of P/CP. Then σ can be lifted to $\tau \in Aut(P)$ with the property that τ is identity modulo J.*

Proof. Let $\sigma = 1 + \psi_q$, where $\psi \in (P/CP)^*$ and $q \in P/CP$ such that $\psi(q) = 0$. Let $p \in P$ and $\theta \in P^*$ be lifts of q and ψ, respectively. Then we have $\theta(p) = c$, for some

$c \in C$.

We first consider the case when q is a unimodular element of P/CP. Then there exists $\varphi \in P^*$ such that $\varphi(p) = 1 + d$, for some $d \in C$.

Set $\phi' = (1+d)\theta - c\varphi$. Then $\phi'(p) = 0$ and ϕ' is a lift of ψ. Therefore $1 + \phi'_p \in \text{Aut}\,(P)$ and it lifts σ. We have $J + C = S$. Therefore, there exist $a \in J$ and $b \in C$ such that $a + b = 1$. Finally we consider $\tau = 1 + a\phi'_p$. Then again $\tau \in \text{Aut}\,(P)$, $\tau = \text{Id}$ modulo J and τ is a lift of σ.

Next we consider the case when $\psi \in \text{Um}((P/CP)^*)$. Then there exists $p' \in P$ such that $\theta(p') = 1 + e$, for some $e \in C$. Consider the element $q' = (1+e)p + cp'$. Then $\theta(q') = 0$. Therefore, $\tau = 1 + a\theta_{q'}$ will work. □

Lemma 2.1.1. *Let $R \hookrightarrow S$ be an extension of rings and C be the conductor of R in S. Let $J \subset R$ be an ideal such that $J + C = R$ and $JS \neq S$. Then the natural map $f : J \otimes_R S \twoheadrightarrow JS$ is an isomorphism (of S-modules).*

Proof. We use a local-global argument to prove that f is an isomorphism. To see this, let $\mathfrak{m}$ be a maximal ideal of S and let $\mathfrak{p} = \mathfrak{m} \cap R$. First note that

$$(J \otimes_R S)_{\mathfrak{m}} = J \otimes_R S_{\mathfrak{m}} = (J \otimes_R R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{m}} = J_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{m}}.$$

If $C \not\subset \mathfrak{m}$, then $C \not\subset \mathfrak{p}$ as well, and $R_{\mathfrak{p}} = S_{\mathfrak{p}} = S_{\mathfrak{m}}$ and in this case $J_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{m}}$ and $J_{\mathfrak{p}} S_{\mathfrak{m}}$ are both isomorphic to $J_{\mathfrak{p}}$, and the isomorphism is induced by f. If $C \subset \mathfrak{m}$, then $JS \not\subset \mathfrak{m}$, $J \not\subset \mathfrak{p}$, and therefore $J_{\mathfrak{p}} = R_{\mathfrak{p}}$, $(JS)_{\mathfrak{m}} = S_{\mathfrak{m}}$. Again, it can be easily seen that f induces the isomorphism. □

The proof of the following lemma can be found in [B-RS 4, Corollary 2.13]. This is a consequence of a result of Eisenbud-Evans [E-E], as stated in [P, p. 1420]. .

Lemma 2.1.2. *Let A be a ring and P be a projective A-module of rank n. Let $(\alpha, a) \in (P^* \oplus A)$. Then there exists an element $\beta \in P^*$ such that $\text{ht}\,(I_a) \geq n$, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geq n$ then $\text{ht}\, I \geq n$. Further, if $(\alpha(P), a)$ is an ideal of height $\geq n$ and I is a proper ideal of A, then $\text{ht}\, I = n$.*

The following lemma is standard. We give a proof for the sake of completeness.

Lemma 2.1.3. *Let R be a ring and $J \subset R$ be an ideal of R. Let $K \subset J$ and $L \subset J^2$ be two ideals of R such that $K + L = J$. Then $J = K + (e)$ for some $e \in L$ with $e(1-e) \in K$ and $K = J \cap J'$ where $J' + L = R$.*

Proof. Let bar denote reduction modulo the ideal K. Since $\overline{J}^2 = \overline{J}$, by Nakayama lemma there exists $\overline{e} \in \overline{J}$ such that $(1 - \overline{e})\overline{J} = 0$. It then follows that $\overline{J} = (\overline{e})$ and $\overline{e}^2 = \overline{e}$. Since $K + L = J$, we can assume that $e \in L$. Now $J = K + (e)$. Since $\overline{e}^2 = \overline{e}$, we have $e - e^2 \in K$. Take $J' = K + (1 - e)$. Then $L + J' = R$, since $e \in L$. We claim that $K = J \cap J'$.

Let $a \in J \cap J'$. Then $a = b + ed = b_1 + (1-e)d_1$, where $b, b_1 \in K$ and $d, d_1 \in R$. This implies that $ed - (1 - e)d_1 \in K$. But $e - e^2 \in K$. Therefore $e^2 d \in K$ and consequently $ed \in K$. Therefore $a \in K$. This proves $K = J \cap J'$. □

The next lemma, which is an application of Lemma (2.1.3) and Lemma (2.1.2), is a synthesis of [B-RS 4, Corollary 2.14] and [B-RS 5, Corollary 2.4]. We shall call this lemma as the "moving lemma".

Lemma 2.1.4. (Moving Lemma) *Let R be a ring of dimension d and let P be a projective R-module of rank n, where $2n \geq d + 2$. Let $J \subset R$ be an ideal of height n and let $\overline{\alpha} : P/JP \twoheadrightarrow J/J^2$ be a surjection. Then there exists an ideal $J' \subset R$ and a surjection $\beta : P \twoheadrightarrow J \cap J'$ such that:*

(i) $J + J' = R$.

(ii) $\beta \otimes R/J = \overline{\alpha}$.

(iii) $ht(J') \geq n$.

(iv) *Given finitely many ideals $J_1, \cdots, J_r$ of R, each of height $\geq d - n + 1$, the ideal J' can be chosen with the additional property that it is comaximal with J_i for each $i = 1, \cdots, r$.*

Proof. As P is a projective module, $\overline{\alpha}$ can be lifted to an R-linear map $\beta : P \longrightarrow J$. Then $\beta(P) + J^2 = J$. By Lemma 2.1.3, there exists $b \in J^2$ such that $\beta(P) + (b) = J$. Let $K = J^2 \cap J_1 \cap \cdots \cap J_r$ and bar denote reduction modulo the ideal K. Note that $\dim(\overline{R}) \leq n - 1$.

Applying Lemma 2.1.2 to the element $(\overline{\beta}, \overline{b}) \in \overline{P}^* \oplus \overline{R}$, we see that there exists $\beta_1 \in P^*$ such that if $N = (\beta + b\beta_1)(P)$ then $\text{ht}(\overline{N}_{\overline{b}}) \geq n$.

Since $\beta + b\beta_1$ is also a lift of $\overline{\alpha}$, replacing β with $\beta + b\beta_1$, we may assume that $N = \beta(P)$. Now $N + (b) = J$ and $b \in J^2$. By Lemma 2.1.3, $N = J \cap J_1$, where $J_1 + (b) = R$. Since $N_b = (J_1)_b$ and $\overline{N} = \overline{J} \cap \overline{J_1}$, we get

$$\text{ht}(\overline{J_1}) = \text{ht}(\overline{J_1})_{\overline{b}} = \text{ht}(\overline{N}_{\overline{b}}) \geq n.$$

But then we have $n \leq \text{ht}(\overline{J_1}) \leq \dim(\overline{R}) \leq n - 1$. Hence we get $\overline{J_1} = \overline{R}$. Therefore, $\overline{N} = \overline{J}$ and hence $\beta(P) + K = J$.

By Lemma 2.1.3, there exists $c \in K$ such that $\beta(P) + (c) = J$. By Lemma 2.1.2, replacing β by $\beta + c\beta_2$ for some $\beta_2 \in P^*$, we may assume that $\beta(P) = J \cap J'$, where $\text{ht}(J') \geq n$ and $J' + (c) = R$. This proves the lemma. □

The following lemma is again an easy application of (2.1.3) and (2.1.2). We give a proof for the sake of completeness.

Lemma 2.1.5. *Let A be a ring of dimension d and I be an ideal of A. Let P be a projective A module with rank$(P) = n \geq d + 1$. Assume that there exists a surjection $\alpha : P/IP \twoheadrightarrow I/I^2$. Then α can be lifted to a surjection $\beta : P \twoheadrightarrow I$.*

Proof. Let $\beta : P \longrightarrow I$ be a lift of α. Then $\beta(P) + I^2 = I$ and therefore, by Lemma 2.1.3, there exists $b \in I^2$ such that $\beta(P) + (b) = I$. Applying Lemma 2.1.2 to the element $(\beta, b) \in P^* \oplus A$, we see that there exists $\gamma \in P^*$ such that if $N = (\beta + b\gamma)(P)$ then $\text{ht}(N_b) \geq n$. Since $\dim(A) = d$ and $n \geq d + 1$, it follows that $b^r \in N$ for some positive integer r.

Since $\beta + b\gamma$ is also a lift of α, it is enough to show that $N = I$. We prove this by showing that $N_\mathfrak{p} = I_\mathfrak{p}$ for all $\mathfrak{p} \in \text{Spec}(A)$. If $N \not\subset \mathfrak{p}$, then clearly $I \not\subset \mathfrak{p}$ and $N_\mathfrak{p} = I_\mathfrak{p} = A_\mathfrak{p}$. If $N \subset \mathfrak{p}$, then as $N + (b) = I$ and $b^r \in N$, it follows that $I \subset \mathfrak{p}$. Note that $b(1-b) \in N$ and $1 - b \in {A_\mathfrak{p}}^*$. Therefore $N_\mathfrak{p} = I_\mathfrak{p}$. This completes the proof. □

The following proposition is implicit in the proof of [B-RS 2, Proposition 2.5].

Proposition 2.1.5. *Let A be a ring of dimension $d \geq 1$ and I be an ideal of $A[T]$ of height ≥ 2. Assume that $I = (f_1, \cdots, f_n) + I^2$, where $n \geq d + 1$. Then there exist*

$g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ with $f_i - g_i \in I^2$ for $i = 1, \cdots, n$.

We improve the above proposition in the following form to suit our needs. The proof is similar to the one given in [B-RS 2].

Proposition 2.1.6. *Let A be a ring of dimension $d \geq 2$. Let I be an ideal of $A[T]$ of height ≥ 2 and P be a projective $A[T]$-module of rank $n \geq d+1$. Suppose that there exists $\phi : P/IP \twoheadrightarrow I/I^2$. Then ϕ can be lifted to a surjection $\psi : P \twoheadrightarrow I$.*

Proof. Since $\text{rank}(P) = n \geq \dim(A) + 1$, by a result of Plumstead [P, Corollary 2 of Section 3], P has a free summand of rank one. Let $P = Q \oplus A[T]$, where Q is a projective $A[T]$ module. Let $J = I \cap A$. Since $\text{ht}(I) \geq 2$, we have $ht(J) \geq 1$. Therefore we can choose $b \in J^2$ such that $\text{ht}(b) = 1$. Now

$$n \geqq \dim(A) + 1 = \dim(A[T]) - 1 + 1 \geq \dim(A[T]/bA[T]) + 1$$

Therefore, by Lemma 2.1.5 we have $\Phi : P/bP \twoheadrightarrow I/(b)$, and Φ is a lift of $\phi \otimes A[T]/bA[T]$. Let $\gamma \in \text{Hom}_{A[T]}(P, I)$ be a lift of Φ (not necessarily surjective). Then, as $I/bA[T] = \Phi(P/bP)$, we have $\gamma(P) + bA[T] = I$. By applying Lemma 2.1.2 to the pair $(\gamma, b) \in \text{Hom}_{A[T]}(P, A[T]) \oplus A[T]$, we see that there exists $\beta \in \text{Hom}_{A[T]}(P, A[T])$ such that $\text{ht}(K_b) \geq n$, where $K = (\gamma + b\beta)(P)$.

Note that $K + bA[T] = I$ and $b \in I^2$. By Lemma 2.1.3, there exists an ideal I' of $A[T]$ such that $K = I \cap I'$ and $I' + bA[T] = A[T]$. Now $\text{ht}(I') = \text{ht}(I'_b) = \text{ht}(K_b) \geq n$. Setting $\mu = \gamma + b\beta$ we further observe:

(i) $\mu : P \twoheadrightarrow I \cap I'$ is a surjection;

(ii) $\mu \otimes A[T]/I = \phi$,

If $I' = A[T]$, then we are done since then $\mu : P \twoheadrightarrow I \cap I' = I$ and μ is a lift of ϕ (recall that $b \in I^2$). If I' is a proper ideal then clearly I' contains a monic polynomial and by [La, Lemma 1.1, p. 79] we have $I' \cap A + bA = A$. Therefore I' contains an element of the form $1 + ba$ for some $a \in A$. Hence $I'_{1+bA} = A[T]_{1+bA}$, and therefore we have a surjection $\mu_{1+bA} : P_{1+bA} \twoheadrightarrow I_{1+bA}$.

Now $P = Q \oplus A[T]$. Let $\theta_b : Q_b \oplus A_b[T] \twoheadrightarrow A_b[T]$ be the projection onto the second factor. Now consider the following surjections:

$$\mu_{b(1+bA)} : P_{b(1+bA)} \twoheadrightarrow A_{b(1+bA)}[T]$$

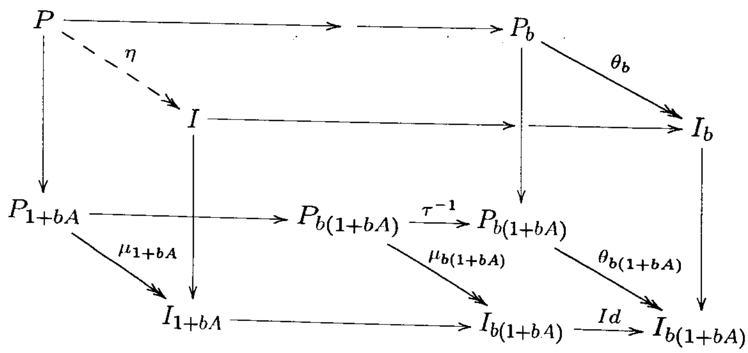
$$\theta_{b(1+bA)} : P_{b(1+bA)} \twoheadrightarrow A_{b(1+bA)}[T]$$

So we have two unimodular elements of $P^*_{b(1+bA)}$.

Observe that $\text{rank}(Q_{b(1+bA)}) = n - 1 \geq \max(2, d)$ and $d - 1 = \dim(A_{b(1+bA)})$. Therefore, by Theorem 2.1.3 we have a transvection $\tau \in \mathcal{E}(P_{b(1+bA)})$ such that

$$\mu_{b(1+bA)}\tau = \theta_{b(1+bA)}$$

Now consider the following fiber product diagram:



By a standard patching argument we have a surjection $\eta : P \twoheadrightarrow I$. It is easy to see that η is a lift of ϕ.

□

Lemma 2.1.6. *Let A be a ring and let I and K be two ideals of $A[T]$ such that $K \subset I^2$. Let P be a projective $A[T]$-module and $\mathfrak{n}$ be the nilradical of A. Let bar denote reduction modulo $\mathfrak{n}[T]$. Suppose that $\alpha : P \twoheadrightarrow I/K$ is a surjection such that the induced map $\overline{\alpha} : P \twoheadrightarrow \overline{I}/\overline{K}$ can be lifted to a surjection $\beta : P \twoheadrightarrow \overline{I}$. Then α can also be lifted to a surjection $\phi : P \twoheadrightarrow I$.*

Proof. We have $\beta : P \twoheadrightarrow \overline{I}$, which is a lift of $\overline{\alpha}$. Therefore, we have

$$\beta : P \twoheadrightarrow (I + \mathfrak{n}[T])/\mathfrak{n}[T] \xrightarrow{\sim} I/(I \cap \mathfrak{n}[T]).$$

We note that $I/(K \cap \mathfrak{n}[T])$ is the fiber product of I/K and $I/(I \cap \mathfrak{n}[T])$ over $I/(K, I \cap \mathfrak{n}[T])$:

$$\begin{array}{ccc} I/(K \cap \mathfrak{n}[T]) & \longrightarrow & I/K \\ \downarrow & & \downarrow \\ I/(I \cap \mathfrak{n}[T]) & \longrightarrow & I/(K, I \cap \mathfrak{n}[T]) \end{array}$$

Therefore α and β will patch to yield a surjection $\gamma : P \twoheadrightarrow I/(K \cap \mathfrak{n}[T])$. Let $\phi : P \longrightarrow I$ be a lift of γ. We prove that ϕ is surjective. We have $\phi(P) + (K \cap \mathfrak{n}[T]) = I$. Since $\mathfrak{n}$ is the nil radical of A, it follows that $V(I) = V(\phi(P))$ (For an ideal $J \subset A[T]$, by $V(J)$, we mean the subset of $Spec(A[T])$, consisting of those prime ideals which contain J) and hence $I = \phi(P)$. Since ϕ lifts α, we are done. □

The proof of the following lemma can be found in [B-RS 1, Remark 3.9].

Lemma 2.1.7. *Let A be a ring, $I \subset A[T]$ be an ideal and P be a projective A-module. Let $\tilde{\varphi} : P[T] \twoheadrightarrow I/I^2$ be a surjection. Assume further that either $I(0) = A$ or there is a surjection $\psi : P \twoheadrightarrow I(0)$ such that $\tilde{\varphi}(0) = \psi \otimes A/I(0)$. Then we can lift $\tilde{\varphi}$ to a surjection $\overline{\varphi} : P[T] \twoheadrightarrow I/(I^2T)$.*

The following theorem is due to Mandal and Raja Sridharan [M-RS].

Theorem 2.1.4. *Let A be a ring and $R = A[T]$. Let $I = I' \cap I''$ be the intersection of two ideals I' and I'' in R such that*

(i) *I' contains a monic polynomial,*

(ii) *$I'' = I''(0)R$ is an extended ideal and*

(iii) *$I' + I'' = R$.*

Suppose that P is a projective A-module of rank $r \geq dim(R/I') + 2$ and $f : P \twoheadrightarrow I(0)$ and $\phi : P[T]/I'P[T] \twoheadrightarrow I'/I'^2$ are two surjective linear maps such that $\phi(0) = f$ mod $I'(0)^2$. Then there is a surjective map $\psi : P[T] \twoheadrightarrow I$ such that $\psi(0) = f$.

2.2 The Euler class group and related results

In this section we quickly recall the generalities of the Euler class group theory. We first accumulate some basic definitions, namely, the definitions of the Euler class group, the Euler class of a projective module, and then state some results which are relevant to this thesis. Detailed accounts of these topics can be found in [B-RS 4, D 1].

We start with the definition (from [B-RS 4]) of the n-th Euler class group $E^n(R, L)$ of a commutative Noetherian ring of dimension n with respect to a projective R-module L of rank one. For brevity of notation, we shall denote $E^n(R, L)$ by $E(R, L)$.

Definition 2.2.1. (*The Euler class group $E(R, L)$*): Write $F = L \oplus R^{n-1}$. Let $J \subset R$ be an ideal of height n such that J/J^2 is generated by n elements. Two surjections α, β from F/JF to J/J^2 are said to be *related* if there exists $\sigma \in SL(F/JF)$ such that $\alpha\sigma = \beta$. Clearly this is an equivalence relation on the set of surjections from F/JF to J/J^2. Let $[\alpha]$ denote the equivalence class of α. Such an equivalence class $[\alpha]$ is called a *local L-orientation* of J. By abuse of notation, we shall identify an equivalence class $[\alpha]$ with α. A local L-orientation α is called a *global L-orientation* if $\alpha : F/JF \twoheadrightarrow J/J^2$ can be lifted to a surjection $\theta : F \twoheadrightarrow J$.

Let G be the free abelian group on the set of pairs $(\mathfrak{n}, \omega_{\mathfrak{n}})$ where $\mathfrak{n}$ is an $\mathfrak{m}$-primary ideal for some maximal ideal $\mathfrak{m}$ of height n such that $\mathfrak{n}/\mathfrak{n}^2$ is generated by n elements and $\omega_{\mathfrak{n}}$ is a local L-orientation of $\mathfrak{n}$. Let $J \subset R$ be an ideal of height n such that J/J^2 is generated by n elements and ω_J is a local L-orientation of J. Let $J = \cap_i \mathfrak{n}_i$ be the (irredundant) primary decomposition of J. We associate to the pair (J, ω_J), the element $\sum_i (\mathfrak{n}_i, \omega_{\mathfrak{n}_i})$ of G where $\omega_{\mathfrak{n}_i}$ is the local L-orientation of $\mathfrak{n}_i$ induced by ω_J. By abuse of notation, we denote $\sum_i (\mathfrak{n}_i, \omega_{\mathfrak{n}_i})$ by (J, ω_J). Let H be the subgroup of G generated by set of pairs (J, ω_J), where J is an ideal of height n and ω_J is a global L-orientation of J. The Euler class group of R with respect to L is $E(R, L) \stackrel{\text{def}}{=} G/H$.

Remark 2.2.1. When $L \simeq R$, the Euler class group $E(R, R)$ is simply denoted by $E(R)$.

Remark 2.2.2. In [M-Y 2, Section 3] Mandal-Yang proved certain interesting functorial properties of the Euler class groups. Here we quote one of their results which is most relevant to this thesis. Let A, B be commutative Noetherian rings, each of dimension

$n \geq 2$. Let $f : A \longrightarrow B$ be a morphism of rings which satisfies a special property: *for any ideal I of A with $ht(I) = n$ and $\mu(I/I^2) = n$, the ideal $IB := f(I)B$ has height $\geq n$.* Let L be a projective A-module of rank one. Then they show that [M-Y 2, Definition 3.3] there is a group homomorphism $E(f) : E(A, L) \longrightarrow E(B, L \otimes_A B)$. Further, if $g : B \longrightarrow C$ is another morphism of rings satisfying the same property as above, one has the following commutative diagram (see [M-Y 2, Proposition 3.4]):

$$\begin{array}{ccc} E(A,L) & \xrightarrow{E(f)} & E(B, L \otimes B) \\ & \searrow^{E(gf)} & \downarrow{\scriptstyle E(g)} \\ & & E(C, L \otimes C) \end{array}$$

For example, if $f : A \longrightarrow B$ is a flat extension of rings, then f satisfies the property specified above. In Chapter 3, after introducing subintegral extension of rings, we shall show that such extensions enjoy the same property (see Remark 3.2.1).

Definition 2.2.2. (*The Euler class of a projective module*): Let P be a projective R-module of rank n such that $L \simeq \wedge^n(P)$ and let $\chi : L \xrightarrow{\sim} \wedge^n P$ be an isomorphism. Let $\varphi : P \twoheadrightarrow J$ be a surjection where J is an ideal of height n. Therefore we obtain an induced surjection $\overline{\varphi} : P/JP \twoheadrightarrow J/J^2$. Let $\overline{\gamma} : F/JF \xrightarrow{\sim} P/JP$ be an isomorphism such that $\wedge^n(\overline{\gamma}) = \overline{\chi}$. Let ω_J be the local L-orientation of J given by the composite surjection $\overline{\varphi}\,\overline{\gamma} : F/JF \twoheadrightarrow J/J^2$. Let $e(P, \chi)$ be the image in $E(R, L)$ of the element (J, ω_J) of G. If $\mathbb{Q} \subset R$, then it is proved in [B-RS 4] that the assignment sending the pair (P, χ) to the element $e(P, \chi)$ of $E(R, L)$ is well defined. The *Euler class* of (P, χ) is defined to be $e(P, \chi)$. We should note that the assumption $\mathbb{Q} \subset R$ is necessary only when we talk about the Euler class of a projective R-module. This assumption was needed to prove [B-RS 4, Proposition 3.1], which essentially shows that the Euler class of a projective module is well defined. But a careful inspection of the proof of [B-RS 4, Proposition 3.1] would reveal that if $\dim(R) = 2$, we do not need to assume that $\mathbb{Q} \subset R$ to define the Euler class. However, in the definition of $E(R[T])$, we need to assume that $\mathbb{Q} \subset R$ to start with.

We recall some results from [B-RS 4] for later use.

Theorem 2.2.1. *[B-RS 4, 4.2, 4.3, 4.4] Let R be a ring of dimension $n \geq 2$ and L be a* *projective R-module of rank* 1. *Let* P *be a projective* R*-module of rank* n *with* $L \simeq \wedge^n(P)$ *and let* $\chi : L \xrightarrow{\sim} \wedge^n P$ *be an isomorphism. Let* $J \subset R$ *be an ideal of height* n *and* ω_J *be a local* L*-orientation of* J.

(i) *Suppose that the image of* (J, ω_J) *is zero in* $E(R, L)$. *Then there exists a surjection* $\alpha : F \twoheadrightarrow J$ *such that* ω_J *is induced by* α *(in other words,* ω_J *is a global* L*-orientation).*

(ii) *Assume that* $\mathbb{Q} \subset R$. *Let* $e(P, \chi) = (J, \omega_J)$ *in* $E(R, L)$. *Then there exists a surjective map* $\alpha : P \twoheadrightarrow J$ *such that* (J, ω_J) *is induced by* (α, χ).

(iii) *Assume that* $\mathbb{Q} \subset R$. *Then,* $P \simeq P_1 \oplus R$ *for some projective* R*-module* P_1 *of rank* $n - 1$ *if and only if* $e(P, \chi) = 0$ *in* $E(R, L)$.

Let R be a commutative Noetherian ring containing $\mathbb{Q}$ with $\dim(R) = n \geq 2$. The notion of the n-th Euler class group $E^n(R[T])$ has been defined in [D 1]. We should note that the definition of $E^n(R[T])$ is different from that of $E^n(R)$ and is not obtained by just replacing R by $R[T]$. Further, for a commutative Noetherian ring A of dimension d and a projective A-module L of rank one, Mandal-Yang [M-Y 1] defined the r-th Euler class groups $E^r(A, L)$ for $1 \leq r \leq n$. The definition of $E^n(R[T])$ given below from [D 1] is not obtained from their definition either (by taking $A = R[T]$, $d = n+1$, $L = R[T]$ and $r = n$). Let us point out the difference. For an ideal I of $R[T]$ of height n with $\mu(I/I^2) = n$, a local orientation of I is defined as an $SL_n(R[T]/I)$-equivalence class of surjections in [D 1] whereas in [M-Y 1] a local orientation of I is defined as an $E_n(R[T]/I)$-equivalence class of surjections. Therefore, a priori the definitions are different and it will be interesting to know whether the groups thus obtained are isomorphic or not.

For brevity we denote $E^n(R[T])$ as $E(R[T])$ and recall its definition from [D 1].

Definition 2.2.3. (The Euler class group $E(R[T])$**)** Let R be a Noetherian ring of dimension $n \geq 3$ containing $\mathbb{Q}$. Let $I \subset R[T]$ be an ideal of height n such that I/I^2 is generated by n elements. Two surjections α and β from $(R[T]/I)^n \twoheadrightarrow I/I^2$ are said to be *related* if there exists $\sigma \in SL_n(R[T]/I)$ such that $\alpha\sigma = \beta$. This is an equivalence relation on the set of surjections from $(R[T]/I)^n$ to I/I^2. Let $[\alpha]$ denote the equivalence class of α. We call such an equivalence class $[\alpha]$ a *local orientation of* I. It was shown in [D 1, Proposition 4.4], that if $\alpha : (R[T]/I)^n \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : R[T]^n \twoheadrightarrow I$ then so can any β equivalent to α. We call a local orientation $[\alpha]$ of I a *global orientation of I* if the surjection $\alpha : (R[T]/I)^n \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : R[T]^n \twoheadrightarrow I$. Let G be the free abelian group on the set of pairs (I, ω_I) where $I \subset R[T]$ is an ideal of height n such that $\text{Spec}(R[T]/I)$ is connected, I/I^2 is generated by n elements and $\omega_I : (R[T]/I)^n \twoheadrightarrow I/I^2$ is a local orientation of I. Let $I \subset R[T]$ be an ideal of height n and ω_I a local orientation of I. Now I can be decomposed uniquely as $I = I_1 \cap \cdots \cap I_r$, where the I_k's are ideals of $R[T]$ of height n, pairwise comaximal, such that $\text{Spec}(R[T]/I_k)$ is connected for each k. Clearly ω_I induces local orientations ω_{I_k} of I_k for $1 \le k \le r$. By (I, ω_I) we mean the element $\Sigma(I_k, \omega_{I_k})$ of G. Let H be the subgroup of G generated by set of pairs (I, ω_I), where I is an ideal of $R[T]$ of height n generated by n elements and ω_I is a global orientation of I given by the set of generators of I. We define the Euler class group of $R[T]$, denoted by $E(R[T])$, to be G/H.

