# TESTS FOR INDEPENDENCE AND SYMMETRY IN MULTIVARIATE NORMAL POPULATIONS

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*SUMMARY.* It is shown that the distribution function of each of the statistics proposed by various authors [Mauchly (1940), Votaw (1948), Wilks (1932, 1935, 1946)] for testing independence and symmetry in multivariate Normal population can be expanded in an asymptotic series involving Chi-square probability integrals. For large samples, the first term of the expansion provides a good approximation by itself, the second term being $O(n^{-2})$, where $n$ is the size of the sample. For moderately large samples, the first two terms may be used and the coefficients required for this purpose are tabulated for each one of the various statistics considered.

## 1. Notation

Given a $p\times p$ matrix $A = ((a_{ij}))$, we shall denote by $\bar{A} = ((\bar{a}_{ij}))$ the $p\times p$ matrix whose elements are

$$\bar{a}_{ii} = a_1, \bar{a}_{ij} = a_2 \text{ if } i \neq j,$$

where
$$a_1 = p^{-1} \sum_{i=1}^{p} a_{ii}, a_2 = p^{-1}(p-1)^{-1} \sum_{i\neq j=1}^{p}\sum a_{ij}. \quad \dots \quad (1.1)$$

Again given a $p\times q$ matrix $B = ((b_{ij}))$, we shall denote by $\bar{\bar{B}} = ((\bar{\bar{b}}_{ij}))$ the $p\times q$ matrix whose elements are

$$\bar{\bar{b}}_{ij} = b, \text{ where } b = (pq)^{-1} \sum_{i=1}^{p} \sum_{j=1}^{q} b_{ij}. \quad \dots \quad (1.2)$$

Given a $p\times p$ matrix $A$ and a partition $(p_1, p_2, \dots, p_m)$ of $p$, we shall write $A$ in the partitioned form

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{2m} \\ \dots & \dots & \dots & \dots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{bmatrix} \quad \dots \quad (1.3)$$

where $A_{\alpha\beta}$ is a $p_\alpha \times p_\beta$ matrix. Thus writing

$$P_\alpha = p_1+p_2+\dots+p_\alpha, (\alpha = 1, 2, \dots, m-1); P_0 = 0, P_m = p \quad \dots \quad (1.4)$$

we have $A_{\alpha\beta} = ((a_{i_\alpha j_\beta}))$ where the ranges of $i_\alpha$ and $j_\beta$ are $P_{\alpha-1} < i_\alpha \leqslant P_\alpha$, $P_{\beta-1} < i_\beta \leqslant P_\beta$ for $\alpha, \beta = 1, 2, ..., m$. Given a $p \times p$ matrix $A$ and a partition $(p_1, p_2, ..., p_m)$ of $p$ we shall denote by $A^*$ the matrix

$$A^* = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \dots & \bar{A}_{1m} \\ \bar{A}_{21} & \bar{A}_{22} & \dots & \bar{A}_{2m} \\ \dots & \dots & \dots & \dots \\ \bar{A}_{m1} & \bar{A}_{m2} & \dots & \bar{A}_{mm} \end{bmatrix} \qquad \dots \quad (1.5)$$

If $p_1 = p_2 = \dots = p_m$, we shall denote by $A^{**}$ the matrix

$$A^{**} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \dots & \bar{A}_{1m} \\ \bar{A}_{21} & \bar{A}_{22} & \dots & \bar{A}_{2m} \\ \dots & \dots & \dots & \dots \\ \bar{A}_{m1} & \bar{A}_{m2} & \dots & \bar{A}_{mm} \end{bmatrix} \qquad \dots \quad (1.6)$$

All tests of significance discussed in later sections relate to a $p$-variate Normal population with means $\mu_i$ and dispersion (or, variance-covariance) matrix $((\sigma_{ij}))$, on the basis of a random sample $x_{i\nu}$ drawn from such a population $i, j = 1, 2, ..., p$; $\nu = 1, 2, ..., n$. The joint probability density function of the $x_{i\nu}$'s is :

$$(2\pi)^{-\frac{1}{2}pn} \, |\sigma_{ij}|^{-\frac{1}{2}n} \exp\left[-\tfrac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \sigma^{ij} \sum_{\nu=1}^{n} (x_{i\nu}-\mu_i)(x_{j\nu}-\mu_j)\right] \qquad \dots \quad (1.7)$$

where $((\sigma^{ij})) = ((\sigma_{ij}))^{-1}$. We shall write

$$\bar{x}_i = n^{-1} \sum_{\nu=1}^{n} x_{i\nu}, \; s_{ij} = \sum_{\nu=1}^{n} (x_{i\nu}-\bar{x}_i)(x_{j\nu}-\bar{x}_j) \text{ and } S = ((s_{ij})). \qquad \dots \quad (1.8)$$

