Wavelet Analysis on Local Fields of Positive Characteristic

Qaiser Jahan

Thesis Supervisor: Dr. Biswaranjan Behera

Thesis Submitted To The Indian Statistical Institute

In Partial Fulfillment Of The Requirements

For The Award Of The Degree Of

Doctor Of Philosophy

In Mathematics

Stat Math Unit

Indian Statistical Institute

Kolkata-700108

India

July 2014

Dedicated

To

My Parents

Acknowledgements

First and foremost I would like to offer my sincere gratitude to my thesis advisor Dr. Biswaranjan Behera for his excellent guidance and constant support. He helped me in each and every step during my research life and made this thesis work successful. I am indebted to him for his expert supervision and to correct all my mistakes. I specially appreciate his patience which taught me how to handle the situation even in the worst scenario. I would also like to mention his family's emotional support which gave me strength all the time.

I am also grateful to Rudra sir and Swagato sir. They made my life easy in this institute. Their unconditional support is unforgettable. I would like to acknowledge Prof. S. C. Bagchi, whose mere presence in ISI was a big motivation for me. His softness and proclivity to work gave me a lesson to improve myself. His unfortunate death was a big loss not only to me but to whole ISI.

I would also like to thank the Dean and the Director of ISI for providing me the necessary facilities for my Ph.D. in the institute. I extend my thank to all members of Stat-Math unit for their cooperation during this period.

My sincere gratitude to Prof. S. Thangavelu, Prof. S. Madan and Prof. P. Mohanty for their valuable suggestions and motivation. I am grateful to the teachers who have taught me during my Ph.D. course work and encouraged me to think widely. I sincerely thank Prof. Amites Dasgupta, Shibuda, Ashisda, Prof. Amartya Dutta, Prof. K. Sreenadh, Prof. Anima Nagar, Prof. R. K. Sharma.

I am also thankful to my college teachers S. S. Tripathi sir, S. Pathak sir and Ashok Pandey sir to introduce the basic mathematics. I am deeply indebted to my university teachers Late Prof. S. C. Shrivastava, Prof. R. P. Shukla and Ramji Lal sir for their fruitful discussions and making us feel what research is all about. I would also like to acknowledge my teachers Nepu bhaiya and Geetudi for teaching me the difference between right and wrong path during my teenage years.

Friends are always an integrated part of our life. Without their support and help our journey of life is impossible. First I would like to thank my school friends Shayna, Nazia, Mahejabeen and Nabeela to be with me during my good and bad times. I sincerely thank my friends Ambuj, Ruchi, Shashank and Sunil. They are my companion since college. I cannot pen down their value in my life. I am thankful to Ritu, Sedigheh, Debasmita, Somindu, Sutriptadi, Sriparna and

Fouzia to make my hostel life beautiful. I am also thankful to Swagatadi, Kavita, Sweta, Shilpa, Suparna, Rajat, Buddha, Debasis, Radhe, Biswarupda, Shubhabrata, Subhajit, Palash, Koushik, Zinna, Ayan, Monika, Munmun, Neena, Mithun and many others for making my research life stress free.

I am thankful to the Council of Scientific and Industrial Research for financial support during my Ph.D. I thank the National Board for Higher Mathematics (NBHM) for providing me the travel support to attend a conference at University of Alberta, Canada in 2011. I am also thankful to CIMPA for providing me fund for CIMPA school in Argentina, 2013. I am grateful to Indian Statistical Institute for providing me the full financial support to attend conferences at Vanderbilt University, USA and Strol14, Austria, 2014. These conferences were very useful for my research work. I could meet many experts in my area and could discuss about the recent research works.

I would like to mention the everlasting support to my family, specially to my late mother and father who allowed me to choose my way by my own. This thesis is dedicated to them. Special thank to my brothers Md. Obaid, Uzair Khan and their family. They are always ready to help me in all situation. My love and regard to my sisters Qutub, Noor, Qamar, Aftab and my brothers Dilshad, Naushad. Without their help and support I cannot imagine my life.

Last but not least, I thank God for the uncountable blessings, the opportunities and strength He showered on me which make my life easy.

Contents

Chapter 1

Introduction

In this chapter, we will give a brief history of wavelet analysis on $\mathbb{R}$. We will also list some basic results on local fields which will be used in subsequent chapters.

1.1 Wavelets on $\mathbb{R}$

We first start with a brief history of wavelets and some basic definitions and results concerning the orthonormal wavelets on $\mathbb{R}$.

1.1.1 A brief history

In the last few decades wavelet theory has grown extensively and has drawn great attention not only in mathematics but also in engineering, physics, computer science and many other fields. In signal and image processing, wavelets play a very important role.

In 1910, A. Haar gave the first example of an orthonormal wavelet on $\mathbb{R}$ but because of the poor frequency localization of the resulting orthonormal basis, they are not of much use in practice. In 1981, while trying to further understand the Hardy spaces, Strömberg [71] obtained a wavelet of $L^2(\mathbb{R})$ by modifying a basis constructed earlier by Franklin in 1927. We refer to [75] for a detailed discussion of the Strömberg wavelet. In the early eighties, Morlet introduced the continuous wavelet transform. Grossman obtained an inversion formula for this transform and along with Morlet explored several applications. Meyer [64] constructed an example of an infinitely differentiable wavelet such that its Fourier transform also had this property. This

construction was generalized to higher dimensions by Lemarié and Meyer [58]. The concept of multiresolution analysis (MRA) was developed by Meyer and Mallat [63, 65]. Daubechies used this concept to construct compactly supported wavelets with arbitrarily high, but fixed, regularity.

The wavelets have poor frequency localization. To overcome this disadvantage, Coifman, Meyer and Wickerhauser [27] constructed wavelet packets from a wavelet associated with an MRA. Cohen, Daubechies and Feauveau in [25] introduced the concept of biorthogonal wavelets. We will discuss these concepts in details in subsequent chapters.

Wavelets and multiresolution analyses were also studied extensively in the higher dimensional cases $\mathbb{R}^n$, see [20, 30, 42, 62, 65, 75] and references therein. The concept of wavelet has been extended to many different setups by several authors. Dahlke [29] introduced it on locally compact abelian groups (see also [32, 43]). It was generalized to abstract Hilbert spaces by Han, Larson, Papadakis and Stavropoulos [39, 70]. Lemarié [56] extended this concept to stratified Lie groups. Recently, R. L. Benedetto and J. J. Benedetto [11] developed a wavelet theory for local fields and related groups. In [12], R. L. Benedetto proved that Haar and Shannon wavelets exist and, in fact, both are the same for such a group.

1.1.2 Basic concepts of wavelets

In this section we will discuss some basic definitions and results which are useful in the theory of wavelets.

Definition 1.1.1. A collection $\{g_n : n \in \mathbb{Z}\}$ of functions in $L^2(\mathbb{R})$ is called an orthonormal system if it satisfies

$$\langle g_m, g_n \rangle = \delta_{m,n}, \qquad m, n \in \mathbb{Z},$$

where

$$\delta_{m,n} = \begin{cases} 1, & if\ m = n, \\ 0, & if\ m \neq n. \end{cases}$$

The inner product is defined by

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx \quad \text{for } f, g \in L^2(\mathbb{R}).$$

Definition 1.1.2. The Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} dx, \quad \xi \in \mathbb{R}.$$

We can define the Fourier transform in $L^2(\mathbb{R})$ by defining it on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and extending it to $L^2(\mathbb{R})$ by using the fact that $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ in the $L^2(\mathbb{R})$-norm. The Plancherel theorem is of the following form:

$$\langle f, g \rangle = \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle, \quad f, g \in L^2(\mathbb{R}).$$

We have the following necessary and sufficient condition for a system of integer translates of a function to be an orthonormal system. The proof is well known, see e.g. [42].

Proposition 1.1.3. *If $g \in L^2(\mathbb{R})$, then $\{g(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal system if and only if*

$$\sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2 = 1 \quad \textit{for a.e. } \xi \in \mathbb{R}. \tag{1.1.1}$$

A complete orthonormal system is called an orthonormal basis.

Two simple operators acting on functions defined on $\mathbb{R}$ play an important role in the theory of wavelets. These are the translation and the dilation operators. The translation operator τ_k and the dilation operator δ_j are defined on $L^2(\mathbb{R})$ as follows:

$$\tau_k f(x) = f(x - k) \quad \text{and} \quad \delta_j f(x) = 2^{j/2} f(2^j x), \quad x \in \mathbb{R}, f \in L^2(\mathbb{R}) \text{ and } j, k \in \mathbb{Z}.$$

Definition 1.1.4. An *orthonormal wavelet* on $\mathbb{R}$ is a function $\psi \in L^2(\mathbb{R})$ such that the system of functions $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(\mathbb{R})$, where

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k), \quad j, k \in \mathbb{Z}.$$

Observe that

$$(\psi_{j,k})^\wedge(\xi) = 2^{-j/2} e^{-i2^{-j}k\xi} \hat{\psi}(2^{-j}\xi), \quad \xi \in \mathbb{R}, \quad j, k \in \mathbb{Z}.$$

The oldest example of such a basis is the Haar basis. Another simple example is the Shannon

wavelet. The Haar wavelet is compactly supported in the time domain but not in the frequency domain whereas the Shannon wavelet is compactly supported in the frequency domain but not in the time domain.

Example 1.1.5. The Haar wavelet is defined by

$$\psi(x) = \begin{cases} 1, & 0 \le x < 1/2, \\ -1, & 1/2 \le x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Example 1.1.6. The Shannon wavelet is defined in terms of the Fourier transform:

$$\hat{\psi} = \mathbf{1}_E, \quad E = [-2\pi, -\pi] \cup [\pi, 2\pi],$$

where $\mathbf{1}_E$ denote the characteristic function of E.

1.1.3 Multiresolution analysis

We will discuss about the construction of orthonormal wavelets from a multiresolution analysis (MRA). The concept of MRA is given by Y. Meyer and S. Mallat [63, 65].

Definition 1.1.7. A multiresolution analysis is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ satisfying the following properties:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$;

(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(d) $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(e) there is a function $\varphi \in L^2(\mathbb{R})$, called the *scaling function*, such that $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ forms an orthonormal basis for V_0.

An example of the spaces $\{V_j : j \in \mathbb{Z}\}$ satisfying (a)-(e) above is

$$V_j = \{f \in L^2(\mathbb{R}) : f|_{[2^{-j}k, 2^{-j}(k+1))} \text{ is constant for all } k \in \mathbb{Z}\}, \quad j \in \mathbb{Z}.$$

This is called the Haar MRA.

We will now briefly discuss how to construct a wavelet from an MRA. Suppose that $\{V_j : j \in \mathbb{Z}\}$ is an MRA with scaling function φ. Since $\varphi \in V_0 \subset V_1$ and $\{\varphi_{1,k} : k \in \mathbb{Z}\}$ is an orthonormal basis of V_1, there exists $\{c_k : k \in \mathbb{Z}\} \in \ell^2(\mathbb{Z})$ such that

$$\varphi(x) = \sum_{k\in\mathbb{Z}} c_k 2^{1/2} \varphi(2x - k).$$

Taking Fourier transform of both sides, we get

$$\hat{\varphi}(\xi) = \sum_{k\in\mathbb{Z}} c_k 2^{-\frac{1}{2}} e^{-ik\frac{\xi}{2}} \hat{\varphi}(\xi/2) = m_0(\xi/2)\hat{\varphi}(\xi/2), \tag{1.1.2}$$

where $m_0(\xi) = 2^{-\frac{1}{2}} \sum_{k\in\mathbb{Z}} c_k e^{-ik\xi}$. The function m_0 is 2π-periodic and is in $L^2(\mathbb{T})$, where $\mathbb{T} = [0, 2\pi]$. It is called the *low-pass filter* associated with the scaling function φ. Using (1.1.2) in (1.1.1), and splitting the sum over $\mathbb{Z}$ into sums over even and odd integers, we get

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{T}. \tag{1.1.3}$$

Let W_j be the orthogonal complement of V_j in V_{j+1}, i.e., $V_{j+1} = V_j \oplus W_j$. By properties (b) and (c) in the definition of an MRA, we have a decomposition

$$L^2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j. \tag{1.1.4}$$

If we can find a function $\psi \in W_0$ such that $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for W_0 then $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ forms an orthonormal basis for W_j by property (d) of MRA. It is clear from the decomposition (1.1.4) that $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(\mathbb{R})$. It turns out that the wavelet ψ can be expressed in terms of the low-pass filter and the Fourier transform of the scaling function. In fact, we have

$$\hat{\psi}(\xi) = e^{i\frac{\xi}{2}} \nu(\xi) \overline{m_0(\xi/2 + \pi)} \hat{\varphi}(\xi/2) \quad \text{for a.e. } \xi \in \mathbb{R}, \tag{1.1.5}$$

where ν is a 2π-periodic measurable function such that $|\nu(\xi)| = 1$ for a.e. ξ in $\mathbb{T}$. We refer to Chapter 2 of [42] and [75] for the details.

A wavelet associated with an MRA described as above is called an MRA-wavelet. Haar and Shannon wavelets are examples of MRA-wavelets.

Two simple equations characterize all wavelets of $L^2(\mathbb{R})$. A function $\psi \in L^2(\mathbb{R})$, with $\|\psi\|_2 = 1$, is a wavelet for $L^2(\mathbb{R})$ if and only if

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{1.1.6}$$

$$\sum_{j\geq 0} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi+2q\pi))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R} \text{ and for all } q \in 2\mathbb{Z}+1. \tag{1.1.7}$$

For the details, we refer to Chapter 7 of [42]. Not all wavelets are associated with an MRA. The first example of a non-MRA wavelet was given by J. L. Journé. Another interesting example is given by Lemarié. The Journé wavelet ψ_J is defined in terms of Fourier transform

$$\hat{\psi}_J = \mathbf{1}_J,$$

where

$$J = [-\tfrac{32}{7}\pi, -4\pi] \cup [-\pi, -\tfrac{4}{7}\pi] \cup [\tfrac{4}{7}\pi, \pi] \cup [4\pi, \tfrac{32}{7}\pi],$$

and Lemarié wavelet ψ_L is defined by

$$\hat{\psi}_L = \mathbf{1}_L,$$

where

$$L = [-\tfrac{8}{7}\pi, -\tfrac{4}{7}\pi] \cup [\tfrac{4}{7}\pi, \tfrac{6}{7}\pi] \cup [\tfrac{24}{7}\pi, \tfrac{32}{7}\pi].$$

We have seen that the scaling function plays an important role in the construction of wavelet from an MRA. The characterization of scaling functions is given by the following theorem.

Theorem 1.1.8. *A function $\varphi \in L^2(\mathbb{R})$ is a scaling function for a multiresolution analysis of $L^2(\mathbb{R})$ if and only if*

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi+2k\pi)|^2 = 1 \quad \textit{for a.e. } \xi \in \mathbb{T}. \tag{1.1.8}$$

$$\lim_{j\to\infty} |\hat{\varphi}(2^{-j}\xi)| = 1 \quad for\ a.e.\ \xi \in \mathbb{R}, \tag{1.1.9}$$

and there exists a 2π*- periodic function* m_0 *in* $L^2(\mathbb{T})$ *such that*

$$\hat{\varphi}(\xi) = m_0(\xi/2)\hat{\varphi}(\xi/2) \quad for\ a.e.\ \xi \in \mathbb{R}. \tag{1.1.10}$$

1.1.4 Dimension function

There is a single equation which tells us when a wavelet is an MRA-wavelet. From (1.1.2), (1.1.3) and (1.1.5), we get

$$|\hat{\varphi}(2\xi)|^2 + |\hat{\psi}(2\xi)|^2 = \Big(|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2\Big)|\hat{\varphi}(\xi)|^2 = |\hat{\varphi}(\xi)|^2.$$

Iterating this equation, we obtain

$$|\hat{\varphi}(\xi)|^2 = |\hat{\varphi}(2^N\xi)|^2 + \sum_{j=1}^{N} |\hat{\psi}(2^j\xi)|^2 \quad \text{for all } N \geq 1.$$

Since, $|\hat{\varphi}(\xi)| \leq 1$, the sequence $\left\{\sum_{j=1}^{N} |\hat{\psi}(2^j\xi)|^2 : N \geq 1\right\}$ is an increasing sequence of real numbers, which is bounded by 1 so that $\lim_{N\to\infty} \sum_{j=1}^{N} |\hat{\psi}(2^j\xi)|^2$ exists. Therefore, $\lim_{N\to\infty} |\hat{\varphi}(2^N\xi)|^2$ also exists. Hence, by using Fatou's lemma, we can show that $\lim_{N\to\infty} |\hat{\varphi}(2^N\xi)|^2 = 0$, as

$$\begin{aligned}\int_{\mathbb{R}} \lim_{N\to\infty} |\hat{\varphi}(2^N\xi)|^2 d\xi &\leq \lim_{N\to\infty} \int_{\mathbb{R}} |\hat{\varphi}(2^N\xi)|^2 d\xi \\ &= \lim_{N\to\infty} \frac{1}{2^N} \int_{\mathbb{R}} |\hat{\varphi}(\xi)|^2 d\xi.\end{aligned}$$

Hence,

$$|\hat{\varphi}(\xi)|^2 = \sum_{j\geq 1} |\hat{\psi}(2^j\xi)|^2 \quad \text{for all } \xi \in \mathbb{R}.$$

Using Proposition 1.1.3, we get for a.e. $\xi \in \mathbb{R}$,

$$1 = \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = \sum_{j\geq 1} \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2 \equiv D_\psi(\xi). \tag{1.1.11}$$

We call D_ψ the *dimension function* of the wavelet ψ. Thus, we observed that if a wavelet is associated with an MRA, then its dimension function is equal to 1 a.e. Gripenberg and Wang have proved independently that this condition is also sufficient. For a proof of the following theorem, we refer to Chapter 7 of [42].

Theorem 1.1.9. *A wavelet ψ is an MRA-wavelet if and only if $D_\psi(\xi) = 1$ for a.e. $\xi \in \mathbb{R}$.*

1.2 Local fields

Before we discuss about a general local field, we provide the construction of p-adic and p-series fields, where p is a prime.

For an integer $n \in \mathbb{Z}$, we define its p-adic norm $|\cdot|_p$ as follows. If $n = 0$, then $|n|_p = 0$. If $n \neq 0$, then we write $n = p^k l$, where k and l are integers and l is relatively prime to p, and define $|n|_p = p^{-k}$. We can easily verify that $|\cdot|_p$ is a norm on the integers which satisfies the stronger triangle inequality $|m+n|_p \leq \max\{|m|_p, |n|_p\}$ for $m, n \in \mathbb{Z}$. If we use the usual arithmetic for the integers and define a metric by $d(m,n) = |m-n|_p$, then $\mathbb{Z}$ is a metric space which is not complete. Its completion is called the p-adic integers. Its field of quotients is called the p-adic numbers. The p-adic numbers can also be obtained directly by extending the definition of the p-adic norm to the rational numbers in a natural way (i.e., write $\frac{m}{n} = p^k \frac{r}{s}$ with r and s relatively prime to p and define $|\frac{m}{n}|_p = p^{-k}$) and then completing the rationals as a metric space with respect to the induced metric. In either case, we obtain a totally disconnected locally compact topological field of characteristic zero.

The elements of this field are identified as formal Laurent series:

$$x = \sum_{k=l}^{\infty} a_k p^k,$$

where $a_k \in \{0, 1, 2, \cdots, p-1\}$ and we carry in the arithmetic. For example, let $p = 7$, $x = 3 + 4.7 + 3.7^2$ and $y = 5 + 3.7 + 5.7^2$, then $x + y = 1 + 1.7 + 2.7^2$.

This field is called the *p-adic field* and is denoted by $\mathbb{Q}_p$.

Now, we again consider the same set of formal Laurent series, but do the addition and multiplication modulo p. For example, if $p = 7$, $x = 3 + 4.7 + 3.7^2$ and $y = 5 + 3.7 + 5.7^2$, then $x + y = 1 + 1.7^2$. If we use the same norm and metric as used for $\mathbb{Q}_p$, then we obtain

another totally disconnected locally compact topological field. This field is of characteristic p and is called the *p-series field*.

We will now discuss about the general local fields.

Let K be a field and a topological space. Then K is called a *locally compact field* or a *local field* if both K^+ and K^* are locally compact abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K respectively.

If K is any field and is endowed with the discrete topology, then K is a local field. So we will only consider non-discrete fields. Further, if a local field K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. If the characteristic of K is positive, then it is a field of formal power series over a finite field $GF(p^c)$. If $c = 1$, then it is a p-series field and if $c \neq 1$, then K is an algebraic extension of degree c of a p-series field. If K is of characteristic zero, then K is either $\mathbb{Q}_p$ for some p or a finite algebraic extension of such a field. So, by a local field, we mean a field K which is locally compact, non-discrete and totally disconnected. We refer to Theorem 4.12 in [66] for a proof of the classification of local fields.

We use the notation of the book by M. H. Taibleson [72]. Proofs of all the results stated in this section can be found in the books [72] and [66].

Let K be a local field. Since K^+ is a locally compact abelian group, we choose a Haar measure dx for K^+. If $\alpha \neq 0, \alpha \in K$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$. We call $|\alpha|$ the *absolute value* or *valuation* of α. We also let $|0| = 0$.

The map $x \to |x|$ has the following properties:

(a) $|x| = 0$ if and only if $x = 0$;

(b) $|xy| = |x||y|$ for all $x, y \in K$;

(c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.

Property (c) is called the *ultrametric inequality*. It follows from this property that

$$|x + y| = \max\{|x|, |y|\} \quad \text{if } |x| \neq |y|. \tag{1.2.1}$$

The set $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is called the *ring of integers* in K. It is the unique maximal compact subring of K. Define $\mathfrak{P} = \{x \in K : |x| < 1\}$. The set $\mathfrak{P}$ is called the *prime ideal* in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$. It is principal and prime.

Since K is totally disconnected, the set of values $|x|$, as x varies over K, is a discrete set of the form $\{s^k : k \in \mathbb{Z}\} \cup \{0\}$ for some $s > 0$. Hence, there is an element of $\mathfrak{P}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{P}$. Such an element is called a *prime element* of K. Note that as an ideal in $\mathfrak{D}$, $\mathfrak{P} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$.

It can be proved that $\mathfrak{D}$ is compact and open. Hence, $\mathfrak{P}$ is compact and open. Since $\mathfrak{P}$ is compact, $\mathfrak{D}/\mathfrak{P}$ is compact. Since $\mathfrak{P}$ is open, $\mathfrak{D}/\mathfrak{P}$ is discrete. Also, since $\mathfrak{P}$ is a maximal ideal in $\mathfrak{D}$, we have $\mathfrak{D}/\mathfrak{P}$ is a field. Thus, $\mathfrak{D}/\mathfrak{P}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$.

For a measurable subset E of K, let $|E| = \int_K \mathbf{1}_E(x)dx$, where $\mathbf{1}_E$ is the characteristic function of E and dx is the Haar measure of K normalized so that $|\mathfrak{D}| = 1$. Then, it is easy to see that $|\mathfrak{P}| = q^{-1}$ and $|\mathfrak{p}| = q^{-1}$. We can decompose $\mathfrak{D}$ into q cosets of $\mathfrak{P}$. Thus $1 = |\mathfrak{D}| = q|\mathfrak{P}|$, this gives $|\mathfrak{P}| = q^{-1}$. Since $\mathfrak{P} = \mathfrak{p}\mathfrak{D}$, we have $|\mathfrak{p}| = q^{-1}$. It follows that if $x \neq 0$, and $x \in K$, then $|x| = q^k$ for some $k \in \mathbb{Z}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{P} = \{x \in K : |x| = 1\}$. $\mathfrak{D}^*$ is the group of units in K^*. If $x \neq 0$, we can write $x = \mathfrak{p}^k x'$, with $x' \in \mathfrak{D}^*$. Let $\mathfrak{P}^k = \mathfrak{p}^k\mathfrak{D} = \{x \in K : |x| \leq q^{-k}\}$, $k \in \mathbb{Z}$. These are called the *fractional ideals*. Each $\mathfrak{P}^k$ is compact and open and is a subgroup of K^+ (see [66]).

If K is a local field, then there is a nontrivial, unitary, continuous character χ on K^+. It can be proved that K^+ is self dual (see [72]). The existence of such a character follows from the Pontryagin duality theorem.

Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is nontrivial on $\mathfrak{P}^{-1}$. We can find such a character by starting with any nontrivial character and rescaling. We will define such a character for a local field of positive characteristic. For $y \in K$, we define $\chi_y(x) = \chi(yx)$, $x \in K$.

Definition 1.2.1. If $f \in L^1(K)$, then the Fourier transform of f is the function $\hat{f}$ defined by

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\, dx.$$

Note that

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi(\xi x)}\, dx = \int_K f(x)\chi(-\xi x)\, dx.$$

Similar to the standard Fourier analysis on the real line, one can prove the following results.

(a) The map $f \to \hat{f}$ is a bounded linear transformation of $L^1(K)$ into $L^\infty(K)$, and

$$\|\hat{f}\|_\infty \leq \|f\|_1.$$

(b) If $f \in L^1(K)$, then $\hat{f}$ is uniformly continuous.

(c) If $f \in L^1(K) \cap L^2(K)$, then $\|\hat{f}\|_2 = \|f\|_2$.

To define the Fourier transform of functions in $L^2(K)$, we introduce the functions Φ_k. For $k \in \mathbb{Z}$, let Φ_k be the characteristic function of $\mathfrak{P}^k$.

Definition 1.2.2. For $f \in L^2(K)$, let $f_k = f\Phi_{-k}$ and

$$\hat{f}(\xi) = \lim_{k\to\infty} \hat{f}_k(\xi) = \lim_{k\to\infty} \int_{|x|\leq q^k} f(x)\overline{\chi_\xi(x)}\, d\xi,$$

where the limit is taken in $L^2(K)$.

We have the following theorem (see Theorem 2.3 in [72]).

Theorem 1.2.3. *The Fourier transform is unitary on* $L^2(K)$.

A set of the form $h + \mathfrak{P}^k$ will be called a *sphere* with centre h and radius q^{-k}. It follows from the ultrametric inequality that if S and T are two spheres in K, then either S and T are disjoint or one sphere contains the other. Also, note that the characteristic function of the sphere $h + \mathfrak{P}^k$ is $\Phi_k(\cdot - h)$ and that $\Phi_k(\cdot - h)$ is constant on cosets of $\mathfrak{P}^k$.

Definition 1.2.4. The set $\mathcal{S}$ is the space of all finite linear combinations of functions of the form $\Phi_k(\cdot - h)$, $h \in K$, $k \in \mathbb{Z}$. This space is called the *space of testing functions*.

This class of functions can also be described in the following way. A function $g \in \mathcal{S}$ if and only if there exist integers k, l such that g is constant on cosets of $\mathfrak{P}^k$ and is supported on $\mathfrak{P}^l$. It follows that $\mathcal{S}$ is closed under Fourier transform and is an algebra of continuous functions with compact support, which is dense in $C_0(K)$ as well as in $L^p(K), 1 \leq p < \infty$. We have the following theorem.

Theorem 1.2.5. *If $g \in \mathcal{S}$ is constant on cosets of $\mathfrak{P}^k$ and is supported on $\mathfrak{P}^l$, then $\hat{g} \in \mathcal{S}$ is constant on cosets of $\mathfrak{P}^{-l}$ and is supported on $\mathfrak{P}^{-k}$.*

We will use the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let χ_u be any character on K^+. Since $\mathfrak{D}$ is a subgroup of K^+, the restriction $\chi_u|_{\mathfrak{D}}$ is a character on $\mathfrak{D}$. Also, as characters on $\mathfrak{D}$, $\chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. That is, $\chi_u = \chi_v$ if $u+\mathfrak{D} = v+\mathfrak{D}$ and $\chi_u \neq \chi_v$ if $(u+\mathfrak{D}) \cap (v+\mathfrak{D}) = \phi$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ is a list of distinct characters on $\mathfrak{D}$. It is proved in [72] that this list is complete. That is, we have the following proposition.

Proposition 1.2.6. *Let $\{u(n) : n \in \mathbb{N}_0\}$ be a complete list of (distinct) coset representatives of $\mathfrak{D}$ in K^+. Then $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ is a complete list of (distinct) characters on $\mathfrak{D}$. Moreover, it is a complete orthonormal system on $\mathfrak{D}$.*

Given such a list of characters $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$, we define the Fourier coefficients of $f \in L^1(\mathfrak{D})$ as

$$\hat{f}(u(n)) = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}dx.$$

The series $\sum_{n=0}^{\infty} \hat{f}(u(n))\chi_{u(n)}(x)$ is called the *Fourier series* of f. From the standard L^2-theory for compact abelian groups we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\int_{\mathfrak{D}} |f(x)|^2 dx = \sum_{n=0}^{\infty} |\hat{f}(u(n))|^2.$$

These results hold irrespective of the ordering of the characters. We now proceed to impose a natural order on the sequence $\{u(n) : n \in \mathbb{N}_0\}$. Note that $\Gamma = \mathfrak{D}/\mathfrak{P}$ is isomorphic to the finite field $GF(q)$ and $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \epsilon_0, \epsilon_1, \epsilon_2, \cdots, \epsilon_{c-1}\} \subset \mathfrak{D}^*$ such that span $\{\epsilon_0, \epsilon_1, \epsilon_2, \cdots, \epsilon_{c-1}\} \cong GF(q)$. For $n \in \mathbb{N}_0$ such that $0 \leq n < q$, we have

$$n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1}, \quad 0 \leq a_k < p, k = 0, 1, \cdots, c-1.$$

Define

$$u(n) = (a_0 + a_1\epsilon_1 + \cdots + a_{c-1}\epsilon_{c-1})\mathfrak{p}^{-1}. \tag{1.2.2}$$

Now, for $n \geq 0$, write

$$n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s, \quad 0 \leq b_k < q, k = 0, 1, 2, \cdots, s,$$

and define

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{1.2.3}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But

$$u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s) \quad \text{if } r \geq 0, k \geq 0 \text{ and } 0 \leq s < q^k. \tag{1.2.4}$$

For brevity, we will write $\chi_n = \chi_{u(n)}, n \in \mathbb{N}_0$. As mentioned before, $\{\chi_n : n \in \mathbb{N}_0\}$ is a complete set of characters on $\mathfrak{D}$.

Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be a fixed set of coset representatives of $\mathfrak{P}$ in $\mathfrak{D}$. Then every $x \in K$ can be expressed uniquely as

$$x = x_0 + \sum_{k=1}^{n} b_k \mathfrak{p}^{-k}, \quad x_0 \in \mathfrak{D}, b_k \in \mathcal{U}.$$

Let K be a local field of characteristic $p > 0$ and $\epsilon_0, \epsilon_1, \ldots, \epsilon_{c-1}$ be as above. We define a character χ on K as follows (see [76]):

$$\chi(\epsilon_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \cdots, c-1 \text{ or } j \neq 1. \end{cases} \tag{1.2.5}$$

Note that χ is trivial on $\mathfrak{D}$ but nontrivial on $\mathfrak{P}^{-1}$.

We have the following result for χ. We refer to [44] for a proof of this fact.

Proposition 1.2.7. *For all $l, k \in \mathbb{N}_0$, we have $\chi(u(k)u(l)) = 1$.*

In order to be able to define the concepts of MRA and wavelets on local fields, we need analogous notions of translation and dilation. Since $\bigcup_{j \in \mathbb{Z}} \mathfrak{p}^{-j}\mathfrak{D} = K$, we can regard $\mathfrak{p}^{-1}$ as the dilation (note that $|\mathfrak{p}^{-1}| = q$) and since $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representatives of $\mathfrak{D}$ in K, the set $\{u(n) : n \in \mathbb{N}_0\}$ can be treated as the translation set. We make the following definition.

Definition 1.2.8. A finite set $\{\psi_m : m = 1, 2, \cdots, M\} \subset L^2(K)$ is called a *set of basic wavelets* of $L^2(K)$ if the system $\{q^{j/2}\psi_m(\mathfrak{p}^{-j} \cdot -u(k)) : 1 \leq m \leq M, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ forms an orthonormal basis for $L^2(K)$.

As we mentioned earlier, the local fields are essentially of two types (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). The local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$. Albeverio, Kozyrev, Khrennikov, Shelkovich, Skopina and their collaborators have discussed about MRA and wavelets on $\mathbb{Q}_p$ in a series of papers [3, 46, 47, 48, 49, 50, 52]. Khrennikov, Shelkovich and Skopina [50] constructed a number of scaling functions generating an MRA of $L^2(\mathbb{Q}_p)$. But later on in [2], Albeverio, Evdokimov and Skopina proved that all these scaling functions lead to the same Haar MRA and that there exist no other orthogonal test scaling functions generating an MRA of $L^2(\mathbb{Q}_p)$ except those described in [50]. Some wavelet bases for $L^2(\mathbb{Q}_p)$ different from the Haar system were constructed in [31] and [1]. These wavelet bases were obtained by relaxing the basis conditions in the definition of an MRA.

A 2-series field is also known as the Cantor dyadic group and p-series fields are also called the Vilenkin groups. Lang [53, 54, 55] constructed several examples of wavelets for Cantor dyadic group. Farkov [32, 33] has constructed many examples of wavelets for the Vilenkin groups. Several examples of biorthogonal wavelets on the Vilenkin groups were constructed by Farkov in [34] and by Farkov and Rodionov in [35].

1.3 Organization of the thesis

The aim of this thesis is to develop a theory of wavelets on local fields K of positive characteristic. The algebraic structure of such local fields is similar to that of real number field and the translation set $\{u(k) : k \in \mathbb{N}_0\}$ of K is a countable discrete subgroup of K. This is analogous to the fact that the translation set $\mathbb{Z}$ of $\mathbb{R}$ is a countable discrete subgroup of $\mathbb{R}$. But, unlike the real line, it is not true in general that $u(k) + u(l) = u(k + l)$ for nonnegative integers k and l. This problem does not show up in the Euclidean case. We have to deal with issues related to this problem separately.

This thesis is based on the five problems discussed in the articles [6], [7], [8], [9] and [10] .

In Chapter 2 we will discuss about the MRA on local fields of positive characteristic. We will show that it is enough to assume that the discrete translates of a single function in

the core subspace of an MRA forms a Riesz basis instead of an orthonormal basis and show how to construct an orthonormal basis from a Riesz basis. We show that the properties in the definition of an MRA on $L^2(K)$ are not independent. We will prove that the intersection triviality condition in the definition of MRA follows from the rest of the properties of an MRA. The union density condition also follows if we assume that the Fourier transform of the scaling function is continuous at 0. At the end of this chapter we will characterize the scaling functions associated with such an MRA. This work is summarized in the publication [7].

The concept of quasi-affine frame in Euclidean spaces was introduced to obtain translation invariance of the discrete wavelet transform. We have extended this concept to a local field of positive characteristic. We have shown that the affine system generated by a finite number of functions is an affine frame if and only if the corresponding quasi-affine system is a quasi-affine frame. In such a case we have proved that the exact frame bounds are equal. This result is obtained by using the properties of an operator associated with two such affine systems. We also characterize the translation invariance of such an operator. A related concept is that of co-affine system. We prove that $L^2(K)$ does not have any co-affine frame. This is the content of Chapter 3. These results are summarized in [10].

In the case of $L^2(\mathbb{R})$, a wavelet ψ is characterized by two basic equations. That is, a function $\psi \in L^2(\mathbb{R})$, with $\|\psi\|_2 = 1$, is a wavelet of $L^2(\mathbb{R})$ if and only if ψ satisfies (1.1.6) and (1.1.7). Bownik [15] gave a new approach to characterize multiwavelets in $L^2(\mathbb{R}^n)$ by means of basic equations. This result is based on results about shift invariant systems in [15] and quasi-affine systems in [22]. Using the results on affine and quasi-affine frames obtained in Chapter 3, we give a characterization of wavelets on local fields of positive characteristic in Chapter 4. Among all the wavelets, the wavelets those arise from an MRA are characterized by a single equation. A wavelet ψ of $L^2(\mathbb{R})$ is an MRA-wavelet if and only if $D_\psi(\xi) = 1$ for a.e. $\xi \in \mathbb{R}$. In the same chapter, we will discuss an analogous result of MRA-wavelets for the case of local fields of positive characteristic. This work is the content of the article [9].

In Chapter 5, we will discuss about wavelet packets and wavelet frame packets. The concept of wavelet packet was introduced by Coifman, Meyer and Wickerhauser. In the context of a local field K of positive characteristic, we first prove a crucial result called the splitting lemma. Using this lemma, we have constructed the wavelet packets associated with an MRA of such a field. We have shown that these wavelet packets generate an orthonormal basis by translations only.

We then proved an analogue of the splitting lemma for wavelet frames on K and constructed the associated wavelet frame packets. This work is summarized in the publication [6].

The concept of biorthogonal wavelets plays an important role in applications. We have generalized this concept to a local field K of positive characteristic. We prove that if φ and $\tilde{\varphi}$ are the scaling functions of two MRAs $\{V_j : j \in \mathbb{Z}\}$ and $\{\tilde{V}_j : j \in \mathbb{Z}\}$ such that their translates are biorthogonal, then the associated families of wavelets are also biorthogonal. Under mild decay conditions on the scaling functions and the wavelets, we also prove that the wavelets generate Riesz bases for $L^2(K)$. This is the content of Chapter 6 and is summarized in the article [8].

Chapter 2

Multiresolution Analysis On Local Fields Of Positive Characteristic

The concepts of MRA and wavelet can be generalized to a local field K of positive characteristic by using a prime element $\mathfrak{p}$ of such a field. An MRA is a sequence of closed subspaces of $L^2(K)$ satisfying certain properties. In this chapter, we will discuss about the interdependency of the properties in the definition of an MRA. We will also characterize the scaling functions associated with an MRA on such a field.

This chapter is organized as follows. In section 2.1, we define the concept of MRA on local fields K of positive characteristic. We will briefly mention how to get a set of basic wavelets from an MRA and provide an example which is analogous to the Haar wavelets on $\mathbb{R}^n$. It is traditional to define an MRA by specifying the five properties that a family of subspaces must satisfy. One of the properties is that the central space or the core space of an MRA has an orthonormal basis consisting of discrete translates of a single function. We show that it is enough to assume that these translates form only a Riesz basis for this space. We also show how to construct an orthonormal basis for the core space from a Riesz basis. In section 2.2, we show that the properties in the definition of an MRA of $L^2(K)$ are not independent. We will prove that the intersection triviality condition follows from the rest of the properties of MRA. Same is the case for the union density condition, if we assume an additional condition on the scaling function. Finally, in section 2.3, we give necessary and sufficient conditions on a function to be the scaling function for an MRA of $L^2(K)$.