Definition 2.2.4. **(The Euler class of a projective $R[T]$-module)** Let R be as in (Definition 2.2.3). Let P be a projective $R[T]$-module of rank n with trivial determinant. Fix a trivialization $\chi : R[T] \xrightarrow{\sim} \wedge^n(P)$. Let $\alpha : P \twoheadrightarrow I$ be a surjection such that I is an ideal of height n. Note that, since P has trivial determinant and $\dim(R[T]/I) \le 1$, P/IP is a free $R[T]/I$-module. Composing $\alpha \otimes R[T]/I$ with an isomorphism $\gamma : (R[T]/I)^n \xrightarrow{\sim} P/IP$ with the property $\wedge^n(\gamma) = \chi \otimes R[T]/I$ we get a local orientation, say ω_I, of I. Let $e(P, \chi)$ be the image in $E(R[T])$ of the element (I, ω_I) of G. (We say that (I, ω_I) is obtained from the pair (α, χ)). It can be proved that the assignment sending the pair (P, χ) to $e(P, \chi)$ is well defined (see [D 1, Lemma 4.6]). The *Euler class* of (P, χ) is defined to be $e(P, \chi)$.

The following results are due to Das [D 1].

Theorem 2.2.2. *[D 1, 4.7, 4.10, 4.11] Let R be a Noetherian ring (containing $\mathbb{Q}$) of dimension $n \ge 3$. Let $I \subset R[T]$ be an ideal of $R[T]$ of height n such that I/I^2 is generated by n elements and ω_I be a local orientation of I. Let P be a rank n projective $R[T]$-module with trivial determinant with a trivialization $\chi : R[T] \simeq \wedge^n(P)$.*

(a) Suppose that the image of (I, ω_I) is zero in $E(R[T])$. Then ω_I is a global orientation of I.

(b) Suppose that $e(P, \chi) = (I, \omega_I)$ *in* $E(R[T])$. *Then there exists a surjection* $\alpha : P \twoheadrightarrow I$ *such that* ω_I *is induced by* α *and* χ *(as described above).*
(c) $P \simeq Q \oplus R[T]$ *for some projective* $R[T]$*-module* Q *if and only if* $e(P, \chi) = 0$ *in* $E(R[T])$.

The following results will be very useful in subsequent chapters.

Proposition 2.2.1. *Let* R *be a ring of dimension* n *and let* $R_{red} = R/\mathfrak{n}$, *where* $\mathfrak{n}$ *denotes the nilradical of* R. *Let* L *be a projective* R*-module of rank one.*

(i) *The groups* $E(R, L)$ *and* $E(R_{red}, L \otimes R_{red})$ *are canonically isomorphic [B-RS 4, Corollary 4.6].*

(ii) *Let* $\mathbb{Q} \subset R$. *Then* $E(R_{red}[T])$ *and* $E(R[T])$ *are canonically isomorphic [D 3, Proposition 2.15].*

We now recall the definition of the weak Euler class group of a ring from [B-RS 4].

Definition 2.2.5. (*The weak Euler class group* $E_0(R, L)$): Let R be a ring of dimension $n \geq 2$. Let L be a projective R-module of rank one. Write $F = L \oplus R^{n-1}$. Let G_0 be the free abelian group on the set of all ideals $\mathfrak{n}$, where $\mathfrak{n}$ is $\mathfrak{m}$-primary for some maximal ideal $\mathfrak{m}$ of height n such that there is a surjection $F \twoheadrightarrow \mathfrak{n}/\mathfrak{n}^2$. Given any ideal J of height n, we take the (irredundant) primary decomposition $J = \cap_i \mathfrak{n}_i$ and associate to J, the element $\sum_i \mathfrak{n}_i$ of G_0. We denote this element by (J). Let H_0 be the subgroup of G_0 generated by all (J) such that J is a surjective image of F. The weak Euler class group of R with respect to L is defined as $E_0(R, L) = G_0/H_0$.

Remark 2.2.3. It is clear from the above definitions that there is an obvious canonical surjective group homomorphism $\Theta_L : E(R, L) \twoheadrightarrow E_0(R, L)$ which sends an element (J, ω_J) of $E(R, L)$ to (J) in $E_0(R, L)$.

We shall need the following result on $E_0(R, L)$.

Proposition 2.2.2. *[B-RS 4, Proposition 6.2] Let* R *be a ring (containing* $\mathbb{Q}$*) of even dimension* n *and* $J \subset R$ *be an ideal of height* n. *Then* $(J) = 0$ *in* $E_0(R, L)$ *if and only if* J *is a surjective image of a projective* R*-module of rank* n *which is stably isomorphic to* $L \oplus R^{n-1}$.

Remark 2.2.4. It has been proved in [B-RS 4, Theorem 6.8] that the groups $E_0(R, L)$ and $E_0(R, R)$ are canonically isomorphic. Therefore, from now on we shall simply denote the weak Euler class group as $E_0(R)$.

2.3 Some Subtraction principles

One of the most important tools for the type of questions we are tackling in this thesis are the so called "subtraction principles". In this section we prove some subtraction principles. These results are used crucially in this thesis. We first state a couple of available subtraction principles and then prove some variants suited to fit our needs.

We first state a simplified version of [B-RS 4, Theorem 3.3].

Proposition 2.3.1. *Let A be a ring of dimension $n \geq 2$ and J, J' be two comaximal ideals of height n. Let $P = Q \oplus A$ be a projective A-module of rank n. Let $\alpha : P \twoheadrightarrow J \cap J'$ and $\beta : P \twoheadrightarrow J'$ be two surjections such that $\alpha \otimes A/J' = \beta \otimes A/J'$. Then there exists a surjection $\theta : P \twoheadrightarrow J$ such that $\theta \otimes A/J = \alpha \otimes A/J$.*

The following variant was proved in [B-K, Theorem 3.7].

Proposition 2.3.2. *Let A be a ring of dimension d and J, J' be two comaximal ideals of A of height n where $2n \geq d + 3$. Let $P = Q \oplus A$ be a projective A-module of rank n. Let $\alpha : P \twoheadrightarrow J \cap J'$ and $\beta : P \twoheadrightarrow J'$ be two surjections such that $\alpha \otimes A/J' = \beta \otimes A/J'$. Then there exists a surjection $\theta : P \twoheadrightarrow J$ such that $\theta \otimes A/J = \alpha \otimes A/J$.*

Modifying the proof of [B-RS 4, Theorem 3.3] we obtain the following subtraction principle.

Proposition 2.3.3. *Let $n \geq 4$ and A be a ring of dimension $n + 1$. Let P and L be projective A-modules of rank n and 1, respectively, such that $P \oplus A \simeq L \oplus A^n$. Write $Q = L \oplus A^{n-2}$. Let $\chi : \wedge^n(P) \xrightarrow{\sim} L$ be an isomorphism. Let $J \subseteq A$ be an ideal of height $\geq n$ and J' be an ideal of height n such that $J + J' = A$. Let $\alpha : P \twoheadrightarrow J \cap J'$ and $\beta : Q \oplus A \twoheadrightarrow J'$ be surjections. Let $\overline{\beta} = \beta \otimes A/J' : (Q \oplus A)/J'(Q \oplus A) \twoheadrightarrow J'/J'^2$ and $\overline{\alpha} = \alpha \otimes A/J' : P/J'P \twoheadrightarrow J'/J'^2$ be the induced surjections. Suppose that there exists an isomorphism $\delta : P/J'P \xrightarrow{\sim} (Q \oplus A)/J'(Q \oplus A)$ such that: (i) $\overline{\beta}\delta = \overline{\alpha}$ and (ii) $\wedge^n(\delta) = \overline{\chi}$. Then there exists a surjection $\theta : P \twoheadrightarrow J$ such that $\theta \otimes A/J = \alpha \otimes A/J$.*

Proof. We write down the proof in two steps as it will turn out to be convenient later.

Step 1. We note that to prove this result we can change β by composing with an element of $SL(Q \oplus A)$. Let $\beta = (\nu, a)$, where $\nu \in Q^*$. Let tilde denote reduction modulo J^2. As $J + J' = A$, it follows that $(\widetilde{\nu}, \widetilde{a}) \in \mathrm{Um}((\widetilde{Q} \oplus \widetilde{A})^*))$. Let ν' be any element of $\mathrm{Um}(\widetilde{Q}^*)$. Since $\dim(A/J^2) \leq 1 < \mathrm{rank}(Q)$, by Theorem 2.1.2 there exists $\widetilde{\sigma} \in \mathcal{E}((\widetilde{Q} \oplus \widetilde{A})^*))$ such that $(\widetilde{\nu}, \widetilde{a})\widetilde{\sigma} = (\nu', 0)$. By Theorem 2.1.3, $\widetilde{\sigma}$ can be lifted to an automorphism $\sigma^* \in SL((Q \oplus A)^*)$. But σ^* induces an automorphism $\sigma \in SL(Q \oplus A)$. Therefore, by replacing β with $\beta\sigma$, we may assume that $\beta = (\nu, a)$ has the property that $a \in J^2$ and $\nu(Q) + J^2 = A$. We can further apply Lemma 2.1.2 to obtain $\tau \in Q^*$ such that if $I = (\nu + a\tau)(Q)$, then $\mathrm{ht}(I_a) \geq n - 1$. Note that $(I, a) = J'$. As $\mathrm{ht}(J') = n$, it follows that $\mathrm{ht}(I) = n - 1$ and thus $\dim(A/I) \leq 2$. As (ν, a) and $(\nu + a\tau, a)$ are connected by a transvection, by replacing (ν, a) by $(\nu + a\tau, a)$, we can assume that:

(i) $\nu(Q) + J^2 = A$.

(ii) $\dim(A(\nu(Q)) \leq 2$.

Using (1) we may further assume that $a = 1$ modulo J^2.

Step 2. Consider the following ideals in $A[Y]$:

$$K_1 = (\nu(Q), Y + a), K_2 = J[Y], K_3 = K_1 \cap K_2.$$

We claim that there is a surjection $\eta(Y) : P[Y] \twoheadrightarrow K_3$ such that $\eta(0) = \alpha$. We first check that the theorem follows from the claim, as it is easier! Putting $Y = 1 - a$ we obtain a surjection $\theta : P \twoheadrightarrow J$. Since $a = 1$ modulo J^2, we have $\theta \otimes A/J = \eta(1 - a) \otimes A/J = \eta(0) \otimes A/J = \alpha \otimes A/J$, which proves the theorem.

Now for the claim, note that $A[Y]/K_1 \simeq A/(\nu(Q))$ and we have $\dim(A[Y]/K_1) \leq 2$. As P and $Q \oplus A$ are stably isomorphic, it is easy to see that there is an isomorphism, say, $\kappa(Y) : P[Y]/K_1P[Y] \xrightarrow{\sim} Q[Y]/K_1Q[Y] \oplus A[Y]/K_1$. We choose $\kappa(Y)$ so that $\wedge^n \kappa(Y) = \chi \otimes A[Y]/K_1$. Since $\wedge^n(\delta) = \chi \otimes A/J'$, it follows that $\kappa(0)$ and δ differ by an element of $SL(Q/J'Q \oplus A/J')$. We can apply Theorem 2.1.3 and alter $\kappa(Y)$ by an element of $SL(Q[Y]/K_1Q[Y] \oplus A[Y]/K_1)$ and assume that $\kappa(0) = \delta$. Now, tensoring the surjection $(\nu \otimes A[Y], Y + a) : Q[Y] \oplus A[Y] \twoheadrightarrow K_1$ with $A[Y]/K_1$, we obtain a surjection $\epsilon(Y)$:

$Q[Y]/K_1Q[Y] \oplus A[Y]/K_1 \twoheadrightarrow K_1/K_1^2$. Therefore, we have a surjection $\pi(Y) = \epsilon(Y)\kappa(Y)$: $P[Y]/K_1P[Y] \twoheadrightarrow K_1/K_1^2$. Since $\overline{\beta}\delta = \overline{\alpha}$, $\epsilon(0) = \overline{\beta}$, and $\kappa(0) = \delta$, we have $\pi(0) = \alpha \otimes A/J'$. Therefore, applying Theorem 2.1.4 we obtain a surjection $\eta(Y) : P[Y] \twoheadrightarrow K_3$ such that $\eta(0) = \alpha$. This proves the claim. □

Using a similar method we have another subtraction principle.

Proposition 2.3.4. *Let R be a ring of dimension $n \geq 3$ and write $A = R[T]$. Assume that $ht\mathcal{J}(R) \geq 2$, where $\mathcal{J}(R)$ is the Jacobson radical of R. Let P and L be projective A-modules of rank n and 1, respectively, together with an isomorphism $\chi : \wedge^n(P) \xrightarrow{\sim} L$. Write $Q = L \oplus A^{n-2}$. Let $J \subseteq A$ be an ideal of height $\geq n$ and J' be an ideal of height n such that $J' + (K^2T) = A$, where $K = \mathcal{J}(R) \cap J$. Let $\alpha : P \twoheadrightarrow J \cap J'$ and $\beta : Q \oplus A \twoheadrightarrow J'$ be surjections such that $\alpha(P) + (K^2T) = J$. Let $\overline{\beta} = \beta \otimes A/J' : (Q \oplus A)/J'(Q \oplus A) \twoheadrightarrow J'/J'^2$ and $\overline{\alpha} = \alpha \otimes A/J' : P/J'P \twoheadrightarrow J'/J'^2$ be the induced surjections. Suppose that there exists an isomorphism $\delta : P/J'P \xrightarrow{\sim} (Q \oplus A)/J'(Q \oplus A)$ such that: (i) $\overline{\beta}\delta = \overline{\alpha}$ and (ii) $\wedge^n(\delta) = \overline{\chi}$. Then there exists a surjection $\theta : P \twoheadrightarrow J$ such that $(\theta - \alpha)(P) \subset (K^2T)$.*

Proof. To be consistent with the above proposition, we write the proof in steps.

Step 1. As in the proof of Proposition 2.3.3, write $\beta = (\nu, a)$, where $\nu \in Q^*$. Let tilde denote reduction modulo (K^2T). Then $(\widetilde{\nu}, \widetilde{a}) \in \mathrm{Um}((\widetilde{Q} \oplus \widetilde{A})^*))$. Let ν' be any element of $\mathrm{Um}(\widetilde{Q}^*)$. Write $D = R[T]/(K^2T)$. Note that KD is contained in the Jacobson radical of D and $D/KD \simeq (R/K)[T]$. With an argument combining Theorem 2.1.3 and Theorem 2.1.3, it is easy to check that, changing β if necessary, we can assume that $\beta = (\nu, a)$ has the property that $a \in (K^2T)$ and $\nu(Q) + (K^2T) = A$. Now apply Lemma 2.1.2 to finally ensure that $\mathrm{ht}(\nu(Q)) = n - 1$, and therefore by [B-RS 1, Lemma 3.1] $\dim(A/(\nu(Q)) \leq 1$. Note that we may further assume that $a = 1$ modulo (K^2T).

Step 2. This step is exactly the same as its counterpart in Proposition 2.3.3. Note that here we have the advantage that $\dim(A/(\nu(Q)) \leq 1$ and therefore we do not need P and $Q \oplus A$ to be stably isomorphic. □

Chapter 3

Subintegral extensions and the Euler class groups

In this chapter, we investigate the relationship between the Euler class group and subintegral extensions.

3.1 Subintegral extensions

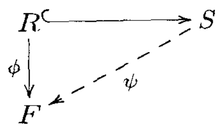
In this section we recall the definitions and some basic properties of subintegral extensions. We may refer to [Sw1] and [I 2] for further details.

Definition 3.1.1. An extension $R \hookrightarrow S$ of rings is called *subintegral* if: (1) it is integral, (2) the induced map $\mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ is bijective, and (3) for each $\mathfrak{P} \in \mathrm{Spec}(S)$ the induced field extension $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \hookrightarrow S_{\mathfrak{P}}/\mathfrak{P}S_{\mathfrak{P}}$ is trivial, where $\mathfrak{p} = \mathfrak{P} \cap R$.

Example 3.1.1. Let k be a field and T be an indeterminate. Then the extension $k[T^2, T^3] \hookrightarrow k[T]$ is subintegral.

The following alternative characterization of subintegral extensions is due to Swan ([Sw1, Lemma 2.1]). Here we give a detailed proof.

Lemma 3.1.1. *$R \hookrightarrow S$ is subintegral if and only if S is integral over R and for any field F and any homomorphism $\phi : R \longrightarrow F$, the diagram*

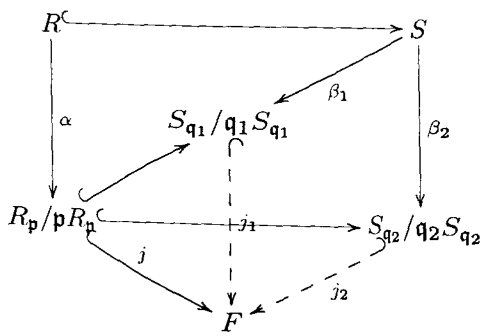


can be filled in uniquely.

Proof. First we assume that $R \hookrightarrow S$ is subintegral. Let F be a field and $\phi : R \longrightarrow F$ be any homomorphism. Then $\mathfrak{p} = \mathrm{Ker}(\phi)$ is a prime ideal of R. By the *lying over theorem*, there exists $\mathfrak{q} \in \mathrm{Spec}(S)$ such that $\mathfrak{q} \cap R = \mathfrak{p}$.

Now ϕ induces a map $\Psi : R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \longrightarrow F$ defined by $\Psi(\overline{x/y}) = \phi(x)/\phi(y)$ for $x, y \in R$ and $y \notin \mathfrak{p}$. This definition does not depend on representatives. Also $\Psi(\overline{x/1}) = \phi(x)$. Since the induced field extension $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \hookrightarrow S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}}$ is trivial. Therefore we have $\Psi : S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}} \longrightarrow F$. Finally we define $\psi : S \longrightarrow F$ by $\psi(s) = \Psi(\overline{s/1})$. From the definition of ψ, it is clear that $\psi|_R = \phi$. Also using the fact that the induced map $\mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ is injective, it is easy to check that ψ is unique.

We now prove the converse. Since S is integral over R, the induced map $\mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ is surjective. So we need to prove the injectivity. Let $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathrm{Spec}(S)$ such that $\mathfrak{q}_1 \cap R = \mathfrak{q}_2 \cap R = \mathfrak{p}$. As S is integral over R, we note that $S_{\mathfrak{q}_1}/\mathfrak{q}_1 S_{\mathfrak{q}_1}$ and $S_{\mathfrak{q}_2}/\mathfrak{q}_2 S_{\mathfrak{q}_2}$ are algebraic extensions of $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Let F be the algebraic closure of $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Then both $S_{\mathfrak{q}_1}/\mathfrak{q}_1 S_{\mathfrak{q}_1}$ and $S_{\mathfrak{q}_2}/\mathfrak{q}_2 S_{\mathfrak{q}_2}$ can be embedded into F.



Now if we take $\phi = j\alpha$, then from the uniqueness of ψ it follows that $\mathfrak{q}_1 = \mathfrak{q}_2$.

Let $\mathfrak{q} \in \mathrm{Spec}(S)$ and $\mathfrak{q} \cap R = \mathfrak{p}$. Then we have a natural homomorphism $\phi : R \longrightarrow R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ defined by $\phi(r) = \overline{r/1}$. From the hypothesis we see that there is a unique homomorphism $\psi : S \longrightarrow R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ such that the following diagram commutes.

$$\begin{array}{ccc} R & \hookrightarrow & S \\ {\scriptstyle\phi}\downarrow & \swarrow {\scriptstyle\psi} & \\ R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} & & \end{array}$$

We can define $\Psi : S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}} \longrightarrow R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ (since the kernel of ψ is $\mathfrak{q}$) such that

$$R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \hookrightarrow S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}} \longrightarrow R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$$

is identity. As the map $S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}} \longrightarrow R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is injective, the induced field extension $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \hookrightarrow S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}}$ is trivial.

This completes the proof. □

Definition 3.1.2. An extension of the form $R \hookrightarrow R[b]$ with $b^2, b^3 \in R$ is subintegral. It is called an *elementarily subintegral extension.*

Remark 3.1.1. We record the following fundamental facts about subintegral extensions.

(i) $R \hookrightarrow S$ is subintegral if and only if S is the filtered union of subrings which can be obtained from R by a finite number of elementarily subintegral extensions.

(ii) If R is a reduced Noetherian ring then any subintegral extension of R is contained in $\overline{R}$, the integral closure of R in its total ring of fractions.

Definition 3.1.3. A ring R is called *seminormal* if it is reduced and whenever $b, c \in R$ satisfy $b^3 = c^2$, there is an $a \in R$ with $a^2 = b$, $a^3 = c$.

Remark 3.1.2. Note that a will be unique if it exists. Conversely, if we assume the uniqueness of a, R will necessarily be reduced.

Notation. Let R be a ring. We denote, by $Q(R)$, the total quotient ring of R.

Definition 3.1.4. Let R be a reduced ring. Then there is a unique maximal R-subalgebra of $Q(R)$ which is subintegral over R. This is called the *seminormalization* (or *subintegral closure*) of R and is denoted by ${}^+(R)$.

The next two lemmas are easy applications of Lemma 3.1.1.

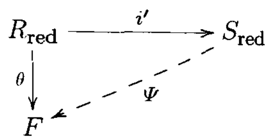
Lemma 3.1.2. *Let $R \hookrightarrow S$ be a subintegral extension. Then $R_{\mathrm{red}} \hookrightarrow S_{\mathrm{red}}$ is also subintegral.*

Proof. First of all, it is easy to check that S_{red} is integral over R_{red}.

Let $i : R \hookrightarrow S$ and $i' : R_{\mathrm{red}} \hookrightarrow S_{\mathrm{red}}$ be the inclusion maps. Let $\pi : R \twoheadrightarrow R_{\mathrm{red}}$ and $\pi' : S \twoheadrightarrow S_{\mathrm{red}}$ be the natural surjections. Let F be a field and $\theta : R_{\mathrm{red}} \longrightarrow F$ be any homomorphism. Then we have the composite $\varphi = \theta\pi : R \longrightarrow F$. Since $R \hookrightarrow S$ is subintegral, by Lemma 3.1.1 there exists a unique homomorphism $\psi : S \longrightarrow F$ such that $\psi i = \theta\pi$.

Now we define a map $\Psi : S_{\mathrm{red}} \longrightarrow F$ by $\Psi([s]) = \psi(s)$. To see that Ψ is well defined, let $s \in S$ be such that $s^r = 0$ for some positive power r. Then $\psi(s^r) = 0$ in F. Since F is a field, $\psi(s) = 0$, implying that $\Psi([s]) = 0$.

It is quite clear that the diagram:



is commutative. Uniqueness of Ψ can be easily checked using the uniqueness of ψ. $\square$

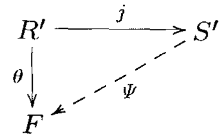
Lemma 3.1.3. *Let $R \hookrightarrow S$ be a subintegral extension and let $R \subset R'$ where R' is flat over R. Then $R' \hookrightarrow R' \otimes_R S$ is also subintegral.*

Proof. Let $S' = R' \otimes_R S$. Note that S' is integral over R'.

Let $i : R \hookrightarrow S$ be the inclusion map. Let $j : R' \hookrightarrow S'$ be the map defined by $j(r') = r' \otimes 1$. Note that the flatness condition ensures that j is injective. Let F be a field and $\theta : R' \longrightarrow F$ be any homomorphism. Now we have the restriction map $\varphi = \theta|_R : R \longrightarrow F$. Since $R \hookrightarrow S$ is subintegral, by Lemma 3.1.1 there exists a unique homomorphism $\psi : S \longrightarrow F$ such that $\psi i = \varphi$.

Now we can define a map $\Psi : S' \longrightarrow F$ by $\Psi(r' \otimes s) = \theta(r')\psi(s)$.

It is quite clear that the following diagram is commutative.



Uniqueness of Ψ can be easily checked using the uniqueness of ψ. □

Remark 3.1.3. In the above lemma if we take R' to be faithfully flat over R then the converse is also true. In other words, let $R \hookrightarrow S$ be an extension of rings and $R \hookrightarrow R'$ be a faithfully flat extension. Write $S' = S \otimes_R R'$. Then $R \hookrightarrow S$ is subintegral if and only if $R' \hookrightarrow S'$ is subintegral. (See [Sw1, Page 215]).

Lemma 3.1.4. *Let $R \hookrightarrow S$ be an elementarily subintegral extension. Let C be the conductor ideal of R in S. Then $(R/C)_{red} = (S/C)_{red}$.*

Proof. Let $R \hookrightarrow R[b]$ with $b^2, b^3 \in R$, be an elementarily subintegral extension. Let $K = \sqrt{C}$, the radical of C in $R[b]$. Then it follows that $(R[b]/C)_{red} = R[b]/K$ and $(R/C)_{red} = R/K \cap R$. Now we will show that $R[b]/K = R/K \cap R$. Note that $b \in K$. Therefore, $R[b]/K = R + Rb/K = R + K/K = R/K \cap R$. □

Lemma 3.1.5. *Let $R \hookrightarrow S$ be a finite ring extension with same total quotient ring. Let C be the conductor ideal of R in S. Then ht(C) ≥ 1. Moreover, if $R \hookrightarrow S$ is a finite subintegral extension of reduced rings then ht(C) ≥ 1.*

Proof. Let T be the multiplicative set of all non-zerodivisors in R. Then $T^{-1}C$ is the conductor ideal of $Q(R) \hookrightarrow Q(S)$. But we have $Q(R) = Q(S)$ and consequently, $T^{-1}C = Q(R)$. Therefore, C contains a non-zerodivisor and hence ht(C) ≥ 1.

We now show that if $R \hookrightarrow S$ is a subintegral extension of reduced rings then $Q(R) = Q(S)$. Let $\mathfrak{p}_i$ and $\mathfrak{P}_i$ be the minimal prime ideals of R and S respectively. Since R and S both are reduced, using the *Chinese remainder theorem* it is easy to see that $Q(R) = \prod R_{\mathfrak{p}_i}/\mathfrak{p}_i R_{\mathfrak{p}_i}$ and $Q(S) = \prod S_{\mathfrak{P}_i}/\mathfrak{P}_i S_{\mathfrak{P}_i}$. From the definition of subintegral extension it follows that $\prod R_{\mathfrak{p}_i}/\mathfrak{p}_i R_{\mathfrak{p}_i} = \prod S_{\mathfrak{P}_i}/\mathfrak{P}_i S_{\mathfrak{P}_i}$. Therefore we have $Q(R) = Q(S)$. □

3.2 Main results

In [I 2] Ischebeck studied the behaviour of certain K-theoretical functors under subintegral extensions. In particular, it is proved in [I 2, Proposition 6] that if $R \hookrightarrow S$ is a (finite) subintegral extension, then the Chow groups $CH_i(S)$ and $CH_i(R)$ are isomorphic. Along the same line, in [G], Gubeladze's object of study is the orbit space of unimodular rows under the natural action of elementary matrices. This orbit space $Um_n(R)/E_n(R)$ carries a group structure for $n = \dim(R) + 1$ (thanks to the work of van der Kallen [VK1]), and as shown in [B-RS 4], is intimately related to the Euler class groups and the weak Euler class groups. Therefore, it is natural to ask the following questions.

Question 3.2.1. *Let $R \hookrightarrow S$ be a subintegral extension of Noetherian rings with $\dim(R) = n \geq 2$. Let L be a projective R-module of rank one. Is $E(R, L) \simeq E(S, L \otimes_R S)$? Also, is $E_0(R, L) \simeq E_0(S, L \otimes_R S)$?*

We have consciously decided to tackle the above questions in the case $L = R$ first. The proofs in this case are much more comprehensible as one is working with generators. We prove that $E(R)$ is isomorphic to $E(S)$. The general case for arbitrary L is proved in Theorem 3.2.4. If the dimension of R is even, then we prove that $E_0(R)$ is isomorphic to $E_0(S)$.

The following remark ensures that for a subintegral extension $R \hookrightarrow S$, there are natural maps from $E(R)$ to $E(S)$ and from $E_0(R)$ to $E_0(S)$.