If the $p$-variates fall into $m$ groups, with $p_\alpha$ variates in the $\alpha$-th group, $\alpha = 1, 2, ..., m$, $p = p_1+p_2+...+p_m$, without loss of generality, we shall assume that $p_1 \geqslant p_2 \geqslant \dots \geqslant p_m \geqslant 1$, and arrange the variates such that the first $p_1$-variates belong to the first group, the next $p_2$-variates belong to the second group and so on and finally the last $p_m$-variates belong to the $m$-th group. We shall then write

$$\bar{\bar{x}}_\alpha = p_\alpha^{-1} \sum_{i=P_{\alpha-1}+1}^{P_\alpha} \bar{x}_i. \qquad \dots \quad (1.9)$$

Further, if the $i$-th variate belongs to the $\alpha$-th group and the $j$-th variate belongs to the $\beta$-th group, we shall write

$$t_{ij} = s_{ij} + n(\bar{x}_i - \bar{\bar{x}}_\alpha)(\bar{x}_j - \bar{\bar{x}}_\beta) \qquad \ldots \quad (1.10)$$

and $T = ((t_{ij}))$ for the $p \times p$ matrix whose elements are $t_{ij}$.

The incomplete Beta-function ratio will be denoted by the symbol

$$I_x(P, Q) = \frac{\Gamma(P+Q)}{\Gamma(P)\ \Gamma(Q)} \int_0^x u^{P-1}(1-u)^{Q-1}\, du. \qquad \ldots \quad (1.11)$$

## 2. An asymptotic expansion of a class of distribution functions

Let $L$ be a random variable with Prob $(0 < L < 1) = 1$ whose $t$-th moment about origin is of the form :

$$E(L^t) = \prod_{j=1}^{s} \frac{\left(\frac{n}{2} + b_j\right)_t}{\left(\frac{n}{2} + c_j\right)_t} \qquad \ldots \quad (2.1)$$

where $n$ is a positive integer, $c_j > b_j$ $(j = 1, 2, \ldots, s)$ are constants not involving $n$ and $(m)_t = \Gamma(m+t)/\Gamma(m)$. It will be seen in later sections that the $\frac{2}{n}$-th power of the various likelihood-ratio statistics for testing hypotheses of independence and symmetry in multivariate Normal populations on the basis of a random sample of size $n$ all have moments of the form (2.1). The probability distribution of specific statistics with moments of this form have been investigated by various authors [Bartlett (1947); Nair (1938); Rao (1948, 1951); Roy (1951); Verma (1951); Wald and Brookner (1941); Wilks and Tukey (1946)]. It is easy to show that asymptotically, for large $n$, $-n \log_e L$ is distributed as a Chi-square with

$$r = 2 \sum_{j=1}^{s} (c_j - b_j) \qquad \ldots \quad (2.2)$$

degrees of freedom, which, however, is not a very good approximation when $n$ is only moderately large. The most convenient is the asymptotic series expansion derived by Box (1949) and Roy (1951) which, to terms of order $n^{-4}$ may be expressed as follows :

Let $\quad r_t = \sum_{j=1}^{s} (c_j^t - b_j^t),\ t = 1, 2, 3;\ a = (r_1 - r_2)/r_1;\ N = n - a;$

then, writing $$X = -N \log_e L \qquad \ldots \quad (2.3)$$

we have for the probability distribution of $X$ the asymptotic expansion :

$$\text{Prob}\,(X > x) = Q_r(x) + \frac{a_2}{N^2}\left[Q_{r+4}(x) - Q_r(x)\right] + \frac{a_3}{N^3}\left[Q_{r+6}(x) - Q_r(x)\right] +$$

$$+ \frac{a_4}{N^4}\left[Q_{r+8}(x) - Q_r(x)\right] - \frac{a_2^2}{N^4}\left[Q_{r+4}(x) - Q_r(x)\right] + O(N^{-5}) \qquad (2.4)$$

where $r = 2r_1$, is defined by (2.3), $Q_r(x)$ stands for the probability that a Chi-square with $r$ degrees of freedom exceeds $x$, that is,

$$Q_r(x) = \int_x^\infty \frac{1}{2^{\frac{1}{2}r}\,\Gamma(\tfrac{1}{2}r)}\, e^{-\frac{1}{2}u}\, u^{\frac{1}{2}r-1}\, du$$

and

$$a_2 = \frac{2}{3}(d_3 - r_1) = \frac{2}{3}(r_2 - r_1) + \frac{1}{2}\,a(r_1 + r_2) \qquad \ldots \quad (2.5)$$

$$a_3 = 2a_2 - \frac{2}{3}(d_4 - r_1)$$

$$a_4 = 3a_3 - 4a_2 + \frac{1}{2}\,a_2^2 + \frac{4}{5}(d_5 - r_1)$$

where

$$d_t = \sum_{j=1}^{s}\left[(c_j + \tfrac{1}{2}a)^t - (b_j + \tfrac{1}{2}a)^t\right], \quad t = 3, 4, 5.$$

The first term of (2.4) by itself provides a good approximation. If the sample size is only moderately large, the first two terms may be used. As it is seldom necessary to use more than the first two terms, we present, for each of the statistics considered in the later sections the values of the constants $r$, $a$ and $a_2$ for use in the expansion (2.4) to terms of order $N^{-2}$.