2.1 MRA on local fields

Similar to $\mathbb{R}^n$, wavelets can be constructed from an MRA. We define an MRA on local fields as follows (see [44]):

Definition 2.1.1. Let K be a local field of characteristic $p > 0$, $\mathfrak{p}$ be a prime element of K and $u(n) \in K$ for $n \in \mathbb{N}_0$, be as defined in (1.2.2) and (1.2.3). A multiresolution analysis (MRA) of $L^2(K)$ is a sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(K)$ satisfying the following properties:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(K)$;

(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(d) $f \in V_j$ if and only if $f(\mathfrak{p}^{-1}\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(e) there is a function $\varphi \in V_0$, called the *scaling function*, such that $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms an orthonormal basis for V_0.

Given an MRA $\{V_j : j \in \mathbb{Z}\}$, we define another sequence $\{W_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(K)$ by

$$W_j = V_{j+1} \ominus V_j.$$

These subspaces also satisfy

$$f \in W_j \text{ if and only if } f(\mathfrak{p}^{-1}\cdot) \in W_{j+1},\ j \in \mathbb{Z}. \tag{2.1.1}$$

Moreover, they are mutually orthogonal, and we have the following orthogonal decompositions:

$$L^2(K) = \bigoplus_{j\in\mathbb{Z}} W_j \tag{2.1.2}$$

$$= V_0 \oplus \Big(\bigoplus_{j\geq 0} W_j\Big). \tag{2.1.3}$$

Observe that the dilation is induced by $\mathfrak{p}^{-1}$ and $|\mathfrak{p}^{-1}| = q$. As in the case of $\mathbb{R}^n$, we expect the existence of $q-1$ number of functions $\{\psi_1, \psi_2, \cdots, \psi_{q-1}\}$ to form a set of basic

wavelets. In view of (2.1.1) and (2.1.2), it is clear that if $\{\psi_1, \cdots, \psi_{q-1}\}$ is a set of functions such that $\{\psi_m(\cdot - u(k)) : 1 \leq m \leq q-1, k \in \mathbb{N}_0\}$ forms an orthonormal basis for W_0, then $\{q^{j/2}\psi_m(\mathfrak{p}^{-j} \cdot -u(k)) : 1 \leq m \leq q-1, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ forms an orthonormal basis for $L^2(K)$.

For $f \in L^2(K)$, we define

$$f_{j,k}(x) = q^{j/2} f(\mathfrak{p}^{-j}x - u(k)), \quad j \in \mathbb{Z}, k \in \mathbb{N}_0.$$

Then it is easy to see that

$$\|f_{j,k}\|_2 = \|f\|_2$$

and

$$(f_{j,k})^\wedge(\xi) = q^{-j/2}\overline{\chi_k(\mathfrak{p}^j\xi)}\hat{f}(\mathfrak{p}^j\xi).$$

We will now briefly mention how one can get a set of basic wavelets from an MRA. Let $\{V_j : j \in \mathbb{Z}\}$ be an MRA of $L^2(K)$. Since $\varphi \in V_0 \subset V_1$, and $\{\varphi_{1,k} : k \in \mathbb{N}_0\}$ forms an orthonormal basis of V_1, there exists $\{\alpha_k^0 : k \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$ such that

$$\varphi(x) = \sum_{k \in \mathbb{N}_0} \alpha_k^0 \varphi_{1,k}(x).$$

Taking Fourier transform, we get

$$\hat{\varphi}(\xi) = q^{-1/2} \sum_{k \in \mathbb{N}_0} \alpha_k^0 \overline{\chi_k(\mathfrak{p}\xi)}\hat{\varphi}(\mathfrak{p}\xi) = m_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi), \tag{2.1.4}$$

where $m_0(\xi) = q^{-1/2} \sum_{k \in \mathbb{N}_0} \alpha_k^0 \overline{\chi_k(\xi)}$.

Definition 2.1.2. A function f on K will be called *integral-periodic* if

$$f(x + u(k)) = f(x) \text{ for all } k \in \mathbb{N}_0.$$

The following property of m_0 is shown in [44] (see Proposition 3).

Theorem 2.1.3. *The function m_0 is integral-periodic and is in $L^2(\mathfrak{D})$.*

As in the case of $\mathbb{R}^n$, it can be shown that if we can find integral-periodic functions $m_i, 1 \leq$

$i \leq q-1$, such that the matrix

$$M(\xi) = \Big[m_i\big(\xi + \mathfrak{p}u(j)\big)\Big]_{i,j=0}^{q-1}$$

is unitary for a.e. $\xi \in \mathfrak{D}$, then $\{\psi^1, \psi^2, \ldots, \psi^{q-1}\}$ forms a set of basic wavelets for $L^2(K)$, where

$$\hat{\psi}^i(\xi) = m_i(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi).$$

In other words, if

$$m_i(\xi) = q^{-1/2} \sum_{k \in \mathbb{N}_0} \alpha_k^i \overline{\chi_k(\xi)},$$

where $\{\alpha_k^i : k \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$, then,

$$\psi^i(x) = q^{1/2} \sum_{k \in \mathbb{N}_0} \alpha_k^i \varphi(\mathfrak{p}^{-1}x - u(k)).$$

We now give an example of an MRA and the associated wavelets.

Example 2.1.4. Let $\varphi = \mathbf{1}_{\mathfrak{D}}$. Define $V_j = \overline{\text{span}}\{\varphi\big(\mathfrak{p}^{-j}\cdot - u(k)\big) : k \in \mathbb{N}_0\}$. Then $\{V_j : j \in \mathbb{Z}\}$ forms an MRA of $L^2(K)$. This will be called the *Haar MRA*. Observe that

$$\varphi(x) = \sum_{k=0}^{q-1} \varphi\big(\mathfrak{p}^{-1}x - u(k)\big).$$

Taking Fourier transform, we get

$$\hat{\varphi}(\xi) = q^{-1} \sum_{k=0}^{q-1} \overline{\chi_k(\mathfrak{p}\xi)}\hat{\varphi}(\mathfrak{p}\xi) = m_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi),$$

where $m_0(\xi) = q^{-1} \sum_{k=0}^{q-1} \overline{\chi_k(\xi)}$.

We define

$$\psi^i(x) = \sum_{j=0}^{q-1} a_{ij} q^{1/2} \varphi\big(\mathfrak{p}^{-1}x - u(j)\big), \quad 1 \leq i \leq q-1, \tag{2.1.5}$$

where $A = (a_{ij})_{i,j=0}^{q-1}$ is an arbitrary unitary matrix such that $a_{0j} = q^{-1/2}$ for $0 \leq j \leq q-1$.

It can be shown that the columns of the corresponding matrix $M(\xi)$ form an orthonormal basis for $\mathbb{C}^q$. Hence the matrix $M(\xi)$ is unitary a.e. This will show that $\{\psi^i : 1 \le i \le q-1\}$ is a set of basic wavelets.

The wavelets constructed above are the analogues of the Haar wavelets on $\mathbb{R}^n$. We will call φ the *Haar scaling function* and the corresponding wavelets will be called the *Haar wavelets*. We would like to point out that the expression for the Haar wavelets given in [44] is not correct. In fact, they are not even orthogonal to each other.

An example of a unitary matrix A with a constant first row is the following. Let $a_{0j} = q^{-1/2}$ for $0 \le j \le q-1$. For $1 \le i \le q-1$, define

$$a_{ij} = \begin{cases} [(q-i)(q-i+1)]^{-1/2}, & j = 0, 1, \ldots, q-i-1, \\ -(q-i)[(q-i)(q-i+1)]^{-1/2}, & j = q-i, \\ 0, & j > q-i. \end{cases}$$

The following elementary properties of $u(n)$ for $n \in \mathbb{N}_0$ will be very useful in the sequel.

Proposition 2.1.5. *For $n \in \mathbb{N}_0$, let $u(n)$ be defined as in (1.2.2) and (1.2.3). Then*

(a) $u(n) = 0$ if and only if $n = 0$, and $|u(n)| = q^r$ if and only if $q^{r-1} \le n < q^r$, $r \ge 1$;

(b) $\{u(k) : k \in \mathbb{N}_0\} = \{-u(k) : k \in \mathbb{N}_0\}$;

(c) For a fixed $l \in \mathbb{N}_0$, $\{u(l) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$.

Proof. It follows from the definition that $u(n) = 0$ if and only if $n = 0$. To prove the second statement in (a), observe that $\{u(n) : n = 0, 1, \ldots, q-1\}$ is a complete set of coset representatives of $\mathfrak{D}$ in $\mathfrak{P}^{-1}$, i.e., $\mathfrak{P}^{-1} = \bigcup_{n=0}^{q-1} (\mathfrak{D} + u(n))$. Using this, for any integer $r \ge 1$, we can write

$$\begin{aligned} \mathfrak{P}^{-r} &= \bigcup_{n=0}^{q^r-1} (\mathfrak{D} + u(n)) \\ &= \bigcup_{n=0}^{q^{r-1}-1} (\mathfrak{D} + u(n)) \bigcup \Big(\bigcup_{n=q^{r-1}}^{q^r-1} (\mathfrak{D} + u(n)) \Big) \\ &= \mathfrak{P}^{-r+1} \bigcup \Big(\bigcup_{n=q^{r-1}}^{q^r-1} (\mathfrak{D} + u(n)) \Big). \end{aligned} \tag{2.1.6}$$

Now, fix $n \in \mathbb{N}_0$ such that $q^{r-1} \leq n < q^r$. Then, by the definition of $u(n)$ and the ultrametric inequality, we have $|u(n)| \leq q^r$. We claim that $|u(n)| > q^{r-1}$. If this is not true, then $u(n) \in \mathfrak{P}^{-r+1}$. But then, any $x \in \mathfrak{D} + u(n)$ will have $|x| \leq q^{r-1}$, by the ultrametric inequality. This would imply that $\mathfrak{D} + u(n) \subset \mathfrak{P}^{-r+1}$. Since the sets appearing in (2.1.6) are disjoint, we get a contradiction. This proves our claim. Since the absolute value of any non zero element in K is a power of q, we get $|u(n)| = q^r$.

Suppose that $0 \leq k < q$. Then $k = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1}$, where $0 \leq a_i < p$ for $i = 0, 1, \cdots, c-1$. For each i, let b_i be such that $0 \leq b_i < p$ and $a_i + b_i = 0$ modulo p. If we let $l = b_0 + b_1 p + \cdots + b_{c-1}p^{c-1}$, then it is clear that $u(k) + u(l) = 0$. Thus, $u(k) = -u(l)$ for some l with $0 \leq l < q$. Now, any $n \geq 0$, let $n = d_0 + d_1 q + d_2 q^2 + \cdots + d_s q^s$, where $0 \leq d_i < q$ for $i = 0, 1, 2, \cdots, s$. Then, by definition

$$u(n) = u(d_0) + u(d_1)\mathfrak{p}^{-1} + \cdots + u(d_s)\mathfrak{p}^{-s}.$$

For each i with $0 \leq i \leq s$, we have $u(d_i) = -u(r_i)$ with $0 \leq r_i < q$. So $u(n) = -u(m)$, where $m = r_0 + r_1 q + r_2 q^2 + \cdots + r_s q^s$ with $0 \leq r_i < q$ for $i = 0, 1, 2, \cdots, s$. This proves (b).

To prove (c), fix $l \in \mathbb{N}_0$, and let $k \in \mathbb{N}_0$. Let

$$l = a_0 + a_1 q + a_2 q^2 + \cdots + a_r q^r, \quad 0 \leq a_i < q, i = 0, 1, 2, \cdots, r,$$

and

$$k = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s, \quad 0 \leq b_i < q, i = 0, 1, 2, \cdots, s.$$

Without loss, we can assume that $r \leq s$. Then

$$u(l) + u(k) = \big(u(a_0) + u(b_0)\big) + \big(u(a_1) + u(b_1)\big)\mathfrak{p}^{-1} + \cdots + \big(u(a_s) + u(b_s)\big)\mathfrak{p}^{-s},$$

where we have taken $a_i = 0$ for $r < i \leq s$. Since K is of characteristic p, we can show that $u(a_i) + u(b_i) = u(l_i)$ for some l_i with $0 \leq l_i < q$. So $u(l) + u(k) = u(m)$, where $m = l_0 + l_1 q + l_2 q^2 + \cdots + l_s q^s$. This shows that $\{u(l) + u(k) : k \in \mathbb{N}_0\} \subseteq \{u(k) : k \in \mathbb{N}_0\}$. To prove the reverse containment, let $k \in \mathbb{N}_0$. We have to find $m \in \mathbb{N}_0$ such that $u(l) + u(m) = u(k)$. By (b), we can find $l' \in \mathbb{N}_0$ such that $u(l') = -u(l)$. Then, $u(k) + u(l') = u(m)$ for some

$m \in \mathbb{N}_0$. Hence, $u(l) + u(m) = u(m) - u(l') = u(k)$. This completes the proof. □

We have the following result for translates of a function to be an orthonormal system. The proof is similar to the proof in the Euclidean case.

Theorem 2.1.6. *Let $\varphi \in L^2(K)$. The system $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ of functions is an orthonormal system in $L^2(K)$ if and only if $\sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = 1$ for a.e. $\xi \in K$.*

In some of our results, we only need that $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ is a Riesz basis of V_0, which is weaker than being an orthonormal basis.

Definition 2.1.7. Let H be a separable Hilbert space. A subset $\{x_n : n \in \mathbb{N}_0\}$ of H is said to be a *Riesz basis* of H if for any $x \in H$, there is a sequence $\{a_k : k \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$ such that $x = \sum_{k\in\mathbb{N}_0} a_k x_k$ with convergence in H, and

$$C_1 \sum_{k\in\mathbb{N}_0} |a_k|^2 \le \Big\| \sum_{k\in\mathbb{N}_0} a_k x_k \Big\|^2 \le C_2 \sum_{k\in\mathbb{N}_0} |a_k|^2,$$

where the constants C_1, C_2 satisfy $0 < C_1 \le C_2 < \infty$ and are independent of x.

The system $\{x_k : k \in \mathbb{N}_0\}$ of functions is said to be a *frame* of H if there exist constants A and B such that

$$A\|x\|_2^2 \le \sum_{k\in\mathbb{N}_0} |\langle x, x_k\rangle|^2 \le B\|x\|_2^2 \quad \text{for all } f \in H.$$

The largest A and smallest B that can be used in the inequalities are called the frame bounds. In the definition of MRA, we can relax the last condition by requiring the system of functions $\{\varphi(\cdot - u(k) : k \in \mathbb{N}_0\}$ to form a Riesz basis instead of an orthonormal basis of V_0. It will follow from the following lemma that we can then find another function φ_1 such that $\{\varphi_1(\cdot - u(k) : k \in \mathbb{N}_0\}$ forms an orthonormal basis of V_0.

Lemma 2.1.8. *Let $\varphi \in L^2(K)$ be such that $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms a Riesz basis of its closed linear span; that is,*

$$C_1 \sum_{k\in\mathbb{N}_0} |a_k|^2 \le \Big\| \sum_{k\in\mathbb{N}_0} a_k \varphi(\cdot - u(k)) \Big\|_2^2 \le C_2 \sum_{k\in\mathbb{N}_0} |a_k|^2, \tag{2.1.7}$$

where $0 < C_1 \le C_2 < \infty$ and they are independent of the sequence $\{a_k : k \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$.

Then for a.e. $\xi \in \mathfrak{D}$, *we have*

$$C_1 \leq \sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 \leq C_2. \tag{2.1.8}$$

Proof. Let

$$A_\alpha = \{\xi \in \mathfrak{D} : \sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 > \alpha\}.$$

Assume that A_α has positive measure. We will prove that $\alpha \leq C_2$. Let $\mathbf{1}_A$ denote the characteristic function of the set A. Consider the sequence $\{a_k : k \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$ such that

$$\mathbf{1}_{A_\alpha}(\xi) = \sum_{k\in\mathbb{N}_0} a_k \overline{\chi_k(\xi)} \quad \text{for a.e. } \xi \in \mathfrak{D}.$$

Then,

$$\begin{aligned}
\Big\| \sum_{k\in\mathbb{N}_0} a_k \varphi(\cdot - u(k)) \Big\|_2^2 &= \int_K \Big| \sum_{k\in\mathbb{N}_0} a_k \overline{\chi_k(\xi)} \Big|^2 |\hat{\varphi}(\xi)|^2 \, d\xi \\
&= \int_{\mathfrak{D}} \Big| \sum_{k\in\mathbb{N}_0} a_k \overline{\chi_k(\xi)} \Big|^2 \sum_{l\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(l))|^2 \, d\xi \\
&= \int_{A_\alpha} \sum_{l\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(l))|^2 \, d\xi \\
&\geq \int_{A_\alpha} \alpha \, d\xi = \alpha |A_\alpha|.
\end{aligned}$$

By Parseval's identity, we have, $\sum_{k\in\mathbb{N}_0} |a_k|^2 = \|\mathbf{1}_{A_\alpha}\|_2^2 = |A_\alpha|$. Hence,

$$\Big\| \sum_{k\in\mathbb{N}_0} a_k \varphi(\cdot - u(k) \Big\|_2^2 \geq \alpha |A_\alpha| = \alpha \sum_{k\in\mathbb{N}_0} |a_k|^2.$$

Comparing with (2.1.7), we see that $\alpha \leq C_2$, as required. Hence, the set $\{\xi \in \mathfrak{D} : \sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 > C_2\}$ has measure zero. Therefore, $\sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 \leq C_2$ for a.e. $\xi \in \mathfrak{D}$. Similarly, considering the set

$$B_\alpha = \{\xi \in \mathfrak{D} : \sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 < \alpha\},$$

we get the left hand inequality of (2.1.8). □

Using this lemma, we can show that if the translates of a function form a Riesz basis for the subspace they span, then it is possible to find another function whose translates form an orthonormal basis for the same subspace.

Proposition 2.1.9. *Let $\varphi \in L^2(K)$ and suppose that $\{\varphi(\cdot - u(k) : k \in \mathbb{N}_0\}$ forms a Riesz basis of its closed linear span V. Then there exists a function φ_1 such that $\{\varphi_1(\cdot - u(k) : k \in \mathbb{N}_0\}$ forms an orthonormal basis for V.*

Proof. Let

$$S_\varphi(\xi) = \Big(\sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2\Big)^{1/2}.$$

Define φ_1 so that $\hat{\varphi}_1 = \frac{\hat{\varphi}}{S_\varphi}$. It follows from (2.1.8) that $\hat{\varphi} \in L^2(K)$. Hence, φ also belongs to $L^2(K)$. Since S_φ and $\frac{1}{S_\varphi}$ belong to $L^2(\mathfrak{D})$, there exist two sequences $\{\alpha_k : k \in \mathbb{N}_0\}$ and $\{\beta_k : k \in \mathbb{N}_0\}$ in $\ell^2(\mathbb{N}_0)$ such that for a.e. $\xi \in \mathfrak{D}$, we have

$$S_\varphi(\xi) = \sum_{k\in\mathbb{N}_0} \alpha_k \overline{\chi_k(\xi)} \quad \text{and} \quad \frac{1}{S_\varphi(\xi)} = \sum_{k\in\mathbb{N}_0} \beta_k \overline{\chi_k(\xi)}.$$

Hence,

$$\hat{\varphi}_1(\xi) = \hat{\varphi}(\xi) \sum_{k\in\mathbb{N}_0} \beta_k \overline{\chi_k(\xi)} \quad \text{and} \quad \hat{\varphi}(\xi) = \hat{\varphi}_1(\xi) \sum_{k\in\mathbb{N}_0} \alpha_k \overline{\chi_k(\xi)}.$$

That is,

$$\varphi_1(x) = \sum_{k\in\mathbb{N}_0} \beta_k \varphi(x - u(k)) \quad \text{and} \quad \varphi(x) = \sum_{k\in\mathbb{N}_0} \alpha_k \varphi_1(x - u(k)),$$

with convergence in $L^2(K)$. Therefore, $\{\varphi(\cdot - u(k) : k \in \mathbb{N}_0\}$ and $\{\varphi_1(\cdot - u(k) : k \in \mathbb{N}_0\}$ span the same subspace of $L^2(K)$. It follows from Proposition 2.1.5 that the function S_φ is integral-periodic. In fact, for each $l \in \mathbb{N}_0$, we have

$$S_\varphi^2(\xi + u(l)) = \sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(l) + u(k))|^2 = \sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = S_\varphi^2(\xi) \quad \textit{for a.e. } \xi \in K.$$

Therefore, for a.e. $\xi \in K$, we have

$$\sum_{l\in\mathbb{N}_0} |\hat{\varphi}_1(\xi + u(l))|^2 = \sum_{l\in\mathbb{N}_0} \frac{|\hat{\varphi}(\xi + u(l))|^2}{S_\varphi^2(\xi + u(l))} = \frac{1}{S_\varphi^2(\xi)} \sum_{l\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(l))|^2 = 1.$$

By Theorem 2.1.6, $\{\varphi_1(\cdot - u(k) : k \in \mathbb{N}_0\}$ is an orthonormal system. □

2.2 Intersection triviality and union density conditions

In this section we show that the intersection triviality condition in the definition of MRA follows from the other properties. For the corresponding results on $\mathbb{R}$ and $\mathbb{R}^n$ we refer to [30, 42, 62].

Theorem 2.2.1. *Let $\{V_j : j \in \mathbb{Z}\}$ be a sequence of closed subspaces of $L^2(K)$ satisfying conditions (a), (d) and (e) of Definition 2.1.1. Then, $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$. This is the case even if, in (e), we only assume that $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ is a Riesz basis.*

Proof. Since $\{\varphi_{0,k} : k \in \mathbb{N}_0\}$ constitutes a Riesz basis for V_0, in particular, it is a frame for V_0. Hence, there exist $A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{k\in\mathbb{N}_0} |\langle f, \varphi_{0,k}\rangle|^2 \leq B\|f\|_2^2 \quad \text{for all } f \in V_0.$$

By condition (d) of Definition 2.1.1, we can write

$$A\|f\|_2^2 \leq \sum_{k\in\mathbb{N}_0} |\langle f, \varphi_{j,k}\rangle|^2 \leq B\|f\|_2^2, \quad \text{for all } f \in V_j, j \in \mathbb{Z}. \tag{2.2.1}$$

Let $f \in \bigcap_{j\in\mathbb{Z}} V_j$ and $\epsilon > 0$. Recall that the space $\mathcal{S}$ of all finite linear combinations of the form $\Phi_k(\cdot - h)$ is dense in $L^2(K)$. So, there exists $g \in \mathcal{S}$ such that

$$\|f - g\|_2 < \epsilon.$$

For $j \in \mathbb{Z}$, let P_j be the orthogonal projection on V_j. That is,

$$P_j f = \sum_{k\in\mathbb{N}_0} \langle f, \varphi_{j,k}\rangle \varphi_{j,k}, \quad f \in L^2(K).$$

Then,

$$\|f - P_j g\|_2 = \|P_j(f - g)\|_2 \leq \|f - g\|_2 < \epsilon.$$

Therefore,

$$\|f\|_2 < \epsilon + \|P_j g\|_2.$$

From (2.2.1), we have

$$\|P_j g\|_2 \leq A^{-1/2}\Big(\sum_{k\in\mathbb{N}_0} |\langle g, \varphi_{j,k}\rangle|^2\Big)^{1/2}. \tag{2.2.2}$$

Since g is compactly supported, we can assume that supp $g \subseteq \{x \in K :| x |\leq q^s\} = \mathfrak{P}^{-s}$ for some $s > 0$. Now

$$\begin{aligned}
\sum_{k\in\mathbb{N}_0} |\langle g, \varphi_{j,k}\rangle|^2 &= \sum_{k\in\mathbb{N}_0} \Big|\int_{|x|\leq q^s} g(x) q^{j/2}\overline{\varphi(\mathfrak{p}^{-j}x - u(k))}dx\Big|^2 \\
&\leq q^j \sum_{k\in\mathbb{N}_0} \Big(\int_{|x|\leq q^s} |g(x)||\varphi(\mathfrak{p}^{-j}x - u(k))|dx\Big)^2 \\
&\leq q^j \|g\|_\infty^2 \sum_{k\in\mathbb{N}_0} \Big(\int_{|x|\leq q^s} |\varphi(\mathfrak{p}^{-j}x - u(k))|dx\Big)^2 \\
&\leq q^j \|g\|_\infty^2 \sum_{k\in\mathbb{N}_0} q^s \int_{|x|\leq q^s} |\varphi(\mathfrak{p}^{-j}x - u(k))|^2 dx \\
&= \|g\|_\infty^2 q^s \sum_{k\in\mathbb{N}_0} \int_{|y+u(k)|\leq q^{j+s}} |\varphi(y)|^2 dy \\
&= \|g\|_\infty^2 q^s \sum_{k\in\mathbb{N}_0} \int_{S_{j,k}} |\varphi(y)|^2 dy,
\end{aligned}$$

where $S_{j,k} = \{y : |y + u(k)| \leq q^{j+s}\}$. Let $S_j = \bigcup_{k\in\mathbb{N}_0} S_{j,k}$.

Note that for j small enough, $\{S_{j,k} : k \in \mathbb{N}_0\}$ is a disjoint collection, since $\{u(k) : k \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K and

$$\{u(k) : k \in \mathbb{N}_0\} = \{-u(k) : k \in \mathbb{N}_0\}.$$

Therefore, for j small enough, we have

$$\begin{aligned}
\sum_{k\in\mathbb{N}_0} |\langle g, \varphi_{j,k}\rangle|^2 &\leq \|g\|_\infty^2 q^s \int_{S_j} |\varphi(y)|^2 dy \\
&= \|g\|_\infty^2 q^s \int_K \mathbf{1}_{S_j}(y)|\varphi(y)|^2 dy.
\end{aligned} \tag{2.2.3}$$

Observe that, if $y \neq -u(k), k \in \mathbb{N}_0$, then $\mathbf{1}_{S_j}(y) \to 0$ as $j \to -\infty$. In fact, there exists $J \in \mathbb{Z}$ such that $\mathbf{1}_{S_j}(y) = 0$ if $j < J$. By Lebesgue dominated convergence theorem, the right hand side of (2.2.3) tends to 0 as $j \to -\infty$. In particular, there exists j such that

$$\sum_{k\in\mathbb{N}_0} |\langle g, \varphi_{j,k}\rangle|^2 < \epsilon^2 A.$$

Substituting in (2.2.2), we get

$$\|P_j g\|_2 < \epsilon.$$

Therefore, $\|f\|_2 < 2\epsilon$. Since ϵ was arbitrary, we get $f = 0$ a.e. □

For the next result, we need the following lemma proved in [59].

Lemma 2.2.2. *If $g \in \mathcal{S}$ and $\varphi \in L^2(K)$, then*

$$\sum_{k\in\mathbb{N}_0} |\langle g, \varphi_{j,k}\rangle|^2 = \int_K \overline{\hat{g}(\xi)}\hat{\varphi}(\mathfrak{p}^j\xi)\Big\{\sum_{l\in\mathbb{N}_0} \hat{g}(\xi + \mathfrak{p}^{-j}u(l))\overline{\hat{\varphi}(\mathfrak{p}^j\xi + u(l))}\Big\}d\xi.$$

We now assume that the function φ is such that $\hat{\varphi}$ is continuous at 0. Under this assumption, we can show that the union density condition follows from conditions (a), (d) and (e) in the definition of MRA.

Theorem 2.2.3. *Let $\{V_j : j \in \mathbb{Z}\}$ be a sequence of closed subspaces of $L^2(K)$ satisfying conditions (a), (d) and (e) of Definition 2.1.1. Assume that the function φ of condition (e) is such that $\hat{\varphi}$ is continuous at $\xi = 0$. Then the following two conditions are equivalent:*

(i) $\hat{\varphi}(0) \neq 0$.

(ii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(K)$.

Moreover, when either is the case, we have $|\hat{\varphi}(0)| = 1$.

Proof. Assume that $\hat{\varphi}(0) \neq 0$. Let $f \in (\bigcup_{j\in\mathbb{Z}} V_j)^{\perp}$ so that $P_j f = 0$ for all $j \in \mathbb{Z}$. We claim that $f = 0$ a. e.

Let $\epsilon > 0$. Since $\mathcal{S}$ is dense in $L^2(K)$, there exists $g \in \mathcal{S}$ such that

$$\|f - g\|_2 < \epsilon. \tag{2.2.4}$$

Hence, for all $j \in \mathbb{Z}$, we have

$$\|P_j g\|_2 = \|P_j(g-f)\|_2 \leq \|g-f\|_2 < \epsilon.$$

Since $\{\varphi_{j,k} : k \in \mathbb{N}_0\}$ forms a Riesz basis, and hence a frame for V_j, there exist $A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{k\in\mathbb{N}_0} |\langle f, \varphi_{j,k}\rangle|^2 \leq B\|f\|_2^2 \quad \text{for all } f \in V_j.$$

In particular, we have

$$\sum_{k\in\mathbb{N}_0} |\langle g, \varphi_{j,k}\rangle|^2 \leq B\|P_j g\|_2^2. \tag{2.2.5}$$

Since $g \in \mathcal{S}$ and $\varphi \in L^2(K)$, by Lemma 2.2.2, we have

$$\begin{aligned}
&\sum_{k\in\mathbb{N}_0} |\langle g, \varphi_{j,k}\rangle|^2 \\
&= \int_K \overline{\hat{g}(\xi)}\hat{\varphi}(\mathfrak{p}^j\xi)\Big\{\sum_{l\in\mathbb{N}_0} \hat{g}(\xi+\mathfrak{p}^{-j}u(l))\overline{\hat{\varphi}(\mathfrak{p}^j\xi+u(l))}\Big\}d\xi \\
&= \int_K |\hat{g}(\xi)|^2|\hat{\varphi}(\mathfrak{p}^j\xi)|^2 d\xi + \int_K \overline{\hat{g}(\xi)}\hat{\varphi}(\mathfrak{p}^j\xi)\Big\{\sum_{l\in\mathbb{N}} \hat{g}(\xi+\mathfrak{p}^{-j}u(l))\overline{\hat{\varphi}(\mathfrak{p}^j\xi+u(l))}\Big\}d\xi \\
&= \int_K |\hat{g}(\xi)|^2|\hat{\varphi}(\mathfrak{p}^j\xi)|^2 d\xi + R_j, \text{ say.}
\end{aligned} \tag{2.2.6}$$

Then

$$\begin{aligned}
|R_j| &\leq \int_K |\hat{g}(\xi)||\hat{\varphi}(\mathfrak{p}^j\xi)|\Big\{\sum_{l\in\mathbb{N}} |\hat{g}(\xi+\mathfrak{p}^{-j}u(l))||\hat{\varphi}(\mathfrak{p}^j\xi+u(l))|\Big\}d\xi \\
&\leq \|\hat{\varphi}\|_\infty^2 \int_K \sum_{l\in\mathbb{N}} |\hat{g}(\xi)||\hat{g}(\xi+\mathfrak{p}^{-j}u(l))|d\xi.
\end{aligned} \tag{2.2.7}$$

Observe that $\hat{\varphi}$ is bounded by (e) of Definition 2.1.1 (see Theorem 2.1.6).

Since $g \in \mathcal{S}$, there exist integers k, l such that g is constant on cosets of $\mathfrak{P}^k$ and is supported on $\mathfrak{P}^l$. Hence, $\hat{g} \in \mathcal{S}$ is constant on cosets of $\mathfrak{P}^{-l}$ and is supported on $\mathfrak{P}^{-k}$ (see Theorem 1.2.5).

We now show that, for large j, each term of the sum that appears on the right of (2.2.7) is 0.

Let $\hat{g}(\xi) \neq 0$. Then $\xi \in \mathfrak{P}^{-k}$. So $|\xi| \leq q^k$. For $j > k$ and for any $l \in \mathbb{N}$, we have

$$|\mathfrak{p}^{-j}u(l)| = q^j|u(l)| \geq q^j > q^k.$$

Therefore, for $j > k$, we have $|\xi| \neq |\mathfrak{p}^{-j}u(l)|$. Hence, by (1.2.1) and Proposition 2.1.5, we have

$$|\xi + \mathfrak{p}^{-j}u(l)| = \max\{|\xi|, |\mathfrak{p}^{-j}u(l)|\} \geq q^j > q^k.$$

That is, $\xi + \mathfrak{p}^{-j}u(l) \notin \mathfrak{P}^{-k}$, and hence, $\hat{g}(\xi + \mathfrak{p}^{-j}u(l)) = 0$ for all $j > k$. This implies that $|R_j| \to 0$ as $j \to \infty$.

Since $\|P_j g\|_2 < \epsilon$, from (2.2.5) and (2.2.6), we have

$$\int_K |\hat{g}(\xi)|^2|\hat{\varphi}(\mathfrak{p}^j\xi)|^2 d\xi + R_j < B\epsilon^2.$$

That is,

$$\int_K |\hat{g}(\xi)|^2|\hat{\varphi}(\mathfrak{p}^j\xi)|^2 d\xi < B\epsilon^2 - R_j.$$

Since $\hat{\varphi}$ is continuous at 0 with $\hat{\varphi}(0) \neq 0$, the left hand side converges to $|\hat{\varphi}(0)|^2\|g\|_2^2$ as $j \to \infty$. It follows that

$$\|g\|_2^2 \leq B\epsilon^2|\hat{\varphi}(0)|^{-2}.$$

Hence,

$$\|g\|_2 \leq B^{1/2}\epsilon|\hat{\varphi}(0)|^{-1}.$$

By (2.2.4)

$$\|f\|_2 \leq (1 + B^{1/2}|\hat{\varphi}(0)|^{-1})\epsilon.$$

Again, since ϵ was arbitrary, we get that $f = 0$ a.e. This proves (ii).

Assume now that $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(K)$. Consider f such that $\hat{f} = \mathbf{1}_{\mathfrak{D}}$. Note that

$$\|f\|_2 = \|\hat{f}\|_2 = 1.$$

We have, $\|f - P_j f\|_2 \to 0$ as $j \to \infty$, due to (a) of Definition 2.1.1 and our assumption. Thus,

$\|P_j f\|_2 \to \|f\|_2 = 1$ as $j \to \infty$. Therefore, we have

$$\begin{aligned} \|P_j f\|_2^2 &= \Big\| \sum_{k\in\mathbb{N}_0} \langle f, \varphi_{j,k}\rangle \varphi_{j,k} \Big\|_2^2 \\ &= \sum_{k\in\mathbb{N}_0} |\langle f, \varphi_{j,k}\rangle|^2, \end{aligned}$$

since $\{\varphi_{j,k} : k \in \mathbb{N}_0\}$ is an orthonormal basis of V_j.

For $j \geq 1$, we have $\mathfrak{D} \subset \mathfrak{P}^{-j}$. From the Plancherel theorem and the fact that $\hat{f} = 1_{\mathfrak{D}}$, we have

$$\begin{aligned} \|P_j f\|_2^2 &= \sum_{k\in\mathbb{N}_0} |\langle f, \varphi_{j,k}\rangle|^2 \\ &= \sum_{k\in\mathbb{N}_0} \Big| \int_K \hat{f}(\xi) q^{-j/2} \overline{\hat{\varphi}(\mathfrak{p}^j\xi)} \chi_k(\mathfrak{p}^j\xi) d\xi \Big|^2 \\ &= \sum_{k\in\mathbb{N}_0} \Big| \int_{\mathfrak{P}^{-j}} \hat{f}(\xi) q^{-j/2} \overline{\hat{\varphi}(\mathfrak{p}^j\xi)} \chi_k(\mathfrak{p}^j\xi) d\xi \Big|^2 \\ &= \sum_{k\in\mathbb{N}_0} q^j \Big| \int_{\mathfrak{D}} \hat{f}(\mathfrak{p}^{-j}\eta) \overline{\hat{\varphi}(\eta)} \chi_k(\eta) d\eta \Big|^2. \end{aligned}$$

By Parseval's identity, we have

$$\begin{aligned} \|P_j f\|_2^2 &= \int_{\mathfrak{D}} q^j |\hat{f}(\mathfrak{p}^{-j}\eta)\hat{\varphi}(\eta)|^2 d\eta \\ &= \int_{\mathfrak{P}^{-j}} |\hat{f}(\xi)\hat{\varphi}(\mathfrak{p}^j\xi)|^2 d\xi \\ &= \int_K |\hat{f}(\xi)\hat{\varphi}(\mathfrak{p}^j\xi)|^2 d\xi. \end{aligned}$$

Since $\|P_j f\|_2 \to 1$ as $j \to \infty$, we have

$$\lim_{j\to\infty} \int_K |\hat{f}(\xi)\hat{\varphi}(\mathfrak{p}^j\xi)|^2 d\xi = 1. \tag{2.2.8}$$

By Lebesgue dominated convergence theorem, the left hand side is equal to $\|\hat{f}\|_2^2 |\hat{\varphi}(0)|^2$. Hence, $|\hat{\varphi}(0)| = 1$. This completes the proof of the theorem. □

2.3 Characterization of scaling functions

In this section we will characterize those functions that are scaling functions for an MRA of $L^2(K)$. But, first we will clarify what we mean when we say that a function is a scaling function for an MRA.

Given a function φ in $L^2(K)$, we define the closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$ as follows:

$$V_0 = \overline{\text{span}}\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}, \quad \text{and}$$

$$V_j = \{f : f(\mathfrak{p}^j \cdot) \in V_0\} \quad \text{if } j \in \mathbb{Z} \setminus \{0\}.$$

We say that $\varphi \in L^2(K)$ is a scaling function for an MRA of $L^2(K)$ if the sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ as defined above constitutes an MRA of $L^2(K)$.