Remark 3.2.1. Let $R \hookrightarrow S$ be a subintegral extension and let $\dim(R) = n$. Then $\dim(S) = n$. The definition of a subintegral extension asserts that the inclusion $i : R \hookrightarrow S$ induces a bijection $i^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$. As S is integral over R, the *going up theorem* holds for this extension. As R, S are both Noetherian, this implies that i^* is a closed map. But since i^* is bijective, it is therefore an open map and as a consequence, the *going down theorem* also holds for this extension. As S is integral over R, the *lying over theorem* is already there. These two theorems imply that for an ideal I of R, we have $\mathrm{ht}(I) = \mathrm{ht}(IS)$. It now follows from [M-Y 2, Definition 3.3] that there is a morphism from $E(R)$ to $E(S)$ (see also Remark 2.2.2). However, we give the details. Let (I, ω_I) be a pair, where I is an ideal of R of height n and $\omega_I : (R/I)^n \twoheadrightarrow I/I^2$ is a local orientation of I. Then by the above discussion, we have $\mathrm{ht}(IS) = n$. Although S may not be flat over R, note that the local orientation ω_I induces a local orientation $\omega_I^* : (S/IS)^n \twoheadrightarrow IS/(IS)^2$ of IS in a natural way. Clearly, if ω_I is a global orientation, then so is ω_I^*. Thus we have a canonical morphism $\Phi : E(R) \to E(S)$ which maps (I, ω_I) to (IS, ω_I^*). It is now easy to observe from the above discussion that there is also a canonical morphism $\Phi_0 : E_0(R) \to E_0(S)$ which sends (I) to (IS).

To answer the question raised at the beginning, we first prove the following result.

Theorem 3.2.1. *Let R be a Noetherian ring of dimension $n \geq 2$. Let $R \hookrightarrow S$ be a finite subintegral extension. Then the induced homomorphism $\Phi : E(R) \longrightarrow E(S)$ is an isomorphism.*

Proof. By Lemma 3.1.2 the extension $R_{\text{red}} \hookrightarrow S_{\text{red}}$ is subintegral. We know that $E(R) \simeq E(R_{\text{red}})$ and $E(S) \simeq E(S_{\text{red}})$ by Proposition 2.2.1, and therefore without loss of generality we assume that R, S are both reduced. Further, we may assume that the extension $R \hookrightarrow S$ is elementarily subintegral. From Lemma 3.1.4, we have $(R/C)_{\text{red}} = (S/C)_{\text{red}}$, where C is the conductor of R in S. Since the extension $R \hookrightarrow S$ is finite and R, S are both reduced rings, it follows from Lemma 3.1.5 that $\mathrm{ht}(C) \geq 1$.

Step 1. In this step we prove that the induced map $\Phi : E(R) \to E(S)$ is injective.

Let $(I, \omega_I) \in E(R)$ such that $(IS, \omega_I^*) = 0$ in $E(S)$. Here $I \subset R$ is an ideal of height n and ω_I is a local orientation of I represented by, say, $I = (a_1, \cdots, a_n) + I^2$. To prove that $(I, \omega_I) = 0$ in $E(R)$, we have to find $v_1, \cdots, v_n$ such that $I = (v_1, \cdots, v_n)$ with $v_i - a_i \in I^2$ for $i = 1, \cdots, n$.

Since $(IS, \omega_I^*) = 0$ in $E(S)$, there exist $\alpha_1, \cdots, \alpha_n \in IS$ such that $IS = (\alpha_1, \cdots, \alpha_n)$ where $a_i - \alpha_i \in (IS)^2$.

Since $(a_1, \cdots, a_n) + I^2 = I$, applying the moving lemma (Lemma 2.1.4), we can find $b_1, \cdots, b_n \in I$ and an ideal I' of R such that

(i) $I \cap I' = (b_1, \cdots, b_n)$ with $a_i - b_i \in I^2$.

(ii) $I' + I \cap C = R$.

(iii) $\mathrm{ht}(I') \geq n$.

If $\mathrm{ht}(I') > n$, then $I' = R$ and we are done by (i). Therefore we assume that $\mathrm{ht}(I') = n$. Now consider the equation $IS \cap I'S = (b_1, \cdots, b_n)$ in S. On the other hand we have $IS = (\alpha_1, \cdots, \alpha_n)$ such that $a_i - \alpha_i \in (IS)^2$. Therefore, $\alpha_i - b_i \in I^2S$. Using the subtraction principle (Proposition 2.3.1), we have $\beta_1, \cdots, \beta_n \in I'S$ such that $I'S = (\beta_1, \cdots, \beta_n)$ with $\beta_i - b_i \in (I'S)^2$.

As $I' + C = R$, we have $I' \otimes S/C \simeq S/C$. Further note that the image of I' in R/C is R/C and the image of $I'S$ in S/C is S/C. This implies that $(\overline{\beta_1}, \cdots, \overline{\beta_n}) \in \mathrm{Um}_n(S/C)$ where bar denotes modulo C in S. Therefore, $(\overline{\beta_1}, \cdots, \overline{\beta_n}) \in \mathrm{Um}_n((S/C)_{\mathrm{red}})$. Note that we have $(R/C)_{\mathrm{red}} = (S/C)_{\mathrm{red}}$. As the canonical map : $\mathrm{Um}_n(R/C) \longrightarrow \mathrm{Um}_n((R/C)_{\mathrm{red}})$ is surjective, there are elements $f_1, \cdots, f_n \in R$ such that $(\widetilde{f_1}, \cdots, \widetilde{f_n}) \in \mathrm{Um}_n(R/C)$ (where tilde denotes reduction modulo C in R). Moreover, for each $i \in \{1, \cdots, n\}$, we have $\overline{f_i} - \overline{\beta_i} \in \mathfrak{n}(S/C)$, where $\mathfrak{n}(S/C)$ is the nil-radical of S/C.

Now we have two unimodular rows $(\overline{\beta_1}, \cdots, \overline{\beta_n})$ and $(\overline{f_1}, \cdots, \overline{f_n})$ in S/C such that $\overline{f_i} - \overline{\beta_i} \in \mathfrak{n}(S/C)$. Therefore, by [MK-M-R, Remark 2.3], there exists a transvection $\overline{\sigma}$ of $(S/C)^n$ such that $(\overline{\beta_1}, \cdots, \overline{\beta_n})\overline{\sigma} = (\overline{f_1}, \cdots, \overline{f_n})$. By Proposition 2.1.4, $\overline{\sigma}$ can be lifted to an automorphism σ of S^n such that $\sigma = \mathrm{Id}$ modulo $I'S$. Let $(\beta_1, \cdots, \beta_n)\sigma = (g_1, \cdots, g_n)$. Then we have $(\overline{g_1}, \cdots, \overline{g_n}) = (\overline{\beta}_1, \cdots, \overline{\beta}_n)\overline{\sigma} = (\overline{f_1}, \cdots, \overline{f_n})$ in S/C. Therefore $f_i - g_i \in C$. Since $f_i \in R$, we have $g_i \in R$.

We now claim that $I' = (g_1, \cdots, g_n)$ and $g_i - b_i \in I'^2$.

Proof of the claim : We first note that as I' is comaximal with C and C is the conductor ideal, we have $I'S \cap C = (I'S)C = I'C = I' \cap C$. Further, $S/I'S = (I'S + C)/I'S \simeq C/(I'S \cap C) = C/(I' \cap C) \simeq (I' + C)/I' = R/I'$. Therefore, $I'S \cap R = I'$.

It now follows from above that $(g_1, \cdots, g_n) \subseteq I'$. As $\sigma = \mathrm{Id}$ modulo $I'S$, it is easy to see that $g_i - b_i \in I'^2S \cap R = I'^2$. As $(\widetilde{f_1}, \cdots, \widetilde{f_n}) \in \mathrm{Um}_n(R/C)$ and $f_i - g_i \in C$, it follows that $(g_1, \cdots, g_n) + C = R$. Recall that we also have $I'S = (g_1, \cdots, g_n)S$.

Let $\mathfrak{m}$ be any maximal ideal of R. If $C \subset \mathfrak{m}$, then $I'R_{\mathfrak{m}} = R_{\mathfrak{m}} = (g_1, \cdots, g_n)R_{\mathfrak{m}}$. On the other hand, if $C \not\subset \mathfrak{m}$, then $R_{\mathfrak{m}} = T^{-1}R = T^{-1}S$, where $T = R \setminus \mathfrak{m}$ (this is true because for any $t \in C$, $R_t = S_t$). In this case,

$$I'R_{\mathfrak{m}} = I'T^{-1}R = I'T^{-1}S = (g_1, \cdots, g_n)T^{-1}S = (g_1, \cdots, g_n)R_{\mathfrak{m}}.$$

Therefore, $I' = (g_1, \cdots, g_n)$ with $g_i - b_i \in I'^2$. This proves the claim.

We have : (i) $I \cap I' = (b_1, \cdots, b_n)$, (ii) $I' = (g_1, \cdots, g_n)$ with $g_i - b_i \in I'^2$. We can now apply the subtraction principle (Proposition 2.3.1) to obtain $v_1, \cdots, v_n \in I$ such that $I = (v_1, \cdots, v_n)$ with $v_i - b_i \in I^2$. As $a_i - b_i \in I^2$, it follows that $v_i - a_i \in I^2$. This proves that $(I, \omega_I) = 0$ in $E(R)$ and that Φ is injective.

Step 2. In this step we show that $\Phi : E(R) \to E(S)$ is surjective.

Let $(I, \omega_I) \in E(S)$. Suppose that ω_I is induced by : $I = (f_1, \cdots, f_n) + J^2$. By using the moving lemma (Lemma 2.1.4), we can find $g_1, \cdots, g_n \in I$ and an ideal $K \subseteq S$ such that

(a) $(g_1, \cdots, g_n) = I \cap K$ where $g_i - f_i \in I^2$

(b) $K + I \cap C = S$ where $\text{ht}(K) \geq n$.

If $\text{ht}(K) > n$, then $K = S$ and $I = (g_1, \cdots, g_n)$ implying that $(I, \omega_I) = 0$ in $E(S)$ and there is nothing to prove. Therefore we assume that $\text{ht}(K) = n$. Let ω_K be the local orientation induced by $g_1, \cdots, g_n$. Then from (a) we have $(I, \omega_I) + (K, \omega_K) = 0$ in $E(S)$. In order to prove that (I, ω_I) has a preimage in $E(R)$ it is enough to prove that (K, ω_K) has a preimage.

Let $K \cap R = J$. As $K + C = S$, we have $J + C = R$, and hence there exists $c \in C$ such that $l = 1 - c \in J$. We can assume that $\text{ht}(l) = 1$. (If $\text{ht}(l) = 0$, choose $l' \in J$ such that l' does not belong to any minimal prime ideal of R. Then $\text{ht}(l + l' - ll') = 1$. Now $(1 - l)(1 - l') = 1 - l - l' + ll'$. If we write $l'' = l + l' - ll'$, then we have $1 - l'' \in C$ and $\text{ht}(l'') = 1$ and we can work with l''). Since $c \in C$, we have $R_c = S_c$. Therefore $R/(1 - c) = S/(1 - c)$ and $R \hookrightarrow S$ is an analytic isomorphism along $l \in J$. Therefore using Proposition 2.1.1, we have

(c) $R/J \simeq S/K$.

(d) $K = JS$

(e) As $l \in J$, we have $J/J^2 \simeq K/K^2$.

Note that we have $K = (g_1, \cdots, g_n) + K^2$. As $J/J^2 \simeq K/K^2$, corresponding to this set of generators of K/K^2 we have a set of generators of J/J^2. Calling them $a_1, \cdots, a_n$

we have $J = (a_1, \cdots, a_n) + J^2$. Let ω_J be the associated local orientation of J. Then $(J, \omega_J) \in E(R)$ and clearly $\Phi((J, \omega_J)) = (K, \omega_K)$. $\square$

To extend the above theorem to all subintegral extensions we first prove that the Euler class group commutes with direct limit in the following sense. Let S be a Noetherian ring such that S is the filtered direct limit of a direct system of Noetherian subrings $\{S_\alpha, \mu_{\alpha\beta}\}$ indexed by Ω. Here, for $\alpha \leq \beta$ the map $\mu_{\alpha\beta} : S_\alpha \hookrightarrow S_\beta$ is the inclusion map and for each $\alpha \in \Omega$, let $\mu_\alpha : S_\alpha \hookrightarrow S$ be the inclusion. Assume that the following conditions hold: (1) $\dim(S) = n = \dim(S_\alpha)$ for each $\alpha \in \Omega$, and (2) for any ideal $I_\alpha \subset S_\alpha$ with $\text{ht}(I_\alpha) = n$ and $\mu(I_\alpha/I_\alpha^2) = n$, one has $\text{ht}(I_\alpha S_\beta) \geq n$ for $\alpha \leq \beta$ and $\text{ht}(I_\alpha S) \geq n$. It is now easy to see that for all $\alpha, \beta \in \Omega$, the map $\mu_{\alpha\beta} : S_\alpha \longrightarrow S_\beta$ induces $\theta_{\alpha\beta} : E(S_\alpha) \longrightarrow E(S_\beta)$ and $\mu_\alpha : S_\alpha \longrightarrow S$ induces $\phi_\alpha : E(S_\alpha) \longrightarrow E(S)$ so that $\{E(S_\alpha), \theta_{\alpha\beta}\}$ is a direct system of groups and $\phi_\beta\theta_{\alpha\beta} = \phi_\alpha$. Then we show below that the Euler class group $E(S)$ is isomorphic to the direct limit $\{\varinjlim E(S_\alpha), \theta_\alpha\}$. A situation as above will occur when, for example, the ring morphisms are all flat extensions (see Remark 2.2.2). In Theorem 3.2.3 we shall soon encounter another set up where it takes place naturally.

Theorem 3.2.2. *With notations as above,* $E(S) = E(\varinjlim S_\alpha) \simeq \varinjlim E(S_\alpha)$.

Proof. As for each α there is a group homomorphism $\phi_\alpha : E(S_\alpha) \longrightarrow E(S)$ with $\phi_\beta\theta_{\alpha\beta} = \phi_\alpha$, by the properties of direct limit there is a group homomorphism $\psi : \varinjlim E(S_\alpha) \longrightarrow E(S)$. We prove that ψ is an isomorphism.

An element x of $\varinjlim E(S_\alpha)$ is of the form $x = \theta_\alpha(x_\alpha)$ for some $\alpha \in \Omega$ and $x_\alpha \in E(S_\alpha)$. Let $x_\alpha = (J_\alpha, \omega_\alpha) \in E(S_\alpha)$, where J_α is an ideal of S_α of height n and ω_α is a local orientation of J_α induced by, say, $J_\alpha = (a_1, \cdots, a_n) + J_\alpha^2$. Assume that $\psi(x) = 0$ in $E(S)$. This implies that $\phi_\alpha((J_\alpha, \omega_\alpha)) = (J_\alpha S, \omega_\alpha^*) = 0$ in $E(S)$, where ω_α^* is the local orientation of $J_\alpha S$ induced by ω_α. Applying Theorem 2.2.1 we obtain $b_1, \cdots, b_n \in J_\alpha S$ such that $J_\alpha S = (b_1, \cdots, b_n)$ with $b_i - a_i = \lambda_i \in J_\alpha^2 S$. Since S is the filtered direct limit of the subrings S_α, it is easy to see that there exists $\beta \in \Omega$ such that $a_1, \cdots, a_n, b_1, \cdots, b_n, \lambda_1, \cdots, \lambda_n \in S_\beta$. Now there is $\gamma \in \Omega$ such that $S_\alpha \hookrightarrow S_\gamma$ and $S_\beta \hookrightarrow S_\gamma$. It follows that in S_γ, we have $J_\alpha S_\gamma = (b_1, \cdots, b_n)$ with $b_i - a_i = \lambda_i \in J_\alpha^2 S_\gamma$. This implies that $(J_\alpha S_\gamma, \omega_\alpha^*) = 0$ in $E(S_\gamma)$ and therefore $x = 0$ in $\varinjlim E(S_\alpha)$, proving that $\psi : \varinjlim E(S_\alpha) \longrightarrow E(S)$ is injective.

Next we prove that ψ is surjective. Let $(I, \omega) \in E(S)$. Then I is an ideal of S of height n and ω is a local orientation of I induced by, say, $I = (f_1, \cdots, f_n) + I^2$. Then by Lemma 2.1.3, there exists $e \in I$ such that $I = (f_1, \cdots, f_n, e)$ where $e(1-e) \in (f_1, \cdots, f_n)$. Suppose that $e(1-e) = k_1 f_1 + \cdots k_n f_n$ where $k_i \in S$. Since S is the filtered direct limit of the subrings S_α, it is easy to see that there exits $\alpha \in \Omega$ such that $f_1, \cdots, f_n, e, k_1, \cdots, k_n \in S_\alpha$. Let $I' = (f_1, \cdots, f_n, e)S_\alpha$. Then $I' = (f_1, \cdots, f_n, e - k_1 f_1 - \cdots - k_n f_n) = (f_1, \cdots, f_n, e^2)$ implying that $I' = (f_1, \cdots, f_n) + I'^2$. If ω' denotes the local orientation of I' induced by this set of generators of I'/I'^2, then $(I', \omega') \in E(S_\alpha)$. Then clearly $\phi_\alpha((I', \omega')) = (I, \omega)$ and it proves that $\psi(\theta_\alpha((I', \omega'))) = (I, \omega)$, implying the surjectivity of ψ. □

Theorem 3.2.3. *Let R be a Noetherian ring of dimension $n \geq 2$. Let $R \hookrightarrow S$ be any subintegral extension. Then the induced homomorphism $\Phi : E(R) \longrightarrow E(S)$ is an isomorphism.*

Proof. Recall that S is the filtered union of subrings S_α where each S_α is obtained from R by a finite sequence of elementarily subintegral extensions. This means that given two subrings S_α, S_β of the above type, there is a subring S_γ of the above type such that $R \hookrightarrow S_\alpha \hookrightarrow S_\gamma$ and $R \hookrightarrow S_\beta \hookrightarrow S_\gamma$. Let $S = \cup_{\alpha \in \Omega} S_\alpha$.

For elements $\alpha, \beta \in \Omega$ define $\alpha \leq \beta$ to mean $S_\alpha \subseteq S_\beta$ and let $\mu_{\alpha\beta} : S_\alpha \hookrightarrow S_\beta$ be the inclusion map. Then S is the filtered direct limit of $\{S_\alpha\}_{\alpha \in \Omega}$, i.e.,

$$S = \varinjlim_{\alpha \in \Omega} S_\alpha.$$

For $\alpha \leq \beta (\in \Omega)$ we have a group homomorphism $\theta_{\alpha\beta} : E(S_\alpha) \longrightarrow E(S_\beta)$ induced by the inclusion map $S_\alpha \hookrightarrow S_\beta$. Note that $S_\alpha \hookrightarrow S_\beta$ is subintegral.

We then have the direct limit $\varinjlim E(S_\alpha)$ of the direct system of groups $\{E(S_\alpha)\}_{\alpha \in \Omega}$ and group homomorphisms $\theta_\alpha : E(S_\alpha) \longrightarrow \varinjlim E(S_\alpha)$.

By Theorem 3.2.2, $E(S) = E(\varinjlim S_\alpha) \simeq \varinjlim E(S_\alpha)$. Since by Theorem 3.2.1, $E(R) \simeq E(S_\alpha)$ for each α, it follows that $E(R) \simeq E(S)$. □

We now proceed to prove that if $R \hookrightarrow S$ is subintegral, then $E(R, L)$ is isomorphic to $E(S, L \otimes_R S)$, where L is a projective R-module of rank one. Again we need to ensure that there is a natural morphism from $\Phi : E(R, L) \longrightarrow E(S, L \otimes_R S)$. As for any ideal $J \subset R$ of height n, by Remark 3.2.1 we have $\mathrm{ht}(JS) = n$, the existence of Φ is ensured by [M-Y 2, Definition 3.3]. However, we present the explicit description of Φ below for the convenience.

We write $F = L \oplus R^{n-1}$. Let J be any ideal of R of height n and $\omega_J : F/JF \twoheadrightarrow J/J^2$ be any surjection. Tensoring with S/JS over R/J we obtain the induced surjection

$$\widetilde{\omega_J} : \frac{(F \otimes_R S)}{JS(F \otimes_R S)} \twoheadrightarrow \frac{(J \otimes_R S)}{JS(J \otimes_R S)}.$$

Now composing $\widetilde{\omega_J}$ with the surjective map $\widetilde{f}$ induced by the natural surjection $f : J \otimes_R S \twoheadrightarrow JS$ we obtain a local orientation of JS. We call it ω_J^*. Thus

$$\omega_J^* : \frac{(F \otimes_R S)}{JS(F \otimes_R S)} \overset{\widetilde{\omega_J}}{\twoheadrightarrow} \frac{(J \otimes_R S)}{JS(J \otimes_R S)} \overset{\widetilde{f}}{\twoheadrightarrow} \frac{JS}{J^2S}.$$

$$\begin{array}{ccc} F/JF & \xrightarrow{\ \omega_J\ } & J/J^2 \\ \downarrow & & \downarrow \\ \frac{(F \otimes_R S)}{JS(F \otimes_R S)} & \xrightarrow{\ \omega_J^*\ } & JS/J^2S \end{array}$$

Note that if ω_J can be lifted to a surjection $\theta : F \twoheadrightarrow J$, then so can be ω_J^*. Therefore, we have a well defined group homomorphism $\Phi : E(R, L) \longrightarrow E(S, L \otimes_R S)$ which takes an element (J, ω_J) of $E(R, L)$ to (JS, ω_J^*) of $E(S, L \otimes_R S)$.

We now prove the following theorem. We shall not give a detailed proof as it is along the same line as the last two theorems. We shall only highlight the crucial deviations.

Theorem 3.2.4. *Let $R \hookrightarrow S$ be a subintegral extension and L be a projective R-module of rank one. Then the map $\Phi : E(R, L) \longrightarrow E(S, L \otimes_R S)$, described above, is an isomorphism.*

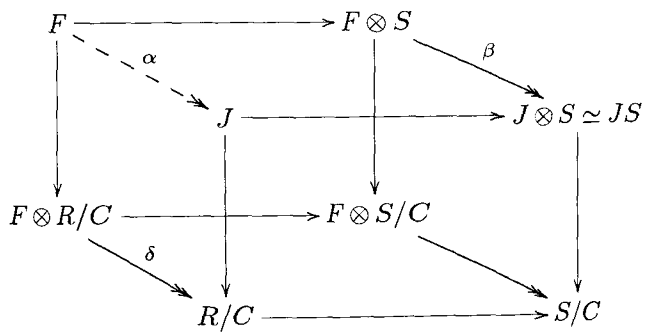
Proof. As the direct limit argument of Theorem 3.2.3 works in this case too, we may assume that S is obtained from R by a finite number of subintegral extensions. Therefore, as before, we may assume that R, S are both reduced and $R \hookrightarrow S$ is elementarily subintegral. If C is the conductor of R in S, then $\mathrm{ht}(C) \geq 1$ and $(R/C)_{\mathrm{red}} = (S/C)_{\mathrm{red}}$.

Let $(J, \omega_J) \in E(R, L)$ be such that $(JS, \omega_J^*) = 0$ in $E(S, L \otimes_R S)$. Applying the moving lemma (Lemma 2.1.4) and the subtraction principle (Proposition 2.3.1), we may assume that $J + C = R$. As $(JS, \omega_J^*) = 0$ in $E(S, L \otimes_R S)$, there exists a surjection $\beta' : F \otimes_R S \twoheadrightarrow JS$ such that β' lifts ω_J^*. As $JS + C = S$, it follows that $\beta'' = \beta' \otimes_S S/C \in \mathrm{Um}((F \otimes_R S/C)^*)$. Now β'' will induce a unimodular element of $(F \otimes_R (S/C)_{\text{red}})^*$ and also, as $(R/C)_{\text{red}} = (S/C)_{\text{red}}$, a unimodular element of $(F \otimes_R (R/C)_{\text{red}})^*$. We lift the latter one to $\delta \in \mathrm{Um}((F \otimes_R R/C)^*)$. Now note that $\delta \otimes_{R/C} S/C$ and β'' are the same modulo the nil radical of S/C. Therefore, applying Proposition 2.1.2, we obtain $\overline{\sigma} \in \mathcal{E}(F \otimes_R S/C)$ such that $\delta \otimes_{R/C} S/C = \beta\overline{\sigma}$. Applying Proposition 2.1.4 we can lift $\overline{\sigma}$ to an automorphism σ of $F \otimes_R S$ such that $\sigma \equiv \mathrm{id}$ modulo JS.

Write $\beta = \beta'\sigma$. Then $\beta : F \otimes_R S \twoheadrightarrow JS$ and β lifts ω_J^* (as σ is identity modulo JS). As $J + C = R$, we have $J \otimes_R S \simeq JS$ (see Lemma 2.1.1), and $J \otimes_R R/C \simeq R/C$, and $J \otimes_R S/C \simeq S/C$. Up to these identifications, the following diagram is Cartesian

$$\begin{array}{ccc} J & \longrightarrow & J \otimes S \simeq JS \\ \downarrow & & \downarrow \\ (R/C) & \longrightarrow & (S/C) \end{array}$$

As β and δ agree over S/C, they will patch to yield a surjection $\alpha : F \twoheadrightarrow J$. Here is the patching diagram:



Identifying $J \otimes_R S$ with JS and using the isomorphism $S/JS \simeq R/J$, we have:

$$\alpha \underset{R}{\otimes} (R/J) = \alpha \underset{R}{\otimes} (S/JS) = (\alpha \underset{R}{\otimes} S) \underset{S}{\otimes} (S/JS) =$$

$$\beta \underset{S}{\otimes} (S/JS) = \omega_J \underset{R/J}{\otimes} (S/JS) = \omega_J \underset{R/J}{\otimes} (R/J) = \omega_J.$$

Thus α lifts ω_J, implying that $(J, \omega_J) = 0$ in $E(R, L)$. This proves that Φ is injective.

The proof that Φ is surjective is similar to the proof in Step 2 of Theorem 3.2.1. □

Applying the above theorem we have the following corollaries.

Corollary 3.2.1. *Let R be a ring of dimension $n \geq 2$ and $^+(R_{red})$ be the seminormalization of R_{red}. Let L be a projective R-module of rank one and write $\widetilde{L} = L \otimes {}^+(R_{red})$. Then $E(R, L) \simeq E({}^+(R_{red}), \widetilde{L})$.*

Proof. Recall that if A is a reduced ring then its seminormalization is the subintegral closure of A in its total ring of fractions.

Here we have $E(R, L) \simeq E(R_{\text{red}}, L \otimes R_{\text{red}})$ by Proposition 2.2.1, and the group $E(R_{\text{red}}, L \otimes R_{\text{red}})$ is isomorphic to $E({}^+(R_{\text{red}}), \widetilde{L})$ by Theorem 3.2.4. □

The (unstated) result of Bhatwadekar from [B 1], as mentioned in the introduction, can now be deduced (although we have the restriction that $\mathbb{Q} \subset R$).

Corollary 3.2.2. *Let $R \hookrightarrow S$ be a subintegral extension of $\mathbb{Q}$-algebras with dim$(R) = n \geq 2$. Let P be a projective R-module of rank n. Then P has a unimodular element if and only if the projective S-module $P \otimes_R S$ has a unimodular element.*

Remark 3.2.2. The above corollary is also true if we take $S = {}^+(R_{\text{red}})$. Further, if $\dim(R) = 2$, then we do not need the assumption that $\mathbb{Q} \subset R$ (see Definition 2.2.2).

Adapting the same method as above, one can similarly prove the following result. We may also note that by Lemma 3.1.3, as $R[T]$ is faithfully flat over R, the extension $R \hookrightarrow S$ is subintegral if and only if so is the extension $R[T] \hookrightarrow S[T]$.

Theorem 3.2.5. *Let R be a ring (containing $\mathbb{Q}$) of dimension $n \geq 3$. Let $R \hookrightarrow S$ be a subintegral extension. Then $E(R[T]) \simeq E(S[T])$. In particular, if R is reduced and if S is the seminormalization of R, then $E(R[T]) \simeq E(S[T])$. Therefore, for an arbitrary R, the groups $E(R[T])$ and $E({}^+(R_{red})[T])$ are isomorphic.*

Proof. Since the method of proof is quite similar to Theorem 3.2.1 above, instead of writing the whole proof, we just work out one key step. We can assume that the rings are reduced and the extension $R \hookrightarrow S$ is elementarily subintegral. Let C be the conductor of R in S. We have $\text{ht}(C) \geq 1$.