## 3. Test for independence

With a given partition $(p_1, p_2, \ldots, p_m)$ of the $p$-variates into $m$ groups, to test the hypothesis $\mathcal{H}_i$ that the $m$ groups of variates are mutually independent, Wilks (1935) showed that the likelihood-ratio statistic is the $\frac{1}{2}n$-th power of

$L_i = |S| / \prod_{a=1}^{m} |S_{aa}|$ where $S_{aa}$ is the $p_a \times p_a$ diagonal sub-matrix of $S$ defined by (1.8).

Asymptotically, $-n \log_e L_i$ is distributed as a Chi-square with $r = p_1(p_2+p_3+\ldots+p_m)+p_2(p_3+\ldots+p_m)+\ldots+p_{m-1}\,p_m$ degrees of freedom. The exact distribution of $L_i$ in closed form has been obtained by Wald and Brookner (1941) and Wilks (1935) in certain special cases, where the number of groups containing an odd number of variates is atmost one. In the two simplest cases, when $m = 2$ we have for $0 < x < 1$,

$$\text{Prob}\ (L_i < x) = \begin{cases} I_x\left(\frac{n-p+1}{2}, \frac{p-1}{2}\right) & \text{if } p_1 = p-1, p_2 = 1 \\ I_{\sqrt{x}}(n-p+1, p-2) & \text{if } p_1 = p-2, p_2 = 2 \end{cases} \qquad \ldots \ (3.1)$$

where $I_x(p, q)$ is the incomplete Beta-function defined by (1.11). An infinite series expansion of the distribution of $-n \log_e L_i$ in the general case was obtained by Wald and Brookner (1941).

The $t$-th moment of $L_i$ computed by Wilks (1935) can be written as

$$E(L_i^t) = \prod_{\alpha=1}^{m-1} \prod_{i_\alpha = P_\alpha+1}^{P_{\alpha+1}} \frac{\left(\frac{n-i_\alpha+1}{2}\right)_t}{\left(\frac{n-i_\alpha+1+P_\alpha}{2}\right)_t} \qquad \ldots \ (3.2)$$

where $P_\alpha$ is defined by (1.4). Since the moments are of the form (2.1) the asymptotic expansion (2.4) is available. The constants $a$, $r$, $a_2$ are tabulated for all partitions of $p$ except $(p-1, 1)$ and $(p-2, 2)$ for $3 < p < 8$ in columns (4)—(6) of Table 1.

## 4. Test for spherical symmetry

Mauchly (1940) proposed the hypothesis $\mathcal{H}_s$ of spherical symmetry, namely that the $p$-variates are mutually independent and have equal variances. He showed that the likelihood-ratio statistic is the $\frac{1}{2}n$-th power of $L_s = |S|/s^p$ where $s = \frac{1}{p}\sum_{i=1}^{p} s_{ii}$.

Asymptotically $-n \log_e L_s$ has the Chi-square distribution with $r = \frac{1}{2}p(p+1)-1$ degrees of freedom. The number of degrees of freedom given by Mauchly (1940) appears to be wrong. The exact probability distribution of $L_s$ when $p = 2$ is worked out by Mauchly (1940) as

$$\text{Prob}\ (L_s < x) = x^{\frac{1}{2}(n-2)} \text{ for } p = 2,\ 0 < x < 1. \qquad \ldots \ (5.1)$$

The $t$-th moment of $L_s$ about origin computed by Mauchly (1940) can be expressed as

$$E(L_s^t) = \prod_{i=1}^{p-1} \frac{\left(\frac{n-i-1}{2}\right)_t}{\left(\frac{n-1}{2}+\frac{i}{p-1}\right)_t} \qquad \ldots \ (5.2)$$

which is of the form (2.1). The constants $a$, $r$, $a_2$ in the asymptotic expansion (2.4) for $3 < p < 8$ are tabulated in columns (2)—(4) of Table 2.

## 5. TEST FOR EQUALITY OF MEANS, EQUALITY OF VARIANCES AND EQUALITY OF COVARIANCES

5.1. *The hypothesis* $\mathcal{H}_{vc}$ *:* Wilks (1946) showed that the likelihood-ratio statistic for testing the hypothesis $\mathcal{H}_{vc}$ that the variances are equal and the covariances are equal is the $\frac{1}{2}n$-th power of $L_{vc} = |S|/|\bar{S}|$ where $\bar{S}$ is derived from $S$ by formula (1.1).