Theorem 2.3.1. *A function $\varphi \in L^2(K)$ is a scaling function for a multiresolution analysis of $L^2(K)$ if and only if*

$$\sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = 1 \quad \text{for a.e. } \xi \in \mathfrak{D}, \tag{2.3.1}$$

$$\lim_{j \to \infty} |\hat{\varphi}(\mathfrak{p}^j \xi)| = 1 \quad \text{for a.e. } \xi \in K, \tag{2.3.2}$$

and there exists an integral-periodic function m_0 in $L^2(\mathfrak{D})$ such that

$$\hat{\varphi}(\xi) = m_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi) \quad \text{for a.e. } \xi \in K. \tag{2.3.3}$$

Proof. Suppose that φ is a scaling function for an MRA of $L^2(K)$. Then $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms an orthonormal system in $L^2(K)$ which is equivalent to (2.3.1) by Theorem 2.1.6. Equality (2.3.3) follows from equation (2.1.4) and Theorem 2.1.3. To prove (2.3.2), we proceed as follows. Since $\{V_j : j \in \mathbb{N}_0\}$ is an MRA of $L^2(K)$, we have $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(K)$. Following the second part of the proof of Theorem 2.2.3 (see equation (2.2.8)), we have

$$\lim_{j \to \infty} \int_{\mathfrak{D}} |\hat{\varphi}(\mathfrak{p}^j \xi)|^2 d\xi = 1.$$

Substituting (2.3.3) into (2.3.1), we get

$$\begin{aligned}
1 &= \sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\mathfrak{p}^{-1}\xi + u(k))|^2 \\
&= \sum_{k\in\mathbb{N}_0} |m_0(\xi + \mathfrak{p}u(k))|^2 |\hat{\varphi}(\xi + \mathfrak{p}u(k))|^2 \\
&= \sum_{l=0}^{q-1} \sum_{k\in\mathbb{N}_0} |m_0(\xi + \mathfrak{p}u(l+kq))|^2 |\hat{\varphi}(\xi + \mathfrak{p}u(l+kq))|^2 \\
&= \sum_{l=0}^{q-1} \sum_{k\in\mathbb{N}_0} |m_0(\xi + u(k) + \mathfrak{p}u(l))|^2 |\hat{\varphi}(\xi + u(k) + \mathfrak{p}u(l))|^2 \\
&= \sum_{l=0}^{q-1} \Big\{ \sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k) + \mathfrak{p}u(l))|^2 \Big\} |m_0(\xi + \mathfrak{p}u(l))|^2.
\end{aligned}$$

In the fourth equality, we have used equation (1.2.4). Using (2.3.1) again, we get

$$\sum_{l=0}^{q-1} |m_0(\xi + \mathfrak{p}u(l))|^2 = 1 \text{ for a.e. } \xi \in K.$$

In particular, we get

$$|m_0(\xi)| \leq 1 \text{ for a.e. } \xi \in K.$$

This inequality and equality (2.3.3) show that $|\hat{\varphi}(\mathfrak{p}^j\xi)|$ is non-decreasing for a.e. $\xi \in K$ as $j \to \infty$. Let

$$g(\xi) = \lim_{j\to\infty} |\hat{\varphi}(\mathfrak{p}^j\xi)|^2.$$

Since $|\hat{\varphi}(\xi)| \leq 1$ a.e. (which follows from Theorem 2.1.6), by Lebesgue dominated convergence theorem it now follows that

$$\int_{\mathfrak{D}} g(\xi)d\xi = 1,$$

and (2.3.2) then follows since $0 \leq g(\xi) \leq 1$ for a.e. $\xi \in K$.

We now prove the converse. Assume that (2.3.1), (2.3.2) and (2.3.3) are satisfied. The orthonormality of $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ is equivalent to (2.3.1), as observed earlier. This fact along with the definition of V_0 gives us (e) of the definition of an MRA.

The definition of the subspaces V_j also shows that $f \in V_j$ if and only if $f(\mathfrak{p}^{-1}\cdot) \in V_{j+1}$,

which is (d) of the definition of an MRA. Now, for each $j \in \mathbb{Z}$, we claim that

$$V_j = \{f : \hat{f}(\mathfrak{p}^{-j}\xi) = \mu_j(\xi)\hat{\varphi}(\xi) \text{ for some integral-periodic } \mu_j \in L^2(\mathfrak{D})\}. \tag{2.3.4}$$

We can show this by expressing $f(\mathfrak{p}^j\cdot) \in V_0$ as a linear combination of $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ and then taking Fourier transforms.

To prove $V_j \subset V_{j+1}$, it is enough to show that $V_0 \subset V_1$. By (2.3.4), given $f \in V_0$, there is an integral-periodic function $\mu_0 \in L^2(\mathfrak{D})$ such that

$$\hat{f}(\mathfrak{p}^{-1}\xi) = \mu_0(\mathfrak{p}^{-1}\xi)\hat{\varphi}(\mathfrak{p}^{-1}\xi).$$

Thus, using (2.3.3), we get

$$\hat{f}(\mathfrak{p}^{-1}\xi) = \mu_0(\mathfrak{p}^{-1}\xi)m_0(\xi)\hat{\varphi}(\xi).$$

It is clear that the function $\mu_0(\mathfrak{p}^{-1}\cdot)m_0$ is integral-periodic. Now

$$\int_{\mathfrak{D}} |\mu_0(\mathfrak{p}^{-1}\xi)|^2|m_0(\xi)|^2 d\xi \le \int_{\mathfrak{D}} |\mu_0(\mathfrak{p}^{-1}\xi)|^2 d\xi < \infty,$$

as $|m_0(\xi)| \le 1$ for a.e. $\xi \in \mathfrak{D}$. Hence, the function $\mu_0(\mathfrak{p}^{-1}\cdot)m_0$ belongs to $L^2(\mathfrak{D})$. Again by (2.3.4), $f \in V_1$.

We have already seen in Theorem 2.2.1 that property (c) in the definition of MRA follows from (a), (d) and (e). Now it remains to prove only one property, i.e., we have to show that $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(K)$.

Let $f \in (\bigcup_{j\in\mathbb{Z}} V_j)^{\perp}$. We claim that $f = 0$ a.e. Let $\epsilon > 0$ be given. Since $\mathcal{S}$ is dense in $L^2(K)$, there exists g in $\mathcal{S}$ such that

$$\|f - g\|_2 < \epsilon.$$

Since $P_j f = 0$ for all $j \in \mathbb{Z}$, we have

$$\|P_j g\|_2 = \|P_j(g - f)\|_2 \le \|g - f\|_2 < \epsilon. \tag{2.3.5}$$

Since $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ is an orthonormal basis for V_0, $\{\varphi_{j,k} : k \in \mathbb{Z}\}$ forms an

orthonormal basis for V_j, by property (d) in the definition of MRA. In particular, it is a frame for V_j. So there exist $A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{k\in\mathbb{N}_0} |\langle f, \varphi_{j,k}\rangle|^2 \leq B\|f\|_2^2 \quad \text{for all } f \in V_j.$$

Taking $f = P_j g$, we have

$$\sum_{k\in\mathbb{N}_0} |\langle g, \varphi_{j,k}\rangle|^2 \leq B\|P_j g\|_2^2. \tag{2.3.6}$$

Let supp $\hat{g} \subset \mathfrak{P}^l$ for some $l \in \mathbb{Z}$. Then for all $j > -l$, we have $\mathfrak{P}^l \subseteq \mathfrak{P}^{-j}$. Hence,

$$\sum_{k\in\mathbb{N}_0} |\langle g, \varphi_{j,k}\rangle|^2 = \sum_{k\in\mathbb{N}_0} \Big| \int_K \hat{g}(\xi) q^{-j/2} \overline{\hat{\varphi}(\mathfrak{p}^j\xi)} \chi_k(\mathfrak{p}^j\xi) d\xi \Big|^2.$$

Now

$$\begin{aligned} \int_K \hat{g}(\xi) q^{-j/2} \overline{\hat{\varphi}(\mathfrak{p}^j\xi)} \chi_k(\mathfrak{p}^j\xi) d\xi &= \int_{\mathfrak{P}^{-j}} \hat{g}(\xi) q^{-j/2} \overline{\hat{\varphi}(\mathfrak{p}^j\xi)} \chi_k(\mathfrak{p}^j\xi) d\xi \\ &= \int_{\mathfrak{D}} \hat{g}(\mathfrak{p}^{-j}\eta) q^{j/2} \overline{\hat{\varphi}(\eta)} \chi_k(\eta) d\eta. \end{aligned}$$

The last expression is the k-th Fourier coefficient of the function $q^{j/2}\hat{g}(\mathfrak{p}^{-j}\cdot)\overline{\hat{\varphi}}$. Hence, by Parseval's identity, we get

$$\begin{aligned} \sum_{k\in\mathbb{N}_0} |\langle g, \varphi_{j,k}\rangle|^2 &= \int_{\mathfrak{D}} q^j |\hat{g}(\mathfrak{p}^{-j}\eta)\hat{\varphi}(\eta)|^2 d\eta \\ &= \int_{\mathfrak{P}^{-j}} |\hat{g}(\xi)\hat{\varphi}(\mathfrak{p}^j\xi)|^2 d\xi \\ &= \int_K |\hat{g}(\xi)\hat{\varphi}(\mathfrak{p}^j\xi)|^2 d\xi. \end{aligned}$$

Therefore, by (2.3.5) and (2.3.6) we have

$$\int_K |\hat{g}(\xi)\hat{\varphi}(\mathfrak{p}^j\xi)|^2 d\xi < B\epsilon^2.$$

Using (2.3.2), we conclude that the left hand side converges to $\|\hat{g}\|_2^2$ as $j \to \infty$. Hence, $g = 0$ a.e., which, in turn, implies that $f = 0$ a.e. This completes the proof of the theorem. □

Chapter 3

Affine, Quasi-affine and Co-affine Frames

The concept of quasi-affine frames in $\mathbb{R}^n$ was introduced by Ron and Shen in [68], where they proved that quasi-affine frames are invariant by translations with respect to elements of $\mathbb{Z}^n$. They also proved that if X is the affine system generated by a finite set $\Psi \subset L^2(\mathbb{R}^n)$ and associated with a dilation matrix A, and $\tilde{X}$ is the corresponding quasi-affine system, then X is an affine frame if and only if $\tilde{X}$ is a quasi-affine frame, provided the Fourier transforms of the functions in Ψ satisfy some mild decay conditions. Later, Chui, Shi, and Stöckler [22] gave an alternative proof of this fact, and more importantly, removed the assumption of the decay conditions. This result was used by Bownik in [15] to provide a new characterization of multiwavelets on $L^2(\mathbb{R}^n)$.

Another concept related to this theme is that of co-affine systems initially defined in [36] for the case of $\mathbb{R}$ where the authors proved that the co-affine system can never be a frame for $L^2(\mathbb{R})$. This result was subsequently extended to $L^2(\mathbb{R}^n)$ by Johnson [45]. Some of the other interesting articles dealing with these concepts are [13, 14, 40, 41]. In this chapter, we extend these concepts to local fields of positive characteristic and prove analogous results.

This chapter is organized as follows. In section 3.1, we define affine and quasi-affine systems on a local field K of positive characteristic and prove that an affine system $X(\Psi)$ is an affine frame for $L^2(K)$ if and only if the corresponding quasi-affine system $\tilde{X}(\Psi)$ is a quasi-affine frame. Moreover, their exact lower and upper bounds are equal. This result also holds for Bessel families. We also characterize the translation invariance of a sesquilinear operator associated

with a pair of affine systems. In section 3.2, we define the affine and quasi-affine duals of a finite subset Ψ of $L^2(K)$ and show that a finite subset Φ of $L^2(K)$ is an affine dual of Ψ if and only if it is a quasi-affine dual. In the last section, we show that $L^2(K)$ cannot have a co-affine frame.

3.1 Affine frames and quasi-affine frames

For $j \in \mathbb{Z}$, and $y \in K$, we define the dilation operator δ_j and the translation operator τ_y on $L^2(K)$ as follows:

$$\delta_j f(x) = q^{j/2} f(\mathfrak{p}^{-j}x) \quad \text{and} \quad \tau_y f(x) = f(x-y), \quad f \in L^2(K).$$

Observe that these operators are unitary and satisfy the following commutation relation:

$$\delta_j \tau_y = \tau_{\mathfrak{p}^j y} \delta_j.$$

In particular, if $j < 0$, then for $k \in \mathbb{N}_0$, we have

$$\delta_j \tau_{u(k)} = \tau_{u(q^{-j}k)} \delta_j. \tag{3.1.1}$$

Let $f_{j,k} = \delta_j \tau_{u(k)} f$. Then

$$f_{j,k}(x) = q^{j/2} f\left(\mathfrak{p}^{-j}x - u(k)\right), \quad j \in \mathbb{Z}, k \in \mathbb{N}_0.$$

We also define

$$\tilde{f}_{j,k} = f_{j,k} = \delta_j \tau_{u(k)} f \quad \text{if } j \geq 0, k \in \mathbb{N}_0,$$

and

$$\tilde{f}_{j,k} = q^{j/2} \tau_{u(k)} \delta_j f \quad \text{if } j < 0, k \in \mathbb{N}_0.$$

Let $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\}$ be a finite family of functions in $L^2(K)$. The *affine system* generated by Ψ is the collection $X(\Psi) = \{\psi^l_{j,k} : 1 \leq l \leq L, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$. The *quasi-affine system* generated by Ψ is $\tilde{X}(\Psi) = \{\tilde{\psi}^l_{j,k} : 1 \leq l \leq L, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$.

Definition 3.1.1. Let $\Psi \subset L^2(K)$ be a finite set. Then $X(\Psi)$ is called an *affine Bessel family* if

there exists a constant $B > 0$ such that

$$\sum_{\eta \in X(\Psi)} |\langle f, \eta \rangle|^2 \leq B\|f\|_2^2 \quad \text{for all } f \in L^2(K). \tag{3.1.2}$$

If, in addition, there exists a constant $A > 0, A \leq B$ such that

$$A\|f\|_2^2 \leq \sum_{\eta \in X(\Psi)} |\langle f, \eta \rangle|^2 \leq B\|f\|_2^2 \quad \text{for all } f \in L^2(K), \tag{3.1.3}$$

then $X(\Psi)$ is called an *affine frame*. The largest A and the smallest B that can be used in the above inequalities are called the lower and upper frame bounds respectively. The affine frame is called *tight* if the lower and upper frame bounds are same.

Similarly, $\tilde{X}(\Psi)$ is called a *quasi-affine Bessel family* if there exists a constant $\tilde{B} > 0$ such that (3.1.2) holds when B is replaced by $\tilde{B}$ and $X(\Psi)$ is replaced by $\tilde{X}(\Psi)$. It is called a *quasi-affine frame* if there exist constants $\tilde{A}$ and $\tilde{B} > 0$ such that (3.1.3) holds when A is replaced by $\tilde{A}$, B is replaced by $\tilde{B}$ and $X(\Psi)$ is replaced by $\tilde{X}(\Psi)$.

For two subsets $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\}$ and $\Phi = \{\varphi^1, \varphi^2, \ldots, \varphi^L\}$ of $L^2(K)$, we define a sesquilinear operator $K_{\Psi,\Phi} : L^2(K) \times L^2(K) \to \mathbb{C}$ by

$$K_{\Psi,\Phi}(f, g) = \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} \langle f, \psi_{j,k}^l \rangle \langle \varphi_{j,k}^l, g \rangle, \quad f, g \in L^2(K). \tag{3.1.4}$$

Note that if $X(\Psi)$ and $X(\Phi)$ are affine Bessel families, then $K_{\Psi,\Phi}$ defines a bounded operator. Similarly, we define the operator $\tilde{K}_{\Psi,\Phi}$ by

$$\tilde{K}_{\Psi,\Phi}(f, g) = \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} \langle f, \tilde{\psi}_{j,k}^l \rangle \langle \tilde{\varphi}_{j,k}^l, g \rangle, \quad f, g \in L^2(K). \tag{3.1.5}$$

It is easy to see that $K_{\Psi,\Phi}$ is dilation invariant, that is, $K_{\Psi,\Phi}(\delta_N f, \delta_N g) = K_{\Psi,\Phi}(f, g)$ for all $N \in \mathbb{Z}$, and $\tilde{K}_{\Psi,\Phi}$ is invariant by translations with respect to $u(k)$, $k \in \mathbb{N}_0$. We write $K_{\Psi,\Psi} = K_\Psi$ and $\tilde{K}_{\Psi,\Psi} = \tilde{K}_\Psi$.

For $j \geq 0$, let $D_j = \{q^j, q^j+1, \ldots, 2q^j - 1\}$. For any $j \in \mathbb{Z}$ and $f, g \in L^2(K)$, we define

$$K_j(f,g) = \sum_{l=1}^{L} \sum_{k \in \mathbb{N}_0} \langle f, \psi_{j,k}^l \rangle \langle \varphi_{j,k}^l, g \rangle$$

and

$$\tilde{K}_j(f,g) = \sum_{l=1}^{L} \sum_{k \in \mathbb{N}_0} \langle f, \tilde{\psi}_{j,k}^l \rangle \langle \tilde{\varphi}_{j,k}^l, g \rangle.$$

We first prove two crucial lemmas before we state and prove the main results of this chapter.

Lemma 3.1.2. *Let $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\}$ and $\Phi = \{\varphi^1, \varphi^2, \ldots, \varphi^L\}$ be two subsets of $L^2(K)$. Fix $J \in \mathbb{N}$. Then, for all $j \geq -J$ and $f, g \in L^2(K)$, we have*

$$\tilde{K}_j(f,g) = q^{-J} \sum_{\nu \in D_J} K_j(\tau_{u(\nu)} f, \tau_{u(\nu)} g).$$

Proof. For $j \geq 0$, $\tilde{K}_j(f,g) = K_j(f,g) = K_j(\tau_{u(\nu)} f, \tau_{u(\nu)} g)$ for any $\nu \in \mathbb{N}_0$. Now, for any integer j such that $-J \leq j < 0$, K_j is invariant with respect to translation by $u(q^{-j}\nu) = \mathfrak{p}^j u(\nu)$, $\nu \in \mathbb{N}_0$. That is,

$$K_j(\tau_{\mathfrak{p}^j u(\nu)} f, \tau_{\mathfrak{p}^j u(\nu)} g) = K_j(f,g), \quad \nu \in \mathbb{N}_0.$$

Note that, for any $m \leq J$, we have

$$\sum_{\nu=0}^{q^J-1} a_\nu = \sum_{\lambda=0}^{q^m-1} \sum_{\mu=0}^{q^{J-m}-1} a_{\mu q^m + \lambda}.$$

Hence,

$$\sum_{\nu \in D_J} a_\nu = \sum_{\nu=q^J}^{2q^J-1} a_\nu = \sum_{\lambda=0}^{q^m-1} \sum_{\mu=0}^{q^{J-m}-1} a_{\mu q^m + \lambda + q^J}.$$

Therefore,

$$\begin{aligned} q^{-J} \sum_{\nu \in D_J} K_j(\tau_{u(\nu)} f, \tau_{u(\nu)} g) &= q^{-J} \sum_{\lambda=0}^{q^{-j}-1} \sum_{\mu=0}^{q^J+j-1} K_j(\tau_{u(\mu q^{-j} + \lambda + q^J)} f, \tau_{u(\mu q^{-j} + \lambda + q^J)} g) \\ &= q^{-J} \sum_{\lambda=q^{-j}}^{2q^{-j}-1} \sum_{\mu=q^{J+j}-1}^{2q^{J+j}-2} K_j(\tau_{u(\mu q^{-j} + \lambda)} f, \tau_{u(\mu q^{-j} + \lambda)} g). \end{aligned}$$

Since $\lambda \in \{q^{-j}, q^{-j}+1, \ldots, 2q^{-j}-1\}$ and $\mu \in \{q^{J+j}-1, q^{J+j}, \ldots, 2q^{J+j}-2\}$, we have $\mu = q^{J+j}+r$ and $\lambda = q^{-j}+s$, where $r \in \{-1, 0, 1, \ldots, q^{J+j}-2\}$ and $s \in \{0, 1, \ldots, q^{-j}-1\}$. Hence, $\mu q^{-j} + \lambda = q^J + rq^{-j} + q^{-j} + s = (q^{J+j} + r + 1)q^{-j} + s$ so that $u(\mu q^{-j} + \lambda) = u(q^{J+j} + r + 1)\mathfrak{p}^j + u(\lambda - q^{-j})$, by (1.2.4). Therefore,

$$
\begin{aligned}
q^{-J} \sum_{\nu \in D_J} K_j(\tau_{u(\nu)} f, \tau_{u(\nu)} g) &= q^{-J} \sum_{\lambda=q^{-j}}^{2q^{-j}-1} \sum_{\mu=q^{J+j}-1}^{2q^{J+j}-2} K_j(\tau_{u(\lambda-q^{-j})} f, \tau_{u(\lambda-q^{-j})} g) \\
&= q^j \sum_{\lambda=0}^{q^{-j}-1} K_j(\tau_{u(\lambda)} f, \tau_{u(\lambda)} g) \\
&= q^j \sum_{\lambda=0}^{q^{-j}-1} K_j(\tau_{-u(\lambda)} f, \tau_{-u(\lambda)} g) \\
&= q^j \sum_{\lambda=0}^{q^{-j}-1} \sum_{l=1}^{L} \sum_{k \in \mathbb{N}_0} \langle \tau_{-u(\lambda)} f, \psi^l_{j,k} \rangle \langle \varphi^l_{j,k}, \tau_{-u(\lambda)} g \rangle \\
&= \sum_{\lambda=0}^{q^{-j}-1} \sum_{l=1}^{L} \sum_{k \in \mathbb{N}_0} \langle f, \tilde{\psi}^l_{j,q^{-j}k+\lambda} \rangle \langle \tilde{\varphi}^l_{j,q^{-j}k+\lambda}, g \rangle \\
&= \sum_{l=1}^{L} \sum_{k \in \mathbb{N}_0} \langle f, \tilde{\psi}^l_{j,k} \rangle \langle \tilde{\varphi}^l_{j,k}, g \rangle = \tilde{K}_j(f, g).
\end{aligned}
$$

We have used the following two facts in the series of equalities above:

(i) $\{u(\lambda) : 0 \le \lambda \le q^m - 1\} = \{-u(\lambda) : 0 \le \lambda \le q^m - 1\}$ for any $m \in \mathbb{N}$.

(ii) $q^{j/2} \langle \tau_{-u(\lambda)} f, \psi^l_{j,k} \rangle = \langle f, \tilde{\psi}^l_{j,q^{-j}k+\lambda} \rangle$.

This completes the proof of the lemma. □

In the following lemma we prove two important properties of the operators K_j and $\tilde{K}_j$ when $\Phi = \Psi$.

Lemma 3.1.3. *Let $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subset L^2(K)$. Put $\Phi = \Psi$ in the definitions of K_j and $\tilde{K}_j$. If $f \in L^2(K)$ has compact support, then*

(a) $\lim_{N \to \infty} \sum_{j<0} \tilde{K}_j(\delta_N f, \delta_N f) = 0.$

(b) $\lim_{N \to \infty} q^{-N} \sum_{j<-N} \sum_{\nu \in D_N} K_j(\tau_{u(\nu)} f, \tau_{u(\nu)} f) = 0.$

Proof. We have,

$$\sum_{j<0} \tilde{K}_j(\delta_N f, \delta_N f) = \sum_{j<0} \sum_{l=1}^{L} \sum_{k\in\mathbb{N}_0} |\langle \delta_N f, q^{j/2} \tau_{u(k)} \delta_j \psi^l \rangle|^2.$$

Let $\Omega = \text{supp } f$. Since

$$\langle \delta_N f, \tau_{u(k)} \delta_j \psi^l \rangle = \langle f, \delta_{-N} \tau_{u(k)} \delta_j \psi^l \rangle = \langle f, \tau_{\mathfrak{p}^{-N}u(k)} \delta_{-N} \delta_j \psi^l \rangle = \langle \tau_{-\mathfrak{p}^{-N}u(k)} f, \delta_{j-N} \psi^l \rangle,$$

we have

$$\begin{aligned}
|\langle \delta_N f, \tau_{u(k)} \delta_j \psi^l \rangle|^2 &= \left| \int_K f(x + \mathfrak{p}^{-N}u(k)) \overline{\delta_{j-N} \psi^l(x)} dx \right|^2 \\
&\leq \|f\|_2^2 \int_{\Omega - \mathfrak{p}^{-N}u(k)} |\delta_{j-N} \psi^l(x)|^2 dx \\
&= \|f\|_2^2 \int_{\Omega - \mathfrak{p}^{-N}u(k)} |q^{(j-N)/2} \psi^l(\mathfrak{p}^{-(j-N)}x)|^2 dx \\
&= \|f\|_2^2 \int_{\mathfrak{p}^{-(j-N)}(\Omega - \mathfrak{p}^{-N}u(k))} |\psi^l(x)|^2 dx \\
&= \|f\|_2^2 \int_{\mathfrak{p}^{-(j-N)}\Omega - \mathfrak{p}^{-j}u(k)} |\psi^l(x)|^2 dx.
\end{aligned}$$

Thus,

$$\sum_{j<0} \tilde{K}_j(\delta_N f, \delta_N f) \leq \|f\|_2^2 \sum_{j<0} q^j \sum_{k\in\mathbb{N}_0} \int_{\mathfrak{p}^{-(j-N)}\Omega - \mathfrak{p}^{-j}u(k)} \sum_{l=1}^{L} |\psi^l(x)|^2 dx.$$

Note that $\mathfrak{p}^{-(j-N)}\Omega - \mathfrak{p}^{-j}u(k) = \mathfrak{p}^{-j}(\mathfrak{p}^N\Omega - u(k))$. Since Ω is compact and $|\mathfrak{p}^N\Omega| = q^{-N}|\Omega|$, we can choose N_0 large enough so that $\mathfrak{p}^N\Omega \subseteq \mathfrak{D}$ if $N \geq N_0$. Since $\{\mathfrak{D} + u(k) : k \in \mathbb{N}_0\}$ is a disjoint collection, it follows that $\{\mathfrak{p}^{-j}(\mathfrak{p}^N\Omega - u(k)) : k \in \mathbb{N}_0\}$ is also a disjoint collection. Hence,

$$\begin{aligned}
\sum_{j<0} \tilde{K}_j(\delta_N f, \delta_N f) &\leq \|f\|_2^2 \sum_{j<0} q^j \int_{\bigcup_{k\in\mathbb{N}_0} \mathfrak{p}^{-j}(\mathfrak{p}^N\Omega - u(k))} \sum_{l=1}^{L} |\psi^l(x)|^2 dx \\
&= \|f\|_2^2 \int_K F_N(x) \sum_{l=1}^{L} |\psi^l(x)|^2 dx,
\end{aligned}$$

where

$$F_N = \sum_{j<0} q^j 1_{\bigcup_{k\in\mathbb{N}_0} \mathfrak{p}^{-j}(\mathfrak{p}^N\Omega - u(k))}.$$

Observe that $|F_N(x)| \leq \sum_{j<0} q^j = \frac{1}{q-1}$. Since $\sum_{l=1}^{L} |\psi^l|^2 \in L^1(K)$, if we can show that $F_N \to 0$ a.e. as $N \to \infty$, then by Lebesgue dominated convergence theorem, the last integral above will converge to 0 as $N \to \infty$.

Let $E = \{x \in K : x = -\mathfrak{p}^{-j}u(k) \text{ for some } j < 0 \text{ and } k \in \mathbb{N}_0\}$. If $x \notin E$, then $\mathfrak{p}^j x + u(k) \neq 0$ for any $j < 0$ and $k \in \mathbb{N}_0$ so that $|\mathfrak{p}^j x + u(k)| = q^r$ for some $r \in \mathbb{Z}$. Thus, $\mathfrak{p}^j x + u(k) \notin \mathfrak{p}^N\Omega$ if $N > -r$. That is, $x \notin \mathfrak{p}^{-j}(\mathfrak{p}^N\Omega - u(k))$ if $N > -r$. Since E is a set of measure zero, it follows that $F_N \to 0$ a.e. as $N \to \infty$. This proves part (a) of the lemma.

To prove part (b), we observe that

$$q^{-N} \sum_{j<-N} \sum_{\nu\in D_N} K_j(\tau_{u(\nu)}f, \tau_{u(\nu)}f) = q^{-N} \sum_{\nu=q^N}^{2q^N-1} \sum_{j<-N} \sum_{l=1}^{L} \sum_{k\in\mathbb{N}_0} |\langle \tau_{u(\nu)}f, \psi^l_{j,k}\rangle|^2.$$

An easy calculation as in part (a) gives us

$$\langle \tau_{u(\nu)}f, \psi^l_{j,k}\rangle = \langle \tau_{u(\nu)-p^j u(k)}f, \delta_j\psi^l\rangle,$$

so that

$$\begin{aligned} |\langle \tau_{u(\nu)}f, \psi^l_{j,k}\rangle|^2 &= |\langle \tau_{u(\nu)-p^j u(k)}f, \delta_j\psi^l\rangle|^2 \\ &\leq \|f\|_2^2 \int_{\Omega+u(\nu)-\mathfrak{p}^j u(k)} |\delta_j\psi^l(x)|^2 dx \\ &= \|f\|_2^2 \int_{\mathfrak{p}^{-j}(\Omega+u(\nu)-\mathfrak{p}^j u(k))} |\psi^l(x)|^2 dx. \end{aligned}$$

Hence,

$$\begin{aligned} &q^{-N} \sum_{j<-N} \sum_{\nu\in D_N} K_j(\tau_{u(\nu)}f, \tau_{u(\nu)}f) \\ &\leq q^{-N}\|f\|_2^2 \sum_{\nu=q^N}^{2q^N-1} \sum_{j<-N} \sum_{k\in\mathbb{N}_0} \int_{\mathfrak{p}^{-j}(\Omega+u(\nu)-\mathfrak{p}^j u(k))} \sum_{l=1}^{L} |\psi^l(x)|^2 dx \end{aligned}$$

$$= \|f\|_2^2 \int_K G_N(x) \sum_{l=1}^{L} |\psi^l(x)|^2 dx,$$

where

$$G_N = q^{-N} \sum_{\nu=q^N}^{2q^N-1} \sum_{j<-N} \sum_{k\in\mathbb{N}_0} \mathbf{1}_{\mathfrak{p}^{-j}(\Omega+u(\nu))-u(k)}.$$

As in part (a), to complete the proof, we need to show that $G_N \to 0$ a.e. as $N \to \infty$.

Note that $\mathfrak{p}^N\Omega \subseteq \mathfrak{D}$ if $N \geq N_0$. For such an N, consider the set $\mathfrak{p}^N(\Omega + u(\nu))$, where $\nu \in D_N$. If $x \in \mathfrak{p}^N(\Omega + u(\nu))$, then $x = y + \mathfrak{p}^N u(\nu)$ for some $y \in \mathfrak{p}^N\Omega \subseteq \mathfrak{D}$. Since $|y| \leq 1$ and $|\mathfrak{p}^N u(\nu)| = q^{-N} \cdot q^{N+1} = q$, we have $|x| = q$, by (1.2.1). Thus, $\mathfrak{p}^N(\Omega + u(\nu)) \subset \mathfrak{P}^{-1} \setminus \mathfrak{D} = \mathfrak{p}^{-1}\mathfrak{D} \setminus \mathfrak{D} = \mathfrak{p}^{-1}\mathfrak{D}^*$ so that $\mathfrak{p}^{-j}(\Omega + u(\nu)) \subseteq \mathfrak{p}^{-j-N-1}\mathfrak{D}^*$ for any $j \in \mathbb{Z}$.

For $N \geq N_0$, fix $j < -N$ and $k \in \mathbb{N}_0$. Note that since Ω is compact, each $\Omega + u(k_0)$, for fixed $k_0 \in \mathbb{N}_0$, can intersect with only finitely many sets of the form $\Omega + u(k)$, $k \in \mathbb{N}_0$. So there exists an integer $d \in \mathbb{N}$ such that each $x \in K$ can belong to at most d such sets. Thus, in particular, any $x \in K$ can belong to at most d sets in the collection $\{\mathfrak{p}^{-j}(\Omega + u(\nu)) - u(k) : \nu \in D_N\}$. Each of these sets is contained in $\mathfrak{p}^{-j-N-1}\mathfrak{D}^* - u(k)$, hence so is their union. Thus,

$$\sum_{\nu=q^N}^{2q^N-1} \mathbf{1}_{\mathfrak{p}^{-j}(\Omega+u(\nu))-u(k)} \leq d\mathbf{1}_{\mathfrak{p}^{-j-N-1}\mathfrak{D}^*-u(k)}.$$

Now,

$$\begin{aligned} \sum_{j<-N} \mathbf{1}_{\mathfrak{p}^{-j-N-1}\mathfrak{D}^*-u(k)} &= \sum_{i>0} \mathbf{1}_{\mathfrak{p}^{i-1}\mathfrak{D}^*-u(k)} \\ &= \mathbf{1}_{\bigcup_{i>0} \mathfrak{p}^{i-1}\mathfrak{D}^*-u(k)} = \mathbf{1}_{\mathfrak{D}^*-u(k)}. \end{aligned}$$

Also, we have

$$\sum_{k\in\mathbb{N}_0} \mathbf{1}_{\mathfrak{D}^*-u(k)} = \sum_{k\in\mathbb{N}_0} \mathbf{1}_{\mathfrak{D}^*+u(k)} \leq \sum_{k\in\mathbb{N}_0} \mathbf{1}_{\mathfrak{D}+u(k)} = 1,$$

since $\{\mathfrak{D} + u(k) : k \in \mathbb{N}_0\}$ is a partition of K and $\mathfrak{D}^* \subset \mathfrak{D}$.

Collecting all these estimates, we get

$$G_N(x) \leq q^{-N} \cdot d.$$

Therefore, $G_N(x) \to 0$ as $N \to \infty$ uniformly in x. This completes the proof of the lemma. □

The following theorem shows the relationship between affine and quasi-affine frames.

Theorem 3.1.4. *Let* $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subset L^2(K)$. *Then*

(a) $X(\Psi)$ *is a Bessel family if and only if* $\tilde{X}(\Psi)$ *is a Bessel family. Moreover, their exact upper bounds are equal.*

(b) $X(\Psi)$ *is an affine frame if and only if* $\tilde{X}(\Psi)$ *is a quasi-affine frame. Moreover, their exact lower and upper bounds are equal.*

Proof. Put $\Phi = \Psi$ in the definitions of K_j and $\tilde{K}_j$. Suppose that $X(\Psi)$ is a Bessel family with upper bound $B \geq 0$. Then, by Lemma 3.1.2, for all $f \in L^2(K)$, we have

$$\begin{aligned}
\tilde{K}_\Psi(f,f) &= \sum_{j=-\infty}^{\infty} \tilde{K}_j(f,f) = \lim_{J\to\infty} \sum_{j\geq -J} \tilde{K}_j(f,f) \\
&= \lim_{J\to\infty} q^{-J} \sum_{\nu\in D_J} \sum_{j\geq -J} K_j(\tau_{u(\nu)}f, \tau_{u(\nu)}f) \\
&\leq \lim_{J\to\infty} q^{-J} \sum_{\nu\in D_J} K_\Psi(\tau_{u(\nu)}f, \tau_{u(\nu)}f) \\
&\leq \lim_{J\to\infty} q^{-J} \sum_{\nu\in D_J} B\|\tau_{u(\nu)}f\|_2^2 = B\|f\|_2^2.
\end{aligned}$$

Thus, the quasi-affine system $\tilde{X}(\Psi)$ is also a Bessel family with upper bound B.

Conversely, let us assume that $\tilde{X}(\Psi)$ is a Bessel family with upper bound $C \geq 0$. Further, assume that there exists $f \in L^2(K)$ with $\|f\|_2 = 1$ and $K_\Psi(f,f) > C$. We will get a contradiction. We have

$$\begin{aligned}
\sum_{j=-N}^{\infty} K_j(f,f) &= \sum_{j=-N}^{\infty} \sum_{l=1}^{L} \sum_{k\in\mathbb{N}_0} |\langle f, \psi_{j,k}^l\rangle|^2 \\
&= \sum_{j=0}^{\infty} \sum_{l=1}^{L} \sum_{k\in\mathbb{N}_0} |\langle f, \psi_{j-N,k}^l\rangle|^2 \\
&= \sum_{j=0}^{\infty} \sum_{l=1}^{L} \sum_{k\in\mathbb{N}_0} |\langle \delta_N f, \psi_{j,k}^l\rangle|^2 \\
&= \sum_{j=0}^{\infty} K_j(\delta_N f, \delta_N f).
\end{aligned}$$

Since $K_\Psi(f,f) = \lim_{N\to\infty} \sum_{j=-N}^{\infty} K_j(f,f) > C$, there exists $N \in \mathbb{N}$ such that

$$\sum_{j=-N}^{\infty} K_j(f,f) = \sum_{j=0}^{\infty} K_j(\delta_N f, \delta_N f) > C.$$

Now,

$$\tilde{K}_\Psi(\delta_N f, \delta_N f) \geq \sum_{j=0}^{\infty} \tilde{K}_j(\delta_N f, \delta_N f) = \sum_{j=0}^{\infty} K_j(\delta_N f, \delta_N f) > C.$$

If $g = \delta_N f$, then we have $\|g\|_2 = \|\delta_N f\|_2 = \|f\|_2 = 1$ but $\tilde{K}_\Psi(g,g) > C$. This is a contradiction to the fact that $\tilde{X}(\Psi)$ is a Bessel family with upper bound C. This proves part (a) of the theorem.

We will now prove part (b). We have dealt with the upper bounds in part (a). So we need only to consider the lower bounds $\tilde{A}$ and A. Suppose that $X(\Psi)$ is an affine frame with lower frame bound A. Then, for all $f \in L^2(K)$ with compact support, we have

$$\begin{aligned}
\tilde{K}_\Psi(f,f) &= \lim_{J\to\infty} q^{-J} \sum_{\nu\in D_J} \sum_{j\geq -J} K_j(\tau_{u(\nu)} f, \tau_{u(\nu)} f) \\
&= \lim_{J\to\infty} q^{-J} \sum_{\nu\in D_J} \sum_{j\in\mathbb{Z}} K_j(\tau_{u(\nu)} f, \tau_{u(\nu)} f) \quad \text{(by Lemma 3.1.3(b))} \\
&= \lim_{J\to\infty} q^{-J} \sum_{\nu\in D_J} K_\Psi(\tau_{u(\nu)} f, \tau_{u(\nu)} f) \\
&\geq \lim_{J\to\infty} q^{-J} \sum_{\nu\in D_J} A\|\tau_{u(\nu)} f\|_2^2 = A\|f\|_2^2.
\end{aligned}$$

The set of all such f is dense in $L^2(K)$. So this holds for all $f \in L^2(K)$. Hence, $\tilde{A} \geq A$.

To show that $\tilde{A} \leq A$, we assume that it is not true and get a contradiction. Thus, there exists $\epsilon > 0$, $f \in L^2(K)$ with $\|f\|_2 = 1$ such that

$$K_\Psi(f,f) \leq \tilde{A} - \epsilon.$$

Without loss of generality, we can assume that f has compact support (otherwise, for any compact set Ω, we consider $f\mathbf{1}_\Omega$). Since K_Ψ is dilation invariant, we also get

$$K_\Psi(\delta_N f, \delta_N f) \leq \tilde{A} - \epsilon.$$

By Lemma 3.1.3(a), there exists $N \in \mathbb{N}$ such that $\sum_{j<0} \tilde{K}_j(\delta_N f, \delta_N f) < \epsilon/2$. Hence,

$$\begin{aligned} \tilde{K}_\Psi(\delta_N f, \delta_N f) &< \sum_{j\geq 0} \tilde{K}j(\delta_N f, \delta_N f) + \epsilon/2 \\ &= \sum_{j\geq 0} Kj(\delta_N f, \delta_N f) + \epsilon/2 \\ &\leq K_\Psi(\delta_N f, \delta_N f) + \epsilon/2 \leq \tilde{A} - \epsilon/2. \end{aligned}$$

This contradicts the definition of the lower bound $\tilde{A}$ of $\tilde{X}(\Psi)$ and completes the proof of the theorem. □

We have observed earlier that $K_{\Psi,\Phi}$ is dilation invariant and $\tilde{K}_{\Psi,\Phi}$ is invariant by translations with respect to $u(k)$, $k \in \mathbb{N}_0$. In the next theorem, we show that a necessary and sufficient condition for the translation invariance of $K_{\Psi,\Phi}$ is that the operators $K_{\Psi,\Phi}$ and $\tilde{K}_{\Psi,\Phi}$ coincide.