Let $I = (f_1, \cdots, f_n) + I^2$. We just show how to 'move away' from I to obtain a suitable residual ideal I' of height n so that I' is comaximal with both I and $C[T]$. Let $J = I^2 \cap C \subset R$. Then $\text{ht}(J) \geq 1$. Let $b \in J$ be such that $\text{ht}(b) = 1$. Let bar denote reduction modulo (b). We have, $\overline{I} = (\overline{a_1}, \cdots, \overline{a_n}) + \overline{I}^2$ in $\overline{R}[T]$. As $\dim(\overline{R}) \leq n-1$, it follows from Proposition 2.1.5 that there exist $g_1, \cdots, g_n \in I$ such that $\overline{I} = (\overline{g_1}, \cdots, \overline{g_n})$, where $\overline{g_i} - \overline{f_i} \in \overline{I}^2$. Therefore, $I = (g_1, \cdots, g_n, b)$ such that $g_i - f_i \in I^2$. One can now apply Lemma 2.1.2 and Lemma 2.1.3 to find an ideal I' such that (possibly after some renaming the g_i's): (1) $I \cap I' = (g_1, \cdots, g_n)$, (2) $I' + I \cap C[T] = R[T]$, (3) $\text{ht}(I') = n$.

Note that $I' = (g_1, \cdots, g_n) + I'^2$. Now we can work with I' and apply the subtraction principle [D 1, Proposition 4.3] at appropriate places to prove the results. □

Let R be a ring of dimension $n \geq 2$. Given a pair (J, ω_J); where $J \subset R$ is an ideal of height ≥ 2 and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ a surjection, the *Segre class* $s(J, \omega_J)$ has been defined in [D-RS] in the following way:

Suppose that ω_J induces $J = (a_1, \cdots, a_n) + J^2$. Applying a variant of the moving lemma [D-RS, Lemma 2.7], we can find $c_1, \cdots, c_n \in J$ such that $(c_1, \cdots, c_n) = J \cap J_1$ where $\text{ht}\, J_1 \geq n$, $J_1 + J = R$ and $c_i = a_i$ modulo J^2. If J_1 is a proper ideal then $J_1 = (c_1, \cdots, c_n) + J_1^2$ and it induces a local orientation $\omega_{J_1} : (R/J_1)^n \twoheadrightarrow J_1/J_1^2$. The Segre class of the pair (J, ω_J) is defined as: $s(J, \omega_J) = -(J_1, \omega_{J_1}) \in E(R)$. If $J_1 = R$ then $J = (c_1, \cdots, c_n)$ and the Segre class is defined to be zero. It is proved that the definition of the Segre class does not depend on the choice of J_1. Further, when $\text{ht}(J) = n$, the Segre class coincides with the Euler class of (J, ω_J). We now recall the following result on Segre classes.

Theorem 3.2.6. *[D-RS, Theorem 3.3] Let R be a ring of dimension $n \geq 2$. Let $J \subset R$ be an ideal of height ≥ 2 and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ be a surjection. Suppose that $s(J, \omega_J) = 0$ in $E(R)$. Then ω_J can be lifted to a surjection $\theta : R^n \twoheadrightarrow J$.*

We now have

Theorem 3.2.7. *Let R be a ring of dimension $n \geq 2$ and $R \hookrightarrow S$ be a subintegral extension. Let $J \subset R$ be an ideal of height ≥ 2 and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ be a surjection. Assume that the induced surjection $\omega_J^* : (S/JS)^n \twoheadrightarrow JS/J^2S$ has a lift to a surjection $\theta : S^n \twoheadrightarrow JS$. Then ω_J can be lifted to a surjection $\alpha : R^n \twoheadrightarrow J$.*

Proof. The hypothesis tells that the Segre class $s(JS, \omega_J^*) = 0$ in $E(S)$. It is now obvious from the definition of the Segre class and Theorem 3.2.3 that $s(J, \omega_J) = 0$ in $E(R)$. Therefore, by the above theorem, ω_J can be lifted to a surjection $\theta : R^n \twoheadrightarrow J$. □

We now mention another immediate consequence of Theorem 3.2.3. We need to recall some generalities from [B-RS 4, Section 7]. Let A be a ring of dimension 2. Let $\widetilde{K_0}Sp(A)$ be the set of isometry classes of (P, s), where P is a projective A-module of rank 2 and $s : P \times P \to A$ a non-degenerate skew-symmetric bilinear form. In [B-RS 4], a group structure is defined on $\widetilde{K_0}Sp(A)$, where the pair (A^2, h) plays the role of the identity element, where h is the unique (up to isometry) non-degenerate alternating form on A^2. It is then remarked that this group coincides with the usual notion of $\widetilde{K_0}Sp(A)$. Further, it is proved in [B-RS 4, Theorem 7.2] that $\widetilde{K_0}Sp(A)$ is isomorphic to the Euler class group $E(A)$.

Corollary 3.2.3. *Let $R \hookrightarrow S$ be a subintegral extension with dim$(R) = 2 =$ dim(S). Then the groups $\widetilde{K_0}Sp(R)$ and $\widetilde{K_0}Sp(S)$ are isomorphic.*

Proof. We have $\widetilde{K_0}Sp(R) \simeq E(R)$ and $\widetilde{K_0}Sp(S) \simeq E(S)$, the result is obvious from Theorem 3.2.3. □

We now consider the weak Euler class groups. We shall first prove the invariance of the weak Euler class group under finite subintegral extension (for even dimensional rings) and then generalize it to arbitrary subintegral extension by a direct limit argument. Before proceeding we first clarify a notation.

Remark 3.2.3. Let R be a ring of dimension n and take $(I, \omega_I) \in E(R)$. Let $u \in R$ be a unit modulo I. By $\overline{u}\omega_I$ we mean the local orientation obtained from the composition

$$(R/I)^n \overset{\delta}{\simeq} (R/I)^n \overset{\omega_I}{\twoheadrightarrow} I/I^2,$$

where $\delta \in GL_n(R/I)$ has determinant $\overline{u}$ (here "bar" means modulo I). It follows from [B 2, Lemma 2.2] that if ω_1 and ω_2 are two local orientations of I, then there exists $v \in R$ such that v is a unit modulo I and $\omega_2 = \overline{v}\omega_1$.

Theorem 3.2.8. *Let R be a ring (containing $\mathbb{Q}$) with dim$(R) = n$, where n is even. Let $R \hookrightarrow S$ be a finite subintegral extension. Then $E_0(R) \simeq E_0(S)$.*

Proof. Let $\Phi_0 : E_0(R) \longrightarrow E_0(S)$ be the group homomorphism induced by the inclusion $R \hookrightarrow S$. We have already proved that there is an isomorphism $\Phi : E(R) \stackrel{\sim}{\to} E(S)$. Recall that there are canonical surjective morphisms $\psi : E(R) \twoheadrightarrow E_0(R)$ and $\psi' : E(S) \twoheadrightarrow E_0(S)$. Furthermore, $\psi'\Phi = \Phi_0\psi$. Therefore it easily follows that Φ_0 is surjective. Note that this is true even if we do not assume that n is even.

We may assume that R is reduced. Let C be the conductor of R in S. Then $\text{ht}(C) \geq 1$. To prove that Φ_0 is injective, let $(I) \in E_0(R)$ be such that $\Phi_0((I)) = (IS) = 0$ in $E_0(S)$. Let ω be any local orientation of I and ω^* be the local orientation of IS induced by ω. Then $(I,\omega) \in E(R)$, $(IS,\omega^*) \in E(S)$ and $\psi((I,\omega)) = (I)$, $\psi'((IS,\omega^*)) = (IS)$. Applying moving lemma (Lemma 2.1.4) we can find $(J,\omega_J) \in E(R)$ such that $(I,\omega) + (J,\omega_J) = 0$ in $E(R)$ and $J + I \cap C = R$. Then $(JS) = 0$ in $E_0(S)$ and if we can prove that $(J) = 0$ in $E_0(R)$ then it implies that $(I) = 0$ in $E_0(R)$. Therefore, without any loss of generality we may assume that I is comaximal with C. Consequently, $S/IS \simeq R/I$. We will need this information in the latter part.

Since $(IS) = 0$ in $E_0(S)$, by Proposition 2.2.2 there exists a stably free S-module P' of rank n together with an isomorphism $\chi' : S \stackrel{\sim}{\to} \wedge^n P'$ such that $e(P',\chi') = (IS, \omega \otimes S)$ in $E(S)$.

We now claim that there is a stably free R-module P of rank n such that $P \otimes S \simeq P'$. *Proof of the claim:* As P' is a stably free S-module of rank $n = \dim(S)$, there is a unimodular row $(a_0, a_1, \cdots, a_n) \in \text{Um}_{n+1}(S)$ corresponding to P'. Then $(\overline{a_0}, \overline{a_1}, \cdots, \overline{a_n}) \in \text{Um}_{n+1}(S/C)$, where bar denotes reduction modulo C. Since $\dim(S/C) \leq n-1$, the row $(\bar{a}_0, \cdots, \bar{a}_n)$ is elementarily completable. Say $(\bar{a}_1, \cdots, \bar{a}_n)\overline{\theta} = (\bar{1}, \cdots, \bar{0})$. Let $\theta \in \mathcal{E}_{n+1}(S)$ be a lift of $\overline{\theta}$. Write $(a_0, \cdots, a_n)\theta = (b_0, \cdots, b_n)$. Then $(\bar{b}_0, \cdots, \bar{b}_n) = (\bar{1}, \bar{0}, ..., \bar{0})$. Therefore we have $b_0 - 1 \in C \subset R$ and $b_1, \cdots, b_n \in C \subset R$. Hence $b_0, \cdots, b_n \in R$. Let $\mathfrak{m}$ be any maximal ideal of R. If $C \subset \mathfrak{m}$, then $b_0 \notin \mathfrak{m}$. If $C \not\subset \mathfrak{m}$, then $b_i \notin \mathfrak{m}$ for some $i = 1, \cdots, n$. In any case, the ideal in R generated by $b_0, \cdots, b_n$ is not contained in any maximal ideal of R. Therefore, $(b_0, \cdots, b_n) \in \text{Um}_{n+1}(R)$.

Let P be the stably free R-module of rank n corresponding to $(b_0, \cdots, b_n)$. Then it is easy to check that $P \otimes S \simeq P'$.

Let $\chi : R \stackrel{\sim}{\to} \wedge^n P$ be an isomorphism. Consider the Euler class $e(P \otimes S, \chi \otimes S) = e(P', \chi \otimes S)$. Now χ' and $\chi \otimes S$ differ by a unit of S, say, u. Therefore, $e(P \otimes S, \chi \otimes S) =$

$(IS, u\omega)$ in $E(S)$. As $S/IS \simeq R/I$, the image of u in S/IS has a lift to $\overline{v} \in (R/I)^*$ (where bar denotes reduction modulo I). We then have $\Phi(e(P,\chi)) = e(P \otimes S, \chi \otimes S) = (IS, u\omega) = (IS, \overline{u}\omega)$. On the other hand, $\Phi((I, \overline{v}\omega)) = (IS, \overline{u}\omega)$. As Φ is injective, it follows that $e(P,\chi) = (I, \overline{v}\omega)$. By Theorem 2.2.1 there is a surjection $\alpha : P \twoheadrightarrow I$. As P is stably free, it follows from Proposition 2.2.2 that $(I) = 0$ in $E_0(R)$. This proves that $\Phi_0 : E_0(R) \to E_0(S)$ is injective and completes the proof. □

Now we give an easy proof of Theorem 3.2.8.

Proof. Let $R \hookrightarrow S$ be a finite subintegral extension of $\mathbb{Q}$-algebras with $\dim(R) = n = \dim(S)$, where n is even. By [B-RS 4, Theorem 7.6] we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\frac{Um_{n+1}(R)}{\mathcal{E}_{n+1}(R)} & \longrightarrow & E(R) & \longrightarrow & E_0(R) & \longrightarrow & 0 \\
\downarrow \phi & & \downarrow \Phi & & \downarrow \Phi_0 & & \\
\frac{Um_{n+1}(S)}{\mathcal{E}_{n+1}(S)} & \longrightarrow & E(S) & \longrightarrow & E_0(S) & \longrightarrow & 0
\end{array}$$

Now Φ is an isomorphism and it is easy to see that ϕ is surjective (see the proof of the claim above). Therefore, it follows that Φ_0 is injective. We may note that ϕ is also injective due to a result by Gubeladze [G]. □

Remark 3.2.4. If $\dim(R) = 2$, we do not need to assume that $\mathbb{Q} \subset R$ in the above theorem.

We now prove that the weak Euler class group also commutes with direct limit. Let S be a Noetherian ring which is the filtered direct limit of a direct system of Noetherian subrings $\{S_\alpha, \mu_{\alpha\beta}\}$ and the set up be exactly as in Theorem 3.2.2. Then it is easy to see that the weak Euler class groups form a direct system $\{E_0(S_\alpha), f_{\alpha\beta}\}$. Let $\{\varinjlim E_0(S_\alpha), f_\alpha\}$ be its direct limit. For each $\alpha \in \Omega$, we also have group homomorphism $h_\alpha : E_0(S_\alpha) \longrightarrow E_0(S)$ induced by the inclusion $\mu_\alpha : S_\alpha \hookrightarrow S$ with the property that $h_\beta f_{\alpha\beta} = h_\alpha$ for $\alpha \leq \beta$. We prove below that $E_0(S)$ is isomorphic to $\varinjlim E_0(S_\alpha)$. Note that we do not put any restriction on the dimension of the ring.

Theorem 3.2.9. *With notations as above,* $E_0(S) = E_0(\varinjlim S_\alpha) \simeq \varinjlim E_0(S_\alpha)$.

Proof. In this proof we shall apply Theorem 3.2.2 and freely use the notations from there. As for each α there is a group homomorphism $h_\alpha : E_0(S_\alpha) \longrightarrow E_0(S)$ with $h_\beta f_{\alpha\beta} = h_\alpha$ for $\alpha \leq \beta$, by the properties of direct limit there is a map $g : \varinjlim E_0(S_\alpha) \longrightarrow E_0(S)$. We prove that g is an isomorphism.

By Remark 2.2.3 there is a canonical surjective morphism $E(S_\alpha) \twoheadrightarrow E_0(S_\alpha)$ for each $\alpha \in \Omega$. They will induce a surjection $\Phi : \varinjlim E(S_\alpha) \twoheadrightarrow \varinjlim E_0(S_\alpha)$. We also have a canonical surjection $\Psi : E(S) \twoheadrightarrow E_0(S)$. We then have the following commutative diagram.

$$\begin{array}{ccc} \varinjlim E(S_\alpha) & \xrightarrow{\ \psi\ } & E(S) \\ \Phi \downarrow & & \downarrow \Psi \\ \varinjlim E_0(S_\alpha) & \xrightarrow{\ g\ } & E_0(S) \end{array}$$

Therefore, g is surjective. To prove that g is injective, note that an element x of $\varinjlim E_0(S_\alpha)$ is of the form $x = f_\alpha(x_\alpha)$ for some $\alpha \in \Omega$ and $x_\alpha \in E_0(S_\alpha)$. Let $x_\alpha = (J_\alpha) \in E_0(S_\alpha)$, where J_α is an ideal of S_α of height n. Assume that $g(x) = 0$ in $E_0(S)$. This implies that $h_\alpha((J_\alpha)) = (J_\alpha S) = 0$ in $E_0(S)$.

Let $(J_\alpha, \omega_\alpha) \in E(S_\alpha)$ be a preimage of (J_α). By a slight abuse of notations let us view $(J_\alpha, \omega_\alpha)$ as an element of $\varinjlim E(S_\alpha)$ and write $\psi((J_\alpha, \omega_\alpha)) = (J_\alpha S, \omega_\alpha^*)$, where ω_α^* is induced by ω_α. By the commutativity of the above diagram, $\Psi((J_\alpha S, \omega_\alpha^*)) = (J_\alpha S) = 0$ in $E_0(S)$. Applying [B-RS 3, Lemma 3.3] (which works for a commutative Noetherian ring) it follows that

$$(J_\alpha S, \omega_\alpha^*) + \sum_{i=1}^{k}(I_i, \omega_i) = \sum_{j=k+1}^{l}(I_j, \omega_j)$$

in $E(S)$, where each of $I_1, \cdots, I_l$ is generated by n elements.

We now want to lift the above equation in $\varinjlim E(S_\alpha)$. Let us describe the process with one element, say, (I_1, ω_1). Let $I_1 = (a_1, \cdots, a_n)$ and let ω denote the global orientation of I_1 induced by these generators. Then by Remark 3.2.3, $(I_1, \omega_1) = (I_1, \overline{u}\omega)$ for some $u \in S$ which is unit modulo I_1. There exists $v \in S$ such that $uv - 1 = a_1 b_1 + \cdots + a_n b_n$. As S is the filtered direct limit of $\{S_\alpha\}_{\alpha \in \Omega}$, we can find some $\beta_1 \in \Omega$ such that $a_1, \cdots, a_n, b_1 \cdots, b_n, u, v \in S_{\beta_1}$. Let $K_1 = (a_1, \cdots, a_n)S_{\beta_1}$ and let σ denote the global orientation of K_1 induced by these generators. Composing σ with an automorphism of

$(S_{\beta_1}/K_1)^n$ with determinant u modulo K_1 we get a local orientation, say, σ_1 of K_1. It is then clear that $\phi_{\beta_1}((K_1, \sigma_1)) = (I_1, \omega_1)$.

Applying the above process for each of $I_1, \cdots, I_l$ we can find a suitable $\beta \in \Omega$ and elements $(K_i, \sigma_i) \in E(S_\beta)$, $1 \leq i \leq l$ such that $\phi_\beta((K_i, \sigma_i)) = (I_i, \omega_i)$ for each i. Moreover, applying Theorem 3.2.2 it is easy to see that the following equation holds in $E(S_\beta)$.

$$(J_\alpha S_\beta, \omega_\alpha \otimes S_\beta) + \sum_{i=1}^{k} (K_i, \sigma_i) = \sum_{j=k+1}^{l} (K_j, \sigma_j)$$

As each K_i is generated by n elements, it follows that $(J_\alpha S_\beta) = 0$ in $E_0(S_\beta)$ and as a consequence, $x = 0$ in $\varinjlim E_0(S_\alpha)$. □

Theorem 3.2.10. *Let R be a ring (containing $\mathbb{Q}$) with dim$(R) = n$, where n is even. Let $R \hookrightarrow S$ be a subintegral extension. Then $E_0(R) \simeq E_0(S)$.*

Proof. The proof is along the same line as in Theorem 3.2.3 and is obtained by using Theorem 3.2.8 and Theorem 3.2.9. □

Remark 3.2.5. Similarly one can prove that if $R \hookrightarrow S$ is a subintegral extension of even dimensional $\mathbb{Q}$-algebras, then $E_0(R[T]) \simeq E_0(S[T])$.

Remark 3.2.6. Let $R \hookrightarrow S$ be a subintegral extension of even dimensional rings and L be a projective R-module of rank one. By Remark 2.2.4 we know that the weak Euler class group does not depend on L. Therefore, applying Remark 2.2.4 and Theorem 3.2.10 above, we have $E_0(R, L) \simeq E_0(R) \simeq E_0(S) \simeq E_0(S, L \otimes S)$.

When dim(R) is not necessarily even, we have the following affirmative result. Recall that for a module M, the notation $\mu(M)$ stands for the minimal number of generators of M.

Theorem 3.2.11. *Let R be an affine algebra over a C_1-field k of characteristic zero and $R \hookrightarrow S$ be a subintegral extension with dim$(R) = n \geq 2$. Then $E_0(R) \simeq E_0(S)$. In particular, if J is an ideal of R of height n such that $\mu(J/J^2) = n$, then $\mu(J) = n$ if and only if $\mu(JS) = n$.*

Proof. For any affine algebra of dimension $n \geq 2$ over a C_1-field of characteristic zero, the Euler class group is isomorphic to the weak Euler class group. A proof for $n \geq 3$ is given in [D 2, Lemma 5.2], whereas the case $n = 2$ can be worked out easily. Therefore, we have $E(R) \simeq E_0(R)$ and $E(S) \simeq E_0(S)$ and the first assertion follows from Theorem 3.2.3.

To prove the second part, let $J \subset R$ be an ideal of height n. If J is generated by n elements, then obviously so is JS.

Conversely, let $\mu(J/J^2) = n$ and suppose it is given that $\mu(JS) = n$. Let $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ be any surjection and let $\omega_J^* : (S/JS)^n \twoheadrightarrow JS/J^2S$ be the surjection induced by ω_J. As $E(S) \simeq E_0(S)$, we have $(JS, \omega_J^*) = 0$ in $E(S)$ and therefore by Theorem 3.2.3, $(J, \omega_J) = 0$ in $E(R)$ and J is generated by n elements. $\square$

Chapter 4

Integral extensions and the Euler class groups

The aim of this short chapter is to explore what happens when $R \hookrightarrow S$ is an integral extension. We shall give an example to show that even if $R \hookrightarrow S$ is a finite (hence integral) birational extension, $E(R)$ may not be isomorphic to $E(S)$. Before giving this example, we engage ourselves in a more delicate investigation and prove a result which generalizes Theorem 3.2.3 and improves the understanding further.

First, to ensure that there is a group homomorphism from $E(R)$ to $E(S)$ when $R \hookrightarrow S$ is integral, we need to prove some generalities on the Euler class group. In fact we define a group which is very similar to the Euler class group.

Definition 4.0.1. Let A be a Noetherian ring of dimension $n \geq 2$. Let $\mathcal{G}$ be the free abelian group on the pairs (J, ω_J), where: (1) J is an $\mathfrak{m}$-primary ideal for some maximal ideal $\mathfrak{m}$ of A (*not necessarily of height* n), (2) ω_J is an $E_n(A/J)$-equivalence class of surjections from $(A/J)^n \twoheadrightarrow J/J^2$ (i.e., a *local orientation* of J). Given any zero dimensional ideal I of A and a surjection $\omega_I : (A/I)^n \twoheadrightarrow I/I^2$, one can associate an element of $\mathcal{G}$ in an obvious manner; we call it (I, ω_I). Let $\mathcal{H}$ be the subgroup of $\mathcal{G}$ generated by all elements of the type (I, ω_I) where $\dim(A/I) = 0$ and ω_I can be lifted to a surjection from A^n to I. We define $\widetilde{E}(A) = \mathcal{G}/\mathcal{H}$.

We can follow the theory of Euler class groups as developed in [B-RS 1, B-RS 4] and adapting similar methods can prove the following results. Recall that in [B-RS 1, B-RS 4] the results are proved for ideals of height n in an n-dimensional ring whereas here we need the same results for zero dimensional ideals. Since the methods of proofs are exactly the same, we do not repeat them here.

Lemma 4.0.1. *Let A be a Noetherian ring of dimension $n \geq 2$ and $J \subset A$ be a zero dimensional ideal. Suppose that $J = (f_1, \cdots, f_n) + J^2$. Then there exist $g_1, \cdots, g_n \in J$ and an ideal $J' \subset A$ such that*

(i) $J \cap J' = (g_1, \cdots, g_n)$ *with* $f_i - g_i \in J^2$.

(ii) $J + J' = A$ *with* $ht(J') \geq n$.

(iii) *Given finitely many zero dimensional ideals $J_1, \cdots, J_r$ of A, the ideal J' can be chosen with the additional property that it is comaximal with J_i for $i = 1, \cdots, r$.*

Proposition 4.0.1. *(Addition principle) Let A be a Noetherian ring of dimension $n \geq 2$. Let J_1 and J_2 be two zero dimensional ideals which are comaximal. Suppose that $J_1 = (a_1, \cdots, a_n)$ and $J_2 = (b_1, \cdots, b_n)$. Then $J_1 \cap J_2 = (c_1, \cdots, c_n)$, where $a_i - c_i \in J_1^2$ and $b_i - c_i \in J_2^2$.*

Proposition 4.0.2. *(Subtraction principle) Let A be a Noetherian ring of dimension $n \geq 2$. Let J_1 and J_2 be two zero dimensional ideals which are comaximal. Suppose that $J_1 = (a_1, \cdots, a_n)$ and $J_1 \cap J_2 = (c_1, \cdots, c_n)$ with $a_i - c_i \in J_1^2$. Then $J_2 = (b_1, \cdots, b_n)$ with $b_i - c_i \in J_2^2$.*

Theorem 4.0.1. *Let A be a Noetherian ring of dimension $n \geq 2$. Let $J \subset A$ be a zero dimensional ideal such that J/J^2 is generated by n elements and let $\omega_J : (A/J)^n \twoheadrightarrow J/J^2$ be a local orientation of J. Suppose that the image of (J, ω_J) is zero in $\widetilde{E}(A)$. Then ω_J can be lifted to a surjection $\alpha : A^n \twoheadrightarrow J$.*

From the above results it is natural to suspect that $\widetilde{E}(A)$ is possibly isomorphic to $E(A)$. Such is indeed the case, as proved in the proposition below.

Proposition 4.0.3. *Let A be a Noetherian ring of dimension $n \geq 2$. The canonical map from $E(A)$ to $\widetilde{E}(A)$ is an isomorphism.*

Proof. It is obvious that there is a canonical map, say, $\theta : E(A) \longrightarrow \widetilde{E}(A)$, which takes an element (J, ω_J) of $E(A)$ to (J, ω_J) in $\widetilde{E}(A)$. It is also clear that θ is a group homomorphism.

We now define a map in the reverse direction. Let $J \subset A$ be an ideal such that $\dim(A/J) = 0$ and let ω_J be a local orientation of J given by: $J = (a_1, \cdots, a_n) + J^2$. By Lemma 2.1.3 there is $e \in J^2$ such that $J = (a_1, \cdots, a_n, e)$, where $e(1-e) \in (a_1, \cdots, a_n)$. Using a standard general position argument (see [D-RS, Lemma 2.4]) it follows that there are elements $\gamma_1, \cdots, \gamma_n \in A$ such that the ideal $I = (a_1 + \gamma_1 e, \cdots, a_n + \gamma_n e)$ has the property that $\text{ht}(I_e) \geq n$. Note that $I + (e) = J$ and $(e) \subset J^2$. Applying Lemma 2.1.3 we see that there is an ideal J' such that

$$(a_1 + \gamma_1 e, \cdots, a_n + \gamma_n e) = J \cap J'$$

where $J' + (e) = A$. Now it is easy to deduce that $\text{ht}(J') \geq n$. The case when $J' = A$ being trivial, we assume that $\text{ht}(J') = n$. Let us write $b_i = a_i + \gamma_i e$. Clearly $b_1, \cdots, b_n$ induce ω_J. Let $\omega_{J'}$ be the local orientation of J' induced by $b_1, \cdots, b_n$. We then have $(J, \omega_J) + (J', \omega_{J'}) = 0$ in $\widetilde{E}(A)$. One can repeat the above procedure for J' and $\omega_{J'}$ to obtain an ideal J'' of height n and a local orientation $\omega_{J''}$ such that $(J', \omega_{J'}) + (J'', \omega_{J''}) = 0$ in $\widetilde{E}(A)$. Therefore, $(J, \omega_J) = (J'', \omega_{J''})$ in $\widetilde{E}(A)$.

We define $\eta : \widetilde{E}(A) \longrightarrow E(A)$ by sending (J, ω_J) to $(J'', \omega_{J''})$ in $E(A)$.

We need to prove that η is well-defined.

We show that our definition of η does not depend on the choice of J''. Since $(J', \omega_{J'}) + (J'', \omega_{J''}) = 0$ in $E(A)$, we only need to check that our definition is independent of the choice of J'. Let I' be an ideal of A of height n such that (i) $J + I' = A$ and (ii) $(d_1, \cdots, d_n) = J \cap I'$, where $d_i - a_i \in I'^2$.

If $I' = A$ then it is easy to check using subtraction principles (Proposition 4.0.2) that $(J', \omega_{J'}) = 0$ in $E(A)$. Therefore assume that I' is a proper ideal. In fact, in the proof we will assume all the ideals to be proper.