Asymptotically $-n \log_e L_{vc}$ follows the Chi-square distribution with $r = \frac{1}{2}p(p+1)-2$ degrees of freedom. The exact probability distributions for the cases $p = 2$, 3 obtained by Wilks (1946) are as follows :

$$\text{Prob}\ (L_{vc} < x) = \begin{cases} I_x\left(\frac{n-2}{2}, \frac{1}{2}\right) \text{ for } p = 2 \\ I_{\sqrt{x}}(n-3,\ 2) \quad \text{for } p = 3 \end{cases} \qquad \ldots \quad (5.1)$$

for $0 < x < 1$. Verma (1951) has obtained infinite series expansions for the distribution of $L_{vc}$ for other small values of $p$.

The $t$-th moment of $L_{vc}$ derived by Wilks (1946) is given by

$$E(L_{vc}^t) = \prod_{i=1}^{p-1} \frac{\left(\frac{n-1+i-p}{2}\right)_t}{\left(\frac{n-1}{2}+\frac{i-1}{p-1}\right)_t} \qquad \ldots \quad (5.2)$$

which is of the form (2.1). The constants $a, r, a_2$ in the asymptotic expansion (2.4) are tabulated in columns (5)—(7) of Table 2 for $4 \leqslant p \leqslant 8$.

5.2. *The hypothesis* $\mathcal{H}_{mvc}$ *:* For testing the hypothesis $\mathcal{H}_{mvc}$ that the means are equal, the variances are equal and the covariances are equal, Wilks (1946) showed that the likelihood-ratio criterion is the $\frac{1}{2}n$-th power of $L_{mvc} = |S|/|\bar{V}|$ where $\bar{V}$ is derived from the matrix $V = ((v_{ij}))$ by formula (1.1) where $v_{ij} = s_{ij}+n(\bar{x}_i-\bar{\bar{x}})(\bar{x}_j-\bar{\bar{x}})$ and $\bar{\bar{x}} = \frac{1}{p}\sum_{i=1}^{p} \bar{x}_i$.

Asymptotically, $-n \log_e L_{mvc}$ is distributed as Chi-square with $r = \frac{1}{2}p(p+3)-3$ degrees of freedom. The exact probability distributions for $p = 2$, 3 are as follows :

$$\text{Prob}\ (L_{mvc} < x) = \begin{cases} I_x\left(\frac{n-2}{2}, 1\right) \text{ for } p = 2 \\ I_{\sqrt{x}}(n-3,\ 3) \quad \text{for } p = 3 \end{cases} \qquad \ldots \quad (5.3)$$

An infinite series expansion of the distribution of $L_{mvc}$ for other small values of $p$ has been obtained by Verma (1951).

The $t$-th moment of $L_{mvc}$ computed by Wilks (1946) is

$$E(L^t_{mvc}) = \prod_{i=1}^{p-1} \frac{\left(\frac{n+i-p-1}{2}\right)_t}{\left(\frac{n}{2}+\frac{i-1}{p-1}\right)_t} \qquad \dots \quad (5.4)$$

which is of the form (2.1). The asymptotic expansion (2.4) is therefore available and the constants $a$, $r$, $a_2$ involved are tabulated in columns (8)–(10) of Table 2 for $4 \leqslant p \leqslant 8$.

## 6. Test for compound symmetry

In whatever follows, unless anything to the contrary is explicitly stated, we shall consider a partition $(p_1, p_2, \dots, p_m)$ of the $p$-variates into $m$ groups. If there are $q$ groups of variates each containing just one variate, we shall write $h = m-q$ $(h \geqslant 1)$ so that $p_\alpha \geqslant 2$ for $\alpha = 1, 2, \dots h$ and $p_\alpha = 1$ for $\alpha = h+1, h+2, \dots, m$. For brevity, we shall denote such a partition by $(p_1, \dots p_h, 1^q)$. We are now in a position to state various hypotheses of compound symmetry proposed by Votaw (1948) who obtained the respective likelihood-ratio statistics and their moments.

6.1. *The hypothesis* $\mathcal{H}_{1(m)}$ : This hypothesis states that within each group of variates, the variances are equal and the covariances are equal, and between groups, the covariances are equal for each pair of groups. The likelihood-ratio criterion is the $\frac{1}{2}n$-th power of $L_{1(m)} = |S|/|S^*|$ where $S^*$ is derived from $S$ by the formula (1.5) corresponding to the partition $(p_1, p_2, \dots, p_h, 1^q)$ of $p$.