Theorem 3.1.5. *Let $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\}$ and $\Phi = \{\varphi^1, \varphi^2, \ldots, \varphi^L\}$ generate two affine Bessel families. Then $K_{\Psi,\Phi}$ is translation invariant if and only if $K_{\Psi,\Phi} = \tilde{K}_{\Psi,\Phi}$.*

Proof. Suppose that $K_{\Psi,\Phi}$ is translation invariant. Then, as in the proof of Theorem 3.1.4, for all $f, g \in L^2(K)$ with compact support, we have

$$\begin{aligned} \tilde{K}_{\Psi,\Phi}(f,g) &= \lim_{J\to\infty} q^{-J} \sum_{\nu\in D_J} K_{\Psi,\Phi}(\tau_{u(\nu)} f, \tau_{u(\nu)} g) \\ &= \lim_{J\to\infty} q^{-J} \sum_{\nu\in D_J} K_{\Psi,\Phi}(f,g) = K_{\Psi,\Phi}(f,g), \end{aligned}$$

where we have used the translation invariance of $K_{\Psi,\Phi}$. By density and the boundedness of the operators $K_{\Psi,\Phi}$ and $\tilde{K}_{\Psi,\Phi}$, the equality holds for all $f, g \in L^2(K)$.

Conversely, assume that $K_{\Psi,\Phi} = \tilde{K}_{\Psi,\Phi}$. Then for $m \in \mathbb{N}_0$, we have

$$\begin{aligned} K_{\Psi,\Phi}(\tau_{u(m)} f, \tau_{u(m)} g) &= \tilde{K}_{\Psi,\Phi}(\tau_{u(m)} f, \tau_{u(m)} g) \\ &= \sum_{l=1}^{L} \sum_{j\geq 0} \sum_{k\in\mathbb{N}_0} \langle \tau_{u(m)} f, \delta_j \tau_{u(k)} \psi^l \rangle \langle \delta_j \tau_{u(k)} \varphi^l, \tau_{u(m)} g \rangle \\ &\quad + \sum_{l=1}^{L} \sum_{j<0} \sum_{k\in\mathbb{N}_0} \langle \tau_{u(m)} f, q^{j/2} \tau_{u(k)} \delta_j \psi^l \rangle \langle q^{j/2} \tau_{u(k)} \delta_j \varphi^l, \tau_{u(m)} g \rangle. \end{aligned}$$

Since $m \in N_0$, by Proposition 2.1.5(b), there exists a unique $m_0 \in \mathbb{N}_0$ such that $-u(m) = u(m_0)$. Hence, in the first sum, we have

$$\begin{aligned}\langle \tau_{u(m)} f, \delta_j \tau_{u(k)} \psi^l \rangle &= \langle f, \tau_{u(m_0)} \delta_j \tau_{u(k)} \psi^l \rangle \\ &= \langle f, \delta_j \tau_{u(q^j m_0)+u(k)} \psi^l \rangle.\end{aligned}$$

Similarly, $\langle \delta_j \tau_{u(k)} \varphi^l, \tau_{u(m)} g \rangle = \langle \delta_j \tau_{u(q^j m_0)+u(k)} \varphi^l, g \rangle$.

For a fixed $j \geq 0$, we have $\{u(k) + u(q^j m_0) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$, by Proposition 2.1.5(c). Hence, for a fixed $j \geq 0$, we have

$$\begin{aligned}\sum_{k\in\mathbb{N}_0} \langle \tau_{u(m)} f, \delta_j \tau_{u(k)} \psi^l \rangle \langle \delta_j \tau_{u(k)} \varphi^l, \tau_{u(m)} g \rangle &= \sum_{k\in\mathbb{N}_0} \langle f, \delta_j \tau_{u(k)} \psi^l \rangle \langle \delta_j \tau_{u(k)} \varphi^l, g \rangle \\ &= \sum_{k\in\mathbb{N}_0} \langle f, \psi^l_{j,k} \rangle \langle \varphi^l_{j,k}, g \rangle.\end{aligned}$$

In the second sum, we have

$$\langle \tau_{u(m)} f, q^{j/2} \tau_{u(k)} \delta_j \psi^l \rangle = \langle f, q^{j/2} \tau_{u(m_0)+u(k)} \delta_j \psi^l \rangle.$$

By a similar argument as above, we get for each $j < 0$,

$$\sum_{k\in\mathbb{N}_0} \langle \tau_{u(m)} f, q^{j/2} \tau_{u(k)} \delta_j \psi^l \rangle \langle q^{j/2} \tau_{u(k)} \delta_j \varphi^l, \tau_{u(m)} g \rangle = \sum_{k\in\mathbb{N}_0} \langle f, \tilde{\psi}^l_{j,k} \rangle \langle \tilde{\varphi}^l_{j,k}, g \rangle.$$

Hence, we have

$$K_{\Psi,\Phi}(\tau_{u(m)} f, \tau_{u(m)} g) = \tilde{K}_{\Psi,\Phi}(f, g) = K_{\Psi,\Phi}(f, g).$$

This proves that $K_{\Psi,\Phi}$ is invariant by translations with respect to $u(m)$, where $m \in \mathbb{N}_0$. Since $K_{\Psi,\Phi}$ is invariant with respect to dilations, it follows that it is invariant with respect to all x of the form $x = \mathfrak{p}^j u(m)$, $m \in \mathbb{N}_0$, $j \in \mathbb{Z}$. But such elements are dense in K. This can be seen as follows. Since $\{\mathfrak{D} + u(m) : m \in \mathbb{N}_0\}$ is a partition of K, $\{\mathfrak{p}^j \mathfrak{D} + \mathfrak{p}^j u(m) : m \in \mathbb{N}_0\}$ is also a partition of K for any $j \in \mathbb{Z}$. Hence, if $x \in K$, then for each $j \in \mathbb{Z}$, there exists a unique $m \in \mathbb{N}_0$ and $y \in \mathfrak{D}$ such that $x = \mathfrak{p}^j y + \mathfrak{p}^j u(m)$ so that $|x - \mathfrak{p}^j u(m)| = |\mathfrak{p}^j y| = q^{-j}|y|$. Since $|y| \leq 1$, we can choose j sufficiently large to make $|x - \mathfrak{p}^j u(m)|$ as small as we want.

Now, since $K_{\Psi,\Phi}$ is a bounded operator and translation is a continuous operation, it follows

that $K_{\Psi,\Phi}$ is invariant with respect to translation by all elements of K. This completes the proof of the theorem. □

3.2 Affine and quasi-affine duals

In this section, we define the affine dual and quasi-affine dual of a finite subset Ψ of $L^2(K)$ generating a Bessel family and show that a finite subset Φ of $L^2(K)$ with cardinality same as that of Ψ is an affine dual of Ψ if and only if it is a quasi-affine dual of Ψ.

Definition 3.2.1. Let $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\}$ and $\Phi = \{\varphi^1, \varphi^2, \ldots, \varphi^L\}$ be two subsets of $L^2(K)$ such that $X(\Psi)$ and $X(\Phi)$ are Bessel families. Then Φ is called an *affine dual* of Ψ if $K_{\Psi,\Phi}(f,g) = \langle f, g\rangle$ for all $f, g \in L^2(K)$, that is,

$$\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\langle f, \psi_{j,k}^l\rangle\langle\varphi_{j,k}^l, g\rangle = \langle f, g\rangle \quad \text{for all } f, g \in L^2(K). \tag{3.2.1}$$

We say that Φ is a *quasi-affine dual* of Ψ if $\tilde{K}_{\Psi,\Phi}(f,g) = \langle f, g\rangle$ for all $f, g \in L^2(K)$, that is,

$$\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\langle f, \tilde{\psi}_{j,k}^l\rangle\langle\tilde{\varphi}_{j,k}^l, g\rangle = \langle f, g\rangle \quad \text{for all } f, g \in L^2(K). \tag{3.2.2}$$

Since $K_{\Psi,\Phi}$ and $\tilde{K}_{\Psi,\Phi}$ are sesquilinear operators, it follows from the polarization identity that (3.2.1) or (3.2.2) holds if and only if it holds for all $f = g$ in $L^2(K)$.

Theorem 3.2.2. *Let $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subset L^2(K)$ generate an affine Bessel family. Then $\Phi = \{\varphi^1, \varphi^2, \ldots, \varphi^L\} \subset L^2(K)$ is an affine dual of Ψ if and only if it is a quasi-affine dual of Ψ.*

Proof. We first assume that Φ is an affine dual of Ψ. So $K_{\Psi,\Phi}(f,g) = \langle f, g\rangle$ for all $f, g \in L^2(K)$. Since $\langle \tau_y f, \tau_y g\rangle = \langle f, g\rangle$ for all $y \in K$ and for all $f, g \in L^2(K)$, it follows that $K_{\Psi,\Phi}$ is translation invariant. Hence, by Theorem 3.1.5, we have

$$\tilde{K}_{\Psi,\Phi}(f,g) = K_{\Psi,\Phi}(f,g) = \langle f, g\rangle \quad \text{for all } f, g \in L^2(K).$$

Therefore, Φ is a quasi-affine dual of Ψ.

Conversely, assume that Φ is a quasi-affine dual of Ψ. Let $f \in L^2(K)$ be a function with compact support. By Lemma 3.1.3(a), we have

$$\sum_{j<0} \tilde{K}_j(\delta_N f, \delta_N f) \to 0 \text{ as } N \to \infty.$$

That is,

$$\sum_{j<0} \sum_{l=1}^{L} \sum_{k\in\mathbb{N}_0} \langle \delta_N f, \tilde{\psi}^l_{j,k}\rangle \langle \tilde{\varphi}^l_{j,k}, \delta_N f\rangle \to 0 \text{ as } N \to \infty. \tag{3.2.3}$$

Now, since Φ is a quasi-affine dual of Ψ, we have

$$\|f\|_2^2 = \|\delta_N f\|_2^2 = \langle \delta_N f, \delta_N f\rangle = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} \langle \delta_N f, \tilde{\psi}^l_{j,k}\rangle \langle \tilde{\varphi}^l_{j,k}, \delta_N f\rangle$$

$$= \sum_{l=1}^{L} \sum_{j\geq 0} \sum_{k\in\mathbb{N}_0} \langle \delta_N f, \tilde{\psi}^l_{j,k}\rangle \langle \tilde{\varphi}^l_{j,k}, \delta_N f\rangle + \sum_{l=1}^{L} \sum_{j<0} \sum_{k\in\mathbb{N}_0} \langle \delta_N f, \tilde{\psi}^l_{j,k}\rangle \langle \tilde{\varphi}^l_{j,k}, \delta_N f\rangle.$$

The second term in the last equality goes to 0 as $N \to \infty$, by (3.2.3). Hence,

$$\sum_{l=1}^{L} \sum_{j\geq 0} \sum_{k\in\mathbb{N}_0} \langle \delta_N f, \tilde{\psi}^l_{j,k}\rangle \langle \tilde{\varphi}^l_{j,k}, \delta_N f\rangle \to \|f\|_2^2 \text{ as } N \to \infty. \tag{3.2.4}$$

But,

$$\begin{aligned}
\sum_{l=1}^{L} \sum_{j\geq 0} \sum_{k\in\mathbb{N}_0} \langle \delta_N f, \tilde{\psi}^l_{j,k}\rangle \langle \tilde{\varphi}^l_{j,k}, \delta_N f\rangle &= \sum_{l=1}^{L} \sum_{j\geq 0} \sum_{k\in\mathbb{N}_0} \langle \delta_N f, \psi^l_{j,k}\rangle \langle \varphi^l_{j,k}, \delta_N f\rangle \\
&= \sum_{l=1}^{L} \sum_{j\geq 0} \sum_{k\in\mathbb{N}_0} \langle f, \psi^l_{j-N,k}\rangle \langle \varphi^l_{j-N,k}, f\rangle \\
&= \sum_{l=1}^{L} \sum_{j\geq -N} \sum_{k\in\mathbb{N}_0} \langle f, \psi^l_{j,k}\rangle \langle \varphi^l_{j,k}, f\rangle.
\end{aligned}$$

Hence, by (3.2.4), we have

$$\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} \langle f, \psi^l_{j,k}\rangle \langle \varphi^l_{j,k}, f\rangle = \|f\|_2^2.$$

This shows that (3.2.1) holds for all $f = g$ with compact support. Since such functions are dense

in $L^2(K)$, (3.2.1) holds for all $f = g$ in $L^2(K)$. This completes the proof of the theorem. □

3.3 Co-affine systems

Recall that the quasi-affine system $\tilde{X}(\Psi)$ was obtained from the affine system $X(\Psi)$ by reversing the dilation and translation operations for negative scales $j < 0$ and then by renormalizing. It is a natural question to ask what happens if we reverse these operations for each scale $j \in \mathbb{Z}$. We make the following definition.

Definition 3.3.1. Let $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subset L^2(K)$ and let $c = \{c_{l,j} : 1 \leq l \leq L, j \in \mathbb{Z}\}$ be a sequence of scalars. The weighted *co-affine system* $X^*(\Psi, c)$ generated by Ψ and c is the collection

$$X^*(\Psi, c) = \{\psi^{*l}_{j,k} = c_{l,j}\tau_{u(k)}\delta_j\psi^l : 1 \leq l \leq L, j \in \mathbb{Z}, k \in \mathbb{N}_0\}.$$

This concept was first defined by Gressman, Labate, Weiss and Wilson in [36] for the case of the real line when Ψ consists of a single function and the dilation is a real number greater than 1. They proved that in this case the weighted co-affine system can never be a frame for $L^2(\mathbb{R})$. Johnson [45] extended this result to $L^2(\mathbb{R}^n)$ for finitely generated co-affine systems associated with expansive dilation matrices. In this section we will extend this result to the case of a local field of positive characteristic.

Let $X^*(\Psi, c)$ be a weighted co-affine system generated by Ψ and c. For $f \in L^2(K)$, define

$$w_f(x) = \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |\langle \tau_x f, \psi^{*l}_{j,k}\rangle|^2.$$

By Proposition 2.1.5(b) and (c), it follows that $w_f(x + u(n)) = w_f(x)$ for all $n \in \mathbb{N}_0$, that is, w_f is integral-periodic. We first prove a lemma.

Lemma 3.3.2. *If $X^*(\Psi, c)$ is a Bessel system for $L^2(K)$, then for each $f \in L^2(K)$, we have*

$$\int_{\mathfrak{D}} w_f(x)dx = \int_K \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |c_{l,j}|^2 q^{-j} |\hat{\psi}^l(\mathfrak{p}^j\xi)|^2 |\hat{f}(\xi)|^2 d\xi.$$

Proof. The result follows from the Plancherel theorem and the fact that

$$(\tau_y \delta_j g)^\wedge(\xi) = q^{-j/2}\hat{g}(\mathfrak{p}^j\xi)\overline{\chi_\xi(y)} \quad \text{for } y \in K, j \in \mathbb{Z}.$$

We have,

$$\begin{aligned}
\int_{\mathfrak{D}} w_f(x)dx &= \int_{\mathfrak{D}} \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |\langle \tau_x f, c_{l,j}\tau_{u(k)}\delta_j\psi^l\rangle|^2 dx \\
&= \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |c_{l,j}|^2 \int_{\mathfrak{D}} \sum_{k\in\mathbb{N}_0} |\langle f, \tau_{-x+u(k)}\delta_j\psi^l\rangle|^2 dx \\
&= \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |c_{l,j}|^2 \int_{K} |\langle f, \tau_{-x}\delta_j\psi^l\rangle|^2 dx \\
&= \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |c_{l,j}|^2 \int_{K} \left|\int_K \hat{f}(\xi)q^{-j/2}\overline{\hat{\psi}^l(\mathfrak{p}^j\xi)}\chi_x(\xi)d\xi\right|^2 dx \\
&= \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |c_{l,j}|^2 q^{-j} \int_{K} \left|\left(\hat{f}\,\overline{\hat{\psi}^l(\mathfrak{p}^j\cdot)}\right)^\vee(x)\right|^2 dx \\
&= \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |c_{l,j}|^2 q^{-j} \int_{K} |\hat{f}(\xi)|^2|\hat{\psi}^l(\mathfrak{p}^j\xi)|^2 d\xi.
\end{aligned}$$

□

We now use this lemma to show that there do not exist any co-affine frame in $L^2(K)$.

Theorem 3.3.3. *Let $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subset L^2(K)$ and $c = \{c_{l,j} : 1 \le l \le L, j \in \mathbb{Z}\}$ be a sequence of scalars. Then $X^*(\Psi, c)$ cannot be a frame for $L^2(K)$.*

Proof. Suppose that $X^*(\Psi, c)$ is a frame with bounds A^* and B^*. That is,

$$A^*\|f\|_2^2 \le \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |\langle f, \psi^{*l}_{j,k}\rangle|^2 \le B^*\|f\|_2^2 \quad \text{for all } f \in L^2(K).$$

Taking $f = \psi^{*l_0}_{j_0,k_0}$ for a fixed $j_0 \in \mathbb{Z}$, $k_0 \in \mathbb{N}_0$ and $1 \le l_0 \le L$, we have

$$\|\psi^{*l_0}_{j_0,k_0}\|_2^4 \le \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |\langle \psi^{*l_0}_{j_0,k_0}, \psi^{*l}_{j,k}\rangle|^2 \le B^*\|\psi^{*l_0}_{j_0,k_0}\|_2^2.$$

This implies $\|\psi^{*l_0}_{j_0,k_0}\|_2^2 \le B^*$. Since $\|\psi^{*l_0}_{j_0,k_0}\|_2 = |c_{l_0,j_0}|\|\psi^{l_0}\|_2$, it follows that

$$|c_{l_0,j_0}|^2 \le B^*\|\psi^{l_0}\|_2^{-2} \quad \text{for all } j_0 \in \mathbb{Z} \text{ and } 1 \le l_0 \le L. \tag{3.3.1}$$

From the definition of w_f, we have

$$A^*\|f\|_2^2 \le w_f(x) \le B^*\|f\|_2^2 \quad \text{for all } f \in L^2(K).$$

Integrating over $\mathfrak{D}$ and applying Lemma 3.3.2, we get

$$A^*\|f\|_2^2 \le \int_K \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |c_{l,j}|^2 q^{-j}|\hat{\psi}^l(\mathfrak{p}^j\xi)|^2|\hat{f}(\xi)|^2 d\xi \le B^*\|f\|_2^2 \quad \text{for all } f \in L^2(K).$$

From this we conclude that

$$A^* \le \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |c_{l,j}|^2 q^{-j}|\hat{\psi}^l(\mathfrak{p}^j\xi)|^2 \le B^* \quad \text{for a.e. } \xi \in K.$$

Now, integrating over $\mathfrak{P}^{-1} \setminus \mathfrak{D}$ after making the substitution $\xi \to \mathfrak{p}^n\xi, n \in \mathbb{Z}$, we have

$$\begin{aligned} A^*|\mathfrak{P}^{-1}\setminus\mathfrak{D}| &\le \int_{\mathfrak{P}^{-1}\setminus\mathfrak{D}} \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |c_{l,j}|^2 q^{-j}|\hat{\psi}^l(\mathfrak{p}^{j+n}\xi)|^2 d\xi \\ &= \int_{\mathfrak{P}^{-1}\setminus\mathfrak{D}} \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |c_{l,j-n}|^2 q^{-j+n}|\hat{\psi}^l(\mathfrak{p}^j\xi)|^2 d\xi \\ &= q^n \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} \int_{\mathfrak{p}^j(\mathfrak{P}^{-1}\setminus\mathfrak{D})} |c_{l,j-n}|^2|\hat{\psi}^l(\xi)|^2 d\xi. \end{aligned}$$

Applying (3.3.1), we have

$$\begin{aligned} A^*(q-1) &\le q^n \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} B^*\|\psi^l\|_2^{-2} \int_{\mathfrak{p}^j(\mathfrak{P}^{-1}\setminus\mathfrak{D})} |\hat{\psi}^l(\xi)|^2 d\xi \\ &= q^n \sum_{l=1}^{L} B^*\|\psi^l\|_2^{-2} \sum_{j\in\mathbb{Z}} \int_{\mathfrak{p}^j(\mathfrak{P}^{-1}\setminus\mathfrak{D})} |\hat{\psi}^l(\xi)|^2 d\xi \\ &= q^n \sum_{l=1}^{L} B^* = q^n L B^*. \end{aligned}$$

That is,

$$A^*(q-1) \leq q^n LB^* \quad \text{for each } n \in \mathbb{Z}.$$

Letting $n \to -\infty$, we see that $A^* = 0$. Hence, $X^*(\Psi, c)$ cannot be a frame for $L^2(K)$. □

Chapter 4

Characterization Of Wavelets

The characterization of all wavelets of $L^2(\mathbb{R})$ has been obtained independently by Wang [74] and Gripenberg [37] (see also [38]). We refer to [42] for an excellent exposition of this result. This characterization was essentially in terms of two basic equations. Calogero [17] obtained the characterization of wavelets of $L^2(\mathbb{R}^n)$ associated with general lattices. Bownik [15] gave a new proof of characterizing multiwavelets in $L^2(\mathbb{R}^n)$ using the results on shift invariant systems and quasi-affine systems in [67, 68] and [22]. In this chapter, we extend the result of Bownik and give a characterization of wavelets on local fields of positive characteristic. To achieve this, we have used the results obtained in Chapter 3.

As we have mentioned in Chapter 2, one can always construct a wavelet from an MRA. But, all wavelets are not obtained in this way. It was an open question for sometime to determine which wavelet could be constructed from an MRA of $L^2(\mathbb{R})$. P. Auscher [4] and P. G. Lemarié [57] gave very mild sufficient conditions for this. It was proved independently by G. Gripenberg [37] and X. Wang [74] that a wavelet arises from an MRA if and only if its dimension function is 1 a.e. Calogero and Garrigós [18] gave a characterization of wavelet families arising from biorthogonal MRAs of multiplicity d. In this chapter, we give a characterization of MRA wavelets on local fields of positive characteristic.

This chapter is structured as follows. In section 4.1, we first prove a result that characterizes the orthonormality of an affine system $X(\Psi)$ in $L^2(K)$ and provide an expression for the dual Gramian of the quasi-affine system $\tilde{X}(\Psi)$. With the help of these results, we will provide a characterization of wavelets on local fields of positive characteristic. We also give another

characterization of wavelets. In section 4.2, we will define the dimension function of a set of basic wavelets and characterize the MRA-wavelets in terms of the dimension function.

4.1 The characterization of wavelets

Let $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\}$ be a finite family of functions in $L^2(K)$. Recall that the affine system generated by Ψ is the collection

$$X(\Psi) = \{\psi^l_{j,k} : 1 \le l \le L, j \in \mathbb{Z}, k \in \mathbb{N}_0\},$$

where $\psi^l_{j,k}(x) = q^{j/2}\psi^l(\mathfrak{p}^{-j}x - u(k)) = \delta_j \tau_{u(k)}\psi^l(x)$. The quasi-affine system generated by Ψ is

$$\tilde{X}(\Psi) = \{\tilde{\psi}^l_{j,k} : 1 \le l \le L, j \in \mathbb{Z}, k \in \mathbb{N}_0\},$$

where

$$\tilde{\psi}^l_{j,k}(x) = \begin{cases} \delta_j \tau_{u(k)}\psi^l(x) = q^{j/2}\psi^l(\mathfrak{p}^{-j}x - u(k)), & j \ge 0, k \in \mathbb{N}_0. \\ q^{j/2}\tau_{u(k)}\delta_j\psi^l(x) = q^j\psi^l(\mathfrak{p}^{-j}(x - u(k))), & j < 0, k \in \mathbb{N}_0. \end{cases} \tag{4.1.1}$$

Observe that Ψ is a set of basic wavelets of $L^2(K)$ if the affine system $X(\Psi)$ forms an orthonormal basis for $L^2(K)$.

Definition 4.1.1. Given $\{t_i : i \in \mathbb{N}\} \subset \ell^2(\mathbb{N}_0)$, define the operator $H : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N})$ by

$$H(v) = \Big(\langle v, t_i\rangle\Big)_{i\in\mathbb{N}}.$$

If H is bounded then $\tilde{G} = H^*H : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ is called the *dual Gramian* of $\{t_i : i \in \mathbb{N}\}$.

Observe that $\tilde{G}$ is a nonnegative definite operator on $\ell^2(\mathbb{N}_0)$. Also, note that for $r, s \in \mathbb{N}_0$, we have

$$\langle \tilde{G}e_r, e_s\rangle = \langle He_r, He_s\rangle = \sum_{i\in\mathbb{N}} \overline{t_i(r)}t_i(s),$$

where $\{e_i : i \in \mathbb{N}_0\}$ is the standard basis of $\ell^2(\mathbb{N}_0)$.

The following result characterizes when the system of translates of a given family of functions is a frame in terms of the dual Gramian. The proof is an easy generalization of the corresponding

results on the Euclidean cases given in [68] and [14].

Theorem 4.1.2. *Let* $\{\varphi_i : i \in \mathbb{N}\} \subset L^2(K)$ *and for a.e.* $\xi \in \mathfrak{D}$, *let* $\tilde{G}(\xi)$ *denote the dual Gramian of* $\{t_i = (\hat{\varphi}_i(\xi + u(k)))_{k\in\mathbb{N}_0} : i \in \mathbb{N}\} \subset \ell^2(\mathbb{N}_0)$. *The system of translates* $\{\tau_{u(k)}\varphi_i : k \in \mathbb{N}_0, i \in \mathbb{N}\}$ *is a frame for* $L^2(K)$ *with constants* A *and* B *if and only if* $\tilde{G}(\xi)$ *is bounded for a.e.* $\xi \in \mathfrak{D}$ *and*

$$A\|v\|^2 \le \langle \tilde{G}(\xi)v, v\rangle \le B\|v\|^2 \text{ for } v \in \ell^2(\mathbb{N}_0) \text{ and for a.e. } \xi \in \mathfrak{D},$$

that is, the spectrum of $\tilde{G}(\xi)$ *is contained in* $[A, B]$ *for a.e.* $\xi \in \mathfrak{D}$.

We first prove a lemma which gives necessary and sufficient conditions for the orthonormality of an affine system.

Lemma 4.1.3. *Suppose that* $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subseteq L^2(K)$. *The affine system* $X(\Psi)$ *is orthonormal in* $L^2(K)$ *if and only if*

$$\sum_{k\in\mathbb{N}_0} \hat{\psi}^l(\xi + u(k))\overline{\hat{\psi}^m(\mathfrak{p}^{-j}(\xi + u(k)))} = \delta_{j,0}\delta_{l,m} \tag{4.1.2}$$

for a.e. $\xi \in K$, $1 \le l, m \le L$, $j \ge 0$.

Proof. Using Proposition 2.1.5 (b) and (c), we observe that

$$\langle \psi^l_{j,k}, \psi^{l'}_{j',k'}\rangle = \delta_{l,l'}\delta_{j,j'}\delta_{k,k'}, \quad 1 \le l, l' \le L,\ j, j' \in \mathbb{Z},\ k, k' \in \mathbb{N}_0$$

is equivalent to

$$\langle \psi^l_{j,k}, \psi^{l'}_{0,0}\rangle = \delta_{l,l'}\delta_{j,0}\delta_{k,0}, \quad 1 \le l, l' \le L, j \ge 0, k \in \mathbb{N}_0.$$

Now, let $1 \le l, l' \le L$, $j \ge 0$, $k \in \mathbb{N}_0$. Then

$$\begin{aligned}
\langle \psi^l_{j,k}, \psi^{l'}_{0,0}\rangle &= \langle \hat{\psi}^l_{j,k}, \hat{\psi}^{l'}_{0,0}\rangle \\
&= \int_K q^{-j/2}\hat{\psi}^l(\mathfrak{p}^j\xi)\overline{\chi_k(\mathfrak{p}^j\xi)}\overline{\hat{\psi}^{l'}(\xi)}d\xi \\
&= \int_K q^{j/2}\hat{\psi}^l(\xi)\overline{\chi_k(\xi)}\overline{\hat{\psi}^{l'}(\mathfrak{p}^{-j}\xi)}d\xi
\end{aligned}$$

$$= q^{j/2}\int_{\mathfrak{D}}\Big\{\sum_{n\in\mathbb{N}_0}\hat{\psi}^l(\xi+u(n))\overline{\hat{\psi}^{l'}(\mathfrak{p}^{-j}(\xi+u(n)))}\Big\}\overline{\chi_k(\xi)}d\xi.$$

If $\langle\psi^l_{j,k},\psi^{l'}_{0,0}\rangle = \delta_{l,l'}\delta_{j,0}\delta_{k,0}$ for all $l,l'\in\{1,2,\ldots,L\}$, $j\geq 0$ and $k\in\mathbb{N}_0$, then the $L^1(\mathfrak{D})$ function F, where $F(\xi)=\sum_{n\in\mathbb{N}_0}\hat{\psi}^l(\xi+u(n))\overline{\hat{\psi}^{l'}(\mathfrak{p}^{-j}(\xi+u(n)))}$, has the property that its Fourier coefficients are all zero except for the coefficient corresponding to $k=0$, which is 1 if $j=0$ and $l=l'$. Hence, $F=\delta_{j,0}\delta_{l,l'}$ a.e. Conversely, if $F=\delta_{j,0}\delta_{l,l'}$ a.e., then the same calculation shows that $\langle\psi^l_{j,k},\psi^{l'}_{0,0}\rangle=\delta_{l,l'}\delta_{j,0}\delta_{k,0}$, since $\{\chi_n : n\in\mathbb{N}_0\}$ is an orthonormal basis of $L^2(\mathfrak{D})$ (see Proposition 1.2.6). □

Define D_j as follows:

$$D_j=\begin{cases}\{0,1,\ldots,q^j-1\}, & j\geq 0,\\ \{0\}, & j<0.\end{cases}$$

Let

$$\begin{aligned}\mathcal{A} &= \{\tilde{\psi}^l_{j,d}: 1\leq l\leq L, j\in\mathbb{Z}, d\in D_j\}\\ &= \{\tilde{\psi}^l_{j,0}: 1\leq l\leq L, j<0\}\cup\{\tilde{\psi}^l_{j,d}: 1\leq l\leq L, j\geq 0, d\in D_j\}.\end{aligned}$$

If $j<0$, then $\tau_{u(k)}\tilde{\psi}^l_{j,0}(x)=\tilde{\psi}^l_{j,0}(x-u(k))=q^j\psi^l(\mathfrak{p}^{-j}(x-u(k)))=\tilde{\psi}^l_{j,k}$. For $j\geq 0$, $0\leq d\leq q^j-1$, $k\geq 0$, we have

$$\begin{aligned}\tau_{u(k)}\tilde{\psi}^l_{j,d}(x) &= \tilde{\psi}^l_{j,d}(x-u(k))=\psi^l_{j,d}(x-u(k))\\ &= q^{j/2}\psi^l(\mathfrak{p}^{-j}(x-u(k))-u(d))\\ &= q^{j/2}\psi^l(\mathfrak{p}^{-j}x-(\mathfrak{p}^{-j}u(k)+u(d)))\\ &= q^{j/2}\psi^l(\mathfrak{p}^{-j}x-u(kq^j+d))\\ &= \psi^l_{j,kq^j+d}(x).\end{aligned}$$

Since it is true that for each $j\geq 0$, every non negative integer m can uniquely be written as $m=kq^j+d$, where $k\in\mathbb{N}_0$, $d\in D_j$, it follows that

$$\tilde{X}(\Psi)=\{\tau_{u(k)}\varphi : k\in\mathbb{N}_0,\varphi\in\mathcal{A}\}.$$

We now define the dual Gramian $\tilde{G}(\xi)$ of the quasi-affine system $\tilde{X}(\Psi)$ at $\xi \in \mathfrak{D}$ to be the dual Gramian of $\{(\hat{\varphi}(\xi + u(k)))_{k\in\mathbb{N}_0} : \varphi \in \mathcal{A}\} \subset \ell^2(\mathbb{N}_0)$. The following lemma will be used later in the computation of $\tilde{G}(\xi)$. To prove this lemma we use a technique used by Zheng in [76].

Lemma 4.1.4. *Let $j \geq 0$. For $p, k \in \mathbb{N}_0$, we have*

$$q^{-j} \sum_{t\in D_j} \chi\Big(\big(u(p) - u(k)\big)\mathfrak{p}^j u(t)\Big) = \begin{cases} 1, & if \quad |p-k| \in q^j\mathbb{N}_0, \\ 0, & otherwise. \end{cases} \tag{4.1.3}$$

Proof. The integers $p, k \in \mathbb{N}_0$ can uniquely be written as $p = r + q^j m_1$ and $k = s + q^j m_2$, where $m_1, m_2 \in \mathbb{N}_0$ and $0 \leq r, s \leq q^j - 1$. Using (1.2.4), we have $u(p) = u(r) + \mathfrak{p}^{-j}u(m_1)$ and $u(k) = u(s) + \mathfrak{p}^{-j}u(m_2)$. Hence,

$$\begin{aligned} \chi\big((u(p) - u(k))\mathfrak{p}^j u(t)\big) &= \chi\big((u(r) - u(s))\mathfrak{p}^j u(t) + (u(m_1) - u(m_2))u(t)\big) \\ &= \chi\big((u(r) - u(s))\mathfrak{p}^j u(t)\big), \end{aligned}$$

since $\chi(u(k)u(l)) = 1$ for $k, l \in \mathbb{N}_0$ (see Proposition 1.2.7). Since $r = s$ if and only if $|p - k| \in q^j\mathbb{N}_0$, it is enough to show that if $0 \leq r, s \leq q^j - 1$, then

$$q^{-j} \sum_{t=0}^{q^j-1} \chi\big((u(r) - u(s))\mathfrak{p}^j u(t)\big) = \delta_{r,s}. \tag{4.1.4}$$

If $r = s$ then $u(r) - u(s) = 0$, hence both sides of the equation (4.1.4) are 1. We now assume that $r \neq s$. Let

$$r = a_0 + a_1 q + \cdots + a_{j-1}q^{j-1}, \qquad s = b_0 + b_1 q + \cdots + b_{j-1}q^{j-1},$$

and

$$t = c_0 + c_1 q + \cdots + c_{j-1}q^{j-1},$$

where $0 \leq a_m, b_m, c_m \leq q - 1$ for $m = 0, 1, \ldots, j - 1$. Then from (1.2.3), we have

$$u(r) = u(a_0) + u(a_1)\mathfrak{p}^{-1} + \cdots + u(a_{j-1})\mathfrak{p}^{-j+1},$$

$$u(s) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_{j-1})\mathfrak{p}^{-j+1},$$

and

$$u(t) = u(c_0) + u(c_1)\mathfrak{p}^{-1} + \cdots + u(c_{j-1})\mathfrak{p}^{-j+1}.$$

Recall that $\mathfrak{D}/\mathfrak{P} \cong GF(q) \cong \text{span}\{\epsilon_j\}_{j=0}^{c-1}$. Since $\{u(n)\mathfrak{p} : n = 0, 1, \ldots, q-1\}$ is a complete set of coset representatives of $\mathfrak{P}$ in $\mathfrak{D}$, for each $n = 0, 1, \ldots, j-1$, we can write

$$u(c_n)\mathfrak{p} = \lambda_0^n\epsilon_0 + \lambda_1^n\epsilon_1 + \cdots + \lambda_{c-1}^n\epsilon_{c-1},$$

where $0 \le \lambda_0^n, \lambda_1^n, \ldots, \lambda_{c-1}^n \le p-1$. It can easily be seen that for each $l = 0, 1, \ldots, c-1$, $\{\epsilon_l u(n)\mathfrak{p} : n = 0, 1, \ldots, q-1\}$ is also a complete set of coset representatives of $\mathfrak{P}$ in $\mathfrak{D}$. Hence, we have

$$\epsilon_l u(a_m)\mathfrak{p} = \alpha_0^{m,l}\epsilon_0 + \alpha_1^{m,l}\epsilon_1 + \cdots + \alpha_{c-1}^{m,l}\epsilon_{c-1}, \quad l = 0, 1, \ldots, c-1,$$

where $0 \le \alpha_0^{m,l}, \alpha_1^{m,l}, \ldots, \alpha_{c-1}^{m,l} \le p-1$. Therefore,

$$\begin{aligned} u(r)\mathfrak{p}^j u(t) &= \sum_{m=0}^{j-1}\sum_{n=0}^{j-1}\big(u(a_m)\mathfrak{p}u(c_n)\mathfrak{p}\big)\mathfrak{p}^{j-m-n-2} \\ &= \sum_{m=0}^{j-1}\sum_{n=0}^{j-1}\Big(\sum_{l=0}^{c-1}\lambda_l^n\epsilon_l u(a_m)\mathfrak{p}\Big)\mathfrak{p}^{j-m-n-2} \\ &= \sum_{m=0}^{j-1}\sum_{n=0}^{j-1}\sum_{l=0}^{c-1}\sum_{k=0}^{c-1}\lambda_l^n\alpha_k^{m,l}\epsilon_k\mathfrak{p}^{j-m-n-2}. \end{aligned}$$

By the definition of the character χ (see (1.2.5)), we have

$$\chi(u(r)\mathfrak{p}^j u(t)) = \exp\Big(\tfrac{2\pi i}{p}\sum_{l=0}^{c-1}(\lambda_l^0\alpha_0^{j-1,l} + \lambda_l^1\alpha_0^{j-2,l} + \cdots + \lambda_l^{j-1}\alpha_0^{0,l})\Big).$$

Similarly, we can write

$$\chi(u(s)\mathfrak{p}^j u(t)) = \exp\Big(\tfrac{2\pi i}{p}\sum_{l=0}^{c-1}(\lambda_l^0\beta_0^{j-1,l} + \lambda_l^1\beta_0^{j-2,l} + \cdots + \lambda_l^{j-1}\beta_0^{0,l})\Big),$$

where $0 \leq \beta_0^{m,l} \leq p-1$ for $m = 0, 1, \ldots, j-1$ and $l = 0, 1, \ldots, c-1$.