Let $\omega_{I'} : (A/I')^n \twoheadrightarrow I'/I'^2$ be the local orientation induced by $d_1, \cdots, d_n$. We have to show that $(J', \omega_{J'}) = (I', \omega_{I'})$ in $E(A)$. Using Lemma 4.0.1 we can find an ideal J_1 of A of height n and a local orientation ω_{J_1} such that: (i) J_1 is comaximal with each of J, J' and I' , (ii) $(J',\omega_{J'})+(J_1,\omega_{J_1})=0$ in $E(A)$. Now it is enough to prove that $(I',\omega_{I'})+(J_1,\omega_{J_1})=0$ in $E(A)$. Again applying Lemma 4.0.1 we can find an ideal J_2 of A of height n such that $J\cap J_2$ is generated by n elements and J_2 is comaximal with each of J , J', I' and J_1. Now the ideals $J'\cap J_1$ and $J\cap J_2$ are both generated by n elements and they are comaximal. Applying the addition principle (Proposition 4.0.1), the ideal $J'\cap J_1\cap J\cap J_2$ is generated by n elements with appropriate set of generators. Since $J'\cap J$ is generated by n elements, by the subtraction principle (Proposition 4.0.2) it follows that $J_1\cap J_2$ is generated by n elements with appropriate set of generators. Since $J\cap I'$ and $J_1\cap J_2$ are both generated by n elements and they are comaximal, by the addition principle $I'\cap J_1\cap J\cap J_2$ is generated by n elements with appropriate set of generators. Again since $J\cap J_2$ is n-generated, it follows using the subtraction principle that $I'\cap J_1$ is n-generated by the appropriate set of generators. Keeping track of the generators, it is easy to see that this implies $(I',\omega_{I'})+(J_1,\omega_{J_1})=0$ in $E(A)$. Therefore, the map is well defined.

Clearly η is a group homomorphism. Further, θ and η are inverses of each other. □

An easier proof of the above proposition can be given, as we see below. However the above proof explicitly demonstrates the map from $\widetilde{E}(A)$ to $E(A)$.

Proof of Proposition 4.0.3: Using Moving lemma 2.1.4 and *Chinese remainder theorem*, it is easy to see that any element of $E(A)$ is of the form (I,ω_I), where $I\subset A$ is an ideal of height n and ω_I is a local orientation of I. Suppose $(I,\omega_I)=0$ in $\widetilde{E}(A)$. By Theorem 4.0.1, ω_I can be lifted to a surjection $\alpha: A^n \twoheadrightarrow I$. But then $(I,\omega_I)=0$ in $E(A)$. Hence θ is injective. Also by Lemma 4.0.1, it is clear that θ is surjective. □

Remark 4.0.1. As $E(A)\simeq E(A_{red})$, it follows that $\widetilde{E}(A)\simeq\widetilde{E}(A_{\text{red}})$.

We now assume that $R\hookrightarrow S$ is an integral extension with $\dim(R)=n\geq 2$. Let $(J,\omega_J)\in\widetilde{E}(R)$, where $\dim(R/J)=0$ and $\omega_J:(R/J)^n\twoheadrightarrow J/J^2$ is a local orientation of J. As $R\hookrightarrow S$ is integral, we have $\dim(S/JS)=0$. Further, ω_J induces $\omega_J^*:(S/JS)^n\twoheadrightarrow JS/(JS)^2$, a local orientation of JS. Therefore, $(JS,\omega_J^*)\in\widetilde{E}(S)$. It is now easy to see that there is a group homomorphism, say, $\Phi:\widetilde{E}(R)\longrightarrow\widetilde{E}(S)$, which takes (J,ω_J) to (JS,ω_J^*). Using Proposition 4.0.3, we have a group homomorphism from $E(R)$ to $E(S)$.

We now prove the following theorem which improves Theorem 3.2.3.

Theorem 4.0.2. *Let $R \hookrightarrow S$ be an integral extension such that the extension $R_{red} \hookrightarrow S_{red}$ is birational. Then the map $\Phi : \widetilde{E}(R) \longrightarrow \widetilde{E}(S)$ is surjective. Further, if $R_{red} \hookrightarrow S_{red}$ is a subintegral extension, then Φ is an isomorphism.*

Proof. We first show that it is enough to prove the result when $R \hookrightarrow S$ is a finite extension. To see this, let $(I, \omega) \in \widetilde{E}(S)$. Then I is an ideal of S with $\dim(S/I) = 0$ and ω is a local orientation of I induced by, say, $I = (f_1, \cdots, f_n) + I^2$. Then by Lemma 2.1.3, there exists $e \in I$ such that $I = (f_1, \cdots, f_n, e)$ where $e(1-e) \in (f_1, \cdots, f_n)$. Suppose that $e(1-e) = k_1 f_1 + \cdots k_n f_n$ where $k_i \in S$. Now consider $R_1 = R[f_1, \cdots, f_n, e, k_1, \cdots, k_n] \hookrightarrow S$ and $R \hookrightarrow R_1$ is finite. Let $I' = (f_1, \cdots, f_n, e)R_1$. Then $I' = (f_1, \cdots, f_n, e - k_1 f_1 - \cdots - k_n f_n) = (f_1, \cdots, f_n, e^2)$ implying that $I' = (f_1, \cdots, f_n) + I'^2$. If ω' denotes the local orientation of I' induced by this set of generators of I'/I'^2, then $(I', \omega') \in \widetilde{E}(R_1)$. The map from $\widetilde{E}(R_1)$ to $\widetilde{E}(S)$ takes (I', ω') to (I, ω). It is enough to find a preimage of (I', ω') in $\widetilde{E}(R)$. Therefore, we may assume the extension to be finite to start with.

Let us write $\overline{R} = R_{\text{red}}$ and $\overline{S} = S_{\text{red}}$. Let C be the conductor of $\overline{R}$ in $\overline{S}$. As $\overline{R} \hookrightarrow \overline{S}$ is birational, using Lemma 3.1.5 we have $\text{ht}(C) \geq 1$.

Let $(I, \omega_I) \in \widetilde{E}(\overline{S})$. By Proposition 4.0.3 we may assume that $\text{ht}(I) = n$. Exactly the same proof as in Step 2 of Theorem 3.2.1 will show that the map from $\widetilde{E}(\overline{R})$ to $\widetilde{E}(\overline{S})$ is surjective. It follows that the map $\Phi : \widetilde{E}(R) \longrightarrow \widetilde{E}(S)$ is surjective.

We now assume that $\overline{R} \hookrightarrow \overline{S}$ is subintegral. Then, by Theorem 3.2.3 $E(\overline{R}) \simeq E(\overline{S})$, and we have,

$$\widetilde{E}(R) \simeq E(R) \simeq E(\overline{R}) \simeq E(\overline{S}) \simeq E(S) \simeq \widetilde{E}(S).$$

□

We now recall some definitions.

Definition 4.0.2. The group of all isomorphism classes of rank one projective R-modules is called the *Picard group* of R and denoted by $\text{Pic}(R)$. The operation is defined by $[P] * [Q] := [P \otimes Q]$, where $[P]$ denotes the isomorphism class of P. Pic is a functor from commutative rings to abelian groups. If $f : R \longrightarrow S$ is a ring homomorphism then $\text{Pic}(f) : \text{Pic}(R) \longrightarrow \text{Pic}(S)$ is a group homomorphism sending L to $L \otimes_R S$.

Definition 4.0.3. Let $\mathfrak{P}(R)$ consists of the isomorphism classes of projective R-modules.

Write (P) for the isomorphism class of P. The *Grothendieck group* of R is

$$\mathrm{K}_0(R) \stackrel{\text{def}}{=} \frac{<(P)|P \in \mathfrak{P}(R)>}{<(P \oplus Q) - (P) - (Q)|P, Q \in \mathfrak{P}(R)>}.$$

Let $[P]$ denotes the image of (P) in $\mathrm{K}_0(R)$. Thus we have $[P \oplus Q] = [P] \oplus [Q]$ in $\mathrm{K}_0(R)$,

Note that if $\alpha : R \longrightarrow S$ is a ring homomorphism, then α induces a group homomorphism $\alpha_* : \mathrm{K}_0(R) \longrightarrow \mathrm{K}_0(S)$ defined by $\alpha_*([P]) = [P \otimes_R S]$.

Definition 4.0.4. Let R be a ring with $Spec(R)$ is connected. $\widetilde{\mathrm{K}}_0(R)$ is the subgroup of $\mathrm{K}_0(R)$ generated by $[P] - [R^n]$, where P has rank n.

The determinant induces an epimorphism

$$\det : \widetilde{\mathrm{K}}_0(R) \longrightarrow \mathrm{Pic}(R)$$

defined by $\det([P]) = \wedge^n(P)$, where $rank(P) = n$. The kernel of this map is denoted by $\mathrm{SK}_0(R)$.

Definition 4.0.5. Let $GL_n(R)$ be the set of all $n \times n$ invertible matrices over R. The group $GL_n(R)$ is embedded in the group $GL_{n+1}(R)$ by identifying a matrix $A \in GL_n(R)$ with the larger matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$ of $GL_{n+1}(R)$. Under such an identification, we have $SL_n(R) \subset SL_{n+1}(R)$ and $E_n(R) \subset E_{n+1}(R)$. Let $GL(R) := \bigcup\limits_{n \geq 1} GL_n(R)$, $SL(R) := \bigcup\limits_{n \geq 1} SL_n(R)$ and $E(R) := \bigcup\limits_{n \geq 1} E_n(R)$.

The *Whitehead group* of R is $\mathrm{K}_1(R) := GL(R)/E(R)$ and the *special Whitehead group* of R is $\mathrm{SK}_1(R) := SL(R)/E(R)$.

The following theorem is due to Milnor.

Theorem 4.0.3. *([Ba, pp. 481]) Let*

$$\begin{array}{ccc} A & \xrightarrow{g_1} & A_1 \\ \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle f_1} \\ A_2 & \xrightarrow{f_2} & B \end{array}$$

be a Cartesian diagram of ring homomorphisms in which f_1 or f_2 is surjective. Then the following Mayer-vietoris sequences are exact

(i)

$$0 \longrightarrow A^* \longrightarrow A_1^* \oplus A_2^* \longrightarrow B^* \longrightarrow Pic(A)$$

$$\longrightarrow Pic(A_1) \oplus Pic(A_2) \longrightarrow Pic(B).$$

(ii)

$$SK_1(A) \longrightarrow SK_1(A_1) \oplus SK_1(A_2) \longrightarrow SK_1(B) \longrightarrow SK_0(A)$$

$$\longrightarrow SK_0(A_1) \oplus SK_0(A_2) \longrightarrow SK_0(B).$$

We end this chapter by the following example. This example was suggested to us by Bhatwadekar (personal communication).

Example 4.0.1. Let $S = \mathbb{C}[X,Y]$ and $f = (X^2 - Y^3)$. Let $S_1 = S/(f) \simeq \mathbb{C}[T^2,T^3]$. Then $(S_1)^* = \mathbb{C}^*$. By [Mu-P, Example 2.5] $\widetilde{K_0}(\mathbb{C}[T^2,T^3]) \simeq \mathbb{C}$. Therefore, by [Mu-P, Corollary 2.2], $SK_1(S_1)$ is of infinite rank. Therefore [Mu-P, Proposition 2.5] ensures that $SK_1(S_1) \neq 0$ (also see [Kr, Section 12]).

Let $R = \mathbb{C}[X] + fS$. Then S is a finite (birational) extension of R and fS is the conductor of R in S. Moreover, $R/fS \simeq \mathbb{C}[X]$.

Now it is easy to see that R is seminormal. From the Cartesian square

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R/f & \longrightarrow & S/fS \simeq S_1 \end{array}$$

we have an exact Mayer-Vietoris sequence (by Theorem 4.0.3):

$$0 \longrightarrow R^* \longrightarrow S^* \oplus (R/f)^* \longrightarrow S_1^* \longrightarrow \text{Pic}(R)$$

$$\longrightarrow \text{Pic}(S) \oplus \text{Pic}(R/f) \longrightarrow \text{Pic}(S_1).$$

Consequently we have

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathbb{C}^* \oplus \mathbb{C}^* \longrightarrow \mathbb{C}^* \longrightarrow \text{Pic}(R) \longrightarrow 0,$$

from which it follows that $\text{Pic}(R) = 0$.

Now using the above Cartesian square, we have another exact sequence

$$SK_1(R) \longrightarrow SK_1(S) \oplus SK_1(R/f) \longrightarrow SK_1(S_1) \longrightarrow SK_0(R)$$

$$\longrightarrow SK_0(S) \oplus SK_0(R/f) \longrightarrow SK_0(S_1).$$

It is easy to see that $SK_1(R/f)$, $SK_0(S)$ and $SK_0(R/f)$ are all trivial. Also by Suslin's stability theorem [Su 2] (or see [La, pp. 220]), we have $SK_1(S) = 0$. Therefore, we have

$$SK_1(S_1) \simeq SK_0(R).$$

As $\text{Pic}(R) = 0$, we have $\widetilde{K_0}(R) \simeq SK_0(R)$ and consequently, $\widetilde{K_0}(R) \simeq SK_1(S_1) \neq 0$. Therefore there exists a projective R-module P of rank two (with trivial determinant) which is not stably free. Fix an isomorphism $\chi : R \simeq \wedge^2(P)$ and consider the Euler class $e(P, \chi) \in E(R)$. As P has trivial determinant, $e(P, \chi) = 0$ would imply that P is free. Therefore, $E(R)$ is not the trivial group whereas $E(S)$ is trivial, showing that the map from $E(R)$ to $E(S)$ is not injective. $\square$

Chapter 5

Subintegral extension of 2-dimensional rings

The following question, mentioned in [I 2, Remark (b), pp 331], is still open.

Question 5.0.1. *Let $R \hookrightarrow S$ be a subintegral extension. Let P and Q be two projective R-modules with $det(P) \simeq det(Q)$ and $P \otimes S \simeq Q \otimes S$. Is $P \simeq Q$?*

In [I 2] it is suggested that perhaps the compatibility of the two isomorphisms is required as an additional hypothesis. Following that suggestion we give an affirmative answer in the case when $\dim(R) = 2$.

We need the following crucial lemma from [B 2].

Lemma 5.0.1. *[B 2, Lemma 3.5] Let A be a ring and P and Q be two projective A-modules of rank 2 such that $det(P) \simeq det(Q)$. Let $\chi : det(P) \xrightarrow{\sim} det(Q)$ be an isomorphism. Let $J \subset A$ be an ideal of height 2. Let $\alpha : P \twoheadrightarrow J$ and $\beta : Q \twoheadrightarrow J$ be two surjections. Let bar denote the reduction modulo J and $\overline{\alpha} : \overline{P} \twoheadrightarrow J/J^2$ and $\overline{\beta} : \overline{Q} \twoheadrightarrow J/J^2$ be the surjections induced from α and β, respectively. Suppose that there exists an isomorphism $\delta : \overline{P} \xrightarrow{\sim} \overline{Q}$ such that : (i) $\overline{\beta}\delta = \overline{\alpha}$ and (ii) $\wedge^2\delta = \overline{\chi}$. Then there exists an isomorphism $\sigma : P \xrightarrow{\sim} Q$ such that $\beta\sigma = \alpha$, σ is a lift of δ and $\wedge^2\sigma = \chi$.*

We now prove the main result of this chapter. Note that in view of Definition 2.2.2, we do not need to assume that $\mathbb{Q} \subset R$.

Theorem 5.0.1. *Let R be a ring of dimension 2 and $R \hookrightarrow S$ be a subintegral extension. Let P and Q be two projective R-modules of rank 2 such that $det(P) \simeq det(Q)$ and $P \otimes S \simeq Q \otimes S$. Let $\chi : det(P) \xrightarrow{\sim} det(Q)$ and $\theta : P \otimes S \xrightarrow{\sim} Q \otimes S$ be isomorphisms. Assume that $\chi \otimes S = \wedge^2\theta$. Then $P \simeq Q$.*

Proof. Now $\det(P) = \wedge^2(P)$ and $\det(Q) = \wedge^2(Q)$ are projective R-modules of rank one. Since they are isomorphic, there is a projective R-module L of rank one which is isomorphic to both. We fix $\chi_1 : L \xrightarrow{\sim} \wedge^2 P$. Let $\chi_2 := (\chi)\chi_1 : L \xrightarrow{\sim} \wedge^2 P \xrightarrow{\sim} \wedge^2 Q$.

We now point out a general fact. Let A be a ring of dimension n and P be a projective A-module of rank n with an isomorphism $\chi : L \xrightarrow{\sim} \wedge^n(P)$. Recall from the definition of the Euler class of the pair (P, χ) that $e(P, \chi)$ is an invariant of the isomorphism class of (P, χ).

From the above paragraph we conclude that the Euler classes $e(P \otimes S, \chi_1 \otimes S)$ and $e(Q \otimes S, \chi_2 \otimes S)$ are equal in the Euler class group $E(S, L \otimes S)$.

The two elements $e(P \otimes S, \chi_1 \otimes S)$ and $(Q \otimes S, \chi_2 \otimes S)$ of $E(S, L \otimes S)$ are the images of the elements $e(P, \chi_1)$ and $e(P, \chi_2)$, respectively, under the natural map $\Phi : E(R, L) \longrightarrow E(S, L \otimes S)$. The map Φ is injective. Therefore $e(P, \chi_1) = e(Q, \chi_2)$ in $E(R, L)$. Let $e(P, \chi_1) = e(Q, \chi_2) = (I, \omega_I)$ in $E(R, L)$.

Now using Theorem 2.2.1, there exist two surjections $f : P \twoheadrightarrow I$ and $g : Q \twoheadrightarrow I$ such that (I, ω_I) is obtained from (f, χ_1) and (g, χ_2).

Let $\mu : (R/I)^2 \xrightarrow{\sim} P/IP$ and $\tau : (R/I)^2 \xrightarrow{\sim} Q/IQ$ be two isomorphisms such that $\wedge^2\mu = \overline{\chi}_1$ and $\wedge^2\tau = \overline{\chi}_2$. From the definition of the Euler class of a projective module, it follows that $\omega_I = \overline{f}\mu = \overline{g}\tau$.

Now consider the isomorphism $\delta = \tau\mu^{-1} : P/IP \xrightarrow{\sim} Q/IQ$. Then we have $\overline{g}\delta = \overline{f}$ and $\wedge^2(\delta) = (\wedge^2\tau)(\wedge^2\mu^{-1}) = \overline{\chi_2\chi_1}^{-1} = \overline{\chi}$.

Therefore we have two surjections $f : P \twoheadrightarrow I$ and $g : Q \twoheadrightarrow I$ and an isomorphism $\delta : P/IP \xrightarrow{\sim} Q/IQ$ such that $\wedge^2\delta = \overline{\chi}$ and $\overline{g}\delta = \overline{f}$. Therefore by the above lemma there exists an isomorphism $\phi : P \xrightarrow{\sim} Q$ such that: (i) $\beta\phi = \alpha$, (ii) ϕ is a lift of δ, and (iii) $\wedge^2\phi = \chi$. Hence the theorem is proved. □

Definition 5.0.1. Let R be a ring and P be a projective R-module. We say that P is cancellative if $P \oplus R \simeq Q \oplus R$ implies $P \simeq Q$.

Corollary 5.0.1. *Let $R \hookrightarrow S$ be a subintegral extension with dim$(R) = 2$. Let L be a rank one projective R-module such that $(L \ominus R) \otimes S$ is cancellative. Then $L \oplus R$ is also cancellative.*

Proof. We give two proofs of this corollary.

Proof 1. Let P be a projective R-module of rank 2 such that $L \oplus R^2 \simeq P \oplus R$. Then $L \simeq \wedge^2(P)$. We fix an isomorphism $\chi : L \xrightarrow{\sim} \wedge^2(P)$.

Let us denote $L \otimes_R S$ by $\widetilde{L}$ and $P \otimes_R S$ as $\widetilde{P}$. As $\widetilde{L}$ is cancellative, there is an isomorphism $\phi : \widetilde{L} \oplus S \xrightarrow{\sim} \widetilde{P}$. Then ϕ induces $\wedge^2(\phi) : \widetilde{L} \xrightarrow{\sim} \wedge^2(\widetilde{P})$. The isomorphism $\chi \otimes S : \widetilde{L} \xrightarrow{\sim} \wedge^2(\widetilde{P})$ (induced by χ) and the isomorphism $\wedge^2(\phi)$ differ by a unit $u \in S$. Define $\tau : \widetilde{L} \oplus S \longrightarrow \widetilde{L} \oplus S$ by sending (l, s) to (l, us). Then τ is an isomorphism and moreover, $\wedge^2(\tau) : \widetilde{L} \xrightarrow{\sim} \widetilde{L}$ is just scalar multiplication by u. Then the composition $\theta = \phi\tau : \widetilde{L} \oplus S \xrightarrow{\sim} \widetilde{P}$ has the property that $\wedge^2(\theta) = \chi \otimes S$. Now we can apply the above theorem to conclude that $L \oplus R \xrightarrow{\sim} P$. Thus, $L \oplus R$ is cancellative.

Proof 2. In this proof we do not use the above theorem. We shall denote $L \otimes_R S$ by $\widetilde{L}$. We use the following observation of Bhatwadekar in [B 2, p. 348] : *for a ring A and a projective A-module L of rank one, $L \oplus A$ is cancellative if and only if $E(A, L) \simeq E_0(A, L)$.*

Now assume that $\widetilde{L} \oplus S$ is cancellative. Then $E(S, \widetilde{L}) \simeq E_0(S, \widetilde{L})$. Since $R \hookrightarrow S$ is subintegral, using Theorem 3.2.4 and Remark 3.2.6 it follows that $E(R, L) \simeq E_0(R, L)$. Therefore $L \oplus R$ is cancellative. □.

The following proposition is due to Bhatwadekar.

Proposition 5.0.1. *[B 2, Proposition 3.7] Let A be ring of dimension 2 and let P be a rank 2 projective A-module. If $\wedge^2 P \oplus A$ is cancellative, then P is cancellative.*

We can now deduce the following corollary.

Corollary 5.0.2. *Let $R \hookrightarrow S$ be a subintegral extension with dim$(R) = 2$. Suppose that all projective S-modules of rank 2 are cancellative. Then all projective R-modules of rank 2 are cancellative.*

Proof. By the above proposition it is sufficient to consider rank 2 projective R-modules of the form $L \oplus R$, where L is a rank one projective R-module. Rest of the proof follows from the above corollary. □.

We can easily extend Theorem 5.0.1 to $R[T]$ for projective modules with trivial determinants, in the following way.

Theorem 5.0.2. *Let R be a ring (containing $\mathbb{Q}$) of dimension 2 and $R \hookrightarrow S$ be a subintegral extension. Let P and Q be two projective $R[T]$-modules of rank 2 with trivial determinants such that $P \otimes S[T] \simeq Q \otimes S[T]$. Let $\chi : det(P) \xrightarrow{\sim} det(Q)$ and $\theta : P \otimes S[T] \xrightarrow{\sim} Q \otimes S[T]$ be isomorphisms. Assume that $\chi \otimes S = \wedge^2\theta$. Then $P \simeq Q$.*

Proof. We just follow the proof of Theorem 5.0.1. We repeat Step 1 word by word, only replacing L by $R[T]$. In Step 2 we only need to use Theorem 2.2.2 in place of Theorem 2.2.1. □

Remark 5.0.1. A nontrivial result that is hidden in the proof of Theorem 5.0.2 is the symplectic cancellation theorem of Bhatwadekar [B 2, Theorem 4.8], which is used to prove [D 1, Theorem 7.6].

Chapter 6

The Euler class group with respect to an extended line bundle

By a ring we shall mean a commutative Noetherian ring containing $\mathbb{Q}$.

Let R be a ring of dimension $n \geq 3$. The aim of this chapter is to extend the theory of the Euler class group $E(R[T])$ of $R[T]$, as developed in [D 1, D 2], to $E(R[T], L[T])$, where L is a projective R-module of rank one. Obviously, when L is free, $E(R[T], L[T])$ should coincide with $E(R[T])$.

Notation. Let A be a ring and let $A[T]$ be the polynomial algebra in one variable T. We denote, by $A(T)$, the ring obtained from $A[T]$ by inverting all monic polynomials. For an ideal I of A$[T]$ and $a \in A$, $I(a)$ denotes the ideal $\{f(a) : f(T) \in I\}$ of A. Let P be a projective A-module. Then $P[T]$ denotes the projective $A[T]$-module $P \otimes_A A[T]$ and $P(T)$ denotes the projective $A(T)$-module $P[T] \otimes_{A[T]} A(T)$.

Definition 6.0.1. Let A be a ring and P be a projective $A[T]$-module. Let $\mathcal{J}(A, P) \subset A$ consist of all those $a \in A$ such that P_a is extended from A_a. It follows from [Qu, Theorem 1] that $\mathcal{J}(A, P)$ is an ideal and $\mathcal{J}(A, P) = \sqrt{\mathcal{J}(A, P)}$. This is called the *Quillen ideal* of P in A.

Remark 6.0.1. It is easy to deduce ht$\mathcal{J}(A, P) \geq 1$ from Quillen-Suslin theorem [Qu, Su 1].

If determinant of P is extended from A, then $\text{ht}\mathcal{J}(A, P) \geq 2$ by [B-R 1, Theorem 3.1].

The proof of the following lemma can be found in [B-RS 1].

Lemma 6.0.1. *Let A be a ring containing an infinite field k and let $I \subset A[T]$ be an ideal of height n. Then there exists $\lambda \in k$ such that either $I(\lambda) = A$ or $I(\lambda)$ is an ideal of height n in A.*

The following theorem was stated without proof in [D 1, Theorem 3.11]. One can actually mimic the proof of [D 1, Theorem 3.10] with necessary modifications to prove this result directly. For the sake of completeness we give here a quick proof, using [D 1, Theorem 3.10]. As [D 1, Theorem 3.10] played a pivotal role in studying tne Euler class group $E(R[T])$, this theorem will do the same for $E(R[T], L[T])$. Recall that $R(T)$ is the ring obtained from $R[T]$ by inverting all the monic polynomials and that $\dim(R(T)) = \dim(R)$.

Theorem 6.0.1. *Let R be a ring of dimension $n \geq 3$ and P be a projective R-module of rank n. Let $I \subset R[T]$ be an ideal of height n such that there is a surjection*

$$\psi : P[T] \twoheadrightarrow I/(I^2T).$$

Assume that $\psi \otimes R(T) : (P[T] \otimes R(T)) \twoheadrightarrow IR(T)/I^2R(T)$ can be lifted to a surjection

$$\widetilde{\theta} : (P[T] \otimes R(T)) \twoheadrightarrow IR(T).$$

Then ψ also has a lift to a surjective map $\theta : P[T] \twoheadrightarrow I$.

Proof. We first note that if I contains a monic polynomial, then the conditions of the theorem are trivially satisfied. In this case, the theorem has been proved by Mandal [M 2, Theorem 2.1]. Therefore, in what follows, we may assume that I does not contain a monic polynomial.

Let $J = I \cap R$. Applying [D 1, Lemma 3.9], we get a lift $\phi \in \text{Hom}_{R[T]}(P[T], I)$ of ψ, such that the ideal $\phi(P[T]) = I''$ satisfies the following properties:

(i) $I'' + (J^2T) = I$.

(ii) $I'' = I \cap I'$, where $\text{ht}(I') \geq n$.

(iii) $I' + (J^2T) = R[T]$.

Let $J' = I' \cap R$. It can be deduced that $\dim(R/(J + J')) = 0$. This was proved in [D 1, Theorem 3.10].

Write $B = R_{1+J}$. Tensoring the surjection $\phi \otimes B : P_{1+J}[T] \twoheadrightarrow (I \cap I')B[T]$ with $B[T]/I'B[T]$ we obtain a surjection

$$\phi_1 : P_{1+J}[T] \twoheadrightarrow I'B[T]/I'^2B[T].$$

Now we note two things. First, as $I' + (J^2T) = A[T]$, it follows that $I'(0)B = B$. Secondly, since JB is contained in the Jacobson radical of B and $\dim(B/JB) \leq 1$, it is easy to see using Theorem 2.1.1 that P_{1+J} has a free summand of rank one and hence there is a surjective map $\alpha : P_{1+J} \twoheadrightarrow I'(0)B(= B)$. Combining these two, it follows from Lemma 2.1.7 that there is a surjection

$$\overline{\beta} : P_{1+J}[T] \twoheadrightarrow I'B[T]/(I'^2T)B[T],$$

which is a lift of ϕ_1.