Asymptotically $-n \log_e L_{1(m)}$ has the Chi-square distribution with $r = \frac{1}{2}[p(p+1)-q(q+1)-h(h+3)]-qh$ degrees of freedom. The exact probability distribution of $L_{1(m)}$ has been worked out in Votaw (1948) for the partitions $(2, 1^{p-2})$ and $(3, 1^{p-3})$ of $p$. Thus :

$$\text{Prob}\ (L_{1(m)} \leqslant x) = \begin{cases} I_x\left(\frac{n-p}{2}, \frac{p-1}{2}\right) \text{ for the partition } (2, 1^{p-2}) \\[1ex] I_{\sqrt{x}}\,(n-p,\ p-1) \text{ for the partition } (3, 1^{p-3}). \end{cases} \qquad \dots \quad (6.1)$$

Some other special cases are derived by Votaw, Kimball and Rafferty (1950).

With $P_a$ defined by (1.4), the $t$-th moment of $L_{1(\pi)}$ can be expressed as follows as shown by Votaw (1948)

$$E(L^t_{1(\pi)}) = \prod_{a=1}^{h} \prod_{i_a=1}^{p_a-1} \frac{\frac{1}{2}(n-m-i_a-P_{a-1}+\alpha-1)_t}{\left(\frac{n-1}{2}+\frac{i_a-1}{p_a-1}\right)_t} \qquad \ldots \quad (6.2)$$

which is of the form (2.1). The values of the constants $a, r, a_s$ in the asymptotic expansion (2.4) are tabulated in columns (7)—(9) of Table 1 for all partitions of $p$ except $(2, 1^{p-2})$ and $(3, 1^{p-3})$ for $4 \leqslant p \leqslant 8$.

6.2. *The hypothesis $\mathcal{H}_{1(m\pi)}$:* This states that within each group of variates the means are equal, the variances are equal and the covariances are equal and between groups, the covariances are equal for each pair of groups. The likelihood-ratio statistic is the $\frac{1}{2}$ $n$-th power of $L_{1(m\pi)} = |S|/|T^*|$ where the matrix $T$ is defined by formula (1.10) for the partition $(p_1, p_2, \ldots, p_h, 1^q)$ of the $p$-variates and $T^*$ is derived from $T$ by formula (1.5).

Asymptotically $-n \log_e L_{1(m\pi)}$ is distributed as a Chi-square with $r = \frac{1}{2}[p(p+3)-q(q+3)-h(h+5)]-hq$ degrees of freedom. The exact probability distribution of $L_{1(m\pi)}$ for the partitions $(2, 1^{p-2})$ and $(3, 1^{p-3})$ obtained by Votaw (1948) are as follows :

$$\text{Prob } (L_{1(m\pi)} \leqslant x) = \begin{cases} I_x\left(\frac{n-p}{2}, \frac{p}{2}\right) \text{ for the partition } (2,\ 1^{p-2}) \\ I_{\sqrt{x}}(n-p,\ p) \text{ for the partition } (3,\ 1^{p-3}) \end{cases} \qquad \ldots \quad (6.3)$$

and certain other cases are examined by Votaw, Kimball and Rafferty (1950). The $t$-th moment of $L_{1(m\pi)}$ computed by Votaw (1948) is

$$E(L^t_{1(m\pi)}) = \prod_{a=1}^{h} \prod_{i_a=1}^{p_a-1} \frac{\frac{1}{2}(n-m-i_a-P_{a-1}+\alpha-1)_t}{\left(\frac{n}{2}+\frac{i_a-1}{p_a-1}\right)_t} \qquad \ldots \quad (6.4)$$

where $P_a$ is defined in (1.4). This is of the form (2.1) and the constants $a, r, a_s$ involved in the asymptotic expansion (2.4) are given in columns (10)—(12) of Table 1 for all partitions of $p$ except $(2, 1^{p-2})$ and $(3, 1^{p-3})$ for $4 \leqslant p \leqslant 8$.

In the special case, $p_1 = p_2 = \ldots = p_m = u$, $u \geqslant 2$, that is when the $p$-variates fall into $m$ groups, each of $u$-variates, for a fixed arrangement of the $u$-variates within each group, two other hypotheses of compound symmetry have been considered by Votaw (1948). These are discussed in the following sections.

6.3. *The hypothesis* $\bar{\mathscr{H}}_{1(vc)}$: This hypothesis states that within each group, the variances are equal and the covariances are equal and between groups, the diagonal covariances are equal and the off-diagonal covariances are equal. The likelihood-ratio criterion is the $\frac{1}{2}n$-th power of $\bar{L}_{1(vc)} = |S|/|S^{**}|$ where $S^{**}$ is derived from $S$ by formula (1.6) corresponding to the partition $(u, u, \ldots, u)$ of $p = mu$.

