ON TESTING THE MEAN OF A DISCRETE LINEAR PROCESS

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SUMMARY. In this paper a method of testing the mean function of a discrete linear stationary process is given when the spectral density function is known. The asymptotic distribution of the proposed test statistic is derived.

1. Introduction

The problem of testing the mean function of a linear process has been solved by Grenander and Rosenblatt (1951) in the case when the process is Gaussian and a simple hypothesis is being tested against a simple alternative. In this paper we consider the problem of testing the null hypothesis that the mean function of an arbitrary linear process is zero against all alternatives. The asymptotic distribution of the proposed test statistic is derived. The test is shown to be consistent against certain alternatives in the class of Gaussian processes.

2. The test

Let $[x_1, ..., x_N]$ be observations on a discrete linear stochastic process $[x_t]$ which is of the form

$$x_t = y_t + m_t$$

$$y_t = \sum_{\nu=-\infty}^{+\infty} a_{t-\nu}\, \xi_\nu \qquad \dots \quad (2.1)$$

$$E|\xi_\nu|^4 < \infty, \quad \sum_{-\infty}^{+\infty} |a_t|^2 < \infty$$

where ξ_ν are independent and identically distributed random variables, each with mean zero and variance one. Let

$$\Sigma = ((\rho_{i-j})) \quad \begin{matrix} 1 \leqslant i \leqslant N \\ 1 \leqslant j \leqslant N \end{matrix}$$

be the dispersion matrix of $[x_1, ..., x_N]$. The function

$$f(\lambda) = \frac{1}{2\pi} \left| \sum_{-\infty}^{+\infty} a_t\, e^{it\lambda} \right|^2 \qquad \dots \quad (2.2)$$

is the spectral density of the process. The problem is to test the null hypothesis

$$H_0 : m_t = 0 (t = \dots -1, 0, 1, \dots). \qquad \dots \quad (2.3)$$

If the process is Gaussian the likelihood ratio test will lead to the statistic $x\Sigma^{-1}x'$, where $x = [x_1 \, x_2 \, ..., \, x_N]$ and large values of the statistic are significant. The test based on this can be shown to be consistent against all alternatives for which

$$\lim_{N \to \infty} \frac{[m_1 \, ... \, m_N] \, \Sigma^{-1} \, [m_1 \, ... \, m_N]}{\sqrt{N}} = \infty \qquad ... \quad (2.4)$$

But this statistic, even though its exact distribution is known, is not very suitable, since for large N it is difficult to compute Σ^{-1}. We suggest here a more convenient alternative procedure which also is consistent against alternatives of the type just considered.

Consider the expression

$$X = \frac{\sum_{t=1}^{N} b_t x_t}{\sqrt{\sum_s \sum_r b_r b_s \rho_{r-s}}} \qquad ... \quad (2.5)$$

where $b_1, b_2, ..., b_N$ are arbitrary constants. If all the x_t's are of expectation zero we expect $|X_b|$ to be small. If we maximise $|X_b|^2$ over all $b_1, ... b_N$ we get $x \, \Sigma^{-1} \, x'$. Instead we rewrite (2.5) as

$$|X_b|^2 = \frac{\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\lambda) \frac{\sum_{r=1}^{N} x_r \, e^{-ir\lambda}}{f(\lambda)} f(\lambda) d\lambda \right|^2}{\int_{-\pi}^{\pi} |h(\lambda)|^2 f(\lambda) d\lambda} \qquad ... \quad (2.6)$$

where

$$h(\lambda) = \sum_{r=1}^{N} b_r e^{ir\lambda}. \qquad ... \quad (2.7)$$

Now we maximise (2.6) over all functions $h(\lambda)$, square integrable with respect to $f(\lambda)$, instead of over $h(\lambda)$ of the type (2.7). Assuming that $1/f(\lambda)$ is integrable over $(-\pi, \pi)$, we obtain thus the statistic

$$Y_N = \max_{h(\lambda)} |X_b|^2 = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \frac{\left| \sum_{1}^{N} x_r \, e^{ir\lambda} \right|^2}{f(\lambda)} \, d\lambda \qquad ... \quad (2.8)$$

by an application of Schwartz's inequality.

The asymptotic null distribution of Y_N is given by the following theorem.

Theorem 1 : *Suppose that* $a_t = 0(t^{-\beta})$, $\beta > 3/2$ *and* $\int_{-\pi}^{\pi} [f(\lambda)]^{-1} \, e^{ir\lambda} \, d\lambda = 0(r^{-1})$. *Then, under the null hypothesis, the distribution of* $\sqrt{N}[(Y_N/_N) - 1]$ *converges to the normal distribution with mean zero and variance* σ^2 *where* $\sigma^2 = E\xi_t^4 - 1$.