Consider the ring $C = B_{1+J'} = R_{1+J+J'}$. As $\dim(R/(J + J')) = 0$, it follows that C is semilocal, and consequently $P_{1+J+J'}$ is a free C-module. Applying the subtraction principle (Proposition 2.3.2) over $C(T)$, we see that there is a surjection $\gamma : P \otimes C(T) \twoheadrightarrow I'C(T)$ which lifts $\overline{\beta} \otimes C(T)$. Since $P_{1+J+J'}$ is a free C-module, it follows from [D 1, Theorem 3.10], that $\overline{\beta} \otimes C[T]$ has a lift to a surjective map $\widetilde{\beta} : P_{1+J+J'}[T] \twoheadrightarrow I'C[T]$. It now follows from [D 1, Lemma 3.8], that $\overline{\beta}$ has a lift to a surjection $\beta : P_{1+J}[T] \twoheadrightarrow I'B[T]$, i.e., $(\beta - \overline{\beta})(P_{1+J}[T]) \subset (I'^2T)B[T]$.

Now we can apply [B-K, Lemma 4.7], and obtain a surjection $\eta : P_{1+J}[T] \twoheadrightarrow IB[T]$, such that $(\eta - \phi)(P_{1+J}[T]) \subset (I^2T)B[T]$. Applying [D 1, Lemma 3.8] again, we are done. □

In this chapter we shall frequently apply the above theorem taking $L[T] \oplus R[T]^{n-1}$ in place of $P[T]$, where L is a projective R-module of rank one. Let us illustrate one such application in the form of following proposition which will be used later.

Notation. Let L be a projective R-module of rank one. Throughout this chapter we shall denote $L \oplus R^{n-1}$ by $\mathcal{L}$ and $L[T] \oplus R[T]^{n-1}$ by $\mathcal{L}[T]$.

Proposition 6.0.1. *Let R be a ring and $I \subset R[T]$ be an ideal of height n. Let α and β be two surjections from $\mathcal{L}[T]/I\mathcal{L}[T]$ to I/I^2 such that there exists $\sigma \in SL(\mathcal{L}[T]/I\mathcal{L}[T])$ with the property that $\alpha\sigma = \beta$. Suppose that α can be lifted to a surjection $\theta : \mathcal{L}[T] \twoheadrightarrow I$. Then β can also be lifted to a surjection $\phi : \mathcal{L}[T] \twoheadrightarrow I$.*

Proof. Since $\mathbb{Q} \subset R$, by Lemma 6.0.1 there exists $\lambda \in \mathbb{Q}$ such that $I(\lambda) = R$ or $I(\lambda)$ is an ideal of R of height n. Without loss of generality we may assume that $\lambda = 0$.

If $I(0) = R$, then by Lemma 2.1.7, we can lift β to a surjection $\widetilde{\beta} : \mathcal{L}[T] \twoheadrightarrow I/(I^2T)$. We now show that the same can be done if $\text{ht}(I(0)) = n$. Let $\alpha(0) : \mathcal{L}/I(0)\mathcal{L} \twoheadrightarrow I(0)/I(0)^2$, $\beta(0) : \mathcal{L}/I(0)\mathcal{L} \twoheadrightarrow I(0)/I(0)^2$ be surjections induced by α, β, respectively. Therefore $\alpha(0)\sigma(0) = \beta(0)$. As $\dim(R/I(0)) = 0$, we have $\sigma(0) \in \mathcal{E}(\mathcal{L}/I(0)\mathcal{L})$. As $\mathcal{E}(\mathcal{L}) \longrightarrow \mathcal{E}(\mathcal{L}/I(0)\mathcal{L})$ is surjective, there exists $\tau \in \mathcal{E}(\mathcal{L})$, which is a lift of $\sigma(0)$. As $\theta(0)$ lifts $\alpha(0)$, the composition $\theta(0)\tau$ lifts β. Again by Lemma 2.1.7, we can lift β to a surjection $\widetilde{\beta} : \mathcal{L}[T] \twoheadrightarrow I/(I^2T)$.

Now consider the ring $R(T)$ and the induced surjections $\alpha \otimes R(T)$ and $\widetilde{\beta} \otimes R(T)$. Again since $\dim(R(T)/IR(T)) = 0$, we have $SL(\mathcal{L} \otimes R(T)/I\mathcal{L} \otimes R(T)) = \mathcal{E}(\mathcal{L} \otimes R(T)/I\mathcal{L} \otimes R(T))$ and as above , $\widetilde{\beta} \otimes R(T)$ can be lifted to a surjection from $\mathcal{L} \twoheadrightarrow IR(T)$. Now we can apply Theorem 6.0.1 and conclude that β can be lifted to a surjection $\phi : \mathcal{L} \twoheadrightarrow I$. □

Remark 6.0.2. The above proposition was proved in [D 1, Proposition 4.4] in the case when $\mathcal{L}$ is free. We may justifiably call the technique involved in the proof as a *"monic inversion technique"*. This was ubiquitous in [D 1]. As we are proving results analogous to [D 1] in this chapter, therefore, whenever we present a result which can be proved by this *monic inversion technique*, either we give a quick sketch or we skip the proof.

The following addition and subtraction principles, like their counterparts in [D 1], can be proved using the *monic inversion technique* illustrated above.

Proposition 6.0.2. (Addition principle) *Let R be a ring of dimension $n \geq 3$ and let $I_1, I_2 \subset R[T]$ be two comaximal ideals, each of height n. Assume that there exist* *surjections* $\theta_1 : \mathcal{L}[T] \twoheadrightarrow I_1$ *and* $\theta_2 : \mathcal{L}[T] \twoheadrightarrow I_2$. *Then there exists surjection* $\theta : \mathcal{L}[T] \twoheadrightarrow I_1 \cap I_2$ *such that* $\theta \otimes R[T]/I_i = \theta_i \otimes R[T]/I_i$, $i = 1, 2$.

Proposition 6.0.3. (Subtraction principle) *Let* R *be a ring of dimension* $n \geq 3$ *and let* $I_1, I_2 \subset R[T]$ *be two comaximal ideals, each of height* n. *Assume that there exist surjections* $\theta_1 : \mathcal{L}[T] \twoheadrightarrow I_1$ *and* $\theta : \mathcal{L}[T] \twoheadrightarrow I_1 \cap I_2$ *such that* $\theta \otimes R[T]/I_1 = \theta_1 \otimes R[T]/I_1$. *Then there exists surjection* $\theta_2 : \mathcal{L}[T] \twoheadrightarrow I_2$ *such that* $\theta \otimes R[T]/I_2 = \theta_2 \otimes R[T]/I_2$.

Let R be a commutative Noetherian ring of dimension $n \geq 3$ containing $\mathbb{Q}$. Let L be a projective R-module of rank one. We now go on to define the (n-th) Euler class group $E^n(R[T], L[T])$. For brevity we denote this group by $E(R[T], L[T])$. As above, we shall denote $L \oplus R^{n-1}$ by $\mathcal{L}$ and $L[T] \oplus R[T]^{n-1}$ as $\mathcal{L}[T]$.

We first define some terms. Let $I \subset R[T]$ be an ideal of height n such that there exists a surjection $\mathcal{L}[T]/I\mathcal{L}[T] \twoheadrightarrow I/I^2$. Two surjections $\alpha, \beta : \mathcal{L}[T]/I\mathcal{L}[T] \twoheadrightarrow I/I^2$ are said to be *related* if there exists $\sigma \in SL(\mathcal{L}[T]/I\mathcal{L}[T])$ such that $\alpha\sigma = \beta$. It easily follows that this defines an equivalence relation on the set of surjections from $\mathcal{L}[T]/I\mathcal{L}[T]$ to I/I^2. Let $[\alpha]$ denote the equivalence class of α. We call such an equivalence class $[\alpha]$ a *local* $L[T]$*-orientation* of I.

We call a local $L[T]$-orientation $[\alpha]$ of I a *global* $L[T]$*-orientation* if the surjection $\alpha : \mathcal{L}[T]/I\mathcal{L}[T] \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : \mathcal{L}[T] \twoheadrightarrow I$. Note that by Proposition 6.0.1, if α can be lifted to a surjection $\theta : \mathcal{L}[T] \twoheadrightarrow I$, then β can also be lifted to a surjection $\eta : \mathcal{L}[T] \twoheadrightarrow I$. Therefore, by a slight abuse of notations, we denote $[\alpha]$ by α.

Let G be the free abelian group on the set of pairs (I, ω_I) where $I \subset R[T]$ is an ideal of height n with the property that $\text{Spec}(R[T]/I)$ is connected and I/I^2 is a surjective image of $\mathcal{L}[T]/I\mathcal{L}[T]$ and $\omega_I : \mathcal{L}[T]/I\mathcal{L}[T] \twoheadrightarrow I/I^2$ is a local $L[T]$-orientation of I.

Let I be any ideal of $R[T]$ of height n such that I/I^2 is surjective image of $\mathcal{L}[T]/I\mathcal{L}[T]$. Then there is a unique decomposition (see [D 1] for details), $I = I_1 \cap \cdots \cap I_k$, where $\text{Spec}(R[T]/I_i)$ is connected and $\text{ht}\, I_i = n$ for each i, and $I_i + I_j = R[T]$ for $i \neq j$. Now if ω_I is a local $L[T]$-orientation of I then it naturally gives rise to $\omega_{I_i} : \mathcal{L}[T]/I_i\mathcal{L}[T] \twoheadrightarrow I_i/I_i^2$ for $1 \leq i \leq k$. By (I, ω_I) we mean the element $\sum (I_i, \omega_{I_i}) \in G$.

Let H be the subgroup of G generated by the set of pairs (I, ω_I) in G such that ω_I

is a global orientation.

Definition 6.0.2. We define the (n-th) Euler class group of $R[T]$ with respect to $L[T]$ as $E(R[T], L[T]) := G/H$.

Let P be a projective $R[T]$-module of rank n having determinant $L[T]$, where L is a projective R-module of rank 1. Let $\chi : L[T] \xrightarrow{\sim} \wedge^n P$ be an isomorphism. To the pair (P, χ), we associate an element $e(P, \chi)$ of $E(R[T], L[T])$ as follows: Let $\lambda_0 : P \twoheadrightarrow I_0$ be a surjection, where I_0 is an ideal of $R[T]$ of height n. Let bar denote reduction modulo I_0. We obtain an induced surjection $\overline{\lambda}_0 : P/I_0P \twoheadrightarrow I_0/I_0^2$. Note that, since P has determinant $L[T]$ and $\dim(R[T]/I_0) \leq 1$, by Serre's splitting theorem (Theorem 2.1.1) we have $P/I_0P \simeq L[T]/I_0L[T] \oplus (R[T]/I_0)^{n-1}$ ($= \mathcal{L}[T]/I_0\mathcal{L}[T]$ in our notation). We choose an isomorphism $\overline{\gamma} : \mathcal{L}[T]/I_0\mathcal{L}[T] \xrightarrow{\sim} P/I_0P$, such that $\wedge^n \overline{\gamma} = \overline{\chi}$. Let ω_{I_0} be the composite surjection

$$\mathcal{L}[T]/I_0\mathcal{L}[T] \overset{\overline{\gamma}}{\xrightarrow{\sim}} P/I_0P \overset{\overline{\lambda_0}}{\twoheadrightarrow} I_0/I_0^2.$$

Let $e(P, \chi)$ be the image in $E(R[T], L[T])$ of the element (I_0, ω_{I_0}). We say that (I_0, ω_{I_0}) is obtained from the pair (λ_0, χ).

As yet another application of the *monic inversion technique*, we have the following

Lemma 6.0.2. *The assignment sending the pair (P, χ) to the element $e(P, \chi)$, as described above, is well defined.*

Proof. (Sketch) Let $\lambda_i : P \twoheadrightarrow I_i$ $(i = 0, 1)$ be two surjections so that (λ_i, χ) induce (I_i, ω_{I_i}). Apply the moving lemma (Lemma 2.1.4) to find an ideal $K \subset R[T]$ and a local $L[T]$-orientation ω_K of K such that $\text{ht}(K) \geq n$, $K + I_i = R[T]$ for $i = 0, 1$ and $(I_0, \omega_{I_0}) + (K, \omega_K) = 0$ in $E(R[T], L[T])$. Now let $I = I_1 \cap K$ and ω_I be the local $L[T]$-orientation of I induced by ω_{I_1} and ω_K. Now use the facts that the Euler class of a projective R-module (resp., $R(T)$-module) is well-defined, and the monic inversion technique as in Theorem 6.0.1 to show that ω_I is a global orientation. This will prove $0 = (I, \omega_I) = (I_1, \omega_{I_1}) + (K, \omega_K)$ in $E(R[T], L[T])$ and therefore, $(I_0, \omega_{I_0}) = (I_1, \omega_{I_1})$. □

Definition 6.0.3. We define the Euler class of (P, χ) to be $e(P, \chi)$.

Remark 6.0.3. It is easy to see from the definition of $E(R,L)$ in [B-RS 4] and the definition of $E(R[T],L[T])$ given above, that there is a canonical group homomorphism $\Phi : E(R,L) \to E(R[T],L[T])$. Following the method of proof of [D 2, Theorem 3.3] with obvious modifications, one can check that there is a surjective group homomorphism $\Psi : E(R[T],L[T]) \twoheadrightarrow E(R,L)$ with the property that if $(I,\omega_I) \in E(R[T],L[T])$ is such that the ideal $I(0)$ is an ideal of R of height n, then $\Psi((I,\omega_I)) = (I_0,\omega_{I(0)})$, where $\omega_{I(0)}$ is the local L-orientation of $I(0)$ induced by ω_I (if $I(0) = R$, then $\Psi((I,\omega_I)) = 0$). Moreover, $\Psi\Phi = \mathrm{id}_{E(R,L)}$ and therefore Φ is injective. On the other hand, as the extension $R[T] \hookrightarrow R(T)$ is flat, we have a canonical group homomorphism $\varphi : E(R[T],L[T]) \to E(R(T), L[T] \otimes R(T))$.

With the above remark in hand, one can prove the following theorem. The method of proof again involves a straightforward *monic inversion technique.*

Theorem 6.0.2. *Let R be a ring of dimension $n \geq 3$. Let $I \subset A[T]$ be an ideal of height n such that I/I^2 is generated by n elements and let $\omega_I : \mathcal{L}[T]/I\mathcal{L}[T] \twoheadrightarrow I/I^2$ be a local $L[T]$-orientation of I. Suppose that the image of (I,ω_I) is zero in $E(A[T],L[T])$. Then ω_I can be lifted to a surjection $\theta : \mathcal{L}[T] \twoheadrightarrow I$ (i.e., ω_I is a global orientation).*

Proof. We leave the proof. □

The following theorem extends [D 1, Theorem 4.8] and a theorem of Mandal [M 2, Theorem 2.1]. The method of proof is similar to [D 1, Theorem 4.8]. As the proof is rather involved, we give an outline.

Theorem 6.0.3. *Let R be a ring of $\dim R = n \geq 3$ and $J \subsetneq R[T]$ be an ideal of height n. Let P be a projective $R[T]$-module of rank n whose determinant is $L[T]$. Assume that we are given a surjection $\psi : P \twoheadrightarrow J/(J^2T)$. Assume further that $\psi \otimes R(T)$ can be lifted to a surjection $\psi' : P \otimes R(T) \twoheadrightarrow JR(T)$. Then, there exists a surjection $\Psi : P \twoheadrightarrow J$ such that Ψ is a lift of ψ .*

Proof. We fix an isomorphism $\chi : L[T] \xrightarrow{\sim} \wedge^n P$. Let $\mathcal{J}(R,P)$ denote the Quillen ideal of P in R and write $K = \mathcal{J}(R,P) \cap J$. Since the determinant of P is extended from R, we have, $\mathrm{ht}(\mathcal{J}(R,P)) \geq 2$. Therefore, $\mathrm{ht}\, K \geq 2$. We can apply [D 1, Lemma 3.9] and obtain a lift $\alpha \in \text{Hom}_{R[T]}(P, J)$ of ψ and an ideal $J' \subset R[T]$ of height n such that (1) $J' + (K^2T) = R[T]$, (2) $\alpha : P \twoheadrightarrow J \cap J'$ is a surjection and (3) $\alpha(P) + (K^2T) = J$.

It follows that $e(P, \chi) = (J \cap J', \omega_{J \cap J'})$ in $E(R[T], L[T])$ where the local orientation $\omega_{J \cap J'}$ is obtained by composing $\alpha \otimes R[T]/(J \cap J')$ with a suitable isomorphism $\overline{\lambda}$: $(R[T]/J \cap J')^n \simeq P/(J \cap J')P$, as described in the definition of an Euler class.

Therefore, $e(P, \chi) = (J, \omega_J) + (J', \omega_{J'})$. We note that since $J'(0) = R$, by Lemma 2.1.7 we can lift $\omega_{J'}$ to a surjection from $\mathcal{L}[T] \twoheadrightarrow J'/(J'^2T)$. Moreover, considering the equation $e(P \otimes R(T), \chi \otimes R(T)) = (JR(T), \omega_J \otimes R(T)) + (J'R(T), \omega_{J'} \otimes R(T))$ in $E(R(T), L[T] \otimes R(T))$ and using the condition of the theorem it is easy to deduce that $(J'R(T), \omega_{J'} \otimes R(T)) = 0$ in $E(R(T), L[T] \otimes R(T))$. (Actually, the condition of the theorem tells that $e(P \otimes R(T), \chi \otimes R(T)) = (JR(T), \omega_J \otimes R(T))$). As $\omega_{J'}$ is induced by a surjection $\mathcal{L}[T] \twoheadrightarrow J'/(J'^2T)$, it follows from Theorem 6.0.1 that there is a surjection $\beta : \mathcal{L}[T] \twoheadrightarrow J'$ which lifts $\omega_{J'}$.

Let us write $B = R_{1+K}$. By [D 1, Lemma 3.8] it is enough to prove that there is a surjection $\theta : \mathcal{L}[T] \otimes B[T] \twoheadrightarrow J$ such that $(\theta - \alpha)(\mathcal{L}[T]) \subset (K^2T)$. We can apply Proposition 2.3.4 to obtain such a θ. □

Remark 6.0.4. Let the notations be as in the above theorem. Note that if J contains a monic polynomial, the conditions of the theorem are trivially satisfied. The conclusion of the theorem asserts that if J contains a monic polynomial, then any surjection $\psi : P \twoheadrightarrow J/(J^2T)$ can be lifted to a surjection $\Psi : P \twoheadrightarrow J$. This improves [M 2, Theorem 2.1], where P was assumed to be extended from R.

To derive some corollaries of the above two theorems, we need the following lemma.

Lemma 6.0.3. *[D 1, Lemma 4.9] Let A be a ring, $I \subset A[T]$ be an ideal and P be a projective $A[T]$-module. Suppose that we are given surjections $\alpha : P \twoheadrightarrow I/I^2$ and $\beta : P \twoheadrightarrow I(0) = I/I \cap (T)$ such that $\alpha \otimes_{A[T]/I} A/I(0) = \beta \otimes_A A/I(0)$. Then there is a surjection $\theta : P \twoheadrightarrow I/(I^2T)$ such that θ lifts both α and β.*

We have the following set of corollaries. These can be derived easily from Theorem 6.0.2, Theorem 6.0.3. For assistance, we may consult [D 1].

Corollary 6.0.1. *Let R be of dimension $n \geq 3$. Let P be a projective $R[T]$-module of* *rank n with determinant isomorphic to $L[T]$. Let $\chi : \wedge^n P \xrightarrow{\sim} L[T]$. Let $e(P,\chi) = (I,\omega_I)$ in $E(R[T], L[T])$. Then, there exists a surjection $\alpha : P \twoheadrightarrow I$ such that (I,ω_I) is obtained from (α,χ).*

Corollary 6.0.2. *Let R be a ring. Let P be a projective $R[T]$-module of rank n with determinant isomorphic to $L[T]$. Let $\chi : \wedge^n P \xrightarrow{\sim} L[T]$. Then, $e(P,\chi) = 0$ if and only if P has a unimodular element. In particular, if P has a unimodular element then P maps onto any ideal of $R[T]$ of height n which is surjective image of $L[T] \oplus R[T]^{n-1}$.*

Corollary 6.0.3. *Let R be a ring and $I \subset R[T]$ be an ideal of height n. Let P be a projective $R[T]$-module of rank n with determinant isomorphic to $L[T]$ and $\alpha : P \twoheadrightarrow I$ be a surjection. Suppose that P has a unimodular element. Then I is surjective image of $L[T] \oplus R[T]^{n-1}$.*

We can also prove the following local-global principle for the Euler class groups.

Theorem 6.0.4. *Let R, L be as above. The following sequence of groups is exact*

$$0 \longrightarrow E(R,L) \longrightarrow E(R[T],L[T]) \longrightarrow \prod_{\mathfrak{m}} E(R_{\mathfrak{m}}[T], L_{\mathfrak{m}}[T]).$$

Proof. Let $(I_1,\omega_{I_1}) \in E(R[T],L[T])$ be such that its image in $E(R_{\mathfrak{m}}[T], L_{\mathfrak{m}}[T])$ is zero for each maximal ideal $\mathfrak{m}$ of R. We show that (I_1,ω_{I_1}) has a preimage in $E(R,L)$.

As $\mathbb{Q} \subset R$, we can assume that either $I_1(0)$ is an ideal of height n or $I_1(0) = R$.
Case 1. Assume that $I_1(0)$ is proper. Apply the moving lemma (Lemma 2.1.4), and obtain an ideal $K \subset R$ of height n which is comaximal with $I_1 \cap R$ and a local $L[T]$-orientation ω_K of K such that $(I_1(0),\omega_{I_1(0)}) + (K,\omega_K) = 0$ in $E(R,L)$.

Let $I = I_1 \cap K[T]$. As I_1 and $K[T]$ are comaximal, ω_{I_1} and ω_K will induce a local $L[T]$-orientation ω_I of I and we have:

$$(I,\omega_I) = (I_1,\omega_{I_1}) + (K[T],\omega_K \otimes R[T]) \text{ in } E(R[T],L[T]).$$

Note that proving $(I,\omega_I) = 0$ will suffice. Observe from the above equation that $(I(0),\omega_{I(0)}) = 0$ in $E(R,L)$ and therefore, by [B-RS 4, Theorem 4.2] $\omega_{I(0)}$ can be lifted to a surjection $\alpha : \mathcal{L} \twoheadrightarrow I(0)$. Therefore, by Lemma 2.1.7 ω_I can be lifted to a surjection

$\psi : \mathcal{L}[T] \twoheadrightarrow I/(I^2T)$. It is now enough to show that ψ can be lifted to a surjection $\theta : \mathcal{L}[T] \twoheadrightarrow I$.

Now we can proceed as in the first half of the proof of Theorem 6.0.1 and reduce the theorem to the case when R is semilocal. But in that case L is free and the proof in this case is given in [D 1, Lemma 5.5].

Case 2. If $I_1(0) = R$, then ω_{I_1} can be lifted to a surjection $\psi_1 : \mathcal{L}[T] \twoheadrightarrow I_1/(I_1^2T)$. We can proceed as in Case 1 to reduce the proof to the semilocal case. □

The following is an analogue of a theorem of Roitman [Ro, Proposition 2], proved for the Euler class groups in [D-RS 2].

Theorem 6.0.5. *Let R, L be as above. Let $S \subset R$ be a multiplicatively closed set. Assume that the canonical map $\Phi : E(R, L) \longrightarrow E(R[T], L[T])$ is given to be surjective. Then the canonical map $\Phi_S : E(S^{-1}R, S^{-1}L) \longrightarrow E(S^{-1}R[T], S^{-1}L[T])$ is also surjective.*

Proof. Write L_S for $L \otimes_R R_S$. By Theorem 6.0.4, we have the following exact sequence of abelian groups

$$0 \longrightarrow E(R_S, L_S) \longrightarrow E(R_S[T], L_S[T]) \longrightarrow \prod_{\mathfrak{m}} E((R_S)_{\mathfrak{m}}[T], (L_S)_{\mathfrak{m}}[T]),$$

where $\mathfrak{m}$ is a maximal ideal of R_S of height n. To prove the theorem, it is enough to show that $E((R_S)_{\mathfrak{m}}[T], (L_S)_{\mathfrak{m}}[T]) = 0$ for each such $\mathfrak{m}$. Since $\mathfrak{m}$ is a maximal ideal of R which avoids S, we are reduced to showing that under the hypothesis of the theorem, $E(R_{\mathfrak{m}}, L_{\mathfrak{m}}) \longrightarrow E(R_{\mathfrak{m}}[T], L_{\mathfrak{m}}[T])$ is surjective. Since $E(R_{\mathfrak{m}}, L_{\mathfrak{m}}) = 0$, we need only prove that $E(R_{\mathfrak{m}}[T], L_{\mathfrak{m}}[T]) = 0$. But $L_{\mathfrak{m}}$ is free and we are done by [D-RS 2, Theorem 4.2]. □

Theorem 6.0.6. *Let R be a regular ring of dimension $n \geq 3$ which is essentially of finite type over a field k such that R has infinite residue fields. Let L be a projective R-module of rank one. Then $E(R, L)$ is isomorphic to $E(R[T], L[T])$.*

Proof. We only need to prove that the canonical map from $E(R, L)$ to $E(R[T], L[T])$ is surjective. In view of the local-global principle Theorem 6.0.4, it is enough to prove that

$E(R_{\mathfrak{m}}[T], L_{\mathfrak{m}}[T]) = 0$ for each maximal ideal $\mathfrak{m}$ of R of height n. But $L_{\mathfrak{m}}$ is free and therefore the result follows from [M-V, Theorem 4], [D 3, 4.9]. □

Theorem 6.0.7. *Let R be an affine algebra of dimension $n \geq 3$ over an algebraically closed field k of characteristic zero. Then the canonical map $E(R[T], L[T]) \to E(R(T), L[T] \otimes R(T))$ is injective.*

Proof. It can be derived by modifying the proof in [D 1, Proposition 5.8] □

Remark 6.0.5. Let R be a regular domain of dimension d containing an infinite field and n be a positive integer such that $2n \geq d+3$. Then the n-th Euler class group $E^n(R[T])$ has been defined in [D-RS 2] and results analogous to those in [D 1, D 2] have been proved. By a result of Lindel [L 1], any line bundle on $R[T]$ is extended from R and hence is of the form $L[T]$, where L is a line bundle on R. The theory can easily be extended to define $E^n(R[T], L[T])$ and many results of this chapter can be proved. For instance, it can be proved that if R is regular and is essentially of finite type over an infinite field, then $E^n(R[T], L[T]) \simeq E^n(R, L)$. A similar result has been proved using different techniques in [M-Y 2]. However we are not going into the details of this setup.

Chapter 7

Some descent lemmas and their applications

In this chapter we prove some technical results which are crucial to the theory and results in Chapter 8. Motivation for the following lemmas came from [B 1, 3.1, 3.2, 3.3]. The basic setup is as follows. We shall try to stick to the notations introduced below throughout this chapter.

Let $R \hookrightarrow S$ be a finite extension of reduced rings and let C be the conductor ideal of R in S. Let $\mathbb{L}$ be a projective $R[T]$-module of rank one and $I \subset R[T]$ be an ideal such that $IS[T]$ is a proper ideal. Write $\mathfrak{L} = \mathbb{L} \oplus R[T]^{n-1}$. Assume that there is a surjection $\alpha : \mathfrak{L}/I\mathfrak{L} \twoheadrightarrow I/I^2$. Then α naturally induces a surjection from $(\mathfrak{L} \otimes S[T])/IS[T](\mathfrak{L} \otimes S[T])$ to $IS[T]/I^2S[T]$, which we shall denote by α^*. We now explicitly describe how α^* is obtained.