Asymptotically $-n \log_e \bar{L}_{1(vc)}$ follows the Chi-square distribution with $r = \frac{1}{2}p(p+1)-m(m+1)$ degrees of freedom. In the case where $m = 2$ and $u = 2$, the exact distribution of $\bar{L}_{1(vc)}$ derived in Votaw (1948) is as follows:

$$\text{Prob}\ (\bar{L}_{1(vc)} \leqslant x) = I_{\sqrt{x}}(n-4, 2) \text{ when } u = 2, m = 2. \qquad \ldots \ (6.5)$$

The $t$-th moment of $\bar{L}_{1(vc)}$ was shown to be

$$E(\bar{L}^t_{1(vc)}) = \prod_{\alpha=1}^{m} \prod_{i_\alpha=1}^{u-1} \frac{\left(\frac{n-m-i_\alpha-(u-1)(\alpha-1)}{2}\right)_t}{\left(\frac{n-1}{2}+\frac{1-\alpha}{2(u-1)}+\frac{i_\alpha-1}{u-1}\right)_t} \qquad \ldots \ (6.6)$$

which is of the form (2.1) and the values of the constants $a, r, a_2$ in the asymptotic expansion (2.4) are presented in columns (4)—(6) of Table 3 for $m = 2$, $u = 3, 4$; $m = 3$, $u = 2$, and $m = 4$, $u = 2$.

6.4. *The hypothesis* $\bar{\mathscr{H}}_{1(mvc)}$: This hypothesis specifies that within each group, the means are equal, the variances are equal and the covariances are equal and between groups, the diagonal covariances are equal and the off-diagonal covariances are equal. The likelihood-ratio criterion was shown (Votaw, 1948) to be the $\frac{1}{2}n$-th power of $\bar{L}_{1(mvc)} = |S|/|T^{**}|$ where the matrix $T$ is defined by (1.10) corresponding to the partition $(u, u, \ldots, u)$ of $p = mu$ and $T^{**}$ is derived from $T$ by means of formula (1.6).

Asymptotically $-n \log_e \bar{L}_{1(mvc)}$ is distributed as a Chi-square with $r = \frac{1}{2}p(p+3)-m(m+2)$ degrees of freedom. The exact distribution in the case $m = 2$, $u = 2$ obtained by Votaw (1948) is as follows:

$$\text{Prob}\ (\bar{L}_{1(mvc)} \leqslant (x)) = I_{\sqrt{x}}(n-4,\ 3) \text{ when } u = 2, m = 2. \qquad \ldots \ (6.7)$$

The $t$-th moment of $\bar{L}_{1(mvc)}$ as derived by Votaw (1948) is :

$$E(\bar{L}^t_{1(mvc)}) = \prod_{\alpha=1}^{m} \prod_{i_\alpha=1}^{u-1} \frac{\left(\frac{n-m-i_\alpha-(u-1)(\alpha-1)}{2}\right)_t}{\left(\frac{n}{2}+\frac{1-\alpha}{2(u-1)}+\frac{i_\alpha-1}{u-1}\right)_t} \qquad \ldots \ (6.8)$$

This is of the form (2.1) and the values of the constants $a, r, a_2$ in the asymptotic expansion (2.4) are presented in columns (7)—(9) of Table 3 for $m = 2$, $u = 3, 4$, $m = 3$, $u = 2$ and $m = 4$, $u = 2$.