Observe that as t varies from 0 to $q^j - 1$, the integers $c_0, \ldots, c_{j-1}$ all vary from 0 to $q-1$. Hence, the integers λ_l^n vary from 0 to $p-1$ for $0 \leq l \leq c-1$ and $0 \leq n \leq j-1$. Therefore,

$$
\begin{aligned}
&\sum_{t=0}^{q^j-1} \chi\big((u(r)-u(s))\mathfrak{p}^j u(t)\big) \\
&= \sum_{t=0}^{q^j-1} \chi\big(u(r)\mathfrak{p}^j u(t)\big)\overline{\chi\big(u(s)\mathfrak{p}^j u(t)\big)} \\
&= \left(\sum_{\lambda_0^0=0}^{p-1} \exp\Big(\tfrac{2\pi i}{p}(\alpha_0^{j-1,0} - \beta_0^{j-1,0})\lambda_0^0\Big)\right) \cdots \left(\sum_{\lambda_0^{j-1}=0}^{p-1} \exp\Big(\tfrac{2\pi i}{p}(\alpha_0^{0,0} - \beta_0^{0,0})\lambda_0^{j-1}\Big)\right) \\
&\quad \times \left(\sum_{\lambda_1^0=0}^{p-1} \exp\Big(\tfrac{2\pi i}{p}(\alpha_0^{j-1,1} - \beta_0^{j-1,1})\lambda_1^0\Big)\right) \cdots \left(\sum_{\lambda_1^{j-1}=0}^{p-1} \exp\Big(\tfrac{2\pi i}{p}(\alpha_0^{0,1} - \beta_0^{0,1})\lambda_1^{j-1}\Big)\right) \\
&\quad \cdots \\
&\quad \times \left(\sum_{\lambda_{c-1}^0=0}^{p-1} \exp\Big(\tfrac{2\pi i}{p}(\alpha_0^{j-1,c-1} - \beta_0^{j-1,c-1})\lambda_{c-1}^0\Big)\right) \\
&\quad \cdots \left(\sum_{\lambda_{c-1}^{j-1}=0}^{p-1} \exp\Big(\tfrac{2\pi i}{p}(\alpha_0^{0,c-1} - \beta_0^{0,c-1})\lambda_{c-1}^{j-1}\Big)\right).
\end{aligned}
$$

Since $r \neq s$, we have $a_m \neq b_m$ for some $m = 0, 1, \ldots, j-1$. We claim that there exists some $l \in \{0, 1, \ldots, c-1\}$ such that $\alpha_0^{m,l} \neq \beta_0^{m,l}$. If $\alpha_0^{m,l} = \beta_0^{m,l}$ for all $l \in \{0, 1, \ldots, c-1\}$, then since $u(a_m)\mathfrak{p} \neq u(b_m)\mathfrak{p}$, we have

$$
\begin{aligned}
GF(q) &= \operatorname{span}\{(u(a_m)\mathfrak{p} - u(b_m)\mathfrak{p})\epsilon_l\}_{l=0}^{c-1} \\
&= \operatorname{span}\{(\alpha_0^{m,l} - \beta_0^{m,l})\epsilon_0 + \cdots + (\alpha_{c-1}^{m,l} - \beta_{c-1}^{m,l})\epsilon_{c-1}\} \\
&\subseteq \operatorname{span}\{\epsilon_1, \epsilon_2, \ldots, \epsilon_{c-1}\}.
\end{aligned}
$$

This is a contradiction which proves the claim. Now for any l such that $\alpha_0^{m,l} \neq \beta_0^{m,l}$, we observe that

$$
\sum_{\lambda_l^{j-1-m}=0}^{p-1} \exp\Big(\tfrac{2\pi i}{p}(\alpha_0^{m,l} - \beta_0^{m,l})\lambda_l^{j-1-m}\Big)
$$

is a factor in the above product. But its value is equal to

$$\frac{1-\exp\left(2\pi i(\alpha_0^{m,l}-\beta_0^{m,l})\right)}{1-\exp\left(\frac{2\pi i}{p}(\alpha_0^{m,l}-\beta_0^{m,l})\right)} = 0,$$

since $\alpha_0^{m,l} - \beta_0^{m,l}$ is an integer with absolute value less than p. This completes the proof of the lemma. □

For $s \in \mathbb{N}_0 \setminus q\mathbb{N}_0$, define the function

$$t_s(\xi) = \sum_{l=1}^{L}\sum_{j=0}^{\infty} \hat{\psi}^l(\mathfrak{p}^{-j}\xi)\overline{\hat{\psi}^l(\mathfrak{p}^{-j}(\xi+u(s)))}. \tag{4.1.5}$$

In the following lemma we compute the dual Gramian $\tilde{G}(\xi)$ of the quasi-affine system $\tilde{X}(\Psi)$ in terms of the Fourier transforms of functions in Ψ.

Lemma 4.1.5. *Let $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subseteq L^2(K)$ and $\tilde{G}(\xi)$ be the dual Gramian of $\tilde{X}(\Psi)$ at $\xi \in \mathfrak{D}$. Then*

$$\langle \tilde{G}(\xi)e_k, e_k\rangle = \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |\hat{\psi}^l(\mathfrak{p}^{-j}(\xi+u(k)))|^2 \qquad \textit{for } k \in \mathbb{N}_0, \tag{4.1.6}$$

and

$$\langle \tilde{G}(\xi)e_{k'}, e_k\rangle = t_s(\mathfrak{p}^m\xi + \mathfrak{p}^m k) \qquad \textit{for } k, k' \in \mathbb{N}_0, k \neq k', \tag{4.1.7}$$

where $m = \max\{j \geq 0 : |k'-k| \in q^j\mathbb{N}_0\}$, $s \in \mathbb{N}_0 \setminus q\mathbb{N}_0$ is such that $u(s) = \mathfrak{p}^m(u(k') - u(k))$, and t_s is the function defined in (4.1.5).

Proof. For $k, k' \in \mathbb{N}_0$, we have

$$\begin{aligned}
\langle \tilde{G}(\xi)e_{k'}, e_k\rangle &= \sum_{\varphi\in\mathcal{A}} \hat{\varphi}(\xi+u(k))\overline{\hat{\varphi}(\xi+u(k'))} \\
&= \sum_{l=1}^{L}\sum_{j<0} \hat{\psi}^l(\mathfrak{p}^j(\xi+u(k)))\overline{\hat{\psi}^l(\mathfrak{p}^j(\xi+u(k')))} \\
&\quad + \sum_{l=1}^{L}\sum_{j\geq 0} \hat{\psi}^l(\mathfrak{p}^j(\xi+u(k)))\overline{\hat{\psi}^l(\mathfrak{p}^j(\xi+u(k')))} \\
&\quad \times \sum_{d\in D_j} q^{-j}\overline{\chi_{\xi+u(k)}(\mathfrak{p}^j u(d))}\chi_{\xi+u(k')}(\mathfrak{p}^j u(d))).
\end{aligned}$$

The sum over D_j is equal to

$$\sum_{d\in D_j} q^{-j}\overline{\chi_{u(k)}(\mathfrak{p}^j u(d))}\chi_{u(k')}(\mathfrak{p}^j u(d))) = \sum_{d\in D_j} q^{-j}\chi\Big(u(d)\mathfrak{p}^j(u(k')-u(k))\Big).$$

By Lemma 4.1.4, this expression is equal to 1 if $|k'-k| \in q^j\mathbb{N}_0$ and 0 otherwise. Therefore, if $k = k'$, then

$$\langle \tilde{G}(\xi)e_k, e_k\rangle = \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} \mid \hat{\psi}^l(\mathfrak{p}^{-j}(\xi+u(k))) \mid^2 \qquad \text{for } k \in \mathbb{N}_0,$$

If $k \neq k'$, let $m = \max\{j \geq 0 : |k'-k| \in q^j\mathbb{N}_0\}$. Then, the sum over D_j will contribute 1 for each $j = 0, 1\ldots, m$ and then 0 from $m+1$ onwards. Thus,

$$\begin{aligned}
\langle \tilde{G}(\xi)e_{k'}, e_k\rangle &= \sum_{l=1}^{L}\sum_{j=-\infty}^{m} \hat{\psi}^l(\mathfrak{p}^j(\xi+u(k)))\overline{\hat{\psi}^l(\mathfrak{p}^j(\xi+u(k')))} \\
&= \sum_{l=1}^{L}\sum_{j=-m}^{\infty} \hat{\psi}^l(\mathfrak{p}^{-j}(\xi+u(k)))\overline{\hat{\psi}^l(\mathfrak{p}^{-j}(\xi+u(k')))} \\
&= \sum_{l=1}^{L}\sum_{j=0}^{\infty} \hat{\psi}^l(\mathfrak{p}^{m-j}(\xi+u(k)))\overline{\hat{\psi}^l(\mathfrak{p}^{m-j}(\xi+u(k')))} \\
&= \sum_{l=1}^{L}\sum_{j=0}^{\infty} \hat{\psi}^l(\mathfrak{p}^{-j}(\mathfrak{p}^m\xi+\mathfrak{p}^m u(k))) \\
&\qquad \times\overline{\hat{\psi}^l(\mathfrak{p}^{-j}(\mathfrak{p}^m\xi+\mathfrak{p}^m u(k)+\mathfrak{p}^m(u(k')-u(k))))}.
\end{aligned}$$

Let $k' = c_0 + c_1 q + \cdots + c_J q^J$ and $k = d_0 + d_1 q + \cdots + d_J q^J$. Since $|k'-k| \in q^m\mathbb{N}_0$ but $|k'-k| \notin q^{m+1}\mathbb{N}_0$, we have $c_i = d_i$ for all $i = 1, 2, \ldots, m-1$ and $c_m \neq d_m$. Hence,

$$\begin{aligned}
\mathfrak{p}^m(u(k')-u(k)) &= u(c_m + c_{m+1}q + \cdots + c_J q^{J-m}) - u(d_m + d_{m+1}q + \cdots + d_J q^{J-m}) \\
&= u(s)
\end{aligned}$$

for some $s \in \mathbb{N}_0$, by Proposition 2.1.5(b) and (c). Note that $s \notin q\mathbb{N}_0$, since, otherwise $u(s) = \mathfrak{p}^{-1}u(n)$ for some $n \in \mathbb{N}_0$. This will imply that $c_m = d_m$, which is false. Therefore,

$$\langle \tilde{G}(\xi)e_{k'}, e_k\rangle = t_s(\mathfrak{p}^m\xi + \mathfrak{p}^m u(k)),$$

where $s \in \mathbb{N}_0 \setminus q\mathbb{N}_0$ is as defined above. This completes the proof of the lemma. □

In the following theorem, we provide necessary and sufficient conditions for the affine system $X(\Psi)$ to be a tight frame in $L^2(K)$. As a consequence, we get a characterization of wavelets.

Theorem 4.1.6. *Suppose $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subseteq L^2(K)$. The affine system $X(\Psi)$ is a tight frame with constant 1 for $L^2(K)$, i.e.,*

$$\|f\|_2^2 = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} |\langle f, \psi_{l,j,k}\rangle|^2 \quad \text{for all } f \in L^2(K)$$

if and only if the functions $\psi^1, \psi^2, \ldots, \psi^L$ satisfy the following two conditions:

$$\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}^l(\mathfrak{p}^{-j}\xi)|^2 = 1 \quad \text{for a.e. } \xi \in K, \tag{4.1.8}$$

and

$$t_m(\xi) = 0 \quad \text{for a.e. } \xi \in K \text{ and for all } m \in \mathbb{N}_0 \setminus q\mathbb{N}_0. \tag{4.1.9}$$

In particular, Ψ is a set of basic wavelets of $L^2(K)$ if and only if $\|\psi^l\|_2 = 1$ for $l = 1, 2, \ldots, L$ and (4.1.8) and (4.1.9) hold.

Proof. It follows from Theorem 3.1.4 that $X(\Psi)$ is a tight frame with constant 1 if and only if $\tilde{X}(\Psi)$ is a tight frame with constant 1. By Theorem 4.1.2, this is equivalent to the spectrum of $\tilde{G}(\xi)$ consisting of a single point 1, i.e., $\tilde{G}(\xi)$ is identity on $\ell^2(\mathbb{N}_0)$ for a.e. $\xi \in \mathfrak{D}$. By Lemma 4.1.5, this is equivalent to the fact that equations (4.1.8) and (4.1.9) hold. The second assertion follows since a tight frame $X(\Psi)$ with constant 1 is an orthonormal basis if and only if $\|\psi^l\|_2 = 1$ for $l = 1, 2, \ldots, L$ (see Theorem 1.8, section 7.1 in [42]). □

The following theorem, initially proved by Bownik [15] for $\mathbb{R}^n$ with an integer dilation matrix, gives a new characterization of tight wavelet frames with constant 1. We extend this result to the case of local fields of positive characteristic.

Theorem 4.1.7. *Suppose $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subseteq L^2(K)$. Assume that $X(\Psi)$ is a Bessel family with constant 1. Then the following are equivalent:*

(a) $X(\Psi)$ is a tight frame with constant 1.

(b) Ψ satisfies (4.1.8).

(c) Ψ *satisfies*

$$\sum_{l=1}^{L}\int_K \frac{|\hat{\psi}^l(\xi)|^2}{|\xi|}d\xi = \frac{q-1}{q}. \tag{4.1.10}$$

Proof. It is obvious from Theorem 4.1.6 that (a) $\Rightarrow$ (b). To show (b) implies (c), assume that (4.1.8) holds. Then, since $\{\mathbf{p}^j\mathfrak{D}^* : j \in \mathbb{Z}\}$ is a partition of K, we have

$$\begin{aligned}
\sum_{l=1}^{L}\int_K |\hat{\psi}^l(\xi)|^2\frac{d\xi}{|\xi|} &= \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\int_{\mathbf{p}^j\mathfrak{D}^*} |\hat{\psi}^l(\xi)|^2\frac{d\xi}{|\xi|} \\
&= \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\int_{\mathfrak{D}^*} |\hat{\psi}^l(\mathbf{p}^{-j}\xi)|^2\frac{q^j d\xi}{|\mathbf{p}^{-j}\xi|} \\
&= \int_{\mathfrak{D}^*}\Big(\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}|\hat{\psi}^l(\mathbf{p}^{-j}\xi)|^2\Big)\frac{d\xi}{|\xi|} \\
&= \int_{\mathfrak{D}^*}\frac{d\xi}{|\xi|} = |\mathfrak{D}^*| \\
&= \frac{q-1}{q}.
\end{aligned}$$

To prove (c) $\Rightarrow$ (a), we assume that (4.1.10) holds. Since $X(\Psi)$ is a Bessel family with constant 1, so is $\tilde{X}(\Psi)$, by Theorem 3.1.4(a). Let $\tilde{G}(\xi)$ be the dual Gramian of $\tilde{X}(\Psi)$ at $\xi \in \mathfrak{D}$. By Theorem 4.1.2, we have $\|\tilde{G}(\xi)\| \leq 1$ for a.e. $\xi \in \mathfrak{D}$. In particular, $\|\tilde{G}(\xi)e_k\| \leq 1$. Hence,

$$\begin{aligned}
1 \geq \|\tilde{G}(\xi)e_k\|^2 &= \sum_{p\in\mathbb{N}_0}|\langle\tilde{G}(\xi)e_k, e_p\rangle|^2 \\
&= |\langle\tilde{G}(\xi)e_k, e_k\rangle|^2 + \sum_{p\in\mathbb{N}_0, p\neq k}|\langle\tilde{G}(\xi)e_k, e_p\rangle|^2.
\end{aligned} \tag{4.1.11}$$

By Lemma (4.1.5), we have

$$\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}|\hat{\psi}^l(\mathbf{p}^{-j}(\xi+u(k)))|^2 = \langle\tilde{G}(\xi)e_k, e_k\rangle \leq 1 \text{ for } k \in \mathbb{N}_0, \text{ a.e. } \xi \in \mathfrak{D}.$$

Hence,

$$\frac{q-1}{q} = \sum_{l=1}^{L}\int_K \frac{|\hat{\psi}^l(\xi)|^2}{|\xi|}d\xi = \int_{\mathfrak{D}^*}\Big(\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}|\hat{\psi}^l(\mathbf{p}^{-j}\xi)|^2\Big)\frac{d\xi}{|\xi|} \leq \int_{\mathfrak{D}^*}\frac{d\xi}{|\xi|} = \frac{q-1}{q}.$$

From this it follows that $\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}|\hat{\psi}^l(\mathfrak{p}^{-j}\xi)|^2 = 1$ for a.e. $\xi \in \mathfrak{D}^*$ and hence for a.e. $\xi \in K$, i.e., equation (4.1.8) holds. By Lemma 4.1.5 and equation (4.1.8), $|\langle\tilde{G}(\xi)e_k, e_k\rangle|^2 = 1$ for all $k \in \mathbb{N}_0$. Hence by (4.1.11), it follows that $\langle\tilde{G}(\xi)e_k, e_{k'}\rangle = 0$ for $k \neq k'$ so that $\tilde{G}(\xi)$ is the identity operator on $\ell^2(\mathbb{N}_0)$. Hence, by Theorem 4.1.2, $\tilde{X}(\Psi)$ is a tight frame with constant 1. Therefore, $X(\Psi)$ is also a tight frame with constant 1, by Theorem 3.1.4. □

As a consequence of the above theorem, we get another characterization of a set of basic wavelets of $L^2(K)$.

Theorem 4.1.8. *Suppose* $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subseteq L^2(K)$. *Then the following are equivalent:*

(a) Ψ *is a set of basic wavelets of* $L^2(K)$.

(b) Ψ *satisfies* (4.1.2) *and* (4.1.8).

(c) Ψ *satisfies* (4.1.2) *and* (4.1.10).

Proof. It follows from Theorem 4.1.7 and Lemma 4.1.5 that (a) ⇒ (b) ⇒ (c). We now prove that (c) implies (a). Assume that Ψ satisfies (4.1.2) and (4.1.10). Equation (4.1.2) implies that $X(\Psi)$ is an orthonormal system, hence it is a Bessel family with constant 1. By Theorem 4.1.7 and equation (4.1.10), $X(\Psi)$ is a tight frame with constant 1. Since each ψ^l has L^2 norm 1, it follows that $X(\Psi)$ is an orthonormal basis for $L^2(K)$. That is, Ψ is a set of basic wavelets of $L^2(K)$. □

4.2 The characterization of MRA-wavelets

Let us recall the definition of an MRA on local fields of positive characteristic (see [44]).

Definition 4.2.1. Let K be a local field of characteristic $p > 0$, $\mathfrak{p}$ be a prime element of K and $u(n) \in K$ for $n \in \mathbb{N}_0$ be as defined in (1.2.2) and (1.2.3). A multiresolution analysis (MRA) of $L^2(K)$ is a sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(K)$ satisfying the following properties:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(K)$;

(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(d) $f \in V_j$ if and only if $f(\mathfrak{p}^{-1}\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(e) there is a function $\varphi \in V_0$, called the *scaling function*, such that $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms an orthonormal basis for V_0.

Let $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\}$ be a set of basic wavelets of $L^2(K)$. We define the spaces W_j, $j \in \mathbb{Z}$, by $W_j = \overline{\text{span}}\{\psi^l_{j,k} : 1 \le l \le L, k \in \mathbb{N}_0\}$. We also define $V_j = \bigoplus_{m<j} W_m$, $j \in \mathbb{Z}$. Then it follows that $\{V_j : j \in \mathbb{Z}\}$ satisfies the properties (a)-(d) in the definition of an MRA. Hence, $\{V_j : j \in \mathbb{Z}\}$ will form an MRA of $L^2(K)$ if we can find a function $\varphi \in L^2(K)$ such that the system $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ is an orthonormal basis for V_0. In this case, we say that Ψ *is associated with an MRA*, or simply that Ψ is an *MRA-wavelet*.

Now suppose that $\{\psi^1, \psi^2, \ldots, \psi^{q-1}\}$ is a set of basic wavelets for $L^2(K)$ associated with an MRA $\{V_j : j \in \mathbb{Z}\}$. Let $\varphi \in L^2(K)$ be the corresponding scaling function. In Theorem 2.3.1, we have characterized the scaling functions for MRAs of $L^2(K)$. In view of this theorem, we have

$$\sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = 1 \quad \text{for a.e. } \xi \in \mathfrak{D}, \tag{4.2.1}$$

$$\lim_{j\to\infty} |\hat{\varphi}(\mathfrak{p}^j\xi)| = 1 \quad \text{for a.e. } \xi \in K, \tag{4.2.2}$$

and

$$\hat{\varphi}(\xi) = m_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi) \quad \text{for a.e. } \xi \in K, \tag{4.2.3}$$

where m_0 is an integral-periodic function in $L^2(\mathfrak{D})$. Also, since $\{\psi^1, \psi^2, \ldots, \psi^{q-1}\}$ are the wavelets associated with an MRA corresponding to the scaling function φ, there exist integral-periodic functions m_l, $1 \le l \le q-1$, such that the matrix

$$M(\xi) = \left[m_{l_1}(\mathfrak{p}\xi + \mathfrak{p}u(l_2))\right]_{l_1,l_2=0}^{q-1}$$

is unitary for a.e. $\xi \in \mathfrak{D}$ (see [44]) and

$$\hat{\psi}^l(\xi) = m_l(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi).$$

Hence, we have

$$|\hat{\varphi}(\xi)|^2 + \sum_{l=1}^{q-1} |\hat{\psi}^l(\xi)|^2 = |m_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi)|^2 + \sum_{l=1}^{q-1} |m_l(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi)|^2 = |\hat{\varphi}(\mathfrak{p}\xi)|^2 \Big(\sum_{l=0}^{q-1} |m_l(\mathfrak{p}\xi)|^2 \Big).$$

Since $M(\xi)$ is unitary, we have

$$|\hat{\varphi}(\xi)|^2 + \sum_{l=1}^{q-1} |\hat{\psi}^l(\xi)|^2 = |\hat{\varphi}(\mathfrak{p}\xi)|^2.$$

This equality holds for a.e. $\xi \in K$. Hence, we have

$$|\hat{\varphi}(\xi)|^2 = |\hat{\varphi}(\mathfrak{p}^{-1}\xi)|^2 + \sum_{l=1}^{q-1} |\hat{\psi}^l(\mathfrak{p}^{-1}\xi)|^2.$$

Iterating, we get, for any integer $N \geq 1$,

$$|\hat{\varphi}(\xi)|^2 = |\hat{\varphi}(\mathfrak{p}^{-N}\xi)|^2 + \sum_{l=1}^{q-1} \sum_{j=1}^{N} |\hat{\psi}^l(\mathfrak{p}^{-j}\xi)|^2.$$

Since $|\hat{\varphi}(\xi)| \leq 1$, the sequence $\{\sum_{j=1}^{N} \sum_{l=1}^{q-1} |\hat{\psi}^l(\mathfrak{p}^{-j}\xi)|^2 : N \geq 1\}$ of real numbers is increasing and is bounded by 1, hence it converges. Therefore, $\lim_{N\to\infty} |\hat{\varphi}(\mathfrak{p}^{-N}\xi)|^2$ also exists. Now,

$$\int_K |\hat{\varphi}(\mathfrak{p}^{-N}\xi)|^2 d\xi = q^{-N} \int_K |\hat{\varphi}(\xi)|^2 d\xi \to 0 \text{ as } N \to \infty.$$

Hence, by Fatou's lemma,

$$\int_K \lim_{N\to\infty} |\hat{\varphi}(\mathfrak{p}^{-N}\xi)|^2 d\xi \leq \lim_{N\to\infty} \int_K |\hat{\varphi}(\mathfrak{p}^{-N}\xi)|^2 d\xi = 0.$$

This shows that $\lim_{N\to\infty} |\hat{\varphi}(\mathfrak{p}^{-N}\xi)|^2 = 0$. Hence, we get

$$|\hat{\varphi}(\xi)|^2 = \sum_{l=1}^{q-1} \sum_{j=1}^{\infty} |\hat{\psi}^l(\mathfrak{p}^{-j}\xi)|^2.$$

Since $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ is an orthonormal system, we get for a.e. $\xi \in K$,

$$1 = \sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = \sum_{l=1}^{q-1}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{N}_0} |\hat{\psi}^l(\mathfrak{p}^{-j}(\xi + u(k)))|^2. \tag{4.2.4}$$

Definition 4.2.2. Suppose $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subseteq L^2(K)$ is a set of basic wavelets for $L^2(K)$. The *dimension function* of Ψ is defined as

$$D_\Psi(\xi) = \sum_{l=1}^{L}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{N}_0} |\hat{\psi}^l(\mathfrak{p}^{-j}(\xi + u(k)))|^2 \quad \text{for a.e. } \xi \in K.$$

Observe that if $\psi^1, \psi^2, \ldots, \psi^L \in L^2(K)$, then

$$\int_{\mathfrak{D}} \sum_{j=1}^{\infty}\sum_{k\in\mathbb{N}_0} |\hat{\psi}^l(\mathfrak{p}^{-j}(\xi + u(k)))|^2 d\xi = \sum_{j=1}^{\infty} q^{-j} \int_K |\hat{\psi}^l(\xi)|^2 d\xi < \infty. \tag{4.2.5}$$

Hence, D_Ψ is well-defined for a.e. $\xi \in K$. In particular, $\sum_{k\in\mathbb{N}_0} |\hat{\psi}^l(\mathfrak{p}^{-j}(\xi + u(k)))|^2 < \infty$ for a.e. $\xi \in K$. Thus, for all $j \geq 1$, $1 \leq l \leq L$, and a.e. $\xi \in K$, we can define the vector $\omega_j^l(\xi)$ in $\ell^2(\mathbb{N}_0)$, where

$$\omega_j^l(\xi) = \{\hat{\psi}^l(\mathfrak{p}^{-j}(\xi + u(k))) : k \in \mathbb{N}_0\}.$$

Note that we can also write D_Ψ as

$$D_\Psi(\xi) = \sum_{l=1}^{L}\sum_{j=1}^{\infty} \|\omega_j^l(\xi)\|^2_{\ell^2(\mathbb{N}_0)}.$$

We have thus proved that if $\Psi = \{\psi^1, \psi^2, \ldots, \psi^{q-1}\}$ is a set of basic wavelets associated with an MRA of $L^2(K)$, then it is necessary that $D_\Psi = 1$ a.e. Our aim is to show that this condition is also sufficient. We will show that if $\Psi = \{\psi^1, \psi^2, \ldots, \psi^{q-1}\}$ is a set of basic wavelets of $L^2(K)$ and $D_\Psi = 1$ a.e., then Ψ is an MRA-wavelet. To prove this we need the following lemma.

Lemma 4.2.3. *For all $j \geq 1$, $l = 1, 2, \ldots, q-1$, and a.e. $\xi \in K$, we have*

$$\omega_j^l(\xi) = \sum_{h=1}^{q-1}\sum_{i=1}^{\infty} \langle \omega_j^l(\xi), \omega_i^h(\xi)\rangle \omega_i^h(\xi). \tag{4.2.6}$$

Proof. The series appearing in the lemma converges absolutely by (4.2.5) for a.e. $\xi \in K$. We first show that

$$\hat{\psi}^l(\mathfrak{p}^{-j}\xi) = \sum_{h=1}^{q-1}\sum_{i=1}^{\infty}\sum_{k\in\mathbb{N}_0} \hat{\psi}^l(\mathfrak{p}^{-j}(\xi+u(k)))\overline{\hat{\psi}^h(\mathfrak{p}^{-i}(\xi+u(k)))}\hat{\psi}^h(\mathfrak{p}^{-i}\xi). \tag{4.2.7}$$

Let us denote the series on the right of (4.2.7) by $G_j^l(\xi)$. Then by using Lemma 4.1.3 and equation (4.1.9), we have

$$\begin{aligned}
G_j^l(\xi) &= \sum_{k\in\mathbb{N}_0} \hat{\psi}^l(\mathfrak{p}^{-j}(\xi+u(k)))\sum_{h=1}^{q-1}\sum_{i=1}^{\infty}\overline{\hat{\psi}^h(\mathfrak{p}^{-i}(\xi+u(k)))}\hat{\psi}^h(\mathfrak{p}^{-i}\xi)\\
&= \sum_{k\in\mathbb{N}_0} \hat{\psi}^l(\mathfrak{p}^{-j}(\xi+u(k)))\Big(t_k(\xi)-\sum_{h=1}^{q-1}\overline{\hat{\psi}^h(\xi+u(k))}\hat{\psi}^h(\xi)\Big)\\
&= \sum_{k\in\mathbb{N}_0} \hat{\psi}^l(\mathfrak{p}^{-j}(\xi+u(k)))t_k(\xi)\\
&\quad -\sum_{h=1}^{q-1}\Big\{\sum_{k\in\mathbb{N}_0}\overline{\hat{\psi}^h(\xi+u(k))}\hat{\psi}^l(\mathfrak{p}^{-j}(\xi+u(k)))\Big\}\hat{\psi}^h(\xi)\\
&= \sum_{k\in q\mathbb{N}_0} \hat{\psi}^l(\mathfrak{p}^{-j}(\xi+u(k)))t_k(\xi)\\
&= \sum_{h=1}^{q-1}\sum_{i=0}^{\infty}\sum_{k\in\mathbb{N}_0} \hat{\psi}^l(\mathfrak{p}^{-j}(\xi+u(qk)))\overline{\hat{\psi}^h(\mathfrak{p}^{-i}(\xi+u(qk)))}\hat{\psi}^h(\mathfrak{p}^{-i}\xi)\\
&= \sum_{h=1}^{q-1}\sum_{i=1}^{\infty}\sum_{k\in\mathbb{N}_0} \hat{\psi}^l(\mathfrak{p}^{-j-1}(\mathfrak{p}\xi+u(k)))\overline{\hat{\psi}^h(\mathfrak{p}^{-i}(\mathfrak{p}\xi+u(k)))}\hat{\psi}^h(\mathfrak{p}^{-i}\mathfrak{p}\xi)\\
&= G_{j+1}^l(\mathfrak{p}\xi).
\end{aligned}$$

This is equivalent to $G_j^l(\xi) = G_{j-1}^l(\mathfrak{p}^{-1}\xi)$. Iterating this equation, we obtain, $G_j^l(\xi) = G_1^l(\mathfrak{p}^{-j+1}\xi)$. We now calculate $G_1^l(\xi)$. We have

$$\begin{aligned}
G_1^l(\xi) &= \sum_{k\in\mathbb{N}_0} \hat{\psi}^l(\mathfrak{p}^{-1}(\xi+u(k)))\sum_{h=1}^{q-1}\sum_{i=1}^{\infty}\overline{\hat{\psi}^h(\mathfrak{p}^{-i}(\xi+u(k)))}\hat{\psi}^h(\mathfrak{p}^{-i}\xi)\\
&= \sum_{k\in\mathbb{N}_0} \hat{\psi}^l(\mathfrak{p}^{-1}\xi+u(qk))\sum_{h=1}^{q-1}\sum_{i=0}^{\infty}\overline{\hat{\psi}^h(\mathfrak{p}^{-i}(\mathfrak{p}^{-1}\xi+u(qk)))}\hat{\psi}^h(\mathfrak{p}^{-i}\mathfrak{p}^{-1}\xi)\\
&= \sum_{k\in q\mathbb{N}_0} \hat{\psi}^l(\mathfrak{p}^{-1}\xi+u(k))\sum_{h=1}^{q-1}\sum_{i=0}^{\infty}\overline{\hat{\psi}^h(\mathfrak{p}^{-i}(\mathfrak{p}^{-1}\xi+u(k)))}\hat{\psi}^h(\mathfrak{p}^{-i}\mathfrak{p}^{-1}\xi).
\end{aligned}$$

In the last equation, we can replace the sum over $k \in q\mathbb{N}_0$ by a sum over $k \in \mathbb{N}_0$ since the additional terms corresponding to $k \in \mathbb{N}_0 \setminus q\mathbb{N}_0$ are all zero by (4.1.9). Hence,

$$\begin{aligned} G_1^l(\xi) &= \sum_{h=1}^{q-1}\sum_{i=0}^{\infty} \hat{\psi}^h(\mathfrak{p}^{-i}\mathfrak{p}^{-1}\xi)\delta_{i,0}\delta_{l,h} \\ &= \hat{\psi}^l(\mathfrak{p}^{-1}\xi). \end{aligned}$$

Thus $G_j^l(\xi) = \hat{\psi}^l(\mathfrak{p}^{-j}\xi)$ for a.e. $\xi \in K$. Since $\langle \omega_j^l(\xi), \omega_i^h(\xi)\rangle$ is integral-periodic, (4.2.6) follows. This completes the proof of the lemma. □

We will also need the following lemma. We refer to [4], [18] and [42] for a proof of this result.

Lemma 4.2.4. *Let $\{v_n : n \geq 1\}$ be a family of vectors in a Hilbert space H such that*

(i) $\sum_{n=1}^{\infty} \|v_n\|^2 = C < \infty$, *and*

(ii) $v_n = \sum_{m=1}^{\infty} \langle v_n, v_m\rangle v_m$ *for all* $n \geq 1$.

Let $\mathcal{F} = \overline{\text{span}}\{v_n : n \geq 1\}$. *Then* $\dim \mathcal{F} = \sum_{n=1}^{\infty} \|v_n\|^2 = C$.

The following theorem characterizes the MRA-wavelets.

Theorem 4.2.5. *A wavelet $\Psi = \{\psi^1, \psi^2, \ldots, \psi^{q-1}\} \subseteq L^2(K)$ is an MRA-wavelet if only if $D_\Psi(\xi) = 1$ for almost every $\xi \in K$.*

Proof. We have already observed that $D_\Psi(\xi) = 1$ for a.e. $\xi \in K$ when Ψ is an MRA-wavelet. We now prove the converse.

Assume that $D_\Psi(\xi) = 1$ for a.e. $\xi \in K$. Let E be the subset of $\mathfrak{D}$ on which $D_\Psi(\xi)$ is finite and (4.2.6) is satisfied. Then ω_j^l are well-defined on E. For $\xi \in E$, we define the space

$$\mathcal{F}(\xi) = \overline{\text{span}}\{\omega_j^l(\xi) : 1 \leq l \leq q-1, j \geq 1\}.$$

Then, by Lemmas 4.2.3 and 4.2.4, we have

$$\dim \mathcal{F}(\xi) = \sum_{l=1}^{q-1}\sum_{j=1}^{\infty} \|\omega_j^l(\xi)\|^2_{\ell^2(\mathbb{N}_0)} = D_\Psi(\xi) = 1. \tag{4.2.8}$$

That is, for each $\xi \in E$, $\mathcal{F}(\xi)$ is generated by a single unit vector $U(\xi)$. We now choose a suitable vector. For $j \geq 1$, let us define

$$X_j = \{\xi \in E : \omega_j^l(\xi) \neq 0 \text{ for some } l, 1 \leq l \leq q-1, \\ \text{and } \omega_m^l(\xi) = 0, \forall m < j \text{ and } 1 \leq l \leq q-1\},$$

and

$$X_0 = \{\xi \in \mathfrak{D} : \omega_j^l(\xi) = 0 \text{ for all } l, 1 \leq l \leq q-1, \text{ and for all } j \geq 1\}.$$

Then $\{X_j : j = 0, 1, 2, \dots\}$ forms a partition of E. Note that $X_0 = \{\xi \in \mathfrak{D} : D_\Psi(\xi) = 0\}$. So for a.e. $\xi \in E \setminus X_0$, there exists $j \geq 1$ such that $\xi \in X_j$. Hence, there exists at least one l, $1 \leq l \leq q-1$, such that $\omega_j^l(\xi) \neq 0$. Choose the smallest such l and define

$$U(\xi) = \frac{\omega_j^l(\xi)}{\|\omega_j^l(\xi)\|_{\ell^2(\mathbb{N}_0)}}.$$

Thus, $U(\xi)$ is well defined and $\|U(\xi)\|_{\ell^2(\mathbb{N}_0)} = 1$ for a.e. $\xi \in \mathfrak{D}$. We write $U(\xi) = \{u_k(\xi) : k \in \mathbb{N}_0\}$. Now, define $\hat{\varphi}(\xi) = u_k(\xi - u(k))$, where k is the unique integer in $\mathbb{N}_0$ such that $\xi \in \mathfrak{D} + u(k)$. This defines $\hat{\varphi}$ on K. We first show that $\varphi \in L^2(K)$ and $\{\varphi(\cdot - u(k) : k \in \mathbb{N}_0)\}$ is an orthonormal system in $L^2(K)$. We have

$$\begin{aligned} \|\hat{\varphi}\|_2^2 &= \int_K |\hat{\varphi}(\xi)|^2 d\xi \\ &= \int_{\mathfrak{D}} \sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 d\xi \\ &= \sum_{k \in \mathbb{N}_0} \int_{\mathfrak{D}} |u_k(\xi)|^2 d\xi \\ &= \int_{\mathfrak{D}} \|U(\xi)\|_{\ell^2(\mathbb{N}_0)}^2 d\xi \\ &= 1. \end{aligned}$$

Thus, $\varphi \in L^2(K)$. Also,

$$\sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = \sum_{k \in \mathbb{N}_0} |u_k(\xi)|^2 = \|U(\xi)\|_{\ell^2(\mathbb{N}_0)}^2 = 1. \tag{4.2.9}$$

This is equivalent to the fact that $\{\varphi(\cdot - u(k) : k \in \mathbb{N}_0)\}$ is an orthonormal system. We now define $V_0^\sharp = \overline{\text{span}}\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$. Let $W_j = \overline{\text{span}}\{\psi_{j,k}^l : 1 \le l \le q-1, k \in \mathbb{N}_0\}$ and $V_0 = \bigoplus_{j<0} W_j$. If we can show that $V_0^\sharp = V_0$, then it will follow that $\{V_j : j \in \mathbb{Z}\}$ is the required MRA (see the discussion just after Definition 4.2.1).

We first show that $V_0 \subset V_0^\sharp$. It is sufficient to verify that $\psi_{j,k}^l \in V_0^\sharp$, $k \in \mathbb{N}_0$, $j < 0$, $1 \le l \le q-1$. For each $j \ge 1$, there exists a measurable function ν_j^l on $\mathfrak{D}$ such that $\omega_j^l(\xi) = \nu_j^l(\xi)U(\xi)$ for a.e. $\xi \in \mathfrak{D}$. That is,

$$\hat{\psi}^l(\mathfrak{p}^{-j}(\xi + u(k))) = \nu_j^l(\xi)\hat{\varphi}(\xi + u(k)) \quad \text{for a.e. } \xi \in \mathfrak{D}, k \in \mathbb{Z}.$$

Hence, by (4.2.9), for a.e. $\xi \in \mathfrak{D}$, we have

$$\sum_{k\in\mathbb{N}_0} |\hat{\psi}^l(\mathfrak{p}^{-j}(\xi + u(k)))|^2 = \sum_{k\in\mathbb{N}_0} |\nu_j^l(\xi)|^2 |\hat{\varphi}(\xi + u(k))|^2 = |\nu_j^l(\xi)|^2. \tag{4.2.10}$$

This shows that $\nu_j^l \in L^2(\mathfrak{D})$ so that we can write its Fourier series expansion. Thus, for $j \ge 1$, there exists $\{a_{j,k}^l : k \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$ such that $\nu_j^l(\xi) = \sum_{k\in\mathbb{N}_0} a_{j,k}^l \overline{\chi_k(\xi)}$, with convergence in $L^2(\mathfrak{D})$. Extending ν_j^l integer periodically, we have

$$\hat{\psi}^l(\mathfrak{p}^{-j}\xi) = \nu_j^l(\xi)\hat{\varphi}(\xi) \text{ for a.e. } \xi \in K, j \ge 1. \tag{4.2.11}$$

Taking inverse Fourier transform, we get

$$\psi_{-j,0}^l(x) = q^{j/2} \sum_{k\in\mathbb{N}_0} a_{j,k}^l \varphi(x - u(k)), \quad j \ge 1.$$

Hence, $\psi_{-j,0}^l \in V_0^\sharp$ for $j \ge 1$. Equivalently, $\psi_{j,0}^l \in V_0^\sharp$ for $j < 0$. Moreover, since $V_0^\sharp$ is invariant under translations by $u(k)$, $k \in \mathbb{N}_0$, we have $\psi_{j,k}^l \in V_0^\sharp$, $j < 0$, $k \in \mathbb{N}_0$, $1 \le l \le q-1$.