Tensoring α with $S[T]/IS[T]$ over $R[T]/I$ we obtain the surjection

$$\widetilde{\alpha} : \frac{(\mathfrak{L} \otimes_{R[T]} S[T])}{IS[T](\mathfrak{L} \otimes_{R[T]} S[T])} \twoheadrightarrow \frac{(I \otimes_{R[T]} S[T])}{IS[T](I \otimes_{R[T]} S[T])}.$$

Composing $\widetilde{\alpha}$ with the surjective map $\widetilde{f}$ induced by the natural surjection $f : I \otimes_{R[T]} S[T] \twoheadrightarrow IS[T]$, we obtain α^*. Thus α^* is the composition $\widetilde{f}\widetilde{\alpha}$

$$\alpha^* : \frac{(\mathfrak{L} \otimes_{R[T]} S[T])}{IS[T](\mathfrak{L} \otimes_{R[T]} S[T])} \overset{\widetilde{\alpha}}{\twoheadrightarrow} \frac{(I \otimes_{R[T]} S[T])}{IS[T](I \otimes_{R[T]} S[T])} \overset{\widetilde{f}}{\twoheadrightarrow} \frac{IS[T]}{I^2S[T]}.$$

Now suppose it is given that α^* has a lift to a surjection $\beta : \mathfrak{L} \otimes S[T] \twoheadrightarrow IS[T]$. In the following three lemmas we investigate, under what additional hypotheses, we may be able to obtain a lift of α to a surjection $\phi : \mathfrak{L} \twoheadrightarrow I$. We fix the above notations for the following lemmas. We shall only mention the additional hypotheses in the statements.

The method of proof of the following lemma is similar to Theorem 3.2.4.

Lemma 7.0.1. *Let $R, S, C, \mathfrak{L}, I, \alpha$ be as above and assume that : (i) $(R/C)_{red} = (S/C)_{red}$, (ii) $I + C[T] = R[T]$, (iii) $n \geq 2$. Then α can be lifted to a surjective map $\phi : \mathfrak{L} \twoheadrightarrow I$.*

Proof. We give the proof in steps.

Step 1. We first note that since $I + C[T] = R[T]$ and C is the conductor ideal of R in S, the following are true:

(i) $I \otimes (R/C)[T] \simeq (R/C)[T]$.

(ii) $I \otimes (S/C)[T] \simeq (S/C)[T]$.

(iii) $R[T]/I \simeq S[T]/IS[T]$.

We have a surjection, $\beta : \mathfrak{L} \otimes S[T] \twoheadrightarrow IS[T]$ which is a lift of α^*. Consider

$$\beta_1 := \beta \otimes (S/C)[T] : \mathfrak{L} \otimes (S/C)[T] \twoheadrightarrow IS[T] \otimes (S/C)[T].$$

From (ii) we have $IS[T] \otimes (S/C)[T] \simeq (S/C)[T]$, implying that the image of $IS[T]$ in $(S/C)[T]$ is $(S/C)[T]$. Therefore, $\beta_1 : \mathfrak{L} \otimes (S/C)[T] \twoheadrightarrow (S/C)[T]$ is a surjection and therefore $\beta_1 \in \mathrm{Um}((\mathfrak{L} \otimes (S/C)[T])^*)$.

Now $\beta_1 \otimes (S/C)_{\mathrm{red}}[T]$ is also a unimodular element of $(\mathfrak{L} \otimes (S/C)_{\mathrm{red}}[T])^*$. Since it is given that $(R/C)_{\mathrm{red}} = (S/C)_{\mathrm{red}}$, it is easy to see that we have a lift of $\beta_1 \otimes (S/C)_{\mathrm{red}}[T]$ to $(R/C)[T]$, say, $\delta : \mathfrak{L} \otimes (R/C)[T] \twoheadrightarrow (R/C)[T]$. In other words, δ is a unimodular element of $(\mathfrak{L} \otimes (R/C)[T])^*$. It is obvious from the way δ is obtained that $\delta \otimes (S/C)[T] = \beta_1$ modulo $\mathfrak{n}((S/C)[T])$, where $\mathfrak{n}((S/C)[T])$ denotes the nil radical of $(S/C)[T]$. So, we have two unimodular elements β_1 and $\delta \otimes (S/C)[T]$ of $(\mathfrak{L} \otimes (S/C)[T])^*$ such that $\delta \otimes (S/C)[T] = \beta_1$ modulo $\mathfrak{n}((S/C)[T])$. Therefore, by Proposition 2.1.2, there exists a transvection σ of

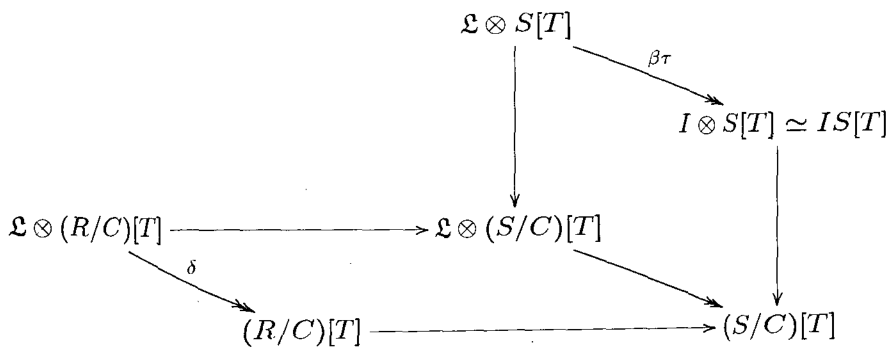
$\mathfrak{L} \otimes (S/C)[T]$ such that $\beta_1\sigma = \delta \otimes (S/C)[T]$. By Proposition 2.1.4, σ can be lifted to an automorphism τ of $\mathfrak{L} \otimes S[T]$ such that τ is identity modulo $IS[T]$.

Step2. Note that $I \otimes (R/C)[T] \simeq (R/C)[T]$ and $I \otimes (S/C)[T] \simeq (S/C)[T]$. Since $I + C[T] = R[T]$, the natural map $f : I \otimes_{R[T]} S[T] \twoheadrightarrow IS[T]$ is actually an isomorphism.

Consider the following Cartesian diagram :

$$\begin{array}{ccc} I & \longrightarrow & IS[T] \simeq I \otimes S[T] \\ \downarrow & & \downarrow \\ (R/C)[T] & \longrightarrow & (S/C)[T] \end{array}$$

Therefore, we have the following commutative diagram:



Since $\delta \otimes (S/C)[T] = \beta_1\sigma = \beta\tau \otimes (S/C)[T]$, the surjective maps δ and $\beta\tau$ will patch to yield a surjection $\phi : \mathfrak{L} \twoheadrightarrow I$.

Step 3. Finally we need to show that $\phi \otimes R[T]/I = \alpha$. Since $\tau = \text{Id}$ (modulo $IS[T]$), we have $\beta\tau = \beta$ modulo $IS[T]$. Identifying $I \otimes_{R[T]} S[T]$ with $IS[T]$ and using the isomorphism $S[T]/IS[T] \simeq R[T]/I$, we have:

$$\phi \underset{R[T]}{\otimes} (R[T]/I) = \phi \underset{R[T]}{\otimes} (S[T]/IS[T]) = (\phi \underset{R[T]}{\otimes} S[T]) \underset{S[T]}{\otimes} (S[T]/IS[T])$$

$$= \beta\tau \underset{S[T]}{\otimes} (S[T]/IS[T]) = \beta \underset{S[T]}{\otimes} (S[T]/IS[T]) = \alpha \underset{R[T]/I}{\otimes} (S[T]/IS[T])$$

$$= \alpha \underset{R[T]/I}{\otimes} (R[T]/I) = \alpha.$$

Thus ϕ lifts α, and the proof of the lemma is complete. □

Lemma 7.0.2. *Let $R, S, C, \mathfrak{L}, I, \alpha$ be as fixed in the beginning of this chapter and assume that:*

(i) $(R/C)_{red} = (S/C)_{red}$;

(ii) $dim(R) = dim(S) = n \geq 4$;

(iii) $ht(C) \geq 1$;

(iv) *for any ideal J of $R[T]$, $ht(J) = ht(JS[T])$.*

Then α can be lifted to a surjective map $\phi : \mathfrak{L} \twoheadrightarrow I$.

Proof. We have $\alpha : \mathfrak{L}/I\mathfrak{L} \twoheadrightarrow I/I^2$. Let $J = I^2 \cap C$. Then $\mathrm{ht}(J) \geq 1$. Therefore, we can choose an element $b \in J$ such that $\mathrm{ht}(b) = 1$. Let bar denote reduction modulo b. Then we have $\overline{\alpha} : \overline{\mathfrak{L}}/\overline{I\mathfrak{L}} \twoheadrightarrow \overline{I}/\overline{I}^2$ and $\dim(R/bR) < \dim(R)$.

Now we can apply Proposition 2.1.6 to get a (surjective) lift $\eta' : \overline{\mathfrak{L}} \twoheadrightarrow \overline{I}$ of $\overline{\alpha}$, and therefore a lift $\eta : \mathfrak{L} \longrightarrow I$ of α such that $(\eta(\mathfrak{L}), b) = I$. Note that $b \in I^2$. Applying Lemma 2.1.2 to the element (η, b) of $\mathfrak{L}^* \oplus R[T]$, we see that there exists $\Psi \in \mathfrak{L}^*$ such that $\mathrm{ht}(K_b) \geq n$, where $K = (\eta + b\Psi)(\mathfrak{L})$. But $(\eta(\mathfrak{L}), b) = I$ has height n and I is a proper ideal. Therefore, by Lemma 2.1.2, $\mathrm{ht}(K) = n$. Since $\eta + b\Psi$ is also a lift of α, we may replace η by $(\eta + b\Psi)$ and write $K = \eta(\mathfrak{L})$. Note that $(K, b) = I$ and $b \in I^2$. It follows, applying Lemma 2.1.3, that there exists an ideal I' of $R[T]$ such that:

(i) $\eta(\mathfrak{L}) = I \cap I'$;

(ii) $\eta \otimes R[T]/I = \alpha$;

(iii) $\mathrm{ht}(I') \geq n$;

(iv) $I' + bR[T] = R[T]$ and therefore, $I' + C[T] = R[T]$.

If $\mathrm{ht}(I') > n$, then $I' = R[T]$ and η is the desired lift of α. So, we assume that $ht(I') = n$.

Let $\beta : \mathfrak{L} \otimes S[T] \twoheadrightarrow IS[T]$ be the lift of α^*. Now consider the surjection

$$\eta^* : \mathfrak{L} \otimes S[T] \overset{\eta \otimes S[T]}{\twoheadrightarrow} (I \cap I') \otimes S[T] \twoheadrightarrow (I \cap I')S[T]$$

As $I' + I = R[T]$, we have $(I \cap I')S[T] = IS[T] \cap I'[T]$. By the subtraction principle (Proposition 2.3.2), there exists $\gamma : \mathfrak{L}\otimes S[T] \twoheadrightarrow I'S[T]$ such that $\gamma\otimes S[T]/I' = \eta^*\otimes S[T]/I'$. Now we can apply the above proposition (taking I' in place of I) to get a surjection $\psi : \mathfrak{L} \twoheadrightarrow I'$ such that $\eta \otimes R[T]/I' = \psi \otimes R[T]/I'$.

Finally, we are going to apply the subtraction principle again to get a lift of α. We have two surjections $\eta : \mathfrak{L} \twoheadrightarrow I \cap I'$ and $\psi : \mathfrak{L} \twoheadrightarrow I'$ such that $\eta \otimes R[T]/I' = \psi \otimes R[T]/I'$. Therefore by the subtraction principle (Proposition 2.3.2), we have a surjection $\phi : \mathfrak{L} \twoheadrightarrow I$ such that $\phi \otimes R[T]/I = \eta \otimes R[T]/I$. As $\eta \otimes R[T]/I = \alpha$, we have $\phi \otimes R[T]/I = \alpha$. Thus the proof is complete. $\square$

Lemma 7.0.3. *Let $R, S, C, \mathfrak{L}, I, \alpha$ be as in the beginning of this chapter and assume that:*

(i) *the canonical map Spec$(S) \longrightarrow$ Spec(R) is bijective;*

(ii) *for every $\mathfrak{q} \in$ Spec(S) the inclusion map $R/(\mathfrak{q} \cap R) \hookrightarrow S/\mathfrak{q}$ is birational;*

(iii) *dim$(R) =$ dim$(S) = n \geq 4$.*

Then α can be lifted to a surjective map $\phi : \mathfrak{L} \twoheadrightarrow I$.

Proof. Let C be the conductor ideal of R in S. By the assumptions of the lemma, ht$(C) \geq 1$ (see Lemma 3.1.5). Further note that $R \hookrightarrow S$ is actually a subintegral extension and therefore by Lemma 3.1.3 $R[T] \hookrightarrow S[T]$ is also subintegral. Further, by Remark 3.2.1 if J is any ideal of $R[T]$, then ht$(J) =$ ht$(JS[T])$.

We prove the lemma by induction on dim(R/C). If dim$(R/C) = 0$, then $(R/C)_{red}$ is also zero-dimensional and $(R/C)_{red}$ does not contain any non-zerodivisor. Let K be the radical of C in S, then we have $(R/C)_{red} = R/K \cap R$ and $(S/C)_{red} = S/K$. Now the total ring of fractions, $Q(S/K) = \prod k(P_i)$, where P_i are minimal prime ideals of S/K. Therefore $Q(R/K \cap R) = \prod k(P_i \cap R)$, since the canonical map Spec$(S) \longrightarrow$ Spec(R) is bijective.

Now by using $k(P_i) = k(P_i \cap R)$, we have $Q(S/K) = Q(R/K \cap R)$. But then

$$R/K \cap R \hookrightarrow S/K \hookrightarrow Q(S/K) = Q(R/K \cap R) = R/K \cap R$$

The last equality holds, since $R/K \cap R$ does not contain any non-zerodivisor.

Therefore it follows that $(R/C)_{red} = (S/C)_{red}$ and we are done by Lemma 7.0.2. So let us assume that $\dim(R/C) > 0$. But then by [B-R 2, Lemma 3.5] there exists a ring S' enjoying the following properties:

(i) $R \hookrightarrow S' \hookrightarrow S$

(ii) $(R/C)_{\text{red}} = (S'/C)_{\text{red}}$

(iii) $\dim(R/C) > \dim(S'/C')$ where C' is the conductor ideal of S' in S.

Now since the extension $S' \hookrightarrow S$ satisfies all the hypotheses of the lemma, by induction hypothesis there exists a surjection $\psi : \mathfrak{L} \otimes S'[T] \twoheadrightarrow IS'[T]$, which is a lift of α^*. But $(R/C)_{red} = (S'/C)_{red}$ and again applying Lemma 7.0.2 we obtain a surjection $\phi : \mathfrak{L} \twoheadrightarrow I$, which is a lift of α. □

Remark 7.0.1. The above lemma is true for $n \geq 2$ if the ideal I is assumed to be comaximal with $C[T]$. To see this, first note that, in the lemma, we need $n \geq 4$ only to be able to apply the subtraction principle which was required to prove Lemma 7.0.2. Now if we start with $I + C[T] = R[T]$, we can apply Lemma 7.0.1 instead. Further note that in the proof of Lemma 7.0.3, the conductor of R in S' is C and C is contained in C' and therefore IS' is comaximal with $C'[T]$. For this last argument one has to go through the proof of [B-R 2, Lemma 3.5].

For the convenience of exposition, we make the following definition.

Definition 7.0.1. Let A be a ring and $\mathbb{L}$ be a projective $A[T]$-module of rank one. A ring extension $A \hookrightarrow B$ will be called *special* $\mathbb{L}$*-regular* if the following conditions are satisfied.

(i) The projective $B[T]$-module $\mathbb{L} \otimes_{A[T]} B[T]$ is extended from B,

(ii) B is module-finite over A,

(iii) the canonical map $\text{Spec}(B) \longrightarrow \text{Spec}(A)$ is bijective, and

(iv) for every $\mathfrak{P} \in \text{Spec}(B)$, the inclusion $A/(\mathfrak{P} \cap A) \hookrightarrow B/\mathfrak{P}$ is birational.

The following result is a special case of [I 1, Theorem 9].

Theorem 7.0.1. *$A \hookrightarrow B$ is seminormal iff $N_rPic(A) \longrightarrow N_rPic(B)$ is injective where $N_rPic(A) := Ker(Pic(A[T_1,\cdots,T_r]) \longrightarrow Pic(A))$ and $N_rPic(B) := Ker(Pic(B[T_1,\cdots,T_r]) \longrightarrow Pic(B))$.*

The following theorem is due to Traverso [T].

Theorem 7.0.2. *If a ring A is seminormal then the canonical map $Pic(A) \longrightarrow Pic(A[X])$ is an isomorphism.*

The following lemma ensures the existence of special $\mathbb{L}$-regular extension.

Lemma 7.0.4. *Let R be a reduced ring and $\mathbb{L}$ be a projective $R[T]$-module of rank one. Then there is a special $\mathbb{L}$-regular extension $R \hookrightarrow S$ with S reduced.*

Proof. Let $\overline{R}$ be the normalisation of R in its total quotient ring. Since $\overline{R}$ is normal, the $\overline{R}[T]$-module $\mathbb{L}\otimes_{R[T]}\overline{R}[T]$ is extended from $\overline{R}$, say, $\mathbb{L}\otimes_{R[T]}\overline{R}[T] = L \otimes_{\overline{R}} \overline{R}[T]$ for some $\overline{R}$-module L. Let $\{f_1,\cdots,f_m\}$ be a set of generators for $\mathbb{L}$ and $\{b_1,\cdots,b_n\}$ be a set of generators for L. Then we get the following relations:

(i) $f_i \otimes 1 = \sum_{j=1}^{n} g_{ij}(b_j \otimes 1)$ for $i = 1,\cdots,m$.

(ii) $b_k \otimes 1 = \sum_{l=1}^{m} h_{kl}(f_l \otimes 1)$ for $k = 1,\cdots,n$.

Where $g_{ij}, h_{kl} \in \overline{R}[T]$. Let R' be the R-subalgebra of $\overline{R}$ generated by the coefficients of $\{g_{ij}, h_{kl}\}$. Consider the R'-module L' generated by $\{b_1,\cdots,b_n\}$. Clearly R' is a finitely generated R-subalgebra of $\overline{R}$. Therefore R' is a finite R-module. Then the equality $\mathbb{L}\otimes_{R[T]}R'[T] = L' \otimes_{R'} R'[T]$ shows that L' is a projective R'-module of rank one. Therefore we have found a ring R' such that (i) $R \hookrightarrow R' \hookrightarrow Q(R)$, (ii) R' is a finite R-module and (iii) $\mathbb{L}\otimes_{R[T]} R'[T]$ is extended from R'.

Let S be the seminormalization of R in R'. Then since S is seminormal in R' and $(\mathbb{L}\otimes_{R[T]} S[T]) \otimes_{S[T]} R'[T]$ is extended from R', by Theorem 7.0.1, the projective $S[T]$-module $\mathbb{L}\otimes_{R[T]} S[T]$ is extended from S. □

Remark 7.0.2. Note that a special $\mathbb{L}$-regular extension $R \hookrightarrow S$ is actually a subintegral extension. If $^+(R)$ denotes the seminormalization of R, then since $^+(R)$ is seminormal, one has $\text{Pic}(^+(R)[T]) \simeq \text{Pic}(^+(R))$ and therefore $\mathbb{L}\otimes(^+(R)[T])$ is extended from $^+(R)$.

The three technical lemmas we just proved above culminate in the following theorem which will be crucially used in the next chapter.

Theorem 7.0.3. *Let R be a reduced ring of dimension $n \geq 4$ and $\mathbb{L}$ be a projective $R[T]$-module of rank one. Let $R \hookrightarrow S$ be a special $\mathbb{L}$-regular extension. Let I be an ideal of $R[T]$ and $\alpha : \mathfrak{L} \twoheadrightarrow I/I^2$ be a surjection. Suppose that the induced surjection $\alpha^* : \mathfrak{L} \otimes S[T] \twoheadrightarrow IS[T]/I^2S[T]$ can be lifted to a surjection $\beta : \mathfrak{L} \otimes S[T] \twoheadrightarrow IS[T]$. Then α can also be lifted to a surjective map $\phi : \mathfrak{L} \twoheadrightarrow I$.*

Proof. Since $R \hookrightarrow S$ is a special $\mathbb{L}$-regular extension, with S reduced and it satisfies all the conditions of Lemma 7.0.3. □

Remark 7.0.3. It follows from Remark 7.0.1 that the above theorem is true for $n \geq 2$ if the ideal I is assumed to be comaximal with $C[T]$. We shall need this observation in Chapter 8.

We now demonstrate an application of Theorem 7.0.3 below. A result of Mandal [M1, Theorem 1.2] is improved, albeit with a stronger hypothesis on the dimension.

Theorem 7.0.4. *Let $A = R[T]$ be a polynomial ring over a commutative Noetherian ring R with $dim(R) = n \geq 4$. Let I be an ideal of A of height n that contains a monic polynomial. let $\mathbb{L}$ be a projective $R[T]$-module of rank 1. Write $\mathfrak{L} = \mathbb{L} \oplus R[T]^{n-1}$. Suppose that there exists $\alpha : \mathfrak{L} \twoheadrightarrow I/I^2$. Then there is a surjection $\beta : \mathfrak{L} \twoheadrightarrow I$ such that β lifts α.*

Proof. By Lemma 2.1.6 we may assume that R is reduced. Let $R \hookrightarrow S$ be a special $\mathbb{L}$-regular extension with S reduced.

Let $\alpha^* : \mathfrak{L} \otimes S[T] \twoheadrightarrow IS[T]/I^2S[T]$ be the surjection induced by α. Now note that $\mathfrak{L} \otimes S[T] = (\mathbb{L} \otimes S[T]) \oplus S[T]^{n-1}$. As $R \hookrightarrow S$ is a special $\mathbb{L}$-regular extension, $\mathbb{L} \otimes S[T]$ is extended from S. It then follows from [B-RS 6, Proposition 3.3], that α^* can be lifted to a surjection $\widetilde{\beta} : \mathfrak{L} \otimes S[T] \twoheadrightarrow I$. The result now follows from Theorem 7.0.3. □

Chapter 8

The Euler class group with respect to an arbitrary line bundle

Our aim in this chapter is to define and study the (n-th) Euler class group of $R[T]$ with respect to a projective $R[T]$-module $\mathbb{L}$ of rank one (which is not necessarily extended from R), and extend the results of Chapter 6.

Remark 8.0.1. We keep it for the record that the top Euler class group $E^{n+1}(R[T])$ is trivial. This case falls in the domain of [B-RS 4]. Let $\phi : \mathbb{L} \oplus R[T]^n \twoheadrightarrow I/I^2$ be any surjection, where I is an ideal of $R[T]$ of height $n+1$. It follows from Proposition 2.1.6 that ϕ can be lifted to a surjective map $\Phi : \mathbb{L} \oplus R[T]^n \twoheadrightarrow I$. Therefore, $E^{n+1}(R[T], \mathbb{L})$ is trivial.

Notation. By a ring R we shall mean a commutative Noetherian ring R containing $\mathbb{Q}$ with $\dim(R) = n \geq 4$. Let us fix a projective $R[T]$-module $\mathbb{L}$ of rank one. Further, we write $\mathfrak{L} = \mathbb{L} \oplus R[T]^{n-1}$.

We now go on to define the n-th Euler class group $E^n(R[T], \mathbb{L})$ (henceforth denoted as $E(R[T], \mathbb{L})$). This definition is simply a verbatim copy of the definition of $E(R[T], L[T])$ that was given in Chapter 6, only replacing $L[T]$ by $\mathbb{L}$. Therefore, we just recall a few terms and point out the differences. The harder part in this chapter is to prove results analogous to those in Chapter 6.

Let $I \subset R[T]$ be an ideal of height n such that I/I^2 is surjective image of $\mathfrak{L}/I\mathfrak{L}$. Let bar denote reduction modulo I. Two surjections $\alpha, \beta : \mathfrak{L}/I\mathfrak{L} \twoheadrightarrow I/I^2$ are said to be *related* if there exists an automorphism $\sigma \in SL(\mathfrak{L}/I\mathfrak{L})$ such that $\alpha\sigma = \beta$. This defines an equivalence relation on the set of surjections from $\mathfrak{L}/I\mathfrak{L}$ to I/I^2. We call such an equivalence class a *local* $\mathbb{L}$*-orientation* of I.

We now prove

Lemma 8.0.1. *Let $\alpha, \beta : \mathfrak{L}/I\mathfrak{L} \twoheadrightarrow I/I^2$ be two surjections belonging to the same eqiuivalence class. Suppose it is given that α can be lifted to a surjection $\phi : \mathfrak{L} \twoheadrightarrow I$. Then β can also be lifted to a surjection $\psi : \mathfrak{L} \twoheadrightarrow I$.*

Proof. In view of Lemma 2.1.6 we may assume that R is reduced.

Let $R \hookrightarrow S$ be a special $\mathbb{L}$-regular extension with S reduced (such an extension exists by Lemma 7.0.4). Consider the two surjections α^*, $\beta^* : (\mathfrak{L} \otimes S[T])/IS[T](\mathfrak{L} \otimes S[T]) \twoheadrightarrow IS[T]/I^2S[T]$, which are induced by α, β, respectively. By the assumption of the lemma, there exists an automorphism $\sigma \in SL(\mathfrak{L}/I\mathfrak{L})$ such that $\alpha\sigma = \beta$. This implies that α^*, β^* are also connected by an automorphism of determinant one. Now note that $\mathbb{L} \otimes S[T]$ is a projective $S[T]$-module of rank one which is extended from S. Therefore, as α^* has a surjective lift $\phi \otimes S[T] : \mathfrak{L} \otimes S[T] \twoheadrightarrow IS[T]$, applying Proposition 6.0.1, it follows that β^* also has a surjective lift, say, $\theta : \mathfrak{L} \otimes S[T] \twoheadrightarrow IS[T]$. Now we can apply Lemma 7.0.2 and conclude that there is a surjection $\psi : \mathfrak{L} \twoheadrightarrow I$ which lifts β. □

Definition 8.0.1. We call a local $\mathbb{L}$-orientation $[\alpha]$ of I a *global orientation* of I if the surjection $\alpha : \mathfrak{L}/I\mathfrak{L} \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : \mathfrak{L} \twoheadrightarrow I$.

Define the groups G and H exactly as in Chapter 6, by only replacing $L[T]$ with $\mathbb{L}$.

Definition 8.0.2. The Euler class group of $R[T]$ with respect to the projective $R[T]$-module $\mathbb{L}$ is defined as $E(R,\mathbb{L}) \stackrel{\text{def}}{=} G/H$.

The following result is crucial for further discussions.

Proposition 8.0.1. *Let R be a reduced ring, $\mathbb{L}$ be a projective $R[T]$-module of rank one and let $R \hookrightarrow S$ be a special $\mathbb{L}$-regular extension. Then there is a canonical injective group homomorphism $\Theta : E(R[T],\mathbb{L}) \longrightarrow E(S[T],\mathbb{L} \otimes S[T])$.*

Proof. We first note that $R \hookrightarrow S$ is a finite subintegral extension. Therefore, $R[T] \hookrightarrow S[T]$ is also subintegral. Further, if $I \subset R[T]$ is an ideal, then $\text{ht}(I) = \text{ht}(IS[T])$ (see Remark (3.2.1)). Given a surjection $\omega_I : \mathfrak{L}/I\mathfrak{L} \twoheadrightarrow I/I^2$, we have the induced surjection

$$\omega_I^* : \frac{(\mathfrak{L} \otimes S[T])}{IS[T](\mathfrak{L} \otimes S[T])} \twoheadrightarrow \frac{IS[T]}{I^2 S[T]},$$

as described at the beginning of Chapter 7. It is now easy to see that there is a canonical group homomorphism $\Theta : E(R[T], \mathbb{L}) \to E(S[T], \mathbb{L} \otimes S[T])$, which takes (I, ω_I) to $(IS[T], \omega_I^*)$. To prove that Θ is injective, let $(I, \omega_I) \in E(R[T], \mathbb{L})$ be such that $\Theta((I, \omega_I)) = 0$ in $E(S[T], \mathbb{L} \otimes S[T])$. In other words, $(IS[T], \omega_I^*) = 0$ in $E(S[T], \mathbb{L} \otimes S[T])$, where ω_I^* is induced by ω_I. As $\mathbb{L} \otimes S[T]$ is extended from S, it follows from Theorem 6.0.2, that ω_I^* has a surjective lift $\eta : \mathfrak{L} \otimes S[T] \twoheadrightarrow IS[T]$. But then Lemma 7.0.2 implies that there is a surjection $\zeta : \mathfrak{L} \twoheadrightarrow I$ lifting ω_I. Therefore ω_I is a global $\mathbb{L}$-orientation and consequently, $(I, \omega_I) = 0$ in $E(R[T], \mathbb{L})$. □

We now prove the following results on the Euler class group $E(R[T], \mathbb{L})$.