TABLE 1. VALUES OF CONSTANTS $a$, $r$, $a_2$ FOR $L_t$, $L_{1(mvc)}$ AND $L_{1(vc)}$ CRITERIA

| | | | independence $L_t$ | | | compound symmetry | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | $L_{1(mvc)}$ | | | $L_{1(vc)}$ | | |
| p | m | partition | $a$ | $r$ | $a_2$ | $a$ | $r$ | $a_2$ | $a$ | $r$ | $a_2$ |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) |
| 3 | 2 | 2,1 | | | | | | | | | |
| | 3 | 1,1,1 | 1.83333 | 3 | -.02083 | • | • | • | • | • | • |
| 4 | 2 | 3,1 | | | | | | | | | |
| | 2 | 2,2 | | | | 2.78571 | 7 | 1.41064 | 3.30000 | 5 | .38750 |
| | 3 | 2,1,1 | 2.30000 | 5 | .38750 | | | | | | |
| | 4 | 1,1,1,1 | 2.16667 | 6 | .45833 | • | • | • | • | • | • |
| 5 | 2 | 4,1 | | | | 2.70635 | 14 | 6.86486 | 3.21717 | 11 | 3.48141 |
| | 2 | 3,2 | | | | 2.88462 | 13 | 5.45673 | 3.40000 | 10 | 2.60000 |
| | 3 | 3,1,1 | 2.78571 | 7 | 1.41064 | | | | | | |
| | 3 | 2,2,1 | 2.75000 | 8 | 1.37500 | 3.27778 | 9 | 3.32630 | 3.78571 | 7 | 4.86606 |
| | 4 | 2,1,1,1 | 2.61111 | 9 | 1.65972 | | | | | | |
| | 5 | 1,1,1,1,1 | 2.50000 | 10 | 1.87500 | • | • | • | • | • | • |
| 6 | 2 | 5,1 | | | | 2.96429 | 21 | 14.36830 | 3.48529 | 17 | 8.24908 |
| | 2 | 4,2 | | | | 3.00444 | 20 | 12.62207 | 3.61806 | 16 | 7.08314 |
| | 2 | 3,3 | 3.50000 | 9 | 2.43750 | 3.10000 | 20 | 12.45000 | 3.62500 | 16 | 6.03750 |
| | 3 | 4,1,1 | 3.27778 | 9 | 3.32639 | 3.10035 | 17 | 11.90778 | 3.70035 | 14 | 6.86486 |
| | 3 | 3,2,1 | 3.22727 | 11 | 3.35796 | 3.37500 | 16 | 9.93750 | 3.86462 | 13 | 5.45673 |
| | 3 | 2,2,2 | 3.16667 | 12 | 3.41667 | 3.56667 | 15 | 7.70583 | 4.08333 | 12 | 3.97917 |
| | 4 | 3,1,1,1 | 3.08333 | 12 | 3.97917 | • | • | • | • | • | • |
| | 4 | 2,2,1,1 | 3.03846 | 13 | 3.99519 | 3.77273 | 11 | 6.35795 | 4.27778 | 9 | 3.32630 |
| | 5 | 2,1,1,1,1 | 2.92857 | 14 | 4.48214 | | | | | | |
| | 6 | 1,1,1,1,1,1 | 2.83333 | 15 | 4.89583 | • | • | • | • | • | • |
| 7 | 2 | 6,1 | | | | 3.24483 | 29 | 26.16543 | 3.77500 | 24 | 16.29625 |
| | 2 | 5,2 | | | | 3.34821 | 28 | 24.02623 | 3.88043 | 23 | 14.79280 |
| | 2 | 4,3 | 4.00000 | 12 | 5.00000 | 3.35317 | 28 | 23.79354 | 3.88647 | 23 | 14.59256 |
| | 3 | 5,1,1 | 3.77273 | 11 | 6.35796 | 3.45000 | 25 | 23.10938 | 3.96429 | 21 | 14.36830 |
| | 3 | 4,2,1 | 3.71429 | 14 | 6.71429 | 3.57870 | 24 | 20.65728 | 4.00444 | 20 | 12.62207 |
| | 3 | 3,3,1 | 3.70000 | 15 | 6.66250 | 3.58333 | 24 | 20.45833 | 4.10000 | 20 | 12.45000 |
| | 3 | 3,2,2 | 3.62500 | 16 | 6.93750 | 3.71739 | 23 | 18.04076 | 4.23684 | 19 | 10.73355 |
| | 4 | 4,1,1,1 | 3.66667 | 15 | 7.70583 | 3.69444 | 20 | 19.25540 | 4.19935 | 17 | 11.99778 |
| | 4 | 3,2,1,1 | 3.50000 | 17 | 7.93750 | 3.86842 | 19 | 16.41776 | 4.37500 | 16 | 9.93750 |
| | 4 | 2,2,2,1 | 3.44444 | 18 | 8.11111 | 4.05556 | 18 | 13.48611 | 4.56667 | 15 | 7.70583 |
| | 5 | 3,1,1,1,1 | 3.38889 | 18 | 8.81944 | | | | | | |
| | 5 | 2,2,1,1,1 | 3.34211 | 19 | 8.94408 | 4.26923 | 13 | 10.76442 | 4.77273 | 11 | 6.35795 |
| | 6 | 2,1,1,1,1,1 | 3.25000 | 20 | 9.68750 | | | | | | |
| | 7 | 1,1,1,1,1,1,1 | 3.16667 | 21 | 10.35417 | • | • | • | • | • | • |
| 8 | 2 | 7,1 | | | | 3.53801 | 38 | 43.58330 | 4.07630 | 32 | 28.78665 |
| | 2 | 6,2 | | | | 3.62432 | 37 | 40.09453 | 4.16452 | 31 | 26.89024 |
| | 2 | 5,3 | 4.50000 | 15 | 9.06250 | 3.62838 | 37 | 40.72255 | 4.16933 | 31 | 26.63272 |
| | 2 | 4,4 | 4.50000 | 16 | 9.00000 | 3.62913 | 37 | 40.67215 | 4.17025 | 31 | 26.60870 |
| | 3 | 6,1,1, | 4.26923 | 13 | 10.76442 | 3.72353 | 34 | 39.65029 | 4.24483 | 29 | 26.16543 |
| | 3 | 5,2,1 | 4.20588 | 17 | 11.81085 | 3.82576 | 33 | 36.74933 | 4.34821 | 28 | 24.02623 |
| | 3 | 4,3,1 | 4.18421 | 19 | 11.83882 | 3.82097 | 33 | 36.48370 | 4.35317 | 28 | 23.79354 |
| | 3 | 4,2,2 | 4.10000 | 20 | 12.45000 | 3.93403 | 32 | 33.68740 | 4.45885 | 27 | 21.74551 |
| | 3 | 3,3,2 | 4.07143 | 21 | 12.47321 | 3.93750 | 32 | 33.46875 | 4.46296 | 27 | 21.55324 |
| | 4 | 5,1,1,1 | 4.05556 | 18 | 13.48611 | 3.93966 | 29 | 34.07360 | 4.45000 | 25 | 23.10938 |
| | 4 | 4,2,1,1 | 3.97619 | 21 | 13.90702 | 4.06746 | 28 | 31.69037 | 4.57870 | 24 | 20.65728 |
| | 4 | 3,3,1,1 | 3.95455 | 22 | 13.98864 | 4.07143 | 28 | 31.46429 | 4.58333 | 24 | 20.45833 |
| | 4 | 3,2,2,1 | 3.89130 | 23 | 14.43207 | 4.20370 | 27 | 28.21991 | 4.71739 | 23 | 17.87409 |
| | 4 | 2,2,2,2 | 3.83333 | 24 | 14.83333 | 4.34615 | 26 | 24.72115 | 4.86364 | 22 | 15.39773 |
| | 5 | 4,1,1,1,1 | 3.86364 | 22 | 15.30773 | 4.19082 | 23 | 29.01285 | 4.69444 | 20 | 19.25540 |
| | 5 | 3,2,1,1,1 | 3.79167 | 24 | 15.73958 | 4.30364 | 23 | 25.27273 | 4.86842 | 19 | 16.41776 |
| | 5 | 2,2,2,1,1 | 3.74000 | 25 | 16.07750 | 4.54762 | 21 | 21.42560 | 5.05556 | 18 | 13.48611 |
| | 6 | 3,1,1,1,1,1 | 3.70000 | 25 | 16.93750 | | | | | | |
| | 6 | 2,2,1,1,1,1 | 3.65385 | 26 | 17.22115 | 4.76667 | 15 | 16.79583 | 5.26923 | 13 | 10.76442 |
| | 7 | 2,1,1,1,1,1,1 | 3.57407 | 27 | 18.27546 | | | | | | |
| | 8 | 1,1,1,1,1,1,1,1 | 3.50000 | 28 | 19.25000 | • | • | • | • | • | • |