To show the reverse inclusion, it suffices to show that $V_0^\sharp \perp W_j$, for $j \ge 0$. For $j \ge 0, k \in \mathbb{N}_0, 1 \le l \le q-1$, we have

$$\begin{aligned} \langle \varphi, \psi_{j,k}^l \rangle = \langle \hat{\varphi}, (\psi_{j,k}^l)^\wedge \rangle &= \int_K \hat{\varphi}(\xi) q^{-j/2} \overline{\hat{\psi}^l(\mathfrak{p}^j\xi)} \chi_k(\mathfrak{p}^j\xi) d\xi \\ &= q^{j/2} \int_K \hat{\varphi}(\mathfrak{p}^{-j}\xi) \overline{\hat{\psi}^l(\xi)} \chi_k(\xi) d\xi \end{aligned}$$

$$= q^{j/2} \int_{\mathfrak{D}} \sum_{n \in \mathbb{N}_0} \hat{\varphi}(\mathfrak{p}^{-j}(\xi + u(n)))\overline{\hat{\psi}^l(\xi + u(n))}\chi_k(\xi)d\xi. \tag{4.2.12}$$

Using equation (4.2.10), we get

$$\sum_{l=1}^{q-1}\sum_{j=1}^{\infty} |\nu_j^l(\xi)|^2 = \sum_{l=1}^{q-1}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{N}_0} |\hat{\psi}^l(\mathfrak{p}^{-j}(\xi + u(k)))|^2 = 1 \text{ for a.e. } \xi \in K.$$

Hence, for such a ξ and for all $j \geq 0$, there exists $j_0 \geq 1$ such that $\nu_{j_0}^l(\mathfrak{p}^{-j}\xi) \neq 0$. Thus, (4.2.11) implies that $\hat{\psi}^l(\mathfrak{p}^{-j-j_0}\xi) = \nu_{j_0}^l(\mathfrak{p}^{-j}\xi)\hat{\varphi}(\mathfrak{p}^{-j}\xi)$. Therefore, for $k \in \mathbb{N}_0$, we get

$$\hat{\psi}^l(\mathfrak{p}^{-j-j_0}(\xi + u(k))) = \nu_{j_0}^l(\mathfrak{p}^{-j}(\xi + u(k)))\hat{\varphi}(\mathfrak{p}^{-j}(\xi + u(k))).$$

Since $\mathfrak{p}^{-j}(\xi + u(k)) = \mathfrak{p}^{-j}\xi + u(q^j k)$ and $\nu_{j_0}^l$ is integral-periodic, we have

$$\hat{\varphi}(\mathfrak{p}^{-j}(\xi + u(k))) = \frac{1}{\nu_{j_0}^l(\mathfrak{p}^{-j}\xi)}\hat{\psi}^l(\mathfrak{p}^{-j-j_0}(\xi + u(k))).$$

Hence, using Lemma 4.1.3, for any h with $1 \leq h \leq q-1$, we have

$$\begin{aligned}
&\sum_{k\in\mathbb{N}_0} \hat{\varphi}(\mathfrak{p}^{-j}(\xi + u(k)))\overline{\hat{\psi}^h(\xi + u(k))} \\
&= \frac{1}{\nu_{j_0}^l(\mathfrak{p}^{-j}\xi)} \sum_{k\in\mathbb{N}_0} \hat{\psi}^l(\mathfrak{p}^{-j-j_0}(\xi + u(k)))\overline{\hat{\psi}^h(\xi + u(k))} \\
&= 0,
\end{aligned}$$

since $j + j_0 \geq 1$. Substituting this in (4.2.12), we get $\langle \varphi, \psi_{j,k}^l \rangle = 0$ for $j \geq 0, k \in \mathbb{N}_0, 1 \leq l \leq q-1$. From this we conclude that $V_0^\sharp \subset V_0$. This completes the proof of the theorem. □

Chapter 5

Wavelet Packets And Frame Packets

In this chapter we will construct the wavelet packets associated with an MRA of a local field of positive characteristic. We will also generalize the concept of wavelet frame packets to this setup. First of all, we will discuss about wavelet packets on $\mathbb{R}$ very briefly.

Let $\{V_j : j \in \mathbb{Z}\}$ be an MRA of $L^2(\mathbb{R})$ with scaling function φ and wavelet ψ. Let W_j be the corresponding wavelet subspaces: $W_j = \overline{\text{span}}\{2^{j/2}\psi(2^j \cdot -k) : k \in \mathbb{Z}\}$. In the construction of a wavelet from an MRA, essentially the space V_1 is split into two orthogonal components V_0 and W_0. Note that V_1 is the closure of the linear span of the functions $\{2^{1/2}\varphi(2 \cdot -k) : k \in \mathbb{Z}\}$, whereas V_0 and W_0 are respectively the closure of the span of $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ and $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$. Since $\varphi(2 \cdot -k) = \varphi\left(2(\cdot - 2^{-1}k)\right)$, we see that the above procedure "splits" the half-integer translates of a function into integer translates of two functions.

In a similar way, we can split W_j, which is the span of $\{\psi(2^j \cdot -k) : k \in \mathbb{Z}\} = \{\psi\left(2^j(\cdot - 2^{-j}k)\right) : k \in \mathbb{Z}\}$, to get two functions whose $2^{-(j-1)}k$ translates will span the same space W_j. Repeating the splitting procedure j times, we get 2^j functions whose integer translates alone span the space W_j. If we apply this to each W_j, then the resulting basis of $L^2(\mathbb{R})$ will consist of integer translates of a countable number of functions (instead of all dilations and translations of the wavelet ψ). This basis is called the "wavelet packet basis". The concept of wavelet packet was introduced by Coifman, Meyer and Wickerhauser [27, 28]. For a nice exposition of wavelet packets of $L^2(\mathbb{R})$ with dilation 2, we refer to [42].

The concept of wavelet packet was subsequently generalized to $\mathbb{R}^n$ by taking tensor products [26]. The non-tensor product versions are due to Shen [69] for dyadic dilation, and

Behera [5] for MRAs associated with a general dilation matrix and several scaling functions. Other notable generalizations are the biorthogonal wavelet packets [24], non-orthogonal version of wavelet packets [21], the wavelet frame packets [19] on $\mathbb{R}$ for dilation 2, and the orthogonal, biorthogonal and frame wavelet packets on $\mathbb{R}^n$ by Long and Chen [61] for the dyadic dilation.

In section 5.1, we prove a crucial lemma called the splitting lemma and construct the wavelet packets associated with an MRA of a local field of positive characteristic. We also prove that the wavelet packets generate an orthonormal basis for $L^2(K)$. In section 5.2, we prove some basic results needed to prove an analogue of the splitting lemma for wavelet frames on K and then construct the wavelet frame packets.

5.1 Construction of wavelet packets

Let $\{V_j : j \in \mathbb{Z}\}$ be an MRA of $L^2(K)$ and φ be the corresponding scaling function. Since $\varphi \in V_0 \subset V_1$, and $\{\varphi_{1,k} : k \in \mathbb{N}_0\}$ is an orthonormal basis in V_1, we have

$$\varphi(x) = \sum_{k\in\mathbb{N}_0} h_k^0 q^{1/2}\varphi(\mathfrak{p}^{-1}x - u(k)),$$

where $h_k^0 = \langle\varphi, \varphi_{1,k}\rangle$ and $\{h_k^0 : k \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$. Taking Fourier transform, we get

$$\begin{aligned}\hat{\varphi}(\xi) &= q^{-1/2}\sum_{k\in\mathbb{N}_0} h_k^0\overline{\chi_k(\mathfrak{p}\xi)}\hat{\varphi}(\mathfrak{p}\xi)\\ &= m_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi),\end{aligned} \tag{5.1.1}$$

where $m_0(\xi) = q^{-1/2}\sum_{k\in\mathbb{N}_0} h_k^0\overline{\chi_k(\xi)}$.

Suppose that there exist $q-1$ integral-periodic functions m_l, $1 \le l \le q-1$, in $L^2(\mathfrak{D})$ such that the matrix

$$M(\xi) = \Big(m_l(\mathfrak{p}\xi + \mathfrak{p}u(k))\Big)_{l,k=0}^{q-1}$$

is unitary. As we mentioned before, it was proved in [44] that $\{\psi_1, \psi_2, \cdots, \psi_{q-1}\}$ is a set of basic wavelets of $L^2(K)$ if we define

$$\hat{\psi}_l(\xi) = m_l(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi).$$

We first prove a lemma, the splitting lemma, which is essential for the construction of wavelet packets. With the help of this lemma, we can decompose a closed subspace of $L^2(K)$ into finitely many mutually orthogonal subspaces in a suitable manner.

Lemma 5.1.1 (The splitting lemma). *Let $\varphi \in L^2(K)$ be such that $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ is an orthonormal system. Let $V = \overline{\operatorname{span}}\{q^{1/2}\varphi(\mathfrak{p}^{-1} \cdot - u(k)) : k \in \mathbb{N}_0\}$. Let $m_l(\xi) = q^{-1/2} \sum_{k \in \mathbb{N}_0} h_k^l \overline{\chi_k}(\xi)$, $0 \le l \le q-1$, where $\{h_k^l : k \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$ for $0 \le l \le q-1$. Define $\hat{\psi}_l(\xi) = m_l(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi)$. Then $\{\psi_l(\cdot - u(k)) : 0 \le l \le q-1, k \in \mathbb{N}_0\}$ is an orthonormal system in V if and only if the matrix*

$$M(\xi) = \Big(m_l(\mathfrak{p}\xi + \mathfrak{p}u(k))\Big)_{l,k=0}^{q-1}$$

is unitary for a.e. $\xi \in \mathfrak{D}$.

Moreover, $\{\psi_l(\cdot - u(k)) : 0 \le l \le q-1, k \in \mathbb{N}_0\}$ is an orthonormal basis of V whenever it is orthonormal.

Proof. Assume that $M(\xi)$ is unitary for a.e. $\xi \in \mathfrak{D}$. Then, for $0 \le s,t \le q-1$ and $k,l \in \mathbb{N}_0$, we have

$$\begin{aligned}
&\Big\langle \psi_s(\cdot - u(k)), \psi_t(\cdot - u(l)) \Big\rangle \\
&= \Big\langle \Big(\psi_s(\cdot - u(k))\Big)^{\wedge}, \Big(\psi_t(\cdot - u(l))\Big)^{\wedge} \Big\rangle \\
&= \int_K \hat{\psi}_s(\xi)\overline{\chi_k(\xi)}\overline{\hat{\psi}_t(\xi)}\chi_l(\xi)\, d\xi \\
&= \int_{\mathfrak{D}} \sum_{n \in \mathbb{N}_0} \hat{\psi}_s(\xi + u(n))\overline{\hat{\psi}_t(\xi + u(n))}\overline{\chi_k(\xi)}\chi_l(\xi)\, d\xi \\
&= \int_{\mathfrak{D}} \sum_{n \in \mathbb{N}_0} m_s(\mathfrak{p}\xi + \mathfrak{p}u(n))\overline{m_t(\mathfrak{p}\xi + \mathfrak{p}u(n))}\,|\hat{\varphi}(\mathfrak{p}\xi + \mathfrak{p}u(n))|^2\overline{\chi_k(\xi)}\chi_l(\xi)\, d\xi \\
&= \int_{\mathfrak{D}} \sum_{\mu=0}^{q-1} \sum_{n \in \mathbb{N}_0} m_s(\mathfrak{p}\xi + \mathfrak{p}u(qn+\mu))\overline{m_t(\mathfrak{p}\xi + \mathfrak{p}u(qn+\mu))} \\
&\qquad \times |\hat{\varphi}(\mathfrak{p}\xi + \mathfrak{p}u(qn+\mu))|^2\overline{\chi_k(\xi)}\chi_l(\xi)\, d\xi \\
&= \int_{\mathfrak{D}} \Big\{ \sum_{\mu=0}^{q-1} m_s(\mathfrak{p}\xi + \mathfrak{p}u(\mu))\overline{m_t(\mathfrak{p}\xi + \mathfrak{p}u(\mu))} \Big\} \overline{\chi_k(\xi)}\chi_l(\xi)\, d\xi \\
&= \int_{\mathfrak{D}} \delta_{s,t}\overline{\chi_k(\xi)}\chi_l(\xi)\, d\xi = \delta_{s,t}\delta_{k,l}.
\end{aligned}$$

Hence, $\{\psi_s(\cdot - u(k)) : 0 \le s \le q-1, k \in \mathbb{N}_0\}$ is an orthonormal system in V. The converse can be proved by reversing the above steps.

To prove the second part, let $f \in V$ be such that f is orthogonal to $\psi_l(\cdot - u(k))$ for all $l = 0, 1, \ldots, q-1$, $k \in \mathbb{N}_0$. We claim that $f = 0$ a.e.

Since $f \in V$, we have

$$f(x) = \sum_{m \in \mathbb{N}_0} q^{1/2} c_m \varphi(\mathfrak{p}^{-1}x - u(m)),$$

for some $\{c_m : m \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$. So there exists an integral-periodic function m_f in $L^2(\mathfrak{D})$ such that

$$\hat{f}(\xi) = m_f(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi).$$

Hence, for all $l = 0, 1, \ldots, q-1$, $k \in \mathbb{N}_0$, we have (by a similar calculation)

$$\begin{aligned} 0 &= \langle f, \psi_l(\cdot - u(k))\rangle \\ &= \int_K \hat{f}(\xi)\overline{\hat{\psi}_l(\xi)}\chi_k(\xi)d\xi \\ &= \int_{\mathfrak{D}} \Big\{\sum_{\mu=0}^{q-1} m_f(\mathfrak{p}\xi + \mathfrak{p}u(\mu))\overline{m_l(\mathfrak{p}\xi + \mathfrak{p}u(\mu))}\Big\}\chi_k(\xi)d\xi. \end{aligned}$$

Therefore, for all $l = 0, 1, \ldots, q-1$, we have

$$\sum_{\mu=0}^{q-1} m_f\big(\mathfrak{p}\xi + \mathfrak{p}u(\mu)\big)\overline{m_l\big(\mathfrak{p}\xi + \mathfrak{p}u(\mu)\big)} = 0.$$

Now, for a.e. ξ, the vector $\Big(m_f(\mathfrak{p}\xi + \mathfrak{p}u(\mu))\Big)_{\mu=0}^{q-1} \in \mathbb{C}^q$, being orthogonal to each row vector of the unitary matrix $M(\xi)$, is the zero vector. In particular, $m_f(\mathfrak{p}\xi) = 0$ a.e. This means $\hat{f} = 0$ a.e. and hence $f = 0$ a.e. □

Applying the splitting lemma to $V = V_1$, we get that $\{\psi_l(\cdot - u(k)) : 0 \le l \le q-1, k \in \mathbb{N}_0\}$ is an orthonormal basis for V_1. Now we will define a sequence $\{\omega_n : n \ge 0\}$ of functions.

Let

$$\omega_0 = \varphi$$

and

$$\omega_n = \psi_n \quad (1 \le n \le q-1),$$

where

$$\hat{\psi}_l(\xi) = m_l(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi) \quad (1 \le l \le q-1). \tag{5.1.2}$$

Suppose ω_m is defined for $m \ge 0$. For $0 \le r \le q-1$, define

$$\omega_{r+qm}(x) = q^{1/2} \sum_{k \in \mathbb{N}_0} h_k^r \omega_m(\mathfrak{p}^{-1}x - u(k)). \tag{5.1.3}$$

Note that this defines ω_n for every integer $n \ge 0$. Taking Fourier transform, we have

$$(\omega_{r+qm})\hat{}(\xi) = m_r(\mathfrak{p}\xi)\hat{\omega}_m(\mathfrak{p}\xi). \tag{5.1.4}$$

Definition 5.1.2. The functions $\{\omega_n : n \ge 0\}$ as defined above will be called the *wavelet packets* corresponding to the MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$.

In the following proposition we find an expression for the Fourier transforms of the wavelet packets in terms of $\hat{\varphi}$.

Proposition 5.1.3. *For an integer $n \ge 1$, consider the unique expansion of n in the base q:*

$$n = \mu_1 + \mu_2 q + \mu_3 q^2 + \cdots + \mu_j q^{j-1}, \tag{5.1.5}$$

where $0 \le \mu_i \le q-1$ for all $i = 1, 2, \ldots, j$ and $\mu_j \ne 0$. Then

$$\hat{\omega}_n(\xi) = m_{\mu_1}(\mathfrak{p}\xi)m_{\mu_2}(\mathfrak{p}^2\xi)\cdots m_{\mu_j}(\mathfrak{p}^j\xi)\hat{\varphi}(\mathfrak{p}^j\xi). \tag{5.1.6}$$

Proof. We will prove it by induction. If n has an expansion as in (5.1.5), then we say that it has length j. Since $\omega_0 = \varphi$, and $\omega_n = \psi_n, 1 \le n \le q-1$, it follows from (5.1.1) and (5.1.2) that the claim is true for length 1. Assume that it is true for length j. Let m be an integer with an expansion of length $j+1$. Then there exist integers $\gamma_1, \gamma_2, \ldots, \gamma_{j+1}$ with $0 \le \gamma_1, \gamma_2, \ldots, \gamma_{j+1} \le q-1$ and $\gamma_{j+1} \ne 0$ such that

$$m = \gamma_1 + \gamma_2 q + \cdots + \gamma_j q^{j-1} + \gamma_{j+1} q^j$$

$$= \gamma_1 + kq,$$

where $k = \gamma_2 + \gamma_3 q + \cdots + \gamma_{j+1} q^{j-1}$. Note that k has length j. Hence,

$$\begin{aligned}
\hat{\omega}_m(\xi) &= (\omega_{\gamma_1+kq})^\wedge(\xi) \\
&= m_{\gamma_1}(\mathfrak{p}\xi)\hat{\omega}_k(\mathfrak{p}\xi) \qquad \text{(by (5.1.4))} \\
&= m_{\gamma_1}(\mathfrak{p}\xi) m_{\gamma_2}(\mathfrak{p}^2\xi) \cdots m_{\gamma_j+1}(\mathfrak{p}^{j+1}\xi)\hat{\varphi}(\mathfrak{p}^{j+1}\xi).
\end{aligned}$$

Hence, the induction is complete. □

We will prove the following theorem regarding the wavelet packets.

Theorem 5.1.4. *Let $\{\omega_n : n \geq 0\}$ be the basic wavelet packets constructed above. Then*

(i) $\{\omega_n(\cdot - u(k)) : q^j \leq n \leq q^{j+1} - 1, k \in \mathbb{N}_0\}$ is an orthonormal basis of W_j, $j \geq 0$.

(ii) $\{\omega_n(\cdot - u(k)) : 0 \leq n \leq q^j - 1, k \in \mathbb{N}_0\}$ is an orthonormal basis of V_j, $j \geq 0$.

(iii) $\{\omega_n(\cdot - u(k)) : n \geq 0, k \in \mathbb{N}_0\}$ is an orthonormal basis of $L^2(K)$.

Proof. Since $\{\omega_n : 1 \leq n \leq q-1\}$ are the wavelets, the case $j = 0$ in (i) is trivial. Assume (i) for j. We will prove for $j+1$. By our assumption, the set of functions $\{\omega_n(\cdot - u(k)) : q^j \leq n \leq q^{j+1} - 1, k \in \mathbb{N}_0\}$ is an orthonormal basis of W_j. Since $f \in W_j$ if and only if $f(\mathfrak{p}^{-1}\cdot) \in W_{j+1}$, $j \in \mathbb{Z}$, we have

$$\{q^{1/2}\omega_n(\mathfrak{p}^{-1}\cdot - u(k)) : q^j \leq n \leq q^{j+1} - 1, k \in \mathbb{N}_0\}$$

is an orthonormal basis of W_{j+1}. Let

$$E_n = \overline{\text{span}}\{q^{1/2}\omega_n(\mathfrak{p}^{-1}\cdot - u(k)) : k \in \mathbb{N}_0\}.$$

Hence,

$$W_{j+1} = \bigoplus_{n=q^j}^{q^{j+1}-1} E_n. \tag{5.1.7}$$

Applying the splitting lemma to E_n, we get the functions

$$g_l^n(x) = \sum_{k \in \mathbb{N}_0} h_k^l q^{1/2}\omega_n(\mathfrak{p}^{-1}x - u(k)), \quad (0 \leq l \leq q-1)$$

such that $\{g_l^n(\cdot - u(k)) : 0 \le l \le q-1, k \in \mathbb{N}_0\}$ forms an orthonormal basis for E_n. Hence, $\{g_l^n(\cdot - u(k)) : 0 \le l \le q-1, q^j \le n \le q^{j+1}-1, k \in \mathbb{N}_0\}$ forms an orthonormal basis for W_{j+1}. But by (5.1.3),

$$g_l^n = \omega_{l+qn}.$$

This fact, together with (5.1.7), shows that

$$\begin{aligned}&\{\omega_{l+qn}(\cdot - u(k)) : 0 \le l \le q-1,\ q^j \le n \le q^{j+1}-1,\ k \in \mathbb{N}_0\}\\ &= \{\omega_n(\cdot - u(k)) : q^{j+1} \le n \le q^{j+2}-1,\ k \in \mathbb{N}_0\}\end{aligned}$$

is an orthonormal basis for W_{j+1}. So (i) is proved. Item (ii) follows from the observation that $V_j = V_0 \oplus W_0 \oplus \cdots \oplus W_{j-1}$ and (iii) follows from the fact that $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(K)$. □

5.2 Wavelet frame packets

Let H be a separable Hilbert space. Recall that a sequence $\{x_k : k \in \mathbb{N}_0\}$ of H is said to be a frame for H if there exist constants C_1 and C_2, $0 < C_1 \le C_2 < \infty$ such that for all $x \in H$

$$C_1\|x\|^2 \le \sum_{k\in\mathbb{N}_0} |\langle x, x_k\rangle|^2 \le C_2\|x\|^2. \tag{5.2.1}$$

The largest C_1 and the smallest C_2 for which (5.2.1) holds are called the frame bounds.

Suppose that $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_N\} \subset L^2(K)$ be such that the system $\{\varphi_l(\cdot - u(k)) : 1 \le l \le N, k \in \mathbb{N}_0\}$ is a frame for its closed linear span $S(\Phi)$. Let $\psi_1, \psi_2, \ldots, \psi_N$ be elements in $S(\Phi)$. A natural question to ask is: when can we say that $\{\psi_l(\cdot - u(k)) : 1 \le l \le N, k \in \mathbb{N}_0\}$ is also a frame for $S(\Phi)$?

If $\psi_j \in S(\Phi)$, then there exists $\{p_{jlk} : k \in \mathbb{N}_0\}$ in $\ell^2(\mathbb{N}_0)$ such that

$$\psi_j(x) = \sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0} p_{jlk}\varphi_l(x - u(k)).$$

Taking Fourier transform, we get

$$\begin{aligned}\hat{\psi}_j(\xi) &= \sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0} p_{jlk}\overline{\chi_k(\xi)}\hat{\varphi}_l(\xi)\\ &= \sum_{l=1}^{N} P_{jl}(\xi)\hat{\varphi}_l(\xi),\end{aligned}$$

where $P_{jl}(\xi) = \sum_{k\in\mathbb{N}_0} p_{jlk}\overline{\chi_k(\xi)}$. Observe that P_{jl} are integral-periodic functions. Let $P(\xi)$ be the $N \times N$ matrix

$$P(\xi) = \Big(P_{jl}(\xi)\Big)_{1\le j,l\le N}.$$

Let S and T be two positive definite matrices of order N. We say that $S \le T$ if $T - S$ is positive definite. The following lemma is the generalization of Lemma 3.1 in [19].

Lemma 5.2.1. *Let φ_l, ψ_l for $1 \le l \le N$, and $P(\xi)$ be as above. Suppose that there exist constants C_1 and C_2, $0 < C_1 \le C_2 < \infty$ such that*

$$C_1 I \le P^*(\xi)P(\xi) \le C_2 I \quad \text{for a.e. } \xi \in \mathfrak{D}. \tag{5.2.2}$$

Then, for all $f \in L^2(K)$, we have

$$\begin{aligned}C_1\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle f,\varphi_l(\cdot - u(k))\rangle|^2 &\le \sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle f,\psi_l(\cdot - u(k))\rangle|^2\\ &\le C_2\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle f,\varphi_l(\cdot - u(k))\rangle|^2.\end{aligned} \tag{5.2.3}$$

Proof. For $f, g \in L^2(K)$, we define

$$[f,g](\xi) = \sum_{l\in\mathbb{N}_0}\hat{f}(\xi + u(l))\overline{\hat{g}(\xi + u(l))}.$$

Then, for $f \in L^2(K)$, we have

$$[f,\psi_j](\xi) = \sum_{l\in\mathbb{N}_0}\hat{f}(\xi + u(l))\overline{\hat{\psi}_j(\xi + u(l))}$$

$$= \sum_{k=1}^{N} \sum_{l \in \mathbb{N}_0} \overline{P_{jk}(\xi + u(l))} \hat{f}(\xi + u(l)) \overline{\hat{\varphi}_k(\xi + u(l))}$$
$$= \sum_{k=1}^{N} \overline{P_{jk}(\xi)} [f, \varphi_k](\xi),$$

since P_{jk} are integral-periodic function. Hence,

$$\sum_{j=1}^{N} |[f, \psi_j]|^2 = \sum_{k,k'=1}^{N} \sum_{j=1}^{N} \overline{P}_{jk} P_{jk'} [f, \varphi_k] \overline{[f, \varphi_{k'}]} = XP^*PX^*,$$

where

$$X = \left([f, \varphi_1], \cdots [f, \varphi_N]\right).$$

By Plancherel Theorem,

$$\sum_{k \in \mathbb{N}_0} \sum_{l=1}^{N} |\langle f, \varphi_l(\cdot - u(k))\rangle|^2 = \sum_{l=1}^{N} \int_{\mathfrak{D}} |[f, \varphi_l](\xi)|^2 \, d\xi.$$

Hence, inequality (5.2.3) is equivalent to

$$C_1 \int_{\mathfrak{D}} XX^* \leq \int_{\mathfrak{D}} XP^*PX^* \leq C_2 \int_{\mathfrak{D}} XX^* \quad \text{for all } f \in L^2(K).$$

This follows from (5.2.2). □

We now introduce a matrix $E(\xi)$. For $0 \leq r, s \leq q-1$, $1 \leq l, j \leq N$, and for a.e. $\xi \in \mathfrak{D}$, define

$$\mathcal{E}_{lj}^{rs}(\xi) = \delta_{lj} q^{-\frac{1}{2}} \overline{\chi\big(u(r)(\xi + \mathfrak{p}u(s))\big)}.$$

Let

$$E^{rs}(\xi) = \left(\mathcal{E}_{lj}^{rs}(\xi)\right)_{1 \leq l,j \leq N}$$

and

$$E(\xi) = \left(E^{rs}(\xi)\right)_{0 \leq r,s \leq q-1}. \tag{5.2.4}$$

So $E(\xi)$ is a block matrix with q blocks in each row and each column, and each block is a square matrix of order N, so that $E(\xi)$ is a square matrix of order qN.

We have the following lemma which will be useful for the splitting trick for frames. The first

part of the lemma is a particular case of Lemma 4.1.4 proved in Chapter 4.

Lemma 5.2.2. *(i) For* $0 \le r, s \le q-1$,

$$\frac{1}{q}\sum_{t=0}^{q-1}\chi\big((u(r)-u(s))\mathfrak{p}u(t)\big) = \delta_{r,s}.$$

(ii) The matrix $E(\xi)$, *defined in* (5.2.4), *is unitary for a.e.* $\xi \in \mathfrak{D}$.

Proof. As we mentioned above, item (i) corresponds to the case $j = 1$ in Lemma 4.1.4 of Chapter 4.

To prove (ii), observe that the (r, s)-th block of the matrix $E(\xi)E^*(\xi)$ is

$$\sum_{t=0}^{q-1} E^{rt}(\xi)\left(E^{ts}(\xi)\right)^*.$$

The (l, j)-th entry in this block is

$$\begin{aligned}
&= \sum_{t=0}^{q-1}\sum_{m=0}^{N} \mathcal{E}_{lm}^{rt}(\xi)\left(\mathcal{E}_{mj}^{ts}(\xi)\right)^* \\
&= \sum_{t=0}^{q-1}\sum_{m=0}^{N} \delta_{lm}q^{-1/2}\overline{\chi\big(u(r)(\xi+\mathfrak{p}u(t))\big)}\cdot\delta_{jm}q^{-1/2}\chi\big(u(s)(\xi+\mathfrak{p}u(t))\big) \\
&= \sum_{m=1}^{N}\delta_{lm}\delta_{jm}q^{-1}\sum_{t=0}^{q-1}\overline{\chi\big(u(r)(\xi+\mathfrak{p}u(t))\big)}\chi\big(u(s)(\xi+\mathfrak{p}u(t))\big) \\
&= \sum_{m=1}^{N}\delta_{lm}\delta_{jm}\chi((u(s)-u(r))\xi)q^{-1}\sum_{t=0}^{q-1}\chi\big((u(s)-u(r))\mathfrak{p}u(t)\big) \\
&= \sum_{m=1}^{N}\delta_{lm}\delta_{jm}\delta_{rs}, \quad \text{(by part (i) of the lemma)} \\
&= \delta_{lj}\delta_{rs}.
\end{aligned}$$

Hence $E(\xi)E^*(\xi) = I$. Similarly, $E(\xi)^*E(\xi) = I$. Therefore, $E(\xi)$ is a unitary matrix. □

Let $\{\varphi_j : 1 \le j \le N\}$ be functions in $L^2(K)$ such that $\{\varphi_j(\cdot - u(k)) : 1 \le j \le N, k \in \mathbb{N}_0\}$ is a frame for its closed linear span V. For $1 \le l \le N, 0 \le r \le q-1$, suppose that there exist

sequences $\{h_{ljk}^r : k \in \mathbb{Z}\} \in \ell^2(\mathfrak{D})$. Define

$$\psi_l^r(x) = q^{1/2} \sum_{j=1}^{N} \sum_{k \in \mathbb{N}_0} h_{ljk}^r \varphi_j(\mathfrak{p}^{-1}x - u(k)).$$

Taking Fourier transform, we get

$$(\psi_l^r)\hat{}(\xi) = \sum_{j=1}^{N} \sum_{k \in \mathbb{N}_0} h_{ljk}^r q^{-1/2} \overline{\chi_k(\mathfrak{p}\xi)} \hat{\varphi}_j(\mathfrak{p}\xi) = \sum_{j=1}^{N} h_{lj}^r \hat{\varphi}_j(\mathfrak{p}\xi),$$

where

$$h_{lj}^r(\xi) = \sum_{k \in \mathbb{N}_0} q^{-1/2} h_{ljk}^r \overline{\chi_k(\xi)}.$$

Let

$$H_r(\xi) = \big(h_{lj}^r(\xi)\big)_{1 \le l,j \le N},$$

and

$$H(\xi) = \Big(H_r(\xi + \mathfrak{p}u(s))\Big)_{0 \le r,s \le q-1}.$$

Note that $H(\xi)$ is a square matrix of order qN. We can write ψ_l^r as

$$\begin{aligned}
\psi_l^r(x) &= \sum_{j=1}^{N} \sum_{k \in \mathbb{N}_0} h_{ljk}^r q^{1/2} \varphi_j\big(\mathfrak{p}^{-1}x - u(k)\big) \\
&= \sum_{j=1}^{N} \sum_{s=0}^{q-1} \sum_{k \in \mathbb{N}_0} h_{lj,qk+s}^r q^{1/2} \varphi_j\big(\mathfrak{p}^{-1}x - u(qk+s)\big) \\
&= \sum_{j=1}^{N} \sum_{s=0}^{q-1} \sum_{k \in \mathbb{N}_0} h_{lj,qk+s}^r \varphi_j^{(s)}(x - u(k)),
\end{aligned}$$

where

$$\varphi_j^{(s)}(x) = q^{1/2} \varphi_j(\mathfrak{p}^{-1}x - u(s)), \quad 0 \le s \le q-1. \tag{5.2.5}$$

Note that $u(qk+s) = \mathfrak{p}^{-1}u(k) + u(s)$ (see eq. (1.2.4)). Taking Fourier transform, we obtain

$$(\psi_l^r)\hat{}(\xi) = \sum_{j=1}^{N} \sum_{s=0}^{q-1} \sum_{k \in \mathbb{N}_0} h_{lj,qk+s}^r \overline{\chi_k(\xi)} (\varphi_j^{(s)})\hat{}(\xi)$$

$$= \sum_{j=1}^{N}\sum_{s=0}^{q-1} p_{lj}^{rs}(\xi)(\varphi_j^{(s)})^{\wedge}(\xi),$$

where $p_{lj}^{rs}(\xi) = \sum_{k\in\mathbb{N}_0} h^r_{lj,qk+s}\overline{\chi_k(\xi)}$. Define the matrices

$$P^{rs}(\xi) = \left(p_{lj}^{rs}(\xi)\right)_{1\le l,j\le N},$$

and

$$P(\xi) = \left(P^{rs}(\xi)\right)_{0\le r,s\le q-1}.$$

Proposition 5.2.3. *$H(\xi) = P(\mathfrak{p}^{-1}\xi)E(\xi)$, where $E(\xi)$ is the unitary matrix defined in* (5.2.4).

Proof. The (r,s)-th block of the matrix $P(\mathfrak{p}^{-1}\xi)E(\xi)$ is the matrix

$$\sum_{t=0}^{q-1} P^{rt}(\mathfrak{p}^{-1}\xi)E^{ts}(\xi).$$

The (l,j)-th entry in this block is equal to

$$\begin{aligned}
&\sum_{t=0}^{q-1}\sum_{m=1}^{N} p_{lm}^{rt}(\mathfrak{p}^{-1}\xi)\mathcal{E}_{mj}^{ts}(\xi)\\
=\ &\sum_{t=0}^{q-1}\sum_{m=1}^{N}\sum_{k\in\mathbb{N}_0} h^r_{l,m.qk+t}\overline{\chi_k(\mathfrak{p}^{-1}\xi)}\delta_{mj}q^{-1/2}\overline{\chi\big(u(t)(\xi+\mathfrak{p}u(s))\big)}\\
=\ &\sum_{t=0}^{q-1}\sum_{k\in\mathbb{N}_0} h^r_{l,j,qk+t}\overline{\chi_k(\mathfrak{p}^{-1}\xi)}q^{-1/2}\overline{\chi\big(u(t)(\xi+\mathfrak{p}u(s))\big)}.
\end{aligned}$$

Now, the (l,j)-th entry in the (r,s)-th block of $H(\xi)$ is

$$\begin{aligned}
&h^r_{lj}(\xi+\mathfrak{p}u(s))\\
=\ &q^{-1/2}\sum_{k\in\mathbb{N}_0} h^r_{ljk}\overline{\chi\big(u(k)(\xi+\mathfrak{p}u(s))\big)}\\
=\ &q^{-1/2}\sum_{t=0}^{q-1}\sum_{k\in\mathbb{N}_0} h^r_{l,j,qk+t}\overline{\chi\big(u(qk+t)(\xi+\mathfrak{p}u(s))\big)}
\end{aligned}$$

$$= q^{-1/2}\sum_{t=0}^{q-1}\sum_{k\in\mathbb{N}_0} h^r_{l,j,qk+t}\overline{\chi(\mathfrak{p}^{-1}u(k)\xi + u(k)u(s) + u(t)\xi + \mathfrak{p}u(t)u(s))}$$

$$= q^{-1/2}\sum_{t=0}^{q-1}\sum_{k\in\mathbb{N}_0} h^r_{l,j,qk+t}\overline{\chi_k(\mathfrak{p}^{-1}\xi)\chi(u(t)(\xi + \mathfrak{p}u(s)))}.$$

□

In particular, we have

$$H^*(\xi)H(\xi) = E^*(\xi)P^*(\mathfrak{p}^{-1}\xi)P(\mathfrak{p}^{-1}\xi)E(\xi).$$

Since $E(\xi)$ is unitary, it follows that $H^*(\xi)H(\xi)$ and $P^*(\mathfrak{p}^{-1}\xi)P(\mathfrak{p}^{-1}\xi)$ are similar matrices.

Let $\lambda(\xi)$ and $\Lambda(\xi)$ respectively be the minimal and maximal eigenvalues of the positive definite matrix $H^*(\xi)H(\xi)$, and let $\lambda = \inf_\xi \lambda(\xi)$ and $\Lambda = \sup_\xi \Lambda(\xi)$. Assume that $0 < \lambda \le \Lambda < \infty$. Then we have

$$\lambda I \le H^*(\xi)H(\xi) \le \Lambda I \quad \text{for a.e. } \xi \in \mathfrak{D}.$$

This is equivalent to say that

$$\lambda I \le P^*(\xi)P(\xi) \le \Lambda I \quad \text{for a.e. } \xi \in \mathfrak{D}.$$

Then by Lemma 5.2.1, for all $g \in L^2(K)$, we have

$$\begin{aligned}\lambda\sum_{s=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}\left|\left\langle g,\varphi_l^{(s)}(\cdot - u(k))\right\rangle\right|^2 &\le \sum_{s=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,\psi_l^s(\cdot - u(k))\rangle|^2 \\ &\le \Lambda\sum_{s=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}\left|\left\langle g,\varphi_l^{(s)}(\cdot - u(k))\right\rangle\right|^2, \quad (5.2.6)\end{aligned}$$

where $\varphi_l^{(s)}$ is defined in (5.2.5). Since

$$\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g, q^{1/2}\varphi_l(\mathfrak{p}^{-1}\cdot - u(k))\rangle|^2 = \sum_{s=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}\left|\left\langle g,\varphi_l^{(s)}(\cdot - u(k))\right\rangle\right|^2,$$

which follows from (5.2.5), inequality (5.2.6) can be written as

$$\begin{aligned}
&\lambda \sum_{l=1}^{N} \sum_{k\in\mathbb{N}_0} \left|\langle g, q^{1/2}\varphi_l(\mathfrak{p}^{-1}\cdot - u(k))\rangle\right|^2 \\
&\leq \sum_{s=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0} \left|\langle g, \psi_l^s(\cdot - u(k))\rangle\right|^2 \\
&\leq \Lambda \sum_{l=1}^{N} \sum_{k\in\mathbb{N}_0} \left|\langle g, q^{1/2}\varphi_l(\mathfrak{p}^{-1}\cdot - u(k))\rangle\right|^2 . \qquad (5.2.7)
\end{aligned}$$

This is the *splitting trick* for frames.