Proof: Let

$$I_N = (1/2\pi N) \left| \sum_1^N x_t e^{it\lambda} \right|^2 \quad \dots \quad (2.9)$$

$$I_{N,\xi} = (1/2\pi N) \left| \sum_1^N \xi_t \; e^{it\lambda} \right|^2$$

Then, by (2.1), if $m_t = 0$ for all t, we have

$$2\pi\sqrt{N} \int_{-\pi}^{\pi} [I_N(\lambda) - 2\pi f(\lambda)\, I_{N,\xi}(\lambda)]\, [f(\lambda)]^{-1}\, d\lambda = N^{-\frac{1}{2}} \sum_{s=-\infty}^{+\infty} \sum_{r=-\infty}^{+\infty} a_r a_s d_{rs} \dots \quad (2.10)$$

where
$$d_{rs} = \sum_{m=1}^{N} \sum_{n=1}^{N} \xi_{n-r}\, \xi_{m-s}\, \rho^{(n-m)} - \sum_{m=1+s}^{N+s} \sum_{n=1+r}^{N+r} \xi_{n-r}\, \xi_{m-s}\, \rho^{(n-m)} \quad \dots \quad (2.11)$$

and
$$\rho^{(m-n)} = \int_{-\pi}^{+\pi} e^{i(n-m)\lambda} [f(\lambda)]^{-1}\, d\lambda. \quad \dots \quad (2.12)$$

There may be a lattice rectangle of points $R_{rs}^{(N)}$ of points (n, m) common to the sums involved in (2.11). The expressions corresponding to those points will cancel. Let $C_{rs}^{(N)}$ be the complement of $R_{rs}^{(N)}$ with respect to the set consisting of all the lattice points in both the summations. We then have

$$E|d_{rs}| \leq \sum_{(n,m)\in C_{rs}^{(N)}} [g(n-m)] \quad \dots \quad (2.13)$$

where
$$g(k) = |\rho^{(k)}|. \quad \dots \quad (2.14)$$

Since $\rho^{(k)} = 0(k^{-1})$ and $a_t = 0(t^{-\beta}), \beta > 3/2$, by the analysis carried out by Grenander and Rosenblatt (1951, chapter 6, pp. 191-192) we have

$$\sqrt{N} \int_{-\pi}^{\pi} [I_N(\lambda) - 2\pi f(\lambda)\, I_{N,\xi}(\lambda)]\, [f(\lambda)]^{-1} d\lambda \to 0$$

in probability as $N \to \infty$. We have

$$\sqrt{N}\, [\int_{-\pi}^{+\pi} I_{N,\xi}(\lambda) d\lambda - 1] = \sqrt{N}\, [N^{-1} \sum_{t=1}^{N} \xi_t^2 - 1]. \quad \dots \quad (2.15)$$

An application of the central limit theorem shows that the limit distribution of (2.15) is normal with mean zero and variance

$$\sigma^2 = E\xi_t^4 - 1.$$

This completes the proof.

3. Consistency of the test when the ξ_t are normally distributed

The proposed test of the null hypothesis H_0 is the following: Reject H_0 if and only if

$$\sigma^{-1}\sqrt{N}[(Y_{N/N})-1] > K_\alpha \qquad \ldots \quad (3.1)$$

where K_α is the upper α percent point of the normal distribution with mean zero and variance unity.

Theorem 2 : *If ξ_t are normally distributed the test given by the critical region*

$$(N/2)^{\frac{1}{2}}\,[(Y_{N/N})-1] > K_\alpha,$$

where K_α is the α percent point of the normal distribution with mean zero and variance unity, is consistent against all alternatives $\{m_t\}$ $(t = \cdots -1, 0, 1, 2 \cdots)$ which satisfy the condition (2.4).

Proof: Since $x\Sigma^{-1}x' \leqslant Y_N$, we have

$$P[\text{rejecting } H_0 | \{m_t\}] = P[\sqrt{N/2}[(Y_{N/N})-1] > K_\alpha | \{m_t\}]$$

$$\geqslant P\left[\sqrt{N/2}\left[\left(\frac{x\Sigma^{-1}x'}{N}\right)-1\right] > K_\alpha|\{m_t\}\right]. \qquad \ldots \quad (3.3)$$

But $$x\,\Sigma^{-1}\,x' = (x-m)\,\Sigma^{-1}\,(x-m)' + 2m\,\Sigma^{-1}\,(x-m)' + m\,\Sigma^{-1}\,m'$$

where $$m = [m_1, m_2, \ldots m_N].$$

Since the limiting distributions of

$$\sqrt{N/2}\left[\frac{(x-m)\,\Sigma^{-1}(x-m)'}{N} - 1\right]$$

and $$\frac{m\,\Sigma^{-1}\,(x-m)'}{(m\,\Sigma^{-1}\,m')^{\frac{1}{2}}}$$

exist as $N \to \infty$, and $$\lim_{N\to\infty} \frac{m\,\Sigma^{-1}\,m'}{\sqrt{N}} = \infty,$$

it is easy to see that, in (3.3), the left side of the inequality within square brackets tends to infinity in probability. Thus the limit of (3.3) as $N \to \infty$ is unity. This completes the proof of Theorem 2.

Reference

Grenander, U. and Rosenblatt, M. (1951): *Statistical Analysis of Stationary Time Series.* John Wiley & Sons, New York.

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