Theorem 8.0.1. *Let R be a reduced ring of dimension $n \geq 4$. Let $I \subset R[T]$ be an ideal of height n such that I/I^2 is surjective image of $\mathfrak{L}$ and let ω_I be a local $\mathbb{L}$-orientation of I. Suppose that the image of (I, ω_I) is zero in $E(R[T], \mathbb{L})$. Then ω_I can be lifted to a surjection $\theta : \mathfrak{L} \twoheadrightarrow I$.*

Proof. This is a direct consequence of Theorem 6.0.2 and Lemma 7.0.2. □

So far we kept assuming that the ring R is reduced. To extend the theory to non-reduced rings, the following proposition is in order.

Proposition 8.0.2. *Let R be a ring and let $R_{red} = R/\mathfrak{n}(R)$, where $\mathfrak{n}(R)$ is the nil-radical of R. Let $\mathbb{L}$ be a projectivc $R[T]$-module of rank one. Then there is a canonical isomorphism $\eta : E(R[T], \mathbb{L}) \xrightarrow{\sim} E(R_{red}[T], \mathbb{L} \otimes R_{red}[T])$.*

Proof. The proof is along the same line as [D 3, Proposition 2.15] and [B-RS 4, Corollary 4.6] and therefore omitted. We may also consult [K, Corollary 4.13]. □

Remark 8.0.2. As a consequence of the above proposition, Theorem 8.0.1 is now valid for non-reduced R and therefore, throughout this chapter we may assume R to be reduced.

We now define the Euler class of a projective $R[T]$-module with determinant $\mathbb{L}$.

Let P be a projective $R[T]$-module of rank n whose determinant is isomorphic to $\mathbb{L}$. Let $\chi : \mathbb{L} \xrightarrow{\sim} \wedge^n(P)$ be an isomorphism. To the pair (P, χ), we associate an element $e(P, \chi)$ of $E(R[T], \mathbb{L})$ as follows:

Let $\alpha : P \twoheadrightarrow I$ be a surjection, where I is an ideal of $R[T]$ of height n. Let bar denote reduction modulo I. Note that, since $\dim(R[T]/I) \leq 1$, by Serre's splitting theorem (Theorem 2.1.1) we have $P/IP \simeq \mathfrak{L}/I\mathfrak{L}$. We choose an isomorphism $\overline{\gamma} : \mathfrak{L}/I\mathfrak{L} \xrightarrow{\sim} P/IP$ such that $\wedge^n \overline{\gamma} = \overline{\chi}$. Let ω_I be the composite surjection

$$\mathfrak{L}/I\mathfrak{L} \overset{\overline{\gamma}}{\xrightarrow{\sim}} P/IP \overset{\overline{\alpha}}{\twoheadrightarrow} I/I^2.$$

Let $e(P, \chi)$ be the image in $E(R[T], \mathbb{L})$ of the element (I, ω_I). We say that (I, ω_I) is obtained from the pair (α, χ).

Definition 8.0.3. We define the Euler class of (P, χ) to be $e(P, \chi)$.

Lemma 8.0.2. *The assignment sending the pair (P, χ) to the element $e(P, \chi)$, as described above, is well defined.*

Proof. Let $\alpha : P \twoheadrightarrow I$ and $\beta : P \twoheadrightarrow J$ be two surjection, where $I, J \subset R[T]$ be two ideals of height n. Let (I, ω_I) and (J, ω_J) be obtained from (α, χ) and (β, χ), respectively.

Applying Lemma 2.1.4, we can find an ideal $K \subset R[T]$ of height n such that K is comaximal with I, J and there is a surjection $\gamma : \mathfrak{L} \twoheadrightarrow I \cap K$ such that $\gamma \otimes R[T]/I = \omega_I$. Since K and I are comaximal, γ induces a local $\mathbb{L}$-orientation ω_K of K. Clearly, $(I, \omega_I) + (K, \omega_K) = 0$ in $E(R[T], \mathbb{L})$.

Let $M = K \cap J$. Note that ω_K and ω_J together will induce a local $\mathbb{L}$-orientation of M. Call it ω_M. Then, $(M, \omega_M) = (K, \omega_K) + (J, \omega_J)$. Therefore, showing $(M, \omega_M) = 0$ in $E(R[T], \mathbb{L})$ is enough to prove the lemma.

We may assume that R is reduced. Let $R \hookrightarrow S$ be a special $\mathbb{L}$-regular extension. Then $\mathbb{L} \otimes S[T]$ is extended from S. As the Euler class of a projective $S[T]$-module (in $E(S[T], \mathbb{L} \otimes S[T])$) is well defined, it follows that $(MS[T], \omega_M^*) = 0$ in $E(S[T], \mathbb{L} \otimes S[T])$, where ω_M^* is induced by ω_M. Therefore, by Theorem 6.0.2, it follows that ω_M^* can be lifted to a surjective map $\theta : \mathfrak{L} \otimes S[T] \twoheadrightarrow MS[T]$. Applying Lemma 7.0.2, we obtain a surjective lift $\phi : \mathfrak{L} \twoheadrightarrow M$ of ω_M. In other words, $(M, \omega_M) = 0$ in $E(R[T], \mathbb{L})$. □

Theorem 8.0.2. *Let R be a ring and $\mathbb{L}, P, \chi$ as above. Then, $e(P, \chi) = 0$ in $E(R[T], \mathbb{L})$ if and only if P has a unimodular element.*

Proof. Without loss of generality we may assume that R is reduced. Let $R \hookrightarrow S$ be a special $\mathbb{L}$-regular extension with S reduced. Let $\alpha : P \twoheadrightarrow I$ be a surjection, where I is an ideal in $R[T]$ of height n. Let $e(P, \chi) = (I, \omega_I)$ in $E(R[T], \mathbb{L})$, where (I, ω_I) is obtained from the pair (α, χ).

We first assume that $e(P, \chi) = 0$ in $E(R[T], \mathbb{L})$. Then, $e(P \otimes S[T], \chi \otimes S[T]) = 0$ in $E(S[T], \mathbb{L} \otimes S[T])$. As $\mathbb{L} \otimes S[T]$ is extended from S, it follows from Corollary 6.0.2 that $P \otimes S[T]$ has a unimodular element. But then by [B 1, Lemma 3.2], P has a unimodular element.

Conversely, assume that P has a unimodular element. Therefore, $P \otimes S[T]$ also has a unimodular element. As $\mathbb{L} \otimes S[T]$ is extended from S, it follows from Corollary 6.0.2 that $e(P \otimes S[T], \chi \otimes S[T]) = (IS[T], \omega_I^*) = 0$ in $E(S[T], \mathbb{L} \otimes S[T])$. But then by Proposition 8.0.1, $e(P, \chi) = 0$ in $E(R[T], \mathbb{L})$. □

Theorem 8.0.3. *Let R be a ring and $\mathbb{L}$ be a projective $R[T]$-module of rank one. Let P be a projective $R[T]$-module of rank n which is stably isomorphic to $\mathbb{L} \oplus R[T]^{n-1}$. Let $\chi : \mathbb{L} \xrightarrow{\sim} \wedge^n P$ be an isomorphism. Let $I \subset R[T]$ be an ideal of height n such that I/I^2 is surjective image of $\mathfrak{L}$ and ω_I be a local $\mathbb{L}$-orientation of I. Suppose that $e(P, \chi) = (I, \omega_I)$ in $E(R[T], \mathbb{L})$. Then, there exists a surjection $\alpha : P \twoheadrightarrow I$ such that (I, ω_I) is obtained from (α, χ).*

Proof. By Theorem 2.1.1, P/IP is isomorphic to $\mathfrak{L}/I\mathfrak{L}$. Choose an isomorphism $\sigma : \mathfrak{L}/I\mathfrak{L} \xrightarrow{\sim} P/IP$ such that $\wedge^n \sigma = \chi \otimes R[T]/I$. Let $\psi : P/IP \twoheadrightarrow I/I^2$ be the composite surjection:

$$P/IP \xrightarrow{\sigma^{-1}} \mathfrak{L}/I\mathfrak{L} \overset{\omega_I}{\twoheadrightarrow} I/I^2$$

Applying Lemma 2.1.4 we obtain a lift of ψ, say, $\phi \in \text{Hom}_{R[T]}(P, I)$ such that $\phi(P) = I \cap I'$, where I' is an ideal of $R[T]$ of height $\geq n$ and $I + I' = R[T]$. If $I' = R[T]$, then obviously ϕ is surjective and we are done in this case. Therefore, we assume that I' is proper and $\text{ht}(I') = n$.

Choose an isomorphism $\delta : \mathfrak{L}/I'\mathfrak{L} \xrightarrow{\sim} P/I'P$ such that $\wedge^n(\delta) = \chi \otimes R[T]/I'$. By the Chinese Remainder Theorem, we have $P/(I \cap I')P \simeq P/IP \oplus P/I'P$. Therefore, σ and δ together will induce an isomorphism $\tau : \mathfrak{L}/(I \cap I')\mathfrak{L} \xrightarrow{\sim} P/(I \cap I')P$ such that $\wedge^n(\tau) = \chi \otimes R[T]/(I \cap I')$. Composing τ with $\phi \otimes R[T]/(I \cap I')$ (as in the definition of the Euler class) one obtains a local orientation of $I \cap I'$, say, $\omega_{I \cap I'}$, and therefore, $e(P, \chi) = (I \cap I', \omega_{I \cap I'})$. Again note that as I and I' are comaximal, $\omega_{I \cap I'}$ induces local $\mathbb{L}$-orientations of I and I' and it is easy to see that the induced local orientation for I is precisely ω_I. If we call the one induced for I' as $\omega_{I'}$ then we have:

$$e(P, \chi) = (I, \omega_I) + (I', \omega_{I'}) \text{ in } E(R[T], \mathbb{L}).$$

From the hypothesis of the theorem it now follows that $(I', \omega_{I'}) = 0$, and therefore by Theorem 8.0.1 there exists a surjection $\beta : \mathfrak{L} \twoheadrightarrow I'$ such that $\beta \otimes R[T]/I' = \phi \otimes R[T]/I'$. Now we can apply Proposition 2.3.3 and conclude the proof of the corollary. □

Theorem 8.0.4. *Let R be a regular ring of dimension n which is essentially of finite type over a field k such that R has infinite residue fields. Let $\mathbb{L}$ be a projective $R[T]$-module of rank one. Then $E(R[T], \mathbb{L}) \simeq E(R, \mathbb{L}/T\mathbb{L})$.*

Proof. By a result of Lindel [L 1], the projective $R[T]$-module $\mathbb{L}$ is extended from R. Therefore, there exists projective R-module L of rank one such that $\mathbb{L} \simeq L[T]$ and $\mathbb{L}/T\mathbb{L} \simeq L$. Therefore, we need to prove that $E(R[T], L[T]) \simeq E(R, L)$ and we are done by Theorem 6.0.6. □

In the following theorem we extend some results from Chapter 3. In Theorem 3.2.5, it has been proved that if $R \hookrightarrow S$ is a subintegral extension then the Euler class groups $E(R[T])$ and $E(S[T])$ are isomorphic. The proofs given below are natural extensions of arguments from Theorem 3.2.5.

Theorem 8.0.5. *Let R be a ring and S be an extension ring. Let $\mathbb{L}$ be a projective $R[T]$-module of rank one. Then $E(R[T], \mathbb{L}) \simeq E(S[T], \mathbb{L} \otimes S[T])$ in the following cases:*

(i) *$R \hookrightarrow S$ is elementarily subintegral.*

(ii) $R \hookrightarrow S$ *is finite subintegral. In particular, when* S *is a special* $\mathbb{L}$*-regular extension.*

(iii) $R \hookrightarrow S$ *is subintegral.*

(iv) $S = {}^+(R_{red})$*, the seminormalization of* R_{red}*.*

Proof. We may assume by Proposition 8.0.2 that the ring R is reduced to start with. Also note that if $R \hookrightarrow S$ is subintegral, then we have a natural group homomorphism $\Theta : E(R[T], \mathbb{L}) \to E(S[T], \mathbb{L} \otimes S[T])$, which sends (J, ω_J) to (JS, ω_J^*), where ω_J^* is the local orientation induced by ω_J.

(1) Let $R \hookrightarrow S$ be elementarily subintegral. Let C be the conductor of R in S. Then by Lemma 3.1.4 we have $(R/C)_{\text{red}} = (S/C)_{\text{red}}$. It now follows from Lemma 7.0.1 that Θ is injective.

To prove that Θ is surjective, let $(I, \sigma) \in E(S[T], \mathbb{L} \otimes S[T])$, where $I \subset S[T]$ is an ideal of height n and $\sigma : (\mathfrak{L} \otimes S[T])/I(\mathfrak{L} \otimes S[T]) \twoheadrightarrow I/I^2$ is a surjection. By using the moving lemma (Lemma 2.1.4), we can find an ideal $K \subsetneq S[T]$ and a surjection $\tau : \mathfrak{L} \otimes S[T] \twoheadrightarrow I \cap K$ such that: (i) $\text{ht}(K) \geq n$, (ii) $K + I \cap C[T] = S[T]$, and (iii) $\tau \otimes S[T]/I = \sigma$.

If $\text{ht}(K) > n$, then $K = S[T]$ and we have $(I, \sigma) = 0$ in $E(S[T])$. Therefore we assume that $\text{ht}(K) = n$. Let $\eta = \tau \otimes S[T]/K$ be the local $\mathbb{L}$-orientation of K. Then we have, $(I, \sigma) + (K, \eta) = 0$ in $E(S[T], \mathbb{L} \otimes S[T])$. It is now enough to prove that (K, η) has a preimage in $E(R[T], \mathbb{L})$.

Let $K \cap R[T] = J$. As $K + C[T] = S[T]$, we have $J + C[T] = R[T]$, and therefore there exists $f \in C[T]$ such that $g = 1 - f \in J$. We can assume that $\text{ht}(g) = 1$. Since $f \in C[T]$, we have $R[T]_f = S[T]_f$. Therefore $R[T]/(1-f) = S[T]/(1-f)$ and $R[T] \hookrightarrow S[T]$ is an analytic isomorphism along $g \in J$. Therefore using Proposition 2.1.1, we have

(a) $R[T]/J \simeq S[T]/K$.

(b) $K = JS[T]$

(c) As $g \in J$, we have $J/J^2 \simeq K/K^2$.

As a consequence of (a) we have, $\mathfrak{L}/J\mathfrak{L} \simeq (\mathfrak{L} \otimes S[T])/K(\mathfrak{L} \otimes S[T])$. It is now easy to see from (c) that η is induced from a surjection $\omega_J : \mathfrak{L}/J\mathfrak{L} \twoheadrightarrow J/J^2$. Therefore, $\Theta((J, \omega_J)) = (K, \eta)$.

(2) Let $R \hookrightarrow S$ be finite subintegral. Then S is obtained from R by a finite sequence of elementarily subintegral extensions and we are done by (1) above.

(3) Here S is the filtered direct limit of subrings S_α such that each S_α can be obtained from R by a finite number of elementarily subintegral extensions. A direct limit argument as in Theorem 3.2.2 can easily be given to conclude the result.

(4) Obvious from (3). $\square$

8.1 Low dimensional rings

In this section we treat the cases when $\dim(R) = 2, 3$. The methods of previous sections do not naturally extend to three dimensional rings due to the lack of a suitable subtraction principle and we need to handle this carefully. The case of two dimensional rings is much simpler but the method is different.

8.1.1 Three dimensional rings

Let R be a ring of dimension 3 (containing $\mathbb{Q}$) and $\mathbb{L}$ be a projective $R[T]$-module of rank one. Assume for the time being that R is reduced. Let $R \hookrightarrow S$ be a special $\mathbb{L}$-regular extension with S reduced and C be the conductor of R in S. We fix S for the following discussion. We have, $\text{ht}(C) \geq 1$. A careful inspection of the theory and the results in Chapter 6 would reveal that if we had $\text{ht}(C) \geq 2$, or more generally, $\text{ht}(J(R, \mathbb{L})) \geq 2$, where $J(R, \mathbb{L})$ is the Quillen ideal of $\mathbb{L}$ in R, then one can similarly develop the theory of $E(R[T], \mathbb{L})$ and prove all the results of Chapters 6, 8. However, we now assume that $\text{ht}(C) \geq 1$ and define a "restricted" Euler class group $\widetilde{E}_S(R[T], \mathbb{L})$ below, which will serve most of our purposes. The definition is exactly the same as those in Chapters 6 and 8, only with one restriction imposed on the ideals concerned. We shall not repeat the whole definition in detail and we shall freely use terms defined in Chapter 8.

Definition 8.1.1. (*The "restricted" Euler class group* $\widetilde{E}_S(R[T], \mathbb{L})$): Let R be reduced. Let $\widetilde{G}$ be the free abelian group on pairs $(\mathcal{I}, \omega_{\mathcal{I}})$, where $\mathcal{I} \subset R[T]$ is an ideal of height n such that $\text{Spec}(R[T]/\mathcal{I})$ is connected and $\mathcal{I} + C[T] = R[T]$ (here is the restriction), and

$$\omega_{\mathcal{I}} : \frac{(\mathbb{L} \oplus R[T]^2)}{\mathcal{I}(\mathbb{L} \oplus R[T]^2)} \twoheadrightarrow \frac{\mathcal{I}}{\mathcal{I}^2}$$

is a local $\mathbb{L}$-orientation of $\mathcal{I}$. Given any ideal I of $R[T]$ such that $I + C[T] = R[T]$ and any local $\mathbb{L}$-orientation ω_I, one can easily associate an element in $\widetilde{G}$, as it was done before. We denote this element as (I, ω_I). Take $\widetilde{H}$ to be the subgroup of $\widetilde{G}$ generated by all those (I, ω_I) of $\widetilde{G}$ such that ω_I is a global $\mathbb{L}$-orientation. Define $\widetilde{E}_S(R[T], \mathbb{L}) = \widetilde{G}/\widetilde{H}$.

We write $\mathfrak{L} = \mathbb{L} \oplus R[T]^2$.

Theorem 8.1.1. *Let R be a reduced ring of dimension 3, $\mathbb{L}$ be a projective $R[T]$-module of rank one and S be as above. Then there is an injective group homomorphism $\Theta : \widetilde{E}_S(R[T], \mathbb{L}) \longrightarrow E(S[T], \mathbb{L} \otimes S[T])$.*

Proof. The definition of Θ is the same as Proposition 8.0.1. Obviously it is a group homomorphism. The injectivity of Θ follows from Remark 7.0.3. □

Corollary 8.1.1. *Let $R, \mathbb{L}, S$ be as above. Let $(I, \omega_I) = 0$ in $\widetilde{E}_S(R[T], \mathbb{L})$. Then ω_I is a global $\mathbb{L}$-orientation of I, i.e., there is a surjective map $\alpha : \mathfrak{L} \twoheadrightarrow I$ such that α lifts ω_I.*

Proof. Clearly follows from the above theorem because Θ is injective. □

Now let P be a projective $R[T]$-module of rank 3 with determinant $\mathbb{L}$ and let $\chi : \mathbb{L} \xrightarrow{\sim} \wedge^3(P)$ be an isomorphism. We can associate an element $e(P, \chi)$, called the Euler class of (P, χ), in the group $\widetilde{E}_S(R[T], \mathbb{L})$ so that it serves as the precise obstruction for P to split off a free summand of rank one. We describe it now.

Let $R \hookrightarrow S$ be a special $\mathbb{L}$-regular extension as above and C be the conductor of R in S. Since $\dim(R/C) \le 2$, it follows from Theorem 2.1.1 that the projective $(R/C)[T]$-module $P/C[T]P$ has a unimodular element. Applying Lemma 2.1.2 it is easy to see that there is an ideal $I \subset R[T]$ of height 3 which is comaximal with $C[T]$ such that there is a surjection $\alpha : P \twoheadrightarrow I$. Choose an isomorphism $\overline{\gamma} : \mathfrak{L}/I\mathfrak{L} \xrightarrow{\sim} P/IP$ such that $\wedge^n \overline{\gamma} = \overline{\chi}$, where bar denotes reduction modulo I. Let ω_I be the composite surjection

$$\mathfrak{L}/I\mathfrak{L} \overset{\bar{\gamma}}{\xrightarrow{\sim}} P/IP \overset{\bar{\alpha}}{\twoheadrightarrow} I/I^2.$$

We define the Euler class of (P, χ) as $e(P, \chi) = (I, \omega_I) \in \widetilde{E}_S(R[T], \mathbb{L})$. Following the same method as in Lemma 8.0.2 and using Theorem 8.1.1 it is easy to prove that the Euler class is well defined.

Theorem 8.1.2. *Let $R, \mathbb{L}$ be as above. Let P be a projective $R[T]$-module of rank 3 with determinant $\mathbb{L}$ and let $\chi : \mathbb{L} \xrightarrow{\sim} \wedge^3(P)$ be an isomorphism. Then $e(P, \chi) = 0$ in $\widetilde{E}_S(R[T], \mathbb{L})$ if and only if P has a unimodular element.*

Proof. Let $R \hookrightarrow S$ be the special $\mathbb{L}$-regular extension fixed above and $\Theta : \widetilde{E}_S(R[T], \mathbb{L}) \longrightarrow E(S[T], \mathfrak{L} \otimes S[T])$ be the group homomorphism from the above theorem. Let $e(P, \chi) = (I, \omega_I)$ in $\widetilde{E}_S(R[T], \mathbb{L})$.

First assume that $e(P, \chi) = 0$ in $\widetilde{E}_S(R[T], \mathbb{L})$. This will imply that $e(P \otimes S[T], \chi \otimes S[T]) = (IS[T], \omega_I^*) = 0$ in $E(S[T], \mathfrak{L} \otimes S[T])$, where ω_I^* is the local orientation of $IS[T]$ induced by ω_I. As $\mathbb{L} \otimes S[T]$ is extended from S, it follows from Corollary 6.0.2 that $P \otimes S[T]$ has a unimodular element. Then by [B 1, Lemma 3.1], P has a unimodular element.

Conversely, if P has a unimodular element then the same is true for $P \otimes S[T]$, and then $(IS[T], \omega_I^*) = 0$ in $E(S[T], \mathfrak{L} \otimes S[T])$. As Θ is injective, it follows that $(I, \omega_I) = 0$ in $\widetilde{E}_S(R[T], \mathbb{L})$. Therefore, $e(P, \chi) = 0$ in $\widetilde{E}_S(R[T], \mathbb{L})$. □

Remark 8.1.1. Now let R be a ring of dimension 3 which is not necessarily reduced. Let P be a projective $R[T]$-module of rank 3 with determinant $\mathbb{L}$. Let $R_{\text{red}} = R/\mathfrak{n}(R)$, where $\mathfrak{n}(R)$ is the nil radical of R. It is easy to derive that P has a unimodular element if and only if $P \otimes R_{\text{red}}$ has a unimodular element. Fix an isomorphism $\chi : \mathbb{L} \xrightarrow{\sim} \wedge^3(P)$. Consider the Euler class $e(P \otimes R_{\text{red}}, \chi \otimes R_{\text{red}}) \in \widetilde{E}_S(R_{\text{red}}, \mathbb{L} \otimes R_{\text{red}})$. Then P has a unimodular element if and only if $e(P \otimes R_{\text{red}}, \chi \otimes R_{\text{red}}) = 0$ in $\widetilde{E}(R_{\text{red}}, \mathbb{L} \otimes R_{\text{red}})$.

8.1.2 Two dimensional rings

Let R be a ring of dimension 2 and $\mathbb{L}$ be a projective $R[T]$-module of rank one. The theory of the Euler class group $E(R[T], \mathbb{L})$ is very much similar to the two-dimensional case developed in [D 1, Section 7]. Unlike the higher dimensional cases treated so far, most of the results for two dimensional rings from [D 1, Section 7] can be extended without much hurdle. We first prove the following easy lemma.

Lemma 8.1.1. *Let R be a ring of dimension 2 and $\mathbb{L}$ be a projective $R[T]$-module of rank one. Let $J \subset R[T]$ be an ideal such that there is a surjection $\overline{\alpha} : \mathbb{L} \oplus R[T] \twoheadrightarrow J/J^2$.*

Then there exists a projective $R[T]$-module P of determinant $\mathbb{L}$ such that P maps onto J.

Proof. Write $\mathfrak{L} = \mathbb{L} \oplus R[T]$. Let $\alpha : \mathfrak{L} \to J$ be a lift of $\overline{\alpha}$. Then $\alpha(\mathfrak{L}) + J^2 = J$. Therefore, by Lemma 2.1.3 there exits $e \in J$ such that $J = (\alpha(\mathfrak{L}), e)$ with $e(1-e) \in \alpha(\mathfrak{L})$. This implies that $\alpha' = \alpha_{1-e} : \mathfrak{L}_{1-e} \twoheadrightarrow J_{1-e}$ is a surjection.

On the other hand, we have a surjection $\beta : \mathfrak{L}_e \twoheadrightarrow J_e = R[T]_e$ which is projection onto the second factor.

Thus we obtain two surjections α'_e, β_{1-e} from $\mathfrak{L}_{e(1-e)}$ to $J_{e(1-e)} = R[T]_{e(1-e)}$, and exact sequences:

$$0 \to \ker(\alpha'_e) \to \mathfrak{L}_{e(1-e)} \twoheadrightarrow J_{e(1-e)} = R[T]_{e(1-e)} \to 0$$

$$0 \to \ker(\beta_{1-e}) \to \mathfrak{L}_{e(1-e)} \twoheadrightarrow J_{e(1-e)} = R[T]_{e(1-e)} \to 0$$

As projective modules of rank one are always cancellative, we have $\ker(\alpha'_e) \simeq \mathbb{L}_{e(1-e)} \simeq \ker(\beta_{1-e})$. Therefore, there is an automorphism ϕ of $\mathfrak{L}_{e(1-e)}$ such that $\det(\phi) = 1$ and $(\beta_{1-e})\phi = \alpha'_e$. By a standard patching argument we obtain a projective $R[T]$-module P of rank 2 and a surjection from P to J. As $\det(\phi) = 1$, it is easy to see that $\wedge^2(P) \simeq \mathbb{L}$. □

Armed with the above lemma one can now easily extend [D 1, Theorem 7.1] in the following manner. We omit the proof as it can be worked out modifying the proof of [D 1, Theorem 7.1].

Theorem 8.1.3. *Let R be a ring of dimension 2 and $\mathbb{L}$ be a projective $R[T]$-module of rank one. Write $\mathfrak{L} = \mathbb{L} \oplus R[T]$. Let $J \subset R[T]$ be an ideal such that there is a surjection $\overline{\alpha} : \mathfrak{L} \twoheadrightarrow J/J^2$. Suppose that there is a surjection $\Gamma : \mathfrak{L} \otimes R(T) \twoheadrightarrow JR(T)$ such that Γ lifts $\alpha \otimes R(T)$. Then there is a surjective map $\beta : \mathfrak{L} \twoheadrightarrow J$ and $\theta \in SL(\mathfrak{L}/J\mathfrak{L})$ such that $\alpha\theta = \beta \otimes R[T]/J$.*

Remark 8.1.2. Let R be a ring of dimension 2 and $\mathbb{L}$ be a projective $R[T]$-module of rank one. Applying Theorem 8.1.3 one can easily extend [D 1, Corollary 7.2, 7.3]. The Euler class group $E(R[T], \mathbb{L})$ can be defined exactly as it has been done before. The only difference is that, a local $\mathbb{L}$-orientation $\alpha : \mathfrak{L}/J\mathfrak{L} \twoheadrightarrow J/J^2$ will be called global if there is a surjection $\theta : \mathfrak{L} \twoheadrightarrow J$ and some $\sigma \in SL(\mathfrak{L}/J\mathfrak{L})$ such that $\alpha\sigma = \theta \otimes R[T]/J$. The Euler class of a projective $R[T]$-module P of rank 2, together with an isomorphism $\chi : \mathbb{L} \xrightarrow{\sim} \wedge^2(P)$ can also be defined as it has been done in previous section. It can be easily checked that the Euler class $e(P, \chi)$ is trivial in $E(R[T], \mathbb{L})$ if and only if $P \simeq \mathbb{L} \oplus R[T]$. We leave all the details as no new technique is involved here. Only result that we could not extend from [D 1] is [D 1, Theorem 7.6]. Note that in the proof of [D 1, Theorem 7.6], the "Symplectic" cancellation theorem of Bhatwadekar [B 2, Theorem 4.8] is crucially used, which is not available in this case.

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