We now apply the splitting trick to the functions $\{\psi_l^s : 1 \leq l \leq N\}$ for each $s, 0 \leq s \leq q-1$. We have

$$\begin{aligned}
&\lambda \sum_{l=1}^{N} \sum_{k\in\mathbb{N}_0} \left|\langle g, q^{1/2}\psi_l^s(\mathfrak{p}^{-1}\cdot - u(k))\rangle\right|^2 \\
&\leq \sum_{r=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0} \left|\langle g, \psi_l^{s,r}(\cdot - u(k))\rangle\right|^2 \\
&\leq \Lambda \sum_{l=1}^{N} \sum_{k\in\mathbb{N}_0} \left|\langle g, q^{1/2}\psi_l^s(\mathfrak{p}^{-1}\cdot - u(k))\rangle\right|^2 , \qquad (5.2.8)
\end{aligned}$$

where $\psi_l^{s,r}, 0 \leq r \leq q-1$ are defined as

$$\psi_l^{s,r}(x) = \sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0} h_{ljk}^s q^{1/2}\psi_j^r(\mathfrak{p}^{-1}x - u(k)),\ 0 \leq s \leq q-1, 1 \leq l \leq N. \qquad (5.2.9)$$

Summing (5.2.8) over $0 \leq s \leq q-1$, we have

$$\begin{aligned}
\lambda \sum_{s=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0} \left|\langle g, q^{1/2}\psi_l^s(\mathfrak{p}^{-1}\cdot - u(k))\rangle\right|^2 &\leq \sum_{s=0}^{q-1}\sum_{r=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0} \left|\langle g, \psi_l^{s,r}(\cdot - u(k))\rangle\right|^2 \\
&\leq \Lambda \sum_{s=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0} \left|\langle g, q^{1/2}\psi_l^s(\mathfrak{p}^{-1}\cdot - u(k))\rangle\right|^2 .
\end{aligned}$$

Using (5.2.7), we obtain

$$\begin{aligned}
&\lambda^2 \sum_{l=1}^{N} \sum_{k\in\mathbb{N}_0} \left|\langle g, q^{2/2}\varphi_l(\mathfrak{p}^2 \cdot -u(k))\rangle\right|^2 \\
&\leq \sum_{s=0}^{q-1}\sum_{r=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0} \left|\langle g, \psi_l^{s,r}(\cdot - u(k))\rangle\right|^2 \\
&\leq \Lambda^2 \sum_{l=1}^{N} \sum_{k\in\mathbb{N}_0} \left|\langle g, q^{2/2}\varphi_l(\mathfrak{p}^2 \cdot -u(k))\rangle\right|^2. \qquad (5.2.10)
\end{aligned}$$

We now define the wavelet frame packets similar to the orthonormal case. We start with the functions $\varphi_1, \varphi_2, \ldots, \varphi_N$. Apply the splitting trick to the space

$$\overline{\text{span}}\{q^{1/2}\varphi_l(\mathfrak{p}^{-1} \cdot -u(k)) : 1 \leq l \leq N, k \in \mathbb{N}_0\}$$

to get the functions $\{\psi_l^s : 1 \leq l \leq N, 0 \leq s \leq q-1\}$ (see (5.2.7)). Now for any integer $n \geq 0$, we define ψ_l^n, $1 \leq l \leq N$, recursively as follows. Suppose that ψ_l^r is already defined for $r \in \mathbb{N}_0$ and $1 \leq l \leq N$. Then for $0 \leq s \leq q-1$ and $1 \leq l \leq N$, define

$$\psi_l^{s+qr} = \sum_{j=1}^{N}\sum_{k\in\mathbb{N}_0} h_{ljk}^s q^{1/2}\psi_j^r(\mathfrak{p}^{-1} \cdot -u(k)).$$

Comparing this with equation (5.2.9), we see that

$$\begin{aligned}
\{\psi_l^{s,r} : 0 \leq r, s \leq q-1\} &= \{\psi_l^{s+qr} : 0 \leq r, s \leq q-1\} \\
&= \{\psi_l^n : 0 \leq n \leq q^2-1\}.
\end{aligned}$$

So (5.2.10) can be written as

$$\begin{aligned}
\lambda^2 \sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0} \left|\langle g, q^{2/2}\varphi_l(\mathfrak{p}^{-2} \cdot -u(k))\rangle\right|^2 &\leq \sum_{n=0}^{q^2-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0} \left|\langle g, \psi_l^n(\cdot - u(k))\rangle\right|^2 \\
&\leq \Lambda^2 \sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0} \left|\langle g, q^{2/2}\varphi_l(\mathfrak{p}^{-2} \cdot -u(k))\rangle\right|^2.
\end{aligned}$$

By induction, we get for each $j \geq 1$,

$$\begin{aligned}
&\lambda^j \sum_{l=1}^{N} \sum_{k\in\mathbb{N}_0} \left|\langle g, q^{j/2}\varphi_l(\mathfrak{p}^{-j}\cdot - u(k))\rangle\right|^2 \\
&\leq \sum_{n=0}^{q^j-1} \sum_{l=1}^{N} \sum_{k\in\mathbb{N}_0} |\langle g, \psi_l^n(\cdot - u(k))\rangle|^2 \\
&\leq \Lambda^j \sum_{l=1}^{N} \sum_{k\in\mathbb{N}_0} \left|\langle g, q^{j/2}\varphi_l(\mathfrak{p}^{-j}\cdot - u(k))\rangle\right|^2 . \qquad (5.2.11)
\end{aligned}$$

We summarize the above discussion in the following theorem.

Theorem 5.2.4. *Let $\{\varphi_l : 1 \leq l \leq N\} \subset L^2(K)$ be such that $\{\varphi_l(\cdot - u(k)) : 1 \leq l \leq N, k \in \mathbb{N}_0\}$ is a frame for its closed linear span V_0, with frame bounds C_1 and C_2. Let $H(\xi), H_r(\xi), \lambda$ and Λ be as above. Assume that all entries of $H(\xi)$ are bounded measurable functions such that $0 < \lambda \leq \Lambda < \infty$. Let $\{\psi_l^n : n \geq 0, 1 \leq l \leq N\}$ be the wavelet frame packets and let $V_j = \{f \in L^2(K) : f(\mathfrak{p}^j\cdot) \in V_0\}$. Then for all $j \geq 0$, the system of functions*

$$\{\psi_l^n(\cdot - u(k)) : 0 \leq n \leq q^j - 1, 1 \leq l \leq N, k \in \mathbb{N}_0\}$$

is a frame of V_j with frame bounds $\lambda^j C_1$ and $\Lambda^j C_2$.

Proof. Since $\{\varphi_l(\cdot - u(k)) : 1 \leq l \leq N, k \in \mathbb{N}_0\}$ is a frame of V_0 with frame bounds C_1 and C_2, it is clear that for all j,

$$\{q^{j/2}\varphi_l(\mathfrak{p}^{-j}\cdot - u(k)) : 1 \leq l \leq N, k \in \mathbb{N}_0\}$$

is a frame of V_j with the same bounds. So from (5.2.11), we have

$$\lambda^j C_1 \|g\|^2 \leq \sum_{n=0}^{q^j-1} \sum_{l=1}^{N} \sum_{k\in\mathbb{N}_0} |\langle g, \psi_l^n(\cdot - u(k))\rangle|^2 \leq \Lambda^j C_2 \|g\|^2 \quad \text{for all } g \in V_j.$$

□

Chapter 6

Biorthogonal Wavelets

The concept of biorthogonal wavelets plays an important role in applications. We refer to [23, 25, 51, 73] for various aspects of this theory on $\mathbb{R}$. For the higher dimensional situation on $\mathbb{R}^n$, we refer to the articles [16, 18, 60]. This chapter is devoted to the study of biorthogonal wavelets on local fields of positive characteristic.

As we mentioned in Chapter 1, Farkov [32] has constructed many examples of wavelets for the Vilenkin groups which are local fields of positive characteristic. Several examples of biorthogonal wavelets on the Vilenkin groups were constructed by Farkov in [34] and by Farkov and Rodionov in [35]. By choosing the parameters appearing in these constructions suitably, we can see that these wavelets are not orthogonal. Also, in [35], the authors have provided an algorithm to construct biorthogonal wavelets on such groups.

In this chapter we generalize the concept of biorthogonal wavelets to a local field K of positive characteristic. We say that two MRAs are dual to each other if the translates of the corresponding scaling functions are biorthogonal. We define the projection operators associated with dual MRAs and show that they are uniformly bounded on $L^2(K)$. We show that if φ and $\tilde{\varphi}$ are the scaling functions of dual MRAs, then the associated families of wavelets are biorthogonal. Under mild decay conditions on the scaling functions and the corresponding wavelets, we also show that the wavelets generate Riesz bases for $L^2(K)$.

6.1 Riesz bases of translates

Definition 6.1.1. Let $\{\psi_n : n \in \mathbb{N}_0\}$ and $\{\tilde{\psi}_n : n \in \mathbb{N}_0\}$ be two collections of functions in $L^2(K)$. We say that they are biorthogonal if

$$\langle \psi_n, \tilde{\psi}_m \rangle = \delta_{n,m} \quad \text{for every } m, n \in \mathbb{N}_0.$$

A collection $\{\psi_n : n \in \mathbb{N}_0\}$ of functions in $L^2(K)$ is said to be *linearly independent* if for any ℓ^2-sequence $\{a_n : n \in \mathbb{N}_0\}$ of coefficients with $\sum_{n\in\mathbb{N}_0} a_n\psi_n = 0$ in $L^2(K)$, we have $a_n = 0$ for all $n \in \mathbb{N}_0$. It is easy to see that biorthogonal sets are linearly independent.

Lemma 6.1.2. *Let $\{\psi_n : n \in \mathbb{N}_0\}$ be a collection of functions in $L^2(K)$. Suppose that there is a collection $\{\tilde{\psi}_n : n \in \mathbb{N}_0\}$ in $L^2(K)$ which is biorthogonal to $\{\psi_n : n \in \mathbb{N}_0\}$. Then $\{\psi_n : n \in \mathbb{N}_0\}$ is linearly independent.*

Proof. Let $\{a_n : n \in \mathbb{N}_0\}$ be an ℓ^2-sequence satisfying $\sum_{n\in\mathbb{N}_0} a_n\psi_n = 0$ in $L^2(K)$. Then for each $m \in \mathbb{N}_0$, we have

$$0 = \langle 0, \tilde{\psi}_m \rangle = \Big\langle \sum_{n=1}^{\infty} a_n\psi_n, \tilde{\psi}_m \Big\rangle = \sum_{n=1}^{\infty} a_n \langle \psi_n, \tilde{\psi}_m \rangle = a_m.$$

Hence, $\{\psi_n : n \in \mathbb{N}_0\}$ is linearly independent. □

We first recall the definition of a Riesz basis. Let $\{x_n : n \in \mathbb{N}_0\}$ be a subset of a separable Hilbert space H. Then $\{x_n : n \in \mathbb{N}_0\}$ is called a Riesz basis of H if

(a) $\overline{\text{span}}\{x_n : n \in \mathbb{N}_0\} = H$, and

(b) there exist constants A and B with $0 < A \le B < \infty$ such that

$$A \sum_{n\in\mathbb{N}_0} |c_n|^2 \le \Big\| \sum_{n\in\mathbb{N}_0} c_n x_n \Big\|^2 \le B \sum_{n\in\mathbb{N}_0} |c_n|^2 \quad \text{for every } \{c_n : n \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0).$$

The above definition is equivalent to the following definition. A subset $\{x_n : n \in \mathbb{N}_0\}$ is a Riesz basis for H if

(i) $\{x_n : n \in \mathbb{N}_0\}$ is linearly independent, and

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_2^2 \leq \sum_{n\in\mathbb{N}_0} |\langle x, x_n\rangle|^2 \leq B\|x\|_2^2 \quad \text{for every } x \in H.$$

Note. The condition in (ii) above is known as the "frame condition".

In the following lemma, we provide a necessary and sufficient condition for the translates of two functions to be biorthogonal.

Lemma 6.1.3. *Let $\varphi, \tilde{\varphi} \in L^2(K)$ be given. Then $\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$ is biorthogonal to $\{\tilde{\varphi}(\cdot - u(n)) : n \in \mathbb{N}_0\}$ if and only if*

$$\sum_{n\in\mathbb{N}_0} \hat{\varphi}(\xi + u(n))\overline{\hat{\tilde{\varphi}}(\xi + u(n))} = 1 \quad \text{for a.e. } \xi \in K. \tag{6.1.1}$$

Proof. For a fixed $l \in \mathbb{N}_0$, we have $\{u(l) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ (see Proposition 2.1.5(c)). Hence, it follows that $\langle \varphi(\cdot - u(n)), \tilde{\varphi}(\cdot - u(m))\rangle = \delta_{n,m}$ if and only if $\langle \varphi, \tilde{\varphi}(\cdot - u(m))\rangle = \delta_{0,m}$. Since

$$\begin{aligned} \langle \varphi, \tilde{\varphi}(\cdot - u(m))\rangle &= \int_K \hat{\varphi}(\xi)\overline{\hat{\tilde{\varphi}}(\xi)}\chi_m(\xi)d\xi \\ &= \int_{\mathfrak{D}} \sum_{l\in\mathbb{N}_0} \hat{\varphi}(\xi + u(l))\overline{\hat{\tilde{\varphi}}(\xi + u(l))}\chi_m(\xi)d\xi, \end{aligned}$$

the result follows from the uniqueness of the Fourier series and the fact that $\{\chi_m : m \in \mathbb{N}_0\}$ is an orthonormal basis for $L^2(\mathfrak{D})$. □

The following lemma provides a sufficient condition for the translates of a function to be linearly independent.

Lemma 6.1.4. *Let $\varphi \in L^2(K)$. Assume that there exist constants $c_1, c_2 > 0$ such that*

$$c_1 \leq \sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 \leq c_2 \quad \text{for a.e. } \xi \in K. \tag{6.1.2}$$

Then $\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$ is linearly independent.

Proof. By Lemma 6.1.2, it suffices to find a function $\tilde{\varphi}$ whose translates are biorthogonal to the

translates of φ. We define $\tilde{\varphi}$ by

$$\widehat{\tilde{\varphi}}(\xi) = \frac{\hat{\varphi}(\xi)}{\sum\limits_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2}.$$

By (6.1.2), this function is well-defined. Now for a.e. $\xi \in K$, we have

$$\begin{aligned}\sum_{m\in\mathbb{N}_0} \hat{\varphi}(\xi + u(m))\overline{\widehat{\tilde{\varphi}}(\xi + u(m))} &= \sum_{m\in\mathbb{N}_0} \hat{\varphi}(\xi + u(m))\frac{\overline{\hat{\varphi}(\xi + u(m))}}{\sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k) + u(m))|^2} \\ &= \frac{\sum\limits_{m\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(m))|^2}{\sum\limits_{l\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(l))|^2} = 1.\end{aligned}$$

By Lemma 6.1.3, $\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$ is biorthogonal to $\{\tilde{\varphi}(\cdot - u(n)) : n \in \mathbb{N}_0\}$. □

Lemma 6.1.5. *Suppose that φ satisfies* (6.1.2). *Any f in* $\operatorname{span}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$ *is of the form $f = \sum\limits_{n\in\mathbb{N}_0} a_n\varphi(\cdot - u(n))$, where $\{a_n\}$ is a finite sequence. Let $\hat{a}$ be its Fourier transform, that is, $\hat{a}(\xi) = \sum\limits_{n\in\mathbb{N}_0} a_n\overline{\chi_n}(\xi)$. Then*

$$c_1 \int_{\mathfrak{D}} |\hat{a}(\xi)|^2\, d\xi \le \|f\|_2^2 \le c_2 \int_{\mathfrak{D}} |\hat{a}(\xi)|^2\, d\xi.$$

Proof. By Placherel's theorem, we have

$$\begin{aligned}\int_K |f(x)|^2\, dx &= \int_K \Big|\sum_{n\in\mathbb{N}_0} a_n\varphi(x - u(n))\Big|^2\, dx \\ &= \int_K \Big|\sum_{n\in\mathbb{N}_0} a_n\hat{\varphi}(\xi)\overline{\chi_n}(\xi)\Big|^2\, d\xi \\ &= \int_K |\hat{\varphi}(\xi)|^2\Big|\sum_{n\in\mathbb{N}_0} a_n\overline{\chi_n}(\xi)\Big|^2\, d\xi \\ &= \int_K |\hat{\varphi}(\xi)|^2|\hat{a}(\xi)|^2\, d\xi \\ &= \int_{\mathfrak{D}} \sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))^2\hat{a}(\xi)|^2\, d\xi.\end{aligned}$$

The result follows by (6.1.2). □

Remark 6.1.6. In particular, for a finite sequence $\{a_n\}$, we have

$$\Big\| \sum_{n\in\mathbb{N}_0} a_n \varphi(\cdot - u(n)) \Big\|_2^2 \le c_2 \sum_{n\in\mathbb{N}_0} |a_n|^2.$$

Theorem 6.1.7. *Let $\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$ be a Riesz basis for its closed linear span. Suppose that there exists a function $\tilde{\varphi}$ such that $\{\tilde{\varphi}(\cdot - u(n)) : n \in \mathbb{N}_0\}$ is biorthogonal to $\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$. Then*

(a) for every $f \in \overline{\text{span}}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$, we have

$$f = \sum_{n\in\mathbb{N}_0} \langle f, \tilde{\varphi}(\cdot - u(n))\rangle \varphi(\cdot - u(n));$$

(b) there exist constants $A, B > 0$ such that for every $f \in \overline{\text{span}}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$,

$$A\|f\|_2^2 \le \sum_{n=1}^{\infty} |\langle f, \tilde{\varphi}(\cdot - u(n))\rangle|^2 \le B\|f\|_2^2. \tag{6.1.3}$$

Proof. Since $\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$ forms a Riesz basis for its closed linear span, there exist constants c_1 and c_2 such that (6.1.2) holds (see Lemma 2.1.8). We will first prove (a) and (b) for $f \in \text{span}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$ and then generalize the results to $\overline{\text{span}}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$.

(a) Let $f \in \text{span}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$, then there exist a finite sequence $\{a_n\}$ such that $f = \sum_{n\in\mathbb{N}_0} a_n \varphi(\cdot - u(n))$. Using biorthogonality, we have

$$\begin{aligned} \langle f, \tilde{\varphi}(\cdot - u(k))\rangle &= \Big\langle \sum_{n\in\mathbb{N}_0} a_n \varphi(\cdot - u(n)), \tilde{\varphi}(\cdot - u(k)) \Big\rangle \\ &= \sum_{n\in\mathbb{N}_0} a_n \langle \varphi(\cdot - u(n)), \tilde{\varphi}(\cdot - u(k))\rangle \\ &= a_k. \end{aligned}$$

(b) Since (6.1.2) is satisfied, by Lemma 6.1.5, for every $f \in \text{span}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$, we have

$$c_2^{-1}\|f\|_2^2 \le \int_{\mathfrak{D}} |\hat{a}(\xi)|^2 d\xi \le c_1^{-1}\|f\|_2^2.$$

By Plancherel formula for Fourier series and the fact that $a_n = \langle f, \tilde{\varphi}(\cdot - u(n))\rangle$, we have

$$\int_{\mathfrak{D}} |\hat{a}(\xi)|^2 d\xi = \sum_{n\in\mathbb{N}_0} |a_n|^2 = \sum_{n\in\mathbb{N}_0} |\langle f, \tilde{\varphi}(\cdot - u(n))\rangle|^2.$$

So (b) is proved.

We now generalize the results to $\overline{\text{span}}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$. First we will prove (b). For $f \in \overline{\text{span}}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$, there exists a sequence $\{f_m : m \in \mathbb{N}_0\}$ in $\text{span}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$ such that $\|f_m - f\|_2 \to 0$ as $m \to \infty$. Hence, for each $n \in \mathbb{N}_0$,

$$\langle f_m, \tilde{\varphi}(\cdot - u(n))\rangle \to \langle f, \tilde{\varphi}(\cdot - u(n))\rangle \text{ as } m \to \infty.$$

The result holds for each f_m, Hence,

$$\begin{aligned}
\sum_{n=0}^{N} |\langle f, \tilde{\varphi}(\cdot - u(n))\rangle|^2 &= \sum_{n=0}^{N} \lim_{m\to\infty} |\langle f_m, \tilde{\varphi}(\cdot - u(n))\rangle|^2 \\
&= \lim_{m\to\infty} \sum_{n=0}^{N} |\langle f_m, \tilde{\varphi}(\cdot - u(n))\rangle|^2 \\
&\le B \lim_{m\to\infty} \|f_m\|_2^2 \\
&= B\|f\|_2^2.
\end{aligned}$$

Letting $N \to \infty$ in the above expression, we get

$$\sum_{n\in\mathbb{N}_0} |\langle f, \tilde{\varphi}(\cdot - u(n))\rangle|^2 \le B\|f\|_2^2.$$

Hence, the upper bound in (6.1.3) holds. Now

$$\begin{aligned}
&\Big(\sum_{n\in\mathbb{N}_0} |\langle f_m, \tilde{\varphi}(\cdot - u(n))\rangle|^2\Big)^{\frac{1}{2}} \\
&\le \Big(\sum_{n\in\mathbb{N}_0} |\langle f_m - f, \tilde{\varphi}(\cdot - u(n))\rangle|^2\Big)^{\frac{1}{2}} + \Big(\sum_{n\in\mathbb{N}_0} |\langle f, \tilde{\varphi}(\cdot - u(n))\rangle|^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

Since the upper bound in (6.1.3) holds for each $f_m - f$ and the lower bound holds for each f_m,

we have

$$A^{\frac{1}{2}}\|f_m\|_2 \leq B^{\frac{1}{2}}\|f_m - f\|_2 + \Big(\sum_{n\in\mathbb{N}_0} |\langle f, \tilde{\varphi}(\cdot - u(n))\rangle|^2\Big)^{\frac{1}{2}}.$$

Taking limit as $m \to \infty$, we get

$$A\|f\|_2^2 \leq \sum_{n\in\mathbb{N}_0} |\langle f, \tilde{\varphi}(\cdot - u(n))\rangle|^2.$$

Now, we will prove (a) for $f \in \overline{\text{span}}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$. Let $\epsilon > 0$ and $g \in \text{span}\{\varphi(\cdot - u(n)) : n \in \mathbb{N}_0\}$ such that $\|f - g\|_2 < \epsilon$. Since (a) holds for g, for large enough $N \in \mathbb{N}_0$, we have

$$\begin{aligned} & f - \sum_{n=0}^{N}\langle f, \tilde{\varphi}(\cdot - u(n))\rangle\varphi(\cdot - u(n)) \\ &= f - g + \sum_{n=0}^{N}\langle g, \tilde{\varphi}(\cdot - u(n))\rangle\varphi(\cdot - u(n)) - \sum_{n=0}^{N}\langle f, \tilde{\varphi}(\cdot - u(n))\rangle\varphi(\cdot - u(n)) \\ &= f - g + \sum_{n=0}^{N}\langle g - f, \tilde{\varphi}(\cdot - u(n))\rangle\varphi(\cdot - u(n)). \end{aligned}$$

Hence,

$$\begin{aligned} & \Big\|f - \sum_{n=0}^{N}\langle f, \tilde{\varphi}(\cdot - u(n))\rangle\varphi(\cdot - u(n))\Big\|_2 \\ &\leq \|f - g\|_2 + \Big\|\sum_{n=0}^{N}\langle g - f, \tilde{\varphi}(\cdot - u(n))\rangle\varphi(\cdot - u(n))\Big\|_2 \\ &\leq \|f - g\|_2 + \sqrt{c_2}\Big(\sum_{n=0}^{N}|\langle g - f, \tilde{\varphi}(\cdot - u(n))\rangle|^2\Big)^{\frac{1}{2}} \text{ (by Remark 6.1.6)} \\ &\leq \|f - g\|_2 + \sqrt{c_2}\sqrt{B}\|f - g\|_2 < (1 + \sqrt{c_2 B})\epsilon. \end{aligned}$$

Since ϵ is arbitrary, the result follows. □

6.2 Projection operators associated with dual MRAs

In the usual definition of an MRA (see Definition 2.1.1), it is required that there exists a function $\varphi \in V_0$ such that $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms an orthonormal basis for V_0. In Chapter 2, we

have proved that if $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms a Riesz basis for V_0, then we can find another function $\varphi_1 \in V_0$ such that $\{\varphi_1(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms an orthonormal basis for V_0 (see Proposition 2.1.9).

Therefore, in the definition of an MRA, we will replace the condition (e) by the following weaker condition:

(e') there is a function $\varphi \in V_0$, called the scaling function, such that $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms a Riesz basis for V_0.

We can use the condition (e') to get Riesz bases for V_j.

Lemma 6.2.1. *Let φ be the scaling function for an MRA $\{V_j : j \in \mathbb{Z}\}$. Then, for each $j \in \mathbb{Z}$, $\{\varphi_{j,k} : k \in \mathbb{N}_0\}$ is a Riesz basis for V_j.*

Proof. If we define $\tilde{\varphi}$ by

$$\hat{\tilde{\varphi}}(\xi) = \frac{\hat{\varphi}(\xi)}{\sum\limits_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2},$$

then $\{\tilde{\varphi}(\cdot - u(k)) : k \in \mathbb{N}_0\}$ is biorthogonal to $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ (see the proof of Lemma 6.1.4). Hence,

$$\langle \varphi_{j,n}, \tilde{\varphi}_{j,m} \rangle = \langle \delta_j \varphi(\cdot - u(n)), \delta_j \tilde{\varphi}(\cdot - u(m)) \rangle = \langle \varphi(\cdot - u(n)), \tilde{\varphi}(\cdot - u(m)) \rangle = \delta_{n,m}.$$

That is, $\{\tilde{\varphi}_{j,k} : k \in \mathbb{N}_0\}$ is biorthogonal to $\{\varphi_{j,k} : k \in \mathbb{N}_0\}$ for every $j \in \mathbb{Z}$. Hence, by Lemma 6.1.2, $\{\varphi_{j,k} : k \in \mathbb{N}_0\}$ is linearly independent.

We need to show that $\{\varphi_{j,k} : k \in \mathbb{N}_0\}$ satisfies the frame condition. For any $f \in V_j$, we have

$$\sum_{k\in\mathbb{N}_0} |\langle f, \varphi_{j,k} \rangle|^2 = \sum_{k\in\mathbb{N}_0} |\langle f, \delta_j \varphi(\cdot - u(k)) \rangle|^2 = \sum_{k\in\mathbb{N}_0} |\langle \delta_{-j} f, \varphi(\cdot - u(k)) \rangle|^2.$$

Since $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ is a Riesz basis for V_0 and $\delta_{-j} f \in V_0$, there are constants $A, B > 0$ such that for every $f \in V_j$,

$$A\|\delta_{-j} f\|_2^2 \le \sum_{k\in\mathbb{N}_0} |\langle \delta_{-j} f, \varphi(\cdot - u(k)) \rangle|^2 \le B\|\delta_{-j} f\|_2^2.$$

This is equivalent to

$$A\|f\|_2^2 \leq \sum_{k\in\mathbb{N}_0} |\langle f, \delta_j\varphi(\cdot - u(k))\rangle|^2 \leq B\|f\|_2^2.$$

Hence, $\{\varphi_{j,k} : k \in \mathbb{N}_0\}$ satisfies the frame condition. □

Lemma 6.2.2. *Suppose that $\{V_j : j \in \mathbb{Z}\}$ is an MRA with scaling function φ. Then there exists an ℓ^2-sequence $\{h_n : n \in \mathbb{N}_0\}$ such that*

$$\varphi(x) = \sum_{n\in\mathbb{N}_0} h_n q^{1/2}\varphi(\mathfrak{p}^{-1}x - u(n)),$$

and an integral-periodic function m_0 in $L^2(\mathfrak{D})$ such that

$$\hat{\varphi}(\xi) = m_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi).$$

Proof. Since $q^{-1}\varphi(\mathfrak{p}\cdot) \in V_{-1} \subset V_0$, by Theorem 6.1.7(a), we have

$$q^{-1}\varphi(\mathfrak{p}x) = \sum_{n\in\mathbb{N}_0} \langle f, \tilde{\varphi}(\cdot - u(n))\rangle\varphi(x - u(n)) = \sum_{n\in\mathbb{N}_0} h_n\varphi(x - u(n)).$$

Taking Fourier transform, we get

$$\hat{\varphi}(\mathfrak{p}^{-1}\xi) = \sum_{n\in\mathbb{N}_0} h_n\overline{\chi_n(\xi)}\hat{\varphi}(\xi) = m_0(\xi)\hat{\varphi}(\xi).$$

This is equivalent to

$$\hat{\varphi}(\xi) = m_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi).$$

By Theorem 6.1.7(b), $\{h_n : n \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$. Hence, $m_0 \in L^2(\mathfrak{D})$. As in Proposition 3 in [44], we can show that m_0 is integral-periodic. □

Definition 6.2.3. A pair of MRAs $\{V_j : j \in \mathbb{Z}\}$ and $\{\tilde{V}_j : j \in \mathbb{Z}\}$ with scaling functions φ and $\tilde{\varphi}$ respectively are said to be dual to each other if $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ and $\{\tilde{\varphi}(\cdot - u(k)) : k \in \mathbb{N}_0\}$ are biorthogonal.

Definition 6.2.4. Let φ and $\tilde{\varphi}$ be scaling functions for dual MRAs. For each $j \in \mathbb{Z}$, define the

operators P_j, $\tilde{P}_j$ on $L^2(K)$ by

$$P_j f(x) = \sum_{k\in\mathbb{N}_0} \langle f, \tilde{\varphi}_{j,k}\rangle \varphi_{j,k}(x), \tag{6.2.1}$$

$$\tilde{P}_j f(x) = \sum_{k\in\mathbb{N}_0} \langle f, \varphi_{j,k}\rangle \tilde{\varphi}_{j,k}(x). \tag{6.2.2}$$

We first note that the series defining these operators are convergent in $L^2(K)$ and that these operators are uniformly bounded on $L^2(K)$.

Lemma 6.2.5. *The operators P_j and $\tilde{P}_j$ are uniformly bounded.*

Proof. Since the translates of φ and $\tilde{\varphi}$ form Riesz bases for their closed linear spans, by Lemma 2.1.8, there exist constants C_1 and C_2 such that

$$C_1 \le \sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 \le C_2 \quad \text{and} \quad C_1 \le \sum_{k\in\mathbb{N}_0} |\hat{\tilde{\varphi}}(\xi + u(k))|^2 \le C_2.$$

Also, there exists a constant $B > 0$ such that for all $\{c_k : k \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$, we have

$$\Big\| \sum_{k\in\mathbb{N}_0} c_k \varphi_{0,k} \Big\|_2^2 \le B \sum_{k\in\mathbb{N}_0} |c_k|^2. \tag{6.2.3}$$

Now, for $f \in L^2(K)$, we have

$$\begin{aligned}
\sum_{k\in\mathbb{N}_0} |\langle f, \varphi_{0,k}\rangle|^2 &= \sum_{k\in\mathbb{N}_0} \Big| \int_K \hat{f}(\xi)\overline{\hat{\varphi}(\xi)}\chi_k(\xi)\, d\xi \Big|^2 \\
&= \sum_{k\in\mathbb{N}_0} \Big| \int_{\mathfrak{D}} \sum_{l\in\mathbb{N}_0} \hat{f}(\xi + u(l))\overline{\hat{\varphi}(\xi + u(l))}\chi_k(\xi)\, d\xi \Big|^2 \\
&= \sum_{k\in\mathbb{N}_0} \Big| \int_{\mathfrak{D}} F(\xi)\chi_k(\xi)\, d\xi \Big|^2 = \sum_{k\in\mathbb{N}_0} |\hat{F}(k)|^2 = \|F\|^2_{L^2(\mathfrak{D})} \\
&= \int_{\mathfrak{D}} \Big| \sum_{l\in\mathbb{N}_0} \hat{f}(\xi + u(l))\overline{\hat{\varphi}(\xi + u(l))} \Big|^2 d\xi \\
&\le \int_{\mathfrak{D}} \Big(\sum_{l\in\mathbb{N}_0} |\hat{f}(\xi + u(l))|^2 \Big) \Big(\sum_{l\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(l))|^2 \Big)\, d\xi \\
&\le C_2 \int_{\mathfrak{D}} \Big(\sum_{l\in\mathbb{N}_0} |\hat{f}(\xi + u(l))|^2 \Big)\, d\xi \\
&= C_2 \int_K |\hat{f}(\xi)|^2\, d\xi = C_2 \|f\|_2^2.
\end{aligned}$$

Similar estimates hold for $\tilde{\varphi}$. Hence, for $f \in L^2(K)$, we have

$$\begin{aligned} \|P_0 f\|_2^2 &= \Big\| \sum_{k\in\mathbb{N}_0} \langle f, \tilde{\varphi}_{0,k}\rangle \varphi_{0,k} \Big\|_2^2 \\ &\leq B \sum_{k\in\mathbb{N}_0} |\langle f, \tilde{\varphi}_{0,k}\rangle|^2 \quad \text{(by6.2.3)} \\ &\leq BC_2 \|f\|_2^2. \end{aligned}$$

Thus, P_0 is a bounded operator on $L^2(K)$ with norm at most $\sqrt{BC_2} = C$, say. Now, since the dilation operators are unitary and since

$$P_j f = \sum_{k\in\mathbb{N}_0} \langle f, \tilde{\varphi}_{j,k}\rangle \varphi_{j,k} = \sum_{k\in\mathbb{N}_0} \langle \delta_{-j} f, \tilde{\varphi}_{0,k}\rangle \delta_{-j}\varphi_{0,k},$$

we conclude that the operator norm of P_j is at most C. Similar arguments work for $\tilde{P}_j$. This finishes the proof of the lemma. □

In the following lemma, we prove some useful properties of the operators P_j and $\tilde{P}_j$.

Lemma 6.2.6. *The operators P_j and $\tilde{P}_j$ satisfy the following properties.*

(a) $P_j f = f$ if and only if $f \in V_j$, $\tilde{P}_j f = f$ if and only if $f \in \tilde{V}_j$;

(b) $\lim_{j\to\infty} \|P_j f - f\|_2 = 0$, and $\lim_{j\to-\infty} \|P_j f\|_2 = 0$ for every $f \in L^2(K)$.

Proof. (a) $P_j f = f$ if and only if $f = \sum_{n\in\mathbb{N}_0} \langle f, \tilde{\varphi}_{j,n}\rangle \varphi_{j,n}$. Since $\{\varphi_{j,n} : n \in \mathbb{N}_0\}$ is a Riesz basis for V_j and $\{\tilde{\varphi}_{j,n}\}$ is biorthogonal to $\{\varphi_{j,n}\}$, the result follows from Theorem 6.1.7. Similar argument works for $\tilde{P}_j f$.

(b) Let $f \in L^2(K)$ and $\epsilon > 0$. Since $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(K)$, there exists $J \in \mathbb{Z}$ and $g \in V_J$ such that $\|f - g\|_2 < \frac{\epsilon}{1+C}$, where C is as in Lemma 6.2.5. If $g \in V_j$, then $P_j g = g$ for every $j \geq J$. Thus, for $j \geq J$,

$$\begin{aligned} \|f - P_j f\|_2 &\leq \|f - g\|_2 + \|P_j(f - g)\|_2 \\ &\leq (1 + \|P_j\|)\|f - g\|_2 \\ &\leq (1 + C)\|f - g\|_2 < \epsilon. \end{aligned}$$

This shows that

$$\lim_{j\to\infty} \|P_j f - f\|_2 = 0.$$

Now consider $h \in \mathcal{S}$ (see Definition 1.2.4). Then

$$\|P_j h\|_2^2 = \Big\| \sum_{k\in\mathbb{N}_0} \langle h, \tilde{\varphi}_{j,k}\rangle \varphi_{j,k} \Big\|_2^2 \le B \sum_{k\in\mathbb{N}_0} |\langle h, \tilde{\varphi}_{j,k}\rangle|^2.$$

In Theorem 2.2.1, we proved that if $h \in \mathcal{S}$, then $\sum_{k\in\mathbb{N}_0} |\langle h, \tilde{\varphi}_{j,k}\rangle|^2 \to 0$ as $j \to -\infty$. Hence, $\|P_j h\|_2 \to 0$ as $j \to -\infty$. Since $\mathcal{S}$ is dense in $L^2(K)$, given $\epsilon > 0$, there exists $h \in \mathcal{S}$ such that $\|f - h\|_2 < \epsilon$. Hence,

$$\|P_j f\|_2 \le \|P_j(f-h)\|_2 + \|P_j h\|_2 \le C\|f-h\|_2 + \|P_j h\|_2.$$

Therefore, $\|P_j f\|_2 \to 0$ as $j \to -\infty$. □

6.3 Biorthogonality of the wavelets

Let $\{V_j : j \in \mathbb{Z}\}$ and $\{\tilde{V}_j : j \in \mathbb{Z}\}$ be dual MRAs with scaling function φ and $\tilde{\varphi}$ respectively. By Lemma 6.2.2, there exist integral-periodic functions m_0 and $\tilde{m}_0$ in $L^2(\mathfrak{D})$ such that $\hat{\varphi}(\xi) = m_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi)$ and $\hat{\tilde{\varphi}}(\xi) = \tilde{m}_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi)$. Assume that there exist integral-periodic functions m_l and $\tilde{m}_l$ in $L^2(\mathfrak{D})$ for $1 \le l \le q-1$ such that

$$M(\xi)\tilde{M}^*(\xi) = I, \tag{6.3.1}$$

where $M(\xi) = \Big(m_l(\mathfrak{p}\xi + \mathfrak{p}u(k))\Big)_{l,k=0}^{q-1}$ and $\tilde{M}(\xi) = \Big(\tilde{m}_l(\mathfrak{p}\xi + \mathfrak{p}u(k))\Big)_{l,k=0}^{q-1}$. Now for $1 \le l \le q-1$, we define the associated wavelets ψ_l and $\tilde{\psi}_l$ as follows:

$$\hat{\psi}_l(\xi) = m_l(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi) \quad \text{and} \quad \hat{\tilde{\psi}}_l(\xi) = \tilde{m}_l(\mathfrak{p}\xi)\hat{\tilde{\varphi}}(\mathfrak{p}\xi).$$

We have the following lemma which shows orthogonality relationships among the translates of the scaling functions and the wavelets.

Lemma 6.3.1. *Let φ and $\tilde{\varphi}$ be the scaling functions for dual MRAs and $\psi_l, \tilde{\psi}_l$, $1 \le l \le q-1$*

be the associated wavelets satisfying the matrix condition (6.3.1). Then the following hold.