TABLE 2. VALUES OF $a$, $r$, $a_2$ FOR $L_s$, $L_{vc}$ AND $L_{mvc}$ CRITERIA

| $p$ | spherical symmetry $L_s$ | | | equality of means, of variances and of covariances $L_{vc}$ | | | $L_{mvc}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $a$ | $r$ | $a_2$ | $a$ | $r$ | $a_2$ | $a$ | $r$ | $a_2$ |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
| 3 | 1.27778 | 5 | 0.45910 | | | | | | |
| 4 | 1.58333 | 9 | 1.85038 | 2.73611 | 8 | 1.47184 | 2.21718 | 11 | 3.48140 |
| 5 | 1.90000 | 14 | 4.86500 | 3.01023 | 13 | 4.24880 | 2.48530 | 17 | 8.24908 |
| 6 | 2.22222 | 20 | 10.30864 | 3.32105 | 19 | 9.41040 | 2.77500 | 24 | 16.29624 |
| 7 | 2.54762 | 27 | 19.19005 | 3.63248 | 26 | 17.95537 | 3.07638 | 32 | 28.78664 |
| 8 | 2.87500 | 35 | 32.67378 | 3.94038 | 34 | 31.04982 | 3.38502 | 41 | 47.05200 |

TABLE 3. VALUES OF $a$, $r$, $a_2$ FOR $\bar{L}_{1(vc)}$ AND $\bar{L}_{1(mvc)}$ CRITERIA

| $p$ | $m$ | partition | compound symmetry $\bar{L}_{1(vc)}$ | | | $\bar{L}_{1(mvc)}$ | | |
|---|---|---|---|---|---|---|---|---|
| | | | $a$ | $r$ | $a_2$ | $a$ | $r$ | $a_2$ |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) |
| 6 | 2 | (3,3) | 3.75000 | 15 | 30.34896 | 3.22308 | 19 | 44.34367 |
| 6 | 3 | (2,2,2) | 4.50000 | 9 | 2.43750 | 4.00000 | 12 | 6.00000 |
| 8 | 2 | (4,4) | 4.25926 | 30 | 89.72119 | 3.71605 | 36 | 37.54596 |
| 8 | 4 | (2,2,2,2) | 5.50000 | 16 | 9.00000 | 6.00000 | 20 | 15.00000 |

An asterisk in the above tables indicate that the hypothesis is not defined, a blank that the exact distribution is available in closed form.

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