(a) $\{\psi_{l,0,n} : n \in \mathbb{N}_0\}$ *is biorthogonal to* $\{\tilde{\psi}_{l,0,n} : n \in \mathbb{N}_0\}$*;*

(b) $\langle \psi_{l,0,n}, \tilde{\varphi}_{0,m} \rangle = \langle \tilde{\psi}_{l,0,n}, \varphi_{0,m} \rangle = 0$ *for all* $m, n \in \mathbb{N}_0$.

Proof. (a) We have

$$\begin{aligned}
&\sum_{n\in\mathbb{N}_0} \hat{\psi}_l(\xi + u(n))\overline{\hat{\tilde{\psi}}_l(\xi + u(n))} \\
&= \sum_{n\in\mathbb{N}_0} m_l(\mathfrak{p}\xi + \mathfrak{p}u(n))\hat{\varphi}(\mathfrak{p}\xi + \mathfrak{p}u(n))\overline{\tilde{m}_l(\mathfrak{p}\xi + \mathfrak{p}u(n))}\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi + \mathfrak{p}u(n))} \\
&= \sum_{s=0}^{q-1}\sum_{k\in\mathbb{N}_0} m_l(\mathfrak{p}\xi + \mathfrak{p}u(qk+s))\hat{\varphi}(\mathfrak{p}\xi + \mathfrak{p}u(qk+s)) \\
&\quad \times \overline{\tilde{m}_l(\mathfrak{p}\xi + \mathfrak{p}u(qk+s))}\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi + \mathfrak{p}u(qk+s))} \\
&= \sum_{s=0}^{q-1}\sum_{k\in\mathbb{N}_0} m_l(\mathfrak{p}\xi + \mathfrak{p}u(s))\hat{\varphi}(\mathfrak{p}\xi + \mathfrak{p}u(s) + u(k)) \\
&\quad \times \overline{\tilde{m}_l(\mathfrak{p}\xi + \mathfrak{p}u(s))}\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi + \mathfrak{p}u(s) + u(k))} \\
&= \sum_{s=0}^{q-1} m_l(\mathfrak{p}\xi + \mathfrak{p}u(s))\overline{\tilde{m}_l(\mathfrak{p}\xi + \mathfrak{p}u(s))} \\
&= 1.
\end{aligned}$$

Hence, by Lemma 6.1.3, $\{\psi_{l,0,n} : n \in \mathbb{N}_0\}$ is biorthogonal to $\{\tilde{\psi}_{l,0,n} : n \in \mathbb{N}_0\}$.

(b) For $m, n \in \mathbb{N}_0$, we have

$$\begin{aligned}
&\langle \psi_{l,0,n}, \tilde{\varphi}_{0,m} \rangle \\
&= \langle \psi_l(\cdot - u(n)), \tilde{\varphi}(\cdot - u(m)) \rangle \\
&= \langle \hat{\psi}_l\overline{\chi_n}, \hat{\tilde{\varphi}}\overline{\chi_m} \rangle \\
&= \int_K m_l(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi)\overline{\chi_n}(\xi)\overline{\tilde{m}_0(\mathfrak{p}\xi)}\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi)}\chi_m(\xi)d\xi \\
&= \int_{\mathfrak{D}} \sum_{k\in\mathbb{N}_0} m_l(\mathfrak{p}\xi + \mathfrak{p}u(k))\hat{\varphi}(\mathfrak{p}\xi + \mathfrak{p}u(k))\overline{\chi_n}(\xi) \\
&\qquad \times \overline{\tilde{m}_0(\mathfrak{p}\xi + \mathfrak{p}u(k))}\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi + \mathfrak{p}u(k))}\chi_m(\xi)d\xi \\
&= \int_{\mathfrak{D}} \sum_{s=0}^{q-1}\sum_{k\in\mathbb{N}_0} m_l(\mathfrak{p}\xi + \mathfrak{p}u(s) + u(k))\hat{\varphi}(\mathfrak{p}\xi + \mathfrak{p}u(s) + u(k))\overline{\chi_n}(\xi) \\
&\qquad \times \overline{\tilde{m}_0(\mathfrak{p}\xi + \mathfrak{p}u(s) + u(k))}\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi + \mathfrak{p}u(s) + u(k))}\chi_m(\xi)d\xi
\end{aligned}$$

$$= \int_{\mathfrak{D}} \sum_{s=0}^{q-1} m_l(\mathfrak{p}\xi + \mathfrak{p}u(s))\overline{\tilde{m}_0(\mathfrak{p}\xi + \mathfrak{p}u(s))}$$
$$= 0.$$

Similarly, we can show that $\langle \tilde{\psi}_{l,0,n}, \varphi_{0,m} \rangle = 0$. □

Our aim is to show that the wavelets associated with dual MRAs are biorthogonal and they form Riesz bases for $L^2(K)$. The following proposition is crucial for the proof of the main result of this chapter.

Proposition 6.3.2. *Let $\varphi, \tilde{\varphi}$ and $\psi_l, \tilde{\psi}_l$ for $1 \le l \le q-1$ be as in Lemma 6.3.1. Denote $\psi_0 = \varphi$ and $\tilde{\psi}_0 = \tilde{\varphi}$. Then for every $f \in L^2(K)$, we have*

$$P_1 f = P_0 f + \sum_{l=1}^{q-1} \sum_{k \in \mathbb{N}_0} \langle f, \tilde{\psi}_{l,0,k} \rangle \psi_{l,0,k} \tag{6.3.2}$$

and

$$\tilde{P}_1 f = \tilde{P}_0 f + \sum_{l=1}^{q-1} \sum_{k \in \mathbb{N}_0} \langle f, \psi_{l,0,k} \rangle \tilde{\psi}_{l,0,k}, \tag{6.3.3}$$

where the series converge in $L^2(K)$.

Proof. It is enough to prove (6.3.2) as the proof of (6.3.3) is similar. Moreover, it is enough to prove (6.3.2) in the weak sense, that is, for all $f, g \in L^2(K)$

$$\begin{aligned} \langle P_1 f, g \rangle &= \langle P_0 f, g \rangle + \sum_{l=1}^{q-1} \sum_{k \in \mathbb{N}_0} \langle f, \tilde{\psi}_{l,0,k} \rangle \overline{\langle g, \psi_{l,0,k} \rangle} \\ &= \sum_{l=0}^{q-1} \sum_{k \in \mathbb{N}_0} \langle f, \tilde{\psi}_{l,0,k} \rangle \overline{\langle g, \psi_{l,0,k} \rangle}. \end{aligned}$$

We have

$$\sum_{l=0}^{q-1} \sum_{k \in \mathbb{N}_0} \langle f, \tilde{\psi}_{l,0,k} \rangle \overline{\langle g, \psi_{l,0,k} \rangle}$$
$$= \sum_{l=0}^{q-1} \sum_{k \in \mathbb{N}_0} \Big(\int_K \hat{f}(\xi) \overline{\hat{\tilde{\psi}}_l(\xi)} \chi_k(\xi) d\xi \Big) \Big(\int_K \overline{\hat{g}(\xi)} \hat{\psi}_l(\xi) \overline{\chi_k(\xi)} d\xi \Big)$$

$$
\begin{aligned}
&= \sum_{l=0}^{q-1}\sum_{k\in\mathbb{N}_0}\Big(\int_{\mathfrak{D}}\sum_{\alpha\in\mathbb{N}_0}\hat{f}(\xi+u(\alpha))\overline{\hat{\tilde{\psi}}_l(\xi+u(\alpha))}\chi_k(\xi)d\xi\Big)\\
&\qquad\times\Big(\int_{\mathfrak{D}}\sum_{\beta\in\mathbb{N}_0}\overline{\hat{g}(\xi+u(\beta))}\hat{\psi}_l(\xi+u(\beta))\overline{\chi_k(\xi)}d\xi\Big)\\
&= \sum_{l=0}^{q-1}\int_{\mathfrak{D}}\Big(\sum_{\alpha\in\mathbb{N}_0}\hat{f}(\xi+u(\alpha))\overline{\hat{\tilde{\psi}}_l(\xi+u(\alpha))}\Big)\Big(\sum_{\beta\in\mathbb{N}_0}\overline{\hat{g}(\xi+u(\beta))}\hat{\psi}_l(\xi+u(\beta))\Big)d\xi\\
&= \int_{\mathfrak{D}}\sum_{l=0}^{q-1}\Big(\sum_{\alpha\in\mathbb{N}_0}\hat{f}(\xi+u(\alpha))\overline{\tilde{m}_l(\mathfrak{p}\xi+\mathfrak{p}u(\alpha))}\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+\mathfrak{p}u(\alpha))}\\
&\qquad\times\sum_{\beta\in\mathbb{N}_0}\overline{\hat{g}(\xi+u(\beta))}m_l(\mathfrak{p}\xi+\mathfrak{p}u(\beta))\hat{\varphi}(\mathfrak{p}\xi+\mathfrak{p}u(\beta))\Big)d\xi\\
&= \int_{\mathfrak{D}}\sum_{l=0}^{q-1}\Big(\sum_{\nu=0}^{q-1}\sum_{\alpha'\in\mathbb{N}_0}\hat{f}(\xi+u(q\alpha')+u(\nu))\overline{\tilde{m}_l(\mathfrak{p}\xi+u(\alpha')+\mathfrak{p}u(\nu))}\\
&\qquad\times\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+u(\alpha')+\mathfrak{p}u(\nu))}\\
&\qquad\times\sum_{\nu'=0}^{q-1}\sum_{\beta'\in\mathbb{N}_0}\overline{\hat{g}(\xi+u(q\beta')+u(\nu'))}m_l(\mathfrak{p}\xi+u(\beta')+\mathfrak{p}u(\nu'))\\
&\qquad\times\hat{\varphi}(\mathfrak{p}\xi+u(\beta')+\mathfrak{p}u(\nu'))\Big)d\xi\\
&= \int_{\mathfrak{D}}\sum_{\alpha'}\sum_{\beta'}\sum_{\nu}\sum_{\nu'}\Big(\sum_{l}\overline{\tilde{m}_l(\mathfrak{p}\xi+\mathfrak{p}u(\nu))}m_l(\mathfrak{p}\xi+\mathfrak{p}u(\nu'))\\
&\qquad\times\hat{f}(\xi+u(q\alpha')+u(\nu))\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+u(\alpha')+\mathfrak{p}u(\nu))}\\
&\qquad\times\overline{\hat{g}(\xi+u(q\beta')+u(\nu'))}\hat{\varphi}(\mathfrak{p}\xi+u(\beta')+\mathfrak{p}u(\nu'))\Big)d\xi\\
&= \int_{\mathfrak{D}}\sum_{\nu}\sum_{\alpha'}\sum_{\beta'}\Big(\hat{f}(\xi+u(q\alpha')+u(\nu))\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+u(\alpha')+\mathfrak{p}u(\nu))}\\
&\qquad\times\overline{\hat{g}(\xi+u(q\beta')+u(\nu'))}\hat{\varphi}(\mathfrak{p}\xi+u(\beta')+\mathfrak{p}u(\nu))\Big)d\xi\\
&= \sum_{\nu}\int_{\mathfrak{D}+u(\nu)}\sum_{\alpha'}\sum_{\beta'}\Big(\hat{f}(\xi+u(q\alpha'))\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+u(\alpha'))}\\
&\qquad\times\overline{\hat{g}(\xi+u(q\beta'))}\hat{\varphi}(\mathfrak{p}\xi+u(\beta'))\Big)d\xi. \qquad (6.3.4)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\langle P_1 f, g\rangle\\
&\quad= \sum_{k\in\mathbb{N}_0}\langle f,\tilde{\varphi}_{1,k}\rangle\overline{\langle g,\varphi_{1,k}\rangle}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{N}_0}\Big(\int_K \hat{f}(\xi)\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi)}\chi_k(\mathfrak{p}\xi)d\xi\Big)\Big(\int_K \overline{\hat{g}(\xi)}\hat{\varphi}(\mathfrak{p}\xi)\overline{\chi_k(\mathfrak{p}\xi)}d\xi\Big)\\
&= \sum_{k\in\mathbb{N}_0}\Big(\int_{\mathfrak{p}^{-1}}\sum_{\alpha}\hat{f}(\xi+\mathfrak{p}^{-1}u(\alpha))\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+u(\alpha))}\chi_k(\mathfrak{p}\xi)d\xi\Big)\\
&\qquad\times\Big(\int_{\mathfrak{p}^{-1}}\sum_{\beta}\overline{\hat{g}(\xi+\mathfrak{p}^{-1}u(\beta))}\hat{\varphi}(\mathfrak{p}\xi+u(\beta))\overline{\chi_k(\mathfrak{p}\xi)}d\xi\Big)\\
&= \int_{\mathfrak{p}^{-1}}\sum_{\alpha}\sum_{\beta}\Big(\hat{f}(\xi+u(q\alpha))\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\xi+u(\alpha))}\\
&\qquad\times\overline{\hat{g}(\xi+u(q\beta))}\hat{\varphi}(\mathfrak{p}\xi+u(\beta))\Big)d\xi. \qquad (6.3.5)
\end{aligned}
$$

Since the right side of (6.3.4) and (6.3.5) are same, the proof is finished. □

Combining Lemma 6.2.6 and Proposition 6.3.2, we have the following proposition.

Proposition 6.3.3. *Let $\varphi, \tilde{\varphi}$ and $\psi_l, \tilde{\psi}_l$ for $1 \le l \le q-1$ be as in Lemma 6.3.1. Then for every $f \in L^2(K)$, we have*

$$
f = \sum_{l=1}^{q-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\langle f, \tilde{\psi}_{l,j,k}\rangle\psi_{l,j,k} = \sum_{l=1}^{q-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\langle f, \psi_{l,j,k}\rangle\tilde{\psi}_{l,j,k}, \qquad (6.3.6)
$$

where the series converge in $L^2(K)$.

We now prove the main result of this chapter.

Theorem 6.3.4. *Let φ and $\tilde{\varphi}$ be the scaling functions for dual MRAs and $\psi_l, \tilde{\psi}_l$, $1 \le l \le q-1$ be the associated wavelets satisfying the matrix condition (6.3.1). Then the collection $\{\psi_{l,j,k} : 1 \le l \le q-1, j \in \mathbb{Z}, k \in N_0\}$ and $\{\tilde{\psi}_{l,j,k} : 1 \le l \le q-1, j \in \mathbb{Z}, k \in N_0\}$ are biorthogonal. In addition, if*

$$
|\hat{\varphi}(\xi)| \le C(1+|\xi|)^{-\frac{1}{2}-\epsilon},\ |\hat{\tilde{\varphi}}(\xi)| \le C(1+|\xi|)^{-\frac{1}{2}-\epsilon},
$$

$$
|\hat{\psi}_l(\xi)| \le C|\xi| \quad \textit{and} \quad |\hat{\tilde{\psi}}_l(\xi)| \le C|\xi|,
$$

for some constant $C > 0$, $\epsilon > 0$ and for a.e. $\xi \in K$, then $\{\psi_{l,j,k} : 1 \le l \le q-1, j \in \mathbb{Z}, k \in N_0\}$ and $\{\tilde{\psi}_{l,j,k} : 1 \le l \le q-1, j \in \mathbb{Z}, k \in N_0\}$ form Riesz bases for $L^2(K)$.

Proof. We begin by proving that $\{\psi_{l,j,k} : 1 \le l \le q-1, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ and $\{\tilde{\psi}_{l,j,k} : 1 \le l \le q-1, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ are biorthogonal to each other. First we will show that, for

$l = 1, 2 \ldots, q-1$ and $j \in \mathbb{Z}$,

$$\langle \psi_{l,j,k}, \tilde{\psi}_{l,j,k'} \rangle = \delta_{k,k'}.$$

We have already proved it for $j = 0$ in Lemma 6.3.1(a). If $j \neq 0$, then

$$\begin{aligned} \langle \psi_{l,j,k}, \tilde{\psi}_{l,j,k'} \rangle &= \langle \delta_{-j}\psi_{l,0,k}, \delta_{-j}\tilde{\psi}_{l,0,k} \rangle \\ &= \langle \psi_{l,0,k}, \tilde{\psi}_{l,0,k} \rangle \\ &= \delta_{k,k'}. \end{aligned}$$

Let $k, k' \in \mathbb{N}_0$ be fixed and let $j, j' \in \mathbb{Z}$. Assume that $j < j'$. We will show that

$$\langle \psi_{l,j,k}, \tilde{\psi}_{l',j',k'} \rangle = 0.$$

It can be shown that $\psi_{l,0,k} \in V_1$. Hence, $\psi_{l,j,k} = \delta_{-j}\psi_{l,0,k} \in V_{j+1} \subseteq V_{j'}$. Therefore, it will be enough to show that $\tilde{\psi}_{l',j',k'}$ is orthogonal to every element of $V_{j'}$. Let $f \in V_{j'}$. By Lemma 6.2.1, $\{\varphi_{j',k} : k \in \mathbb{N}_0\}$ is a Riesz basis for $V_{j'}$. Hence, there exists an ℓ^2-sequence $\{c_k : k \in \mathbb{N}_0\}$ such that $f = \sum_{k \in \mathbb{N}_0} c_k \varphi_{j',k}$ in $L^2(K)$. By Lemma 6.3.1(b),

$$\langle \tilde{\psi}_{l',j',k'}, \varphi_{j',k} \rangle = \langle \delta_{-j'}\tilde{\psi}_{l',0,k'}, \delta_{-j'}\varphi_{0,k} \rangle = \langle \tilde{\psi}_{l',0,k'}, \varphi_{0,k} \rangle = 0.$$

Hence,

$$\langle \tilde{\psi}_{l',j',k'}, f \rangle = \langle \tilde{\psi}_{l',j',k'}, \sum_{k \in \mathbb{N}_0} c_k \varphi_{j',k} \rangle = \sum_{k \in \mathbb{N}_0} \overline{c_k} \langle \tilde{\psi}_{l',j',k'}, \varphi_{j',k} \rangle = 0.$$

In order to show that these two collections form Riesz bases for $L^2(K)$, we must verify that they are linearly independent and satisfy the frame condition. Since they are biorthogonal to each other, both the collections are linearly independent by Lemma 6.1.2.

To show the frame conditions, we must show that there exist constants A, B, $\tilde{A}$, and $\tilde{B} > 0$ such that for every $f \in L^2(K)$,

$$A\|f\|_2^2 \leq \sum_{l=1}^{q-1} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} |\langle f, \psi_{l,j,k} \rangle|^2 \leq B\|f\|_2^2, \tag{6.3.7}$$

and

$$\tilde{A}\|f\|_2^2 \le \sum_{l=1}^{q-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}|\langle f,\tilde{\psi}_{l,j,k}\rangle|^2 \le \tilde{B}\|f\|_2^2. \tag{6.3.8}$$

We first show the existence of upper bounds in (6.3.7) and (6.3.8). We have

$$\begin{aligned}
&\sum_{k\in\mathbb{N}_0}|\langle f,\psi_{l,j,k}\rangle|^2\\
&= \sum_{k\in\mathbb{N}_0}\Big|\int_K \hat{f}(\xi)q^{-j/2}\overline{\hat{\psi}_l(\mathfrak{p}^j\xi)}\chi_k(\mathfrak{p}^j\xi)\,d\xi\Big|^2\\
&= q^{-j}\sum_{k\in\mathbb{N}_0}\Big|\int_{\mathfrak{P}^{-j}}\sum_{m\in\mathbb{N}_0}\hat{f}(\xi+\mathfrak{p}^{-j}u(m))\overline{\hat{\psi}_l(\mathfrak{p}^j\xi+u(m))}\chi_k(\mathfrak{p}^j\xi)\,d\xi\Big|^2\\
&= \int_{\mathfrak{P}^{-j}}\Big|\sum_{m\in\mathbb{N}_0}\hat{f}(\xi+\mathfrak{p}^{-j}u(m))\overline{\hat{\psi}_l(\mathfrak{p}^j\xi+u(m))}\Big|^2\,d\xi\\
&\le \int_{\mathfrak{P}^{-j}}\Big(\sum_{m\in\mathbb{N}_0}|\hat{f}(\xi+\mathfrak{p}^{-j}u(m))|^2|\hat{\psi}_l(\mathfrak{p}^j\xi+u(m))|^{2\delta}\Big)\Big(\sum_{n\in\mathbb{N}_0}|\hat{\psi}_l(\mathfrak{p}^j\xi+u(n))|^{2(1-\delta)}\Big)\,d\xi\\
&= \int_K|\hat{f}(\xi)|^2|\hat{\psi}_l(\mathfrak{p}^j\xi)|^{2\delta}\sum_{n\in\mathbb{N}_0}|\hat{\psi}_l(\mathfrak{p}^j\xi+u(n))|^{2(1-\delta)}\,d\xi,
\end{aligned}$$

where δ is to be chosen suitably.

We have assumed that $|\hat{\varphi}(\xi)| \le C(1+|\xi|)^{-\frac{1}{2}-\epsilon}$. Hence, we have $|\hat{\psi}_l(\xi)| \le C(1+|\mathfrak{p}\xi|)^{-\frac{1}{2}-\epsilon}$. So $\sum_{n\in\mathbb{N}_0}|\hat{\psi}_l(\mathfrak{p}^j\xi+u(n))|^{2(1-\delta)}$ is uniformly bounded if $\delta < 2\epsilon(1+2\epsilon)^{-1}$. Hence, there exists $C>0$ such that

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}|\langle f,\psi_{l,j,k}\rangle|^2\\
&\le C\int_K|\hat{f}(\xi)|^2\sum_{j\in\mathbb{Z}}|\hat{\psi}_l(\mathfrak{p}^j\xi)|^{2\delta}\,d\xi\\
&\le C\sup\Big\{\sum_{j\in\mathbb{Z}}|\hat{\psi}_l(\mathfrak{p}^j\xi)|^{2\delta}:\xi\in\mathfrak{P}^{-1}\setminus\mathfrak{D}\Big\}\|f\|_2^2.
\end{aligned}$$

The last step follows because K is a disjoint union of $\mathfrak{P}^j$, $j\in\mathbb{Z}$, and the function $F(\xi)=\sum_{j\in\mathbb{Z}}|\hat{\psi}_l(\mathfrak{p}^j\xi)|^{2\delta}$ has the property that $F(\xi)=F(\mathfrak{p}\xi)$. Note that $\mathfrak{D}=\mathfrak{P}^0$. Since $\xi\in\mathfrak{P}^{-1}\setminus\mathfrak{D}$, we have $|\xi|=q$. Hence,

$$\sum_{j=-\infty}^{0}|\hat{\psi}_l(\mathfrak{p}^j\xi)|^{2\delta} \le \sum_{j=0}^{\infty}\frac{C^{2\delta}}{(1+|\mathfrak{p}^{-j+1}\xi|)^{\delta(1+2\epsilon)}}$$

$$= \sum_{j=0}^{\infty} \frac{C^{2\delta}}{(1+q^j)^{\delta(1+2\epsilon)}}$$
$$\leq \sum_{j=0}^{\infty} \frac{C^{2\delta}}{q^{j\delta(1+2\epsilon)}} = C^{2\delta} \frac{1}{1-q^{-\delta(1+2\epsilon)}}.$$

Also,

$$\sum_{j=1}^{\infty} |\hat{\psi}_l(\mathfrak{p}^j\xi)|^{2\delta} \leq \sum_{j=1}^{\infty} (Cq^{-j}|\xi|)^{2\delta}$$
$$= C^{2\delta} \sum_{j=1}^{\infty} q^{(-j+1)2\delta} = C^{2\delta} \frac{1}{1-q^{-2\delta}}.$$

These two estimates show that $\sup\{\sum_{j\in\mathbb{Z}} |\hat{\psi}_l(\mathfrak{p}^j\xi)|^{2\delta} : \xi \in \mathfrak{P}^{-1} \setminus \mathfrak{D}\}$ is finite. Hence, there exists $B > 0$ such that the second inequality in (6.3.7) holds. Similarly, we can show that the upper bound in (6.3.8) holds.

Using the existence of the upper bounds, we now show that the lower bounds in (6.3.7) and (6.3.8) also exist. It follows from Proposition 6.3.3 that, if $f \in L^2(K)$, then we have

$$f = \sum_{l=1}^{q-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} \langle f, \tilde{\psi}_{l,j,k}\rangle \psi_{l,j,k} = \sum_{l=1}^{q-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} \langle f, \psi_{l,j,k}\rangle \tilde{\psi}_{l,j,k}.$$

Therefore,

$$\|f\|_2^2 = \langle f, f\rangle$$
$$= \Big\langle \sum_{l=1}^{q-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} \langle f, \tilde{\psi}_{l,j,k}\rangle \psi_{l,j,k}, f \Big\rangle$$
$$= \sum_{l=1}^{q-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} \langle f, \tilde{\psi}_{l,j,k}\rangle \langle \psi_{l,j,k}, f\rangle$$
$$\leq \Big(\sum_{l=1}^{q-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} |\langle f, \tilde{\psi}_{l,j,k}\rangle|^2\Big)^{\frac{1}{2}} \Big(\sum_{l=1}^{q-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} |\langle f, \psi_{l,j,k}\rangle|^2\Big)^{\frac{1}{2}}$$
$$\leq (\tilde{B})^{\frac{1}{2}} \|f\|_2 \Big(\sum_{l=1}^{q-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} |\langle f, \psi_{l,j,k}\rangle|^2\Big)^{\frac{1}{2}}.$$

Hence,

$$\frac{1}{\tilde{B}}\|f\|_2^2 \leq \sum_{l=1}^{q-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}|\langle f,\psi_{l,j,k}\rangle|^2.$$

Similarly, we can show that

$$\frac{1}{B}\|f\|_2^2 \leq \sum_{l=1}^{q-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}|\langle f,\tilde{\psi}_{l,j,k}\rangle|^2.$$

This completes the proof of the theorem. □

Bibliography

[1] S. Albeverio, S. Evdokimov, and M. Skopina. p-adic nonorthogonal wavelet bases. *Proc. Steklov Inst. Math.*, 265(1):1–12, 2009.

[2] S. Albeverio, S. Evdokimov, and M. Skopina. p-adic multiresolution analysis and wavelet frames. *J. Fourier Anal. Appl.*, 16(5):693–714, 2010.

[3] S. Albeverio and S. V. Kozyrev. Multidimensional basis of p-adic wavelets and representation theory. *p-Adic Numbers Ultrametric Anal. Appl.*, 1(3):181–189, 2009.

[4] P. Auscher. Solution of two problems on wavelets. *J. Geom. Anal.*, 5(2):181–236, 1995.

[5] B. Behera. Multiwavelet packets and frame packets of $L^2(\mathbb{R}^d)$. *Proc. Indian Acad. Sci. Math. Sci.*, 111(4):439–463, 2001.

[6] B. Behera and Q. Jahan. Wavelet packets and wavelet frame packets on local fields of positive characteristic. *J. Math. Anal. Appl.*, 395(1):1–14, 2012.

[7] B. Behera and Q. Jahan. Multiresolution analysis on local fields and characterization of scaling functions. *Adv. Pure Appl. Math.*, 3(2):181–202, 2012.

[8] B. Behera and Q. Jahan. Biorthogonal wavelets on local fields of positive characteristic. *Commun. Math. Anal.*, 15(2):52–75, 2013.

[9] B. Behera and Q. Jahan. Characterization of wavelets and MRA wavelets on local fields of positive characteristic. *Collect. Math.*, to appear.

[10] B. Behera and Q. Jahan. Affine, quasi-affine and co-affine frames on local fields of positive characteristic, *preprint*, 2013.

[11] J. J. Benedetto and R. L. Benedetto. A wavelet theory for local fields and related groups. *J. Geom. Anal.*, 14(3):423–456, 2004.

[12] R. L. Benedetto. Examples of wavelets for local fields. In *Wavelets, frames and operator theory*, volume 345 of *Contemp. Math.*, pages 27–47. Amer. Math. Soc., Providence, RI, 2004.

[13] M. Bownik. A characterization of affine dual frames in $L^2(\mathbf{R}^n)$. *Appl. Comput. Harmon. Anal.*, 8(2):203–221, 2000.

[14] M. Bownik. The structure of shift-invariant subspaces of $L^2(\mathbf{R}^n)$. *J. Funct. Anal.*, 177(2):282–309, 2000.

[15] M. Bownik. On characterizations of multiwavelets in $L^2(\mathbb{R}^n)$. *Proc. Amer. Math. Soc.*, 129(11):3265–3274, 2001.

[16] M. Bownik and G. Garrigós. Biorthogonal wavelets, MRA's and shift-invariant spaces. *Studia Math.*, 160(3):231–248, 2004.

[17] A. Calogero. A characterization of wavelets on general lattices. *J. Geom. Anal.*, 10(4):597–622, 2000.

[18] A. Calogero and G. Garrigós. A characterization of wavelet families arising from biorthogonal MRA's of multiplicity d. *J. Geom. Anal.*, 11(2):187–217, 2001.

[19] D.-R. Chen. On the splitting trick and wavelet frame packets. *SIAM J. Math. Anal.*, 31(4):726–739 (electronic), 2000.

[20] C. K. Chui. *An introduction to wavelets*. Academic Press Inc., Boston, MA, 1992.

[21] C. K. Chui and C. Li. Nonorthogonal wavelet packets. *SIAM J. Math. Anal.*, 24(3):712–738, 1993.

[22] C. K. Chui, X. Shi, and J. Stöckler. Affine frames, quasi-affine frames, and their duals. *Adv. Comput. Math.*, 8(1-2):1–17, 1998.

[23] A. Cohen and I. Daubechies. A stability criterion for biorthogonal wavelet bases and their related subband coding scheme. *Duke Math. J.*, 68(2):313–335, 1992.

[24] A. Cohen and I. Daubechies. On the instability of arbitrary biorthogonal wavelet packets. *SIAM J. Math. Anal.*, 24(5):1340–1354, 1993.

[25] A. Cohen, I. Daubechies, and J.-C. Feauveau. Biorthogonal bases of compactly supported wavelets. *Comm. Pure Appl. Math.*, 45(5):485–560, 1992.

[26] R. R. Coifman and Y. Meyer. Orthogonal wave packet bases. 1989.

[27] R. R. Coifman, Y. Meyer, and V. Wickerhauser. Size properties of wavelet-packets. In *Wavelets and their applications*, pages 453–470. Jones and Bartlett, Boston, MA, 1992.

[28] R. R. Coifman, Y. Meyer, and V. Wickerhauser. Wavelet analysis and signal processing. In *Wavelets and their applications*, pages 153–178. Jones and Bartlett, Boston, MA, 1992.

[29] S. Dahlke. Multiresolution analysis and wavelets on locally compact abelian groups. In *Wavelets, images, and surface fitting (Chamonix-Mont-Blanc, 1993)*, pages 141–156. A K Peters, Wellesley, MA, 1994.

[30] I. Daubechies. *Ten lectures on wavelets*, volume 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[31] S. A. Evdokimov and M. Skopina. 2-adic wavelet bases. *Proc. Steklov Inst. Math.*, 266(1):S143–S154, 2009.

[32] Yu. A. Farkov. Orthogonal wavelets on locally compact abelian groups. *Funktsional. Anal. i Prilozhen.*, 31(4):86–88, 1997.

[33] Yu. A. Farkov. Multiresolution analysis and wavelets on vilenkin groups. *Facta Universitatis (NIS), Ser.: Elec. Energ.*, 21:309–325, 2008.

[34] Yu. A. Farkov. Biorthogonal wavelets on Vilenkin groups. *Proc. Steklov Inst. Math.*, 265(1):101–114, 2009.

[35] Yu. A. Farkov and E. A. Rodionov. Algorithms for wavelet construction on Vilenkin groups. *p-Adic Numbers Ultrametric Anal. Appl.*, 3(3):181–195, 2011.

[36] P. Gressman, D. Labate, G. Weiss, and E. Wilson. Affine, quasi-affine and co-affine wavelets. In *Beyond wavelets*, volume 10 of *Stud. Comput. Math.*, pages 215–223. Academic Press/Elsevier, San Diego, CA, 2003.

[37] G. Gripenberg. A necessary and sufficient condition for the existence of a father wavelet. *Studia Math.*, 114(3):207–226, 1995.

[38] Y.-H. Ha, H. Kang, J. Lee, and J. K. Seo. Unimodular wavelets for L^2 and the Hardy space H^2. *Michigan Math. J.*, 41(2):345–361, 1994.

[39] D. Han, D. R. Larson, M. Papadakis, and T. Stavropoulos. Multiresolution analyses of abstract Hilbert spaces and wandering subspaces. In *The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999)*, volume 247 of *Contemp. Math.*, pages 259–284. Amer. Math. Soc., Providence, RI, 1999.

[40] E. Hernández, D. Labate, and G. Weiss. A unified characterization of reproducing systems generated by a finite family. II. *J. Geom. Anal.*, 12(4):615–662, 2002.

[41] E. Hernández, D. Labate, G. Weiss, and E. Wilson. Oversampling, quasi-affine frames, and wave packets. *Appl. Comput. Harmon. Anal.*, 16(2):111–147, 2004.

[42] E. Hernández and G. Weiss. *A first course on wavelets*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1996. With a foreword by Yves Meyer.

[43] M. Holschneider. Wavelet analysis over abelian groups. *Appl. Comput. Harmon. Anal.*, 2(1):52–60, 1995.

[44] H. Jiang, D. Li, and N. Jin. Multiresolution analysis on local fields. *J. Math. Anal. Appl.*, 294(2):523–532, 2004.

[45] B. Johnson. Co-affine systems in $\mathbb{R}^d$. In *Wavelets, frames and operator theory*, volume 345 of *Contemp. Math.*, pages 193–202. Amer. Math. Soc., Providence, RI, 2004.

[46] A. Yu. Khrennikov and S. V. Kozyrev. Wavelets on ultrametric spaces. *Appl. Comput. Harmon. Anal.*, 19(1):61–76, 2005.

[47] A. Yu. Khrennikov and V. M. Shelkovich. An infinite family of p-adic non-Haar wavelet bases and pseudo-differential operators. *p-Adic Numbers Ultrametric Anal. Appl.*, 1(3):204–216, 2009.

[48] A. Yu. Khrennikov and V. M. Shelkovich. Non-Haar p-adic wavelets and their application to pseudo-differential operators and equations. *Appl. Comput. Harmon. Anal.*, 28(1):1–23, 2010.

[49] A. Yu. Khrennikov, V. M. Shelkovich, and M. Skopina. p-adic orthogonal wavelet bases. *p-Adic Numbers Ultrametric Anal. Appl.*, 1(2):145–156, 2009.

[50] A. Yu. Khrennikov, V. M. Shelkovich, and M. Skopina. p-adic refinable functions and MRA-based wavelets. *J. Approx. Theory*, 161(1):226–238, 2009.

[51] H.-O. Kim, R.-Y. Kim, and J.-K. Lim. Characterizations of biorthogonal wavelets which are associated with biorthogonal multiresolution analyses. *Appl. Comput. Harmon. Anal.*, 11(2):263–272, 2001.

[52] S. V. Kozyrev. Wavelet theory as p-adic spectral analysis. *Izv. Math.*, 66(2):367–376, 2002.

[53] W. C. Lang. Orthogonal wavelets on the Cantor dyadic group. *SIAM J. Math. Anal.*, 27(1):305–312, 1996.

[54] W. C. Lang. Fractal multiwavelets related to the Cantor dyadic group. *Internat. J. Math. Math. Sci.*, 21(2):307–314, 1998.

[55] W. C. Lang. Wavelet analysis on the Cantor dyadic group. *Houston J. Math.*, 24(3):533–544, 1998.

[56] P. G. Lemarié. Base d'ondelettes sur les groupes de Lie stratifiés. *Bull. Soc. Math. France*, 117(2):211–232, 1989.

[57] P. G. Lemarié. Projecteurs invariants, matrices de dilatation, ondelettes et analyses multirésolutions. *Rev. Mat. Iberoamericana*, 10(2):283–347, 1994.

[58] P. G. Lemarié and Y. Meyer. Ondelettes et bases hilbertiennes. *Rev. Mat. Iberoamericana*, 2(1-2):1–18, 1986.

[59] D. Li and H. Jiang. The necessary condition and sufficient conditions for wavelet frame on local fields. *J. Math. Anal. Appl.*, 345(1):500–510, 2008.

[60] R. Long and D. Chen. Biorthogonal wavelet bases on $\mathbf{R}^d$. *Appl. Comput. Harmon. Anal.*, 2(3):230–242, 1995.

[61] R. Long and W. Chen. Wavelet basis packets and wavelet frame packets. *J. Fourier Anal. Appl.*, 3(3):239–256, 1997.

[62] W. R. Madych. Some elementary properties of multiresolution analyses of $L^2(\mathbf{R}^n)$. In *Wavelets*, volume 2 of *Wavelet Anal. Appl.*, pages 259–294. Academic Press, Boston, MA, 1992.

[63] S. G. Mallat. Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbf{R})$. *Trans. Amer. Math. Soc.*, 315(1):69–87, 1989.

[64] Y. Meyer. Principe d'incertitude, bases hilbertiennes et algèbres d'opérateurs. *Séminaire Bourbaki*, tome 28:209–223, 1985–86.

[65] Y. Meyer. *Wavelets and operators*. Cambridge University Press, Cambridge, 1992.

[66] D. Ramakrishnan and R. J. Valenza. *Fourier analysis on number fields*. Springer-Verlag, New York, 1999.

[67] A. Ron and Z. Shen. Frames and stable bases for shift-invariant subspaces of $L_2(\mathbf{R}^d)$. *Canad. J. Math.*, 47(5):1051–1094, 1995.

[68] A. Ron and Z. Shen. Affine systems in $L_2(\mathbf{R}^d)$: the analysis of the analysis operator. *J. Funct. Anal.*, 148(2):408–447, 1997.

[69] Z. Shen. Nontensor product wavelet packets in $L_2(\mathbf{R}^s)$. *SIAM J. Math. Anal.*, 26(4):1061–1074, 1995.

[70] T. Stavropoulos and M. Papadakis. On the multiresolution analyses of abstract Hilbert spaces. *Bull. Greek Math. Soc.*, 40:79–92, 1998.

[71] J. O Strömberg. A modified Franklin system and higher-order spline systems on $\mathbf{R}^n$ as unconditional bases for Hardy spaces. In *Conference on harmonic analysis in honor of*

Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981), Wadsworth Math. Ser., pages 475–494. Wadsworth, Belmont, CA, 1983.

[72] M. H. Taibleson. *Fourier analysis on local fields*. Princeton University Press, Princeton, N.J., 1975.

[73] D. F. Walnut. *An introduction to wavelet analysis*. Birkhäuser Boston Inc., Boston, MA, 2002.

[74] X. Wang. *The study of wavelets from the properties of their Fourier transforms*. Ph.D. Thesis, Washington University in St. Louis, 1995.

[75] P. Wojtaszczyk. *A mathematical introduction to wavelets*. Cambridge University Press, Cambridge, 1997.

[76] S. Zheng. Riesz type kernels over the ring of integers of a local field. *J. Math. Anal. Appl.*, 208(2):528–552, 